Continuous dependence and general decay of solutions for a wave equation with a nonlinear memory term

Doan Thi Nhu Quynh[1,2,3,a], Nguyen Huu Nhan[4,b],

Le Thi Phuong Ngoc[5,c], Nguyen Thanh Long[1,2,d,*]

[1]Faculty of Mathematics and Computer Science,
University of Science, Ho Chi Minh City, Vietnam,
227 Nguyen Van Cu Str., Dist. 5, Ho Chi Minh City, Vietnam.
[2]Vietnam National University, Ho Chi Minh City, Vietnam.
[d]E-mail: longnt2@gmail.com
[3]Faculty of Fundamental sciences, Ho Chi Minh City University of Food Industry,
140 Le Trong Tan Str., Tay Thanh Ward, Tan Phu Dist., Ho Chi Minh City,
[a]E-mail: doanthinhuquynh02@gmail.com
[4]Nguyen Tat Thanh University,
300A Nguyen Tat Thanh Str., Dist. 4, Ho Chi Minh City, Viet Nam.
[b]E-mail: nhnhan@ntt.edu.vn
[5]University of Khanh Hoa, 01 Nguyen Chanh Str., Nha Trang City, Vietnam.
[c]E-mail: ngoc1966@gmail.com

Abstract. *This paper is devoted to the study of existence, uniqueness, continuous dependence, general decay of solutions of an initial boundary value problem for a viscoelastic wave equation with strong damping and nonlinear memory term. At first, we state and prove a theorem involving local existence and uniqueness of a weak solution. Next, we establish a sufficient condition to get an estimate of the continuous dependence of the solution with respect to the kernel function and the nonlinear terms. Finally, under suitable conditions to obtain the global solution, we prove the general decay property with positive initial energy for this global solution.*

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(*)**Corresponding author**: *Nguyen Thanh Long.*

1 Introduction

In this paper, we study the following Dirichlet problem for a wave equation with strong damping and nonlinear memory

$$\begin{cases} u_{tt} - \lambda u_{txx} - \dfrac{\partial^2}{\partial x^2}\left(\mu(x,t,u(x,t))\right) + \displaystyle\int_0^t g(t-s)\frac{\partial^2}{\partial x^2}\left(\bar{\mu}(x,s,u(x,s))\right)ds \\ \qquad\qquad\qquad = f(x,t,u,u_t,u_x,u_{tx}),\ 0<x<1,\ 0<t<T, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.1}$$

where $\lambda > 0$ is given constant and $f,\ g,\ \mu,\ \bar{\mu},\ \tilde{u}_0,\ \tilde{u}_1$ are given functions.

Prob. (1.1) is a type of viscoelastic problems, the Volterra integral in the first equation of (1.1) is a memory term, so called viscoelastic term, responsible for viscoelastic damping. The wave equations with memory terms are arised in studies about viscoelastic materials, which possess a capacity of storage and dissipation of mechanical energy. The dynamic properties of viscoelastic materials are great importance and interest as they appear in many applications to natural sciences, for literatures on this topic, we can find in [9]-[12] and references therein.

The viscoelastic problem of the form (1.1) has been studied by many authors, for example, we refer to [3], [17], [18], [22]-[24], [26] - [30], [32], [34]. By using different methods together with various techniques in functional analysis, several results concerning the existence/global existence and the properties of solutions of viscoelastic problems such as blow-up, decay, stability have been established.

For more details, there have been a lot of investigations dedicated to the following viscoelastic wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds - \lambda \Delta u_t + \gamma h\left(u_t\right) = \mathcal{F}(x,t,u). \tag{1.2}$$

In general, the most common forms of the nonlinear damping h and the source $\mathcal{F}$ in Eq. (1.2) are exponential types, especially $h = \left|u_t\right|^{m-2} u_t$ and $\mathcal{F} = |u|^{p-2} u$. In [3], Cavalcanti et al. proved that, as $\lambda = 0,\ \gamma = 0,\ \mathcal{F} = 0$ and together with nonlinear boundary damping, the energy of solutions of the corresponding problem went uniformly to zero at infinity. In [29], Messaoudi considered Eq. (1.2) with $\lambda = 0$, $\gamma = 0$, $\mathcal{F} = |u|^{p-2} u$, and showed that, for certain class of relaxation functions and certain initial data, the solution energy decayed at a similar rate of decay of the relaxation function, which was not necessarily decaying in a polynomial or exponential fashion. In [28], Messaoudi studied Eq. (1.2) in case of $\lambda = 0$, $h = a\left|u_t\right|^{m-2} u_t$, $\mathcal{F} = b\,|u|^{p-2} u$, and proved a blow-up result for solutions with negative initial energy if $p > m$ and a global existence result for $p \leq m$. Latterly, Kafini and Messaoudi [22] also obtained a blow-up result of a Cauchy problem for a nonlinear viscoelastic equation in the form (1.2) with $m = 2$. In [27], Mesloub and Boulaaras studied a viscoelastic equation for more general decaying kernels and established some general decay results, from which the usual exponential and polynomial rates are only special cases. In the presence of the strong damping $-\Delta u_t$ and the linear damping u_t ($m = 2$), Li and He [24] proved the global existence of solutions and established a general decay rate estimate for the corresponding problem given by

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds - \Delta u_t + u_t = u\,|u|^{p-2}. \tag{1.3}$$

On the other hand, the finite-time blow-up results of solutions with both negative initial energy and positive initial energy were also obtained. In [23], with addition the dispersion $-\Delta u_{tt}$, Kafini and Mustafa also investigated Eq. (1.3) on whole space $\mathbb{R}^n$, and the authors proved a blow up result by imposing conditions on the kernel g. For more results related to Eq. (1.2) and Eq. (1.3) such as general decay or blow up in finite time, we can see in [17], [18], [30], [34].

In [26], Long et al. studied a specific form of Eq. (1.2) with $\lambda = 0$, $\gamma = 1$, $h = \left|u_t\right|^{m-2} u_t$, i.e., the authors considered the following viscoelastic equation

$$u_{tt} - u_{xx} + \int_0^t g(t-s)u_{xx}(x,s)ds + \alpha\left|u_t\right|^{p-2} u_t = \mathcal{F}(x,t,u), \tag{1.4}$$

associated with mixed nonhomogeneous conditions. Under a certain local Lipschitzian condition on the source $\mathcal{F}$ and certain class of relaxation functions and suitable initial datum, a global existence was proved and an asymptotic behavior of solutions as $t \to \infty$ was studied. Recently, Quynh et al. [34] has considered Eq. (1.4), in which, an N-order recurrent sequence has been established and its convergence to the unique solution of (1.4) satisfying an estimation of convergent rate in N order has been proved. Furthermore, by using finite-difference approximation, the authors constructed an algorithm to find numerical solutions via the 2-order iterative scheme (as $N = 2$).

However, to the best of our knowledge, there are relatively few works devoted to the study of partial differential equations with nonlinear memory, for example, we can see [6], [7], [20], [21], [32], [35]. In the paper published in 1985 [20], Hrusa considered a one-dimensional nonlinear viscoelastic equation of the form

$$u_{tt} - cu_{xx} + \int_0^t g(t-s)\left(\Psi\left(u_x(x,s)\right)\right)_x ds = f(x,t), \tag{1.5}$$

the author established several global existence results for large data and proved an exponential decay result for strong solutions when $g(s) = e^{-s}$ and Ψ satisfies some conditions. In [35], Shang and Guo proved the existence, uniqueness, and regularity of the global strong solution and gave some conditions of the nonexistence of global solution to the one-dimension pseudoparabolic equation with the nonlinear memory term $\int_0^t g(t-s)\left(\sigma\left(u(x,s), u_x(x,s)\right)\right)_x ds$. In [32], Ngoc et al. proved the local existence of the wave equation with strong damping and nonlinear viscoelastic term as follows

$$\begin{aligned} &u_{tt} - \lambda u_{xxt} - \frac{\partial}{\partial x}\left[\mu_1\left(x,t,u(x,t),\|u(t)\|^2,\|u_x(t)\|^2\right)u_x\right] \\ &\quad + \int_0^t g(t-s)\frac{\partial}{\partial x}\left[\mu_2\left(x,s,u(x,s),\|u(s)\|^2,\|u_x(s)\|^2\right)u_x(x,s)\right]ds \\ &\quad = F\left(x,t,u,u_x,u_t,\|u(t)\|^2,\|u_x(t)\|^2\right),\ 0<x<1,\ 0<t<T, \end{aligned} \tag{1.6}$$

associated with Robin-Dirichlet boundary conditions and initial conditions, where $\lambda > 0$ is a constant, and μ_1, μ_2, g, f are given functions which satisfy some certain conditions. Moreover, the authors established an asymptotic expansion of solutions, i.e., the solutions of (1.6) can be approximated by a $N-$order polynomial in small parameter. Recently, Kaddour and Reissig [21] have proved the global (in time) well-posedness results for Sobolev solutions to the following Cauchy problem for a damped wave equation with nonlinear memory on the right-hand side

$$\begin{cases} u_{tt} - \Delta u + (1+t)^r u_t = \int_0^t (t-\tau)^{-\gamma}\left|u(\tau,x)\right|^p d\tau,\ (t,x)\in(0,\infty)\times\mathbb{R}^n, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ x\in\mathbb{R}^n, \end{cases} \tag{1.7}$$

where $r \in (-1,1)$ and $\gamma \in (0,1)$. Moreover, for another investigation of (1.7) given in [21], they also have proved a blow-up result for local (in time) Sobolev solutions.

On the other hand, it seems that there are no results relating to continuous dependence and general decay of solutions of initial boundary value problems with nonlinear memory term. The topic of continuous dependence on datum has received important attention since 1960, with the earlier works of Douglis [8] and Fritz [12]. After that, P. Benilan and M.G. Crandall [1] discussed the continuous dependence on the nonlinearities of solutions of the Cauchy problem for the equation

$$\begin{cases} u_t - \Delta\phi(u) = 0, \text{ in } \mathbb{R}^n\times\mathbb{R}^+, \\ u(x,0) = u_0(x),\ x\in\mathbb{R}^n. \end{cases} \tag{1.8}$$

The authors defined the continuous dependence of solutions in sense (see [1], p. 162)

$$\|u_n(t) - u_\infty(t)\|_{L^1(\mathbb{R}^n)} \to 0, \text{ as } \phi_n \to \phi_\infty,$$

where $\phi_n : \mathbb{R}\to\mathbb{R}$ are continuous and nondecreasing functions, $\phi_n(0) = 0$, and u_n are solutions of the Cauchy problem (1.8). In [33], Pan proved the following estimation which showed the continuous dependence of solutions for the parabolic equation with exponential nonlinearity

$$\int_0^\infty \int_0^1 \left|u(x,t,m) - u(x,t,m_0)\right| \le C^* \left|m - m_0\right|,$$

where u is solution of the proposed problem, $0 < m,\ m_0 \leq 1$ and C^* is a explicit constant. Recently, Bayraktar and Gür [14] have studied the continuous dependence of solutions on dispersive δ and r and dissipative b coefficients of the damped improved Boussinesq equation

$$u_{tt} - b\Delta u - \delta\Delta u_{tt} - r\Delta u_t = \Delta(-u\,|u|^{p-2}),$$

in which the effects of small perturbations of parameters on solutions have been obtained. For similar results, we refer to [4], [13].

Motivated by the above-mentioned inspiring works, in this paper, we consider Prob. (1.1) and we first prove existence, uniqueness of solutions for this problem (Theorem 3.5) by applying the linearization method together with Faedo-Galerkin method and the weak compact method. Next, we consider the continuous dependence of solutions on the nonlinearities of Prob. (1.1). Precisely, if $u = u(\mu,\ \bar{\mu},\ f,\ g)$ and $u_j = u(\mu_j, \bar{\mu}_j, f_j, g_j)$ are the solutions of Prob. (1.1) respectively depending on the datum $(\mu,\ \bar{\mu},\ f,\ g)$ and $(\mu_j, \bar{\mu}_j, f_j, g_j)$, such that

$$\begin{cases} \sup\limits_{M>0} \max\limits_{|\beta|\leq 3} \left\| D^\beta \mu_j - D^\beta \mu \right\|_{C^0(A_M)} \to 0, \text{ as } j \to \infty, \\ \sup\limits_{M>0} \max\limits_{|\beta|\leq 3} \left\| D^\beta \bar{\mu}_j - D^\beta \bar{\mu} \right\|_{C^0(A_M)} \to 0, \text{ as } j \to \infty, \\ \sup\limits_{M>0} \max\limits_{|\alpha|\leq 1} \left\| D^\alpha f_j - D^\alpha f \right\|_{C^0(\tilde{A}_M)} \to 0, \text{ as } j \to \infty, \\ \left\| g_j - g \right\|_{H^1(0,T^*)} \to 0, \text{ as } j \to \infty, \end{cases} \tag{1.9}$$

where T^* is fixed positive constant; $A_M,\ \tilde{A}_M$ are compact sets depending on a positive constant M; $D^\alpha f$ are partial derivatives with order less than or equal $|\alpha|$, then u_j converges to u in $W_1(T)$, as $j \to \infty$ (Theorem 4.1).

Finally, we consider a specific case of Prob. (1.1) with $\mu = \mu(t,u),\ \bar{\mu} = u,\ f = -\lambda_1 u_t + f(u) - \frac{1}{2} D_2^2 \mu(t,u) u_x^2 + F(x,t)$, and we prove the general decay of solutions of Prob. (1.1) in this case (Theorem 5.6). It is well known that, in order to assure the general decay of solutions, the essential assumption for the relaxation function g usually satisfies a relation of the form

$$g'(t) \leq -\xi(t) g(t), \tag{1.10}$$

where ξ is a differentiable nonincreasing positive function, see [10], [17], [31]. Recently, the condition (1.10) have been relaxed by Mesloub and Boulaaras [27], Boumaza and Boulaaras [2], Conti and Pata [5], in which the kernel g haven't been necessarily decreasing. In the present paper, the relaxation function g also satisfies (1.10), however, it is necessary to set some assumptions for the nonlinear quantity μ, we shall give an example in which μ satisfies a relatively wide class of C^3-functions.

We note more that the decay property is a form of asymptotic behavior/stability in which the energy of solutions tends to zero at infinity. For topic on asymptotic behavior of solutions, there have been many interesting results for models related to (1.1) with memory term, for example, we refer to [16], [19], [24] and the references therein.

The paper consits of five sections. In Section 2, we present some preliminaries. In Section 3, we state and prove the theorem of existence and uniqueness of Prob. (1.1). Sections 4 and 5 are devoted to the continuous dependence and the general decay of solutions of Prob. (1.1). The results obtained here may be considered as relative generalizations of those in [28] -[31], [34].

2 Preliminaries

In this section, we present some notations and materials in order to present main results. Let $\Omega = (0,1)$, $Q_T = (0,1) \times (0,T)$ and we define the scalar product in L^2 by

$$\langle u, v \rangle = \int_0^1 u(x)v(x)dx,$$

and the corresponding norm $\|\cdot\|$, i.e., $\|u\|^2 = \langle u, u\rangle$. Let us denote the standard function spaces by $C^m(\overline{\Omega})$, $L^P = L^p(\Omega)$ and $H^m = H^m(\Omega)$ for $1 \le p \le \infty$ and $m \in \mathbb{N}$. Also, we denote that $\|\cdot\|_X$ is a norm in a Banach space X, and $L^p(0,T;X)$, $1 \le p \le \infty$, is the Banach space of real functions $u : (0,T) \to X$ measurable with the corresponding norm $\|\cdot\|_{L^p(0,T;X)}$ defined by

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p} < \infty \text{ for } 1 \le p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \underset{0<t<T}{ess\sup} \|u(t)\|_X \ \text{ for } p = \infty.$$

On H^1, we use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}. \tag{2.1}$$

The following lemma is known.

Lemma 2.1. *The imbeddings $H^1 \hookrightarrow C^0(\overline{\Omega})$ and $H_0^1 \hookrightarrow C^0(\overline{\Omega})$ are compact and*

$$\begin{aligned} &\text{(i) } \|v\|_{C^0(\overline{\Omega})} \le \sqrt{2}\,\|v\|_{H^1} \ \textit{ for all } v \in H^1, \\ &\text{(ii) } \|v\|_{C^0(\overline{\Omega})} \le \|v_x\| \ \textit{ for all } v \in H_0^1. \end{aligned} \tag{2.2}$$

Remark 2.2. By (2.1) and (2.2), it is easy to prove that, on H_0^1, two norms $v \mapsto \|v\|_{H^1}$ and $v \mapsto \|v_x\|$ are equivalent.

Throughout this paper, we write $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \bigtriangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, to denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively. With $f \in C^k([0,1]\times[0,T^*]\times\mathbb{R}^4)$, $f = f(x,t,y_1,\cdots,y_4)$, we define $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_{2+i} f = \frac{\partial f}{\partial y_i}$, $i = 1,\cdots,4$ and $D^\alpha f = D_1^{\alpha_1}\cdots D_6^{\alpha_6} f$; $\alpha = (\alpha_1,\cdots,\alpha_6) \in \mathbb{Z}_+^6$, $|\alpha| = \alpha_1 + \cdots + \alpha_6 \le k$; $D^{(0,\cdots,0)} f = f$. Similarly, with $\mu \in C^k([0,1]\times[0,T^*]\times\mathbb{R})$, $\mu = \mu(x,t,y)$, we define $D_1\mu = \frac{\partial \mu}{\partial x}$, $D_2\mu = \frac{\partial \mu}{\partial t}$, $D_3\mu = \frac{\partial \mu}{\partial y}$ and $D^\beta \mu = D_1^{\beta_1}\cdots D_3^{\beta_3}\mu$, $\beta = (\beta_1,\cdots,\beta_3) \in \mathbb{Z}_+^3$, $|\beta| = \beta_1 + \cdots + \beta_3 \le k$; $D^{(0,\cdots,0)}\mu = \mu$.

3 Local existence and uniqueness

In this section, we consider the local existence and uniqueness of Prob. (1.1). By using the linearization method together with Faedo-Galerkin method, we prove that there exists a recurrent sequence which converges to the weak solution of (1.1). Let $T^* > 0$, we make the following assumptions:

(H_1) $\tilde{u}_0$, $\tilde{u}_1 \in H^2 \cap H_0^1$;

(H_2) μ, $\bar{\mu} \in C^3([0,1]\times[0,T^*]\times\mathbb{R})$ and
$$D_3\mu(x,t,y) \ge \mu_* > 0, \text{ for all } (x,t,y) \in [0,1]\times[0,T^*]\times\mathbb{R};$$

(H_3) $g \in H^1(0,T^*)$;

(H_4) $f \in C^1([0,1]\times[0,T^*]\times\mathbb{R}^4)$, such that

(i) $f(0,t,0,0,y_3,y_4) = f(1,t,0,0,y_3,y_4) = 0$, for all $(t,y_3,y_4) \in [0,T^*]\times\mathbb{R}^2$,

(ii) There exists a positive constant σ such that
$\sigma < \dfrac{\sqrt{\bar{\mu}_*}}{3\sqrt{2}}$, with $\bar{\mu}_* = \min\{1,\mu_*,2\lambda\}$, and
$\|D_6 f\|_{C^0(\tilde{A}_M)} \le \sigma$, $\forall M > 0$,

where $\tilde{A}_M = [0,1]\times[0,T^*]\times[-M,M]^2\times[-\sqrt{2}M,\sqrt{2}M]^2$.

Concerning the function f satisfying the assumption (H_4), we take

$$f(x,t,y_1,\cdots,y_4)=f_1(x,t,y_1,\cdots,y_3)+\frac{\sigma y_1^2}{1+y_1^2}\sin y_4,\ (x,t,y_1,\cdots,y_4)\in[0,1]\times[0,T^*]\times\mathbb{R}^4,$$

where $f_1\in C^1([0,1]\times[0,T^*]\times\mathbb{R}^3)$, such that

$$f_1(0,t,0,0,y_3)=f_1(1,t,0,0,y_3)=0,\ \forall(t,y,y_3)\in[0,T^*]\times\mathbb{R},$$

and $0<\sigma<\dfrac{\sqrt{\bar{\mu}_*}}{3\sqrt{2}}$, with $\bar{\mu}_*=\min\{1,\mu_*,\lambda\}$.

One can easily verify that $f\in C^1([0,1]\times[0,T^*]\times\mathbb{R}^4)$ and $((H_4),(i))$ is fulfilled.

By

$$|D_6f(x,t,y_1,\cdots,y_4)|=\frac{\sigma y_1^2}{1+y_1^2}|\cos y_4|\le\sigma,\ \forall(x,t,y_1,\cdots,y_4)\in\tilde{A}_M,\ \forall M>0,$$

it follows that $\|D_6f\|_{C^0(\tilde{A}_M)}\le\sigma$, $\forall M>0$. Then, the condition $(H_4)-(ii)$ also holds.

A function u is called a weak solution of the initial-boundary value problem (1.1) if

$$u\in W_T=\{u\in L^\infty(0,T;H^2\cap H_0^1):u'\in L^\infty(0,T;H^2\cap H_0^1),\ u''\in L^2(0,T;H_0^1)\cap L^\infty(0,T;L^2)\},$$

and u satisfies the variational equation

$$\langle u''(t),v\rangle+\lambda\langle u_x'(t),v_x\rangle+a(t;u(t),v)=\int_0^t g(t-s)\bar{a}(s;u(s),v)ds+\langle f[u](t),v\rangle, \tag{3.1}$$

for all $v\in H_0^1$, a.e. $t\in(0,T)$, together with the initial conditions

$$u(0)=\tilde{u}_0,\ u'(0)=\tilde{u}_1, \tag{3.2}$$

where

$$\begin{aligned}
f[u](x,t) &= f(x,t,u(x,t),u'(x,t),u_x(x,t),u_x'(x,t)),\\
a(t;u(t),v) &= \left\langle\frac{\partial}{\partial x}\left(\mu(t,u(t))\right),v_x\right\rangle=\langle D_1\mu(t,u(t))+D_3\mu(t,u(t))u_x(t),v_x\rangle,\\
\bar{a}(t;u(t),v) &= \left\langle\frac{\partial}{\partial x}\left(\bar{\mu}(t,u(t))\right),v_x\right\rangle=\langle D_1\bar{\mu}(t,u(t))+D_3\bar{\mu}(t,u(t))u_x(t),v_x\rangle.
\end{aligned}$$

Let $T^*>0$ be fixed. For $M>0$, we put

$$\begin{cases}
K_M(\mu)=\|\mu\|_{C^3(A_M)}=\max\limits_{|\beta|\le3}\left\|D^\beta\mu\right\|_{C^0(A_M)},\\
K_M(\bar{\mu})=\|\bar{\mu}\|_{C^3(A_M)}=\max\limits_{|\beta|\le3}\left\|D^\beta\bar{\mu}\right\|_{C^0(A_M)},\\
\tilde{K}_M(f)=\|f\|_{C^1(\tilde{A}_M)}=\max\limits_{|\alpha|\le1}\|D^\alpha f\|_{C^0(\tilde{A}_M)},\\
\|\mu\|_{C^0(A_M)}=\sup\limits_{(x,t,y)\in A_M}|\mu(x,t,y)|,\\
\|f\|_{C^0(\tilde{A}_M)}=\sup\limits_{(x,t,y_1,\cdots,y_4)\in\tilde{A}_M}|f(x,t,y_1,\cdots,y_4)|,
\end{cases}$$

where $A_M=[0,1]\times[0,T^*]\times[-M,M]$ and $\tilde{A}_M=[0,1]\times[0,T^*]\times[-M,M]^2\times[-\sqrt{2}M,\sqrt{2}M]^2$.

For any $T\in(0,T^*]$, we consider the set

$$V_T=\{v\in L^\infty(0,T;H^2\cap H_0^1):v'\in L^\infty(0,T;H^2\cap H_0^1),\ v''\in L^2(0,T;H_0^1)\},$$

then V_T is a Banach space with respect to the norm (see Lions [25])

$$\|v\|_{V_T}=\max\{\|v\|_{L^\infty(0,T;H^2\cap H_0^1)},\ \left\|v'\right\|_{L^\infty(0,T;H^2\cap H_0^1)},\ \left\|v''\right\|_{L^2(0,T;H_0^1)}\}.$$

Also, we define the sets

$$\begin{cases} W(M,T) = \{v \in V_T : \|v\|_{V_T} \leq M\}, \\ W_1(M,T) = \{v \in W(M,T) : v'' \in L^\infty(0,T;L^2)\}. \end{cases} \tag{3.3}$$

In the following, we shall establish a linear recurrent sequence $\{u_m\}$ by choosing the first iteration $u_0 \equiv \tilde{u}_0$, and suppose that

$$u_{m-1} \in W_1(M,T), \tag{3.4}$$

then we shall find u_m in $W_1(M,T)$ satisfing the following problem

$$\begin{cases} \langle u_m''(t), v\rangle + \lambda\langle u_{mx}'(t), v_x\rangle + a_m(t; u_m(t), v) \\ \qquad\qquad = \int_0^t g(t-s)\bar{a}_m(s; u_m(s), v)ds + \langle F_m(t), v\rangle, \ \forall v \in H_0^1, \\ u_m(0) = \tilde{u}_0, \ u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.5}$$

in which

$$\begin{aligned} F_m(x,t) &= f[u_{m-1}](x,t) = f(x,t,u_{m-1}(x,t), u_{m-1}'(x,t), \nabla u_{m-1}(x,t), \nabla u_{m-1}'(x,t)), \\ a_m(t;u,v) &= \langle D_1\mu(t, u_{m-1}(t)) + D_3\mu(t, u_{m-1}(t))u_x, v_x\rangle, \\ \bar{a}_m(t;u,v) &= \langle D_1\bar{\mu}(t, u_{m-1}(t)) + D_3\bar{\mu}(t, u_{m-1}(t))u_x, v_x\rangle, \ u, v \in H_0^1. \end{aligned}$$

Note that $a_m(t;u,v)$, $\bar{a}_m(t;u,v)$ can be rewritten in form of

$$\begin{aligned} a_m(t;u,v) &= A_m(t;u,v) + \langle \mu_{1m}(t), v_x\rangle, \\ \bar{a}_m(t;u,v) &= \bar{A}_m(t;u,v) + \langle \bar{\mu}_{1m}(t), v_x\rangle, \ u, v \in H_0^1, \end{aligned}$$

where

$$\begin{aligned} A_m(t;u,v) &= \langle \mu_{3m}(t)u_x, v_x\rangle, \ \bar{A}_m(t;u,v) = \langle \bar{\mu}_{3m}(t)u_x, v_x\rangle, \ u, v \in H_0^1, \\ \mu_{3m}(x,t) &= D_3\mu(x,t,u_{m-1}(x,t)), \ \mu_{1m}(x,t) = D_1\mu(x,t,u_{m-1}(x,t)), \\ \bar{\mu}_{3m}(x,t) &= D_3\bar{\mu}(x,t,u_{m-1}(x,t)), \ \bar{\mu}_{1m}(x,t) = D_1\bar{\mu}(x,t,u_{m-1}(x,t)). \end{aligned}$$

Then, Prob. (3.5) is equivalent to

$$\begin{cases} \langle u_m''(t), v\rangle + \lambda\langle u_{mx}'(t), v_x\rangle + A_m(t; u_m(t), v) \\ \qquad\qquad = \int_0^t g(t-s)\bar{A}_m(s; u_m(s), v)ds + \left\langle \hat{F}_m(t), v\right\rangle, \ \forall v \in H_0^1, \\ u_m(0) = \tilde{u}_0, \ u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.6}$$

where $\hat{F}_m(t) : H_0^1 \to \mathbb{R}$ is a linear continuous functional on H_0^1, which is defined by

$$\left\langle \hat{F}_m(t), v\right\rangle = \langle F_m(t), v\rangle - \langle \mu_{1m}(t), v_x\rangle + \int_0^t g(t-s)\langle \bar{\mu}_{1m}(s), v_x\rangle ds, \ v \in H_0^1. \tag{3.7}$$

The existence of u_m is assured by the following theorem.

Theorem 3.1. *Under assumptions* $(H_1)-(H_4)$, *there exist positive constants* M, T *such that, for* $u_0 \equiv \tilde{u}_0$, *there exists a recurrent sequence* $\{u_m\} \subset W_1(M,T)$ *defined by* (3.4), (3.6) *and* (3.7).

Proof of Theorem 3.1. The proof consists of several steps.

Step 1. The Galerkin approximation. Consider a special orthonormal basis $\{w_j\}$ on $H_0^1 : w_j(x) = \sqrt{2}\sin(j\pi x)$, $j \in \mathbb{N}$, formed by the eigenfunctions of the Laplacian $-\Delta = -\frac{\partial^2}{\partial x^2}$. Put

$$u_m^{(k)}(t) = \sum\nolimits_{j=1}^k c_{mj}^{(k)}(t)w_j,$$

where the coefficients $c_{mj}^{(k)}$ satisfy the following system of linear integrodifferential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j\rangle + \lambda\langle \dot{u}_{mx}^{(k)}(t), w_{jx}\rangle + A_m(t; u_m^{(k)}(t), w_j) \\ \qquad\qquad = \int_0^t g(t-s)\bar{A}_m(s; u_m^{(k)}(s), w_j)ds + \left\langle \hat{F}_m(t), w_j \right\rangle, \ 1 \le j \le k, \\ u_m^{(k)}(0) = \tilde{u}_{0k}, \ \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases} \tag{3.8}$$

in which

$$\begin{cases} \tilde{u}_{0k} = \sum_{j=1}^k \alpha_j^{(k)} w_j \to \tilde{u}_0 \text{ strongly in } H^2 \cap H_0^1, \\ \tilde{u}_{1k} = \sum_{j=1}^k \beta_j^{(k)} w_j \to \tilde{u}_1 \text{ strongly in } H^2 \cap H_0^1. \end{cases} \tag{3.9}$$

Using Banach's contraction principle, it is not difficult to prove that the system (3.8) admits a unique solution $u_m^{(k)}(t)$ on interval $[0, T]$, so let us omit the details.

Step 2. A priori estimate. Put

$$\begin{aligned} S_m^{(k)}(t) &= \left\| \dot{u}_m^{(k)}(t) \right\|^2 + \left\| \dot{u}_{mx}^{(k)}(t) \right\|^2 + \left\| \sqrt{\mu_{3m}(t)} u_{mx}^{(k)}(t) \right\|^2 + \left\| \sqrt{\mu_{3m}(t)} \Delta u_m^{(k)}(t) \right\|^2 \\ &\quad + \lambda \left\| \Delta \dot{u}_m^{(k)}(t) \right\|^2 + 2\lambda \int_0^t \left(\left\| \dot{u}_{mx}^{(k)}(s) \right\|^2 + \left\| \Delta \dot{u}_m^{(k)}(s) \right\|^2 \right) ds + 2 \int_0^t \left\| \ddot{u}_{mx}^{(k)}(s) \right\|^2 ds, \end{aligned}$$

then it follows from (3.8) that

$$\begin{aligned} S_m^{(k)}(t) &= S_m^{(k)}(0) + 2\left\langle \mu_{3mx}(0)\tilde{u}_{0kx}, \Delta\tilde{u}_{0k}\right\rangle + 2\left\langle \frac{\partial}{\partial x}\left(\mu_{3m}(0)\tilde{u}_{0kx}\right), \Delta\tilde{u}_{1k}\right\rangle \\ &\quad + \int_0^t ds \int_0^1 \mu_{3m}'(x,s) \left(\left| u_{mx}^{(k)}(x,s) \right|^2 + \left| \Delta u_m^{(k)}(x,s) \right|^2 \right) dx \\ &\quad + 2\int_0^t g(t-s)\bar{A}_m(s; u_m^{(k)}(s), u_m^{(k)}(t))ds \\ &\quad + 2\int_0^t g(t-s) \left\langle \frac{\partial}{\partial x}\left(\bar{\mu}_{3m}(s)u_{mx}^{(k)}(s)\right), \Delta u_m^{(k)}(t) + \Delta\dot{u}_m^{(k)}(t) \right\rangle ds \\ &\quad - 2g(0)\int_0^t \left\langle \frac{\partial}{\partial x}\left(\bar{\mu}_{3m}(s)u_{mx}^{(k)}(s)\right), \Delta u_m^{(k)}(s) + \Delta\dot{u}_m^{(k)}(s) \right\rangle ds \\ &\quad - 2g(0)\int_0^t \bar{A}_m(s; u_m^{(k)}(s), u_m^{(k)}(s))ds \\ &\quad - 2\int_0^t d\tau \int_0^\tau g'(\tau-s)\bar{A}_m(s; u_m^{(k)}(s), u_m^{(k)}(\tau))ds \\ &\quad - 2\int_0^t d\tau \int_0^\tau g'(\tau-s) \left\langle \frac{\partial}{\partial x}\left(\bar{\mu}_{3m}(s)u_{mx}^{(k)}(s)\right), \Delta u_m^{(k)}(\tau) + \Delta\dot{u}_m^{(k)}(\tau) \right\rangle ds \\ &\quad + 2\int_0^t \left\langle \frac{\partial}{\partial s}\left(\mu_{3mx}(s)u_{mx}^{(k)}(s)\right), \Delta u_m^{(k)}(s) \right\rangle ds \\ &\quad + 2\int_0^t \left\langle \frac{\partial^2}{\partial x \partial s}\left(\mu_{3m}(s)u_{mx}^{(k)}(s)\right), \Delta\dot{u}_m^{(k)}(s) \right\rangle ds \\ &\quad - 2\left\langle \mu_{3mx}(t)u_{mx}^{(k)}(t), \Delta u_m^{(k)}(t) \right\rangle - 2\left\langle \frac{\partial}{\partial x}\left(\mu_{3m}(t)u_{mx}^{(k)}(t)\right), \Delta\dot{u}_m^{(k)}(t) \right\rangle \\ &\quad + 2\int_0^t \left\langle \hat{F}_m(s), \dot{u}_m^{(k)}(s) \right\rangle ds + 2\int_0^t \left\langle \hat{F}_m(s), -\Delta\dot{u}_m^{(k)}(s) \right\rangle ds + 2\int_0^t \left\langle \hat{F}_m(s), -\Delta\ddot{u}_m^{(k)}(s) \right\rangle ds \\ &= S_m^{(k)}(0) + 2\left\langle \mu_{3mx}(0)\tilde{u}_{0kx}, \Delta\tilde{u}_{0k}\right\rangle + 2\left\langle \frac{\partial}{\partial x}\left(\mu_{3m}(0)\tilde{u}_{0kx}\right), \Delta\tilde{u}_{1k}\right\rangle + \sum_{i=1}^{14} J_i. \end{aligned} \tag{3.10}$$

We shall estimate the terms J_i on the right-hand side of (3.10) as follows.
First, we need the following lemma whose proof is easy, hence we omit the details.
Lemma 3.2. *Put*

$$\begin{aligned} \bar{S}_m^{(k)}(t) &= \left\|\dot{u}_m^{(k)}(t)\right\|^2 + \left\|\dot{u}_{mx}^{(k)}(t)\right\|^2 + \left\|\Delta\dot{u}_m^{(k)}(t)\right\|^2 + \left\|u_{mx}^{(k)}(t)\right\|^2 + \left\|\Delta u_m^{(k)}(t)\right\|^2 \\ &\quad + \int_0^t \left(\left\|\dot{u}_{mx}^{(k)}(s)\right\|^2 + \left\|\Delta\dot{u}_m^{(k)}(s)\right\|^2\right) ds + \int_0^t \left\|\ddot{u}_{mx}^{(k)}(s)\right\|^2 ds. \end{aligned} \tag{3.11}$$

Then, the following estimations are admitted

(i) $|\mu'_{im}(x,t)| \leq (1+M)K_M(\mu)$, $i=1,3$,
(ii) $\|\mu'_{im}(t)\| \leq (1+M)K_M(\mu)$, $i=1,3$,
(iii) $|\mu_{imx}(x,t)| \leq (1+2M)K_M(\mu)$, $i=1,3$,
(iv) $\|\mu_{imx}(t)\| \leq (1+M)K_M(\mu)$, $i=1,3$,
(v) $|\mu'_{imx}(x,t)| \leq (1+5M+2M^2)K_M(\mu)$, $i=1,3$,
(vi) $\|\mu'_{imx}(t)\| \leq (1+3M+M^2)K_M(\mu)$, $i=1,3$,
(vii) $\left|\bar{A}_m(s;u_m^{(k)}(s),u_m^{(k)}(t))\right| \leq K_M(\bar{\mu})\left\|u_{mx}^{(k)}(s)\right\|\left\|u_{mx}^{(k)}(t)\right\|$,
(viii) $\left\|\Delta u_m^{(k)}(t) + \Delta\dot{u}_m^{(k)}(t)\right\| \leq \sqrt{2}\sqrt{\bar{S}_m^{(k)}(t)}$,
(ix) $\left\|u_{mx}^{(k)}(t)\right\|^2 \leq 2\|\tilde{u}_{0kx}\|^2 + 2T^*\int_0^t \bar{S}_m^{(k)}(s)ds$,
(x) $\left\|\dfrac{\partial}{\partial x}\left(\mu_{3m}(t)u_{mx}^{(k)}(t)\right)\right\| \leq 2(1+M)K_M(\mu)\sqrt{\bar{S}_m^{(k)}(t)}$,
(xi) $\left\|\dfrac{\partial}{\partial t}\left(\mu_{3mx}(t)u_{mx}^{(k)}(t)\right)\right\| \leq (2+7M+2M^2)K_M(\mu)\sqrt{\bar{S}_m^{(k)}(t)}$,
(xii) $\left\|\dfrac{\partial^2}{\partial x\partial t}\left(\mu_{3m}(t)u_{mx}^{(k)}(t)\right)\right\| \leq 2(2+4M+M^2)K_M(\mu)\sqrt{\bar{S}_m^{(k)}(t)}$.

Moreover, the inequalities (i)-(xii) *are also valid with replacing μ by $\bar{\mu}$.*
By Lemma 3.2, the terms $J_1 - J_9$ on the right-hand side of (3.10) are estimated as follows
Using the inequality

$$S_m^{(k)}(t) \geq \bar{\mu}_*\bar{S}_m^{(k)}(t),$$

where $\bar{\mu}_* = \min\{1, \mu_*, 2\lambda\}$ and

$$2ab \leq \beta a^2 + \frac{1}{\beta}b^2,\ \forall a,b\in\mathbb{R},\ \text{with } \beta = \beta_* = \frac{\bar{\mu}_*}{10},$$

then the terms $J_1 - J_9$ are respectively estimated by

$$\begin{aligned} J_1 &= \int_0^t ds\int_0^1 \mu'_{3m}(x,s)\left(\left|u_{mx}^{(k)}(x,s)\right|^2 + \left|\Delta u_m^{(k)}(x,s)\right|^2\right)dx \\ &\leq (1+M)K_M(\mu)\int_0^t \bar{S}_m^{(k)}(s)ds = C_1(M)\int_0^t \bar{S}_m^{(k)}(s)ds, \\ J_2 &= 2\int_0^t g(t-s)\bar{A}_m(s;u_m^{(k)}(s),u_m^{(k)}(t))ds \\ &\leq \|\tilde{u}_{0kx}\|^2 + \left(T^* + 2K_M^2(\bar{\mu})\|g\|^2_{L^2(0,T^*)}\right)\int_0^t \bar{S}_m^{(k)}(s)ds = \|\tilde{u}_{0kx}\|^2 + C_2(M)\int_0^t \bar{S}_m^{(k)}(s)ds, \\ J_3 &= 2\int_0^t g(t-s)\left\langle\frac{\partial}{\partial x}\left(\bar{\mu}_{3m}(s)u_{mx}^{(k)}(s)\right), \Delta u_m^{(k)}(t) + \Delta\dot{u}_m^{(k)}(t)\right\rangle ds \\ &\leq \beta_*\bar{S}_m^{(k)}(t) + \frac{8}{\beta_*}(1+M)^2K_M^2(\bar{\mu})\|g\|^2_{L^2(0,T^*)}\int_0^t \bar{S}_m^{(k)}(s)ds \end{aligned} \tag{3.12}$$

$$
\begin{aligned}
&= \beta_* \bar{S}_m^{(k)}(t) + C_3(M)\int_0^t \bar{S}_m^{(k)}(s)ds,\\
J_4 &= -2g(0)\int_0^t \left\langle \frac{\partial}{\partial x}\left(\bar{\mu}_{3m}(s)u_{mx}^{(k)}(s)\right), \Delta u_m^{(k)}(s) + \Delta \dot{u}_m^{(k)}(s)\right\rangle ds\\
&\leq 4\sqrt{2}(1+M)K_M(\mu)\left|g(0)\right| \int_0^t \bar{S}_m^{(k)}(s)ds = C_4(M)\int_0^t \bar{S}_m^{(k)}(s)ds,\\
J_5 &= -2g(0)\int_0^t \bar{A}_m(s; u_m^{(k)}(s), u_m^{(k)}(s))ds\\
&\leq 2\left|g(0)\right| K_M(\bar{\mu})\int_0^t \bar{S}_m^{(k)}(s)ds = C_5(M)\int_0^t \bar{S}_m^{(k)}(s)ds,\\
J_6 &= -2\int_0^t d\tau \int_0^\tau g'(\tau - s)\bar{A}_m(s; u_m^{(k)}(s), u_m^{(k)}(\tau))ds\\
&\leq 2K_M(\bar{\mu})\sqrt{T^*}\left\|g'\right\|_{L^2(0,T^*)} \int_0^t \bar{S}_m^{(k)}(s)ds = C_6(M)\int_0^t \bar{S}_m^{(k)}(s)ds,\\
J_7 &= -2\int_0^t d\tau \int_0^\tau g'(\tau - s)\left\langle \frac{\partial}{\partial x}\left(\bar{\mu}_{3m}(s)u_{mx}^{(k)}(s)\right), \Delta u_m^{(k)}(\tau) + \Delta \dot{u}_m^{(k)}(\tau)\right\rangle ds\\
&\leq 4\sqrt{2}(1+M)K_M(\bar{\mu})\sqrt{T^*}\left\|g'\right\|_{L^2(0,T^*)} \int_0^t \bar{S}_m^{(k)}(s)ds = C_7(M)\int_0^t \bar{S}_m^{(k)}(s)ds,\\
J_8 &= 2\int_0^t \left\langle \frac{\partial}{\partial s}\left(\mu_{3mx}(s)u_{mx}^{(k)}(s)\right), \Delta u_m^{(k)}(s)\right\rangle ds\\
&\leq 2(2+7M+2M^2)K_M(\mu)\int_0^t \bar{S}_m^{(k)}(s)ds = C_8(M)\int_0^t \bar{S}_m^{(k)}(s)ds,\\
J_9 &= 2\int_0^t \left\langle \frac{\partial^2}{\partial x \partial s}\left(\mu_{3m}(s)u_{mx}^{(k)}(s)\right), \Delta \dot{u}_m^{(k)}(s)\right\rangle ds\\
&\leq 4(2+4M+M^2)K_M(\mu)\int_0^t \bar{S}_m^{(k)}(s)ds = C_9(M)\int_0^t \bar{S}_m^{(k)}(s)ds.
\end{aligned}
$$

In order to estimate the terms J_{10}, J_{11}, we use the following lemma whose proof is easy, so we omit the details.

Lemma 3.3. *The following estimations are valid*

(i) $\left\|\Delta u_m^{(k)}(t)\right\| \leq \left\|\Delta \tilde{u}_{0k}\right\| + \int_0^t \sqrt{\bar{S}_m^{(k)}(s)}ds,$

(ii) $\left\|\mu_{3mx}(t)u_{mx}^{(k)}(t)\right\| \leq \left\|\mu_{3mx}(0)\tilde{u}_{0kx}\right\| + (2+7M+2M^2)K_M(\mu)\int_0^t \sqrt{\bar{S}_m^{(k)}(s)}ds,$

(iii) $\left\|\frac{\partial}{\partial x}\left(\mu_{3m}(t)u_{mx}^{(k)}(t)\right)\right\| \leq \left\|\frac{\partial}{\partial x}\left(\mu_{3m}(0)\tilde{u}_{0kx}\right)\right\| + 2(2+4M+M^2)K_M(\mu)\int_0^t \sqrt{\bar{S}_m^{(k)}(s)}ds.$

Using Lemma 3.3 and the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, for all a, $b \in \mathbb{R}$, the terms J_{10}, J_{11} are estimated as follows

$$
\begin{aligned}
J_{10} &= -2\left\langle \mu_{3mx}(t)u_{mx}^{(k)}(t), \Delta u_m^{(k)}(t)\right\rangle \qquad (3.13)\\
&\leq 2\left(\left\|\mu_{3mx}(0)\tilde{u}_{0kx}\right\|^2 + \left\|\Delta \tilde{u}_{0k}\right\|^2\right) + 2T^*\left[1 + (2+7M+2M^2)^2 K_M^2(\mu)\ \right]\int_0^t \bar{S}_m^{(k)}(s)ds\\
&= 2\left(\left\|\mu_{3mx}(0)\tilde{u}_{0kx}\right\|^2 + \left\|\Delta \tilde{u}_{0k}\right\|^2\right) + C_{10}(M)\int_0^t \bar{S}_m^{(k)}(s)ds;\\
J_{11} &= -2\frac{\partial}{\partial x}\left(\mu_{3m}(t)u_{mx}^{(k)}(t)\right), \Delta \dot{u}_m^{(k)}(t)
\end{aligned}
$$

$$\begin{aligned}
&\leq \beta_* \bar{S}_m^{(k)}(t) + \frac{2}{\beta_*}\left\|\frac{\partial}{\partial x}\left(\mu_{3m}(0)\tilde{u}_{0kx}\right)\right\|^2 + \frac{8}{\beta_*}(2+4M+M^2)^2 K_M^2(\mu) T^* \int_0^t \bar{S}_m^{(k)}(s)ds \\
&= \beta_* \bar{S}_m^{(k)}(t) + \frac{2}{\beta_*}\left\|\frac{\partial}{\partial x}\left(\mu_{3m}(0)\tilde{u}_{0kx}\right)\right\|^2 + C_{11}(M)\int_0^t \bar{S}_m^{(k)}(s)ds.
\end{aligned}$$

The terms $J_{12} - J_{14}$ are also estimated as below. By the fact that

$$\left\langle \hat{F}_m(t), \dot{u}_m^{(k)}(t)\right\rangle = \left\langle F_m(t), \dot{u}_m^{(k)}(t)\right\rangle - \left\langle \mu_{1m}(t), \dot{u}_{mx}^{(k)}(t)\right\rangle + \int_0^t g(t-s)\left\langle \bar{\mu}_{1m}(s), \dot{u}_{mx}^{(k)}(t)\right\rangle ds,$$

we have

$$\begin{aligned}
\left|\left\langle \hat{F}_m(t), \dot{u}_m^{(k)}(t)\right\rangle\right| &\leq \left(\tilde{K}_M(f) + K_M(\mu) + K_M(\bar{\mu})\left\|g\right\|_{L^1(0,T^*)}\right)\sqrt{\bar{S}_m^{(k)}(t)} \\
&\equiv \sqrt{C_{12}(M)}\sqrt{\bar{S}_m^{(k)}(t)}.
\end{aligned}$$

Then

$$J_{12} = 2\int_0^t \left\langle \hat{F}_m(s), \dot{u}_m^{(k)}(s)\right\rangle ds \leq TC_{12}(M) + \int_0^t \bar{S}_m^{(k)}(s)ds. \tag{3.14}$$

We also have

$$\begin{aligned}
&\left\langle \hat{F}_m(t), -\Delta\dot{u}_m^{(k)}(t)\right\rangle \\
&= \left\langle F_m(t), -\Delta\dot{u}_m^{(k)}(t)\right\rangle - \left\langle \mu_{1m}(t), -\Delta\dot{u}_{mx}^{(k)}(t)\right\rangle + \int_0^t g(t-s)\left\langle \bar{\mu}_{1m}(s), -\Delta\dot{u}_{mx}^{(k)}(t)\right\rangle ds \\
&= \left\langle F_m(t), -\Delta\dot{u}_m^{(k)}(t)\right\rangle - \left\langle \mu_{1mx}(t), \Delta\dot{u}_m^{(k)}(t)\right\rangle + \int_0^t g(t-s)\left\langle \bar{\mu}_{1mx}(s), \Delta\dot{u}_m^{(k)}(t)\right\rangle ds,
\end{aligned}$$

so

$$\begin{aligned}
&\left|\left\langle \hat{F}_m(t), -\Delta\dot{u}_m^{(k)}(t)\right\rangle\right| \\
&\leq \left[\left\|F_m(t)\right\| + \left\|\mu_{1mx}(t)\right\| + \int_0^t \left|g(t-s)\right|\left\|\bar{\mu}_{1mx}(s)\right\| ds\right]\left\|\Delta\dot{u}_m^{(k)}(t)\right\| \\
&\leq \left[\tilde{K}_M(f) + (1+M)\left(K_M(\mu) + K_M(\bar{\mu})\left\|g\right\|_{L^1(0,T^*)}\right)\right]\sqrt{\bar{S}_m^{(k)}(t)} \\
&\equiv \sqrt{C_{13}(M)}\sqrt{\bar{S}_m^{(k)}(t)}.
\end{aligned}$$

Then

$$J_{13} = 2\int_0^t \left\langle \hat{F}_m(s), -\Delta\dot{u}_m^{(k)}(s)\right\rangle ds \leq TC_{13}(M) + \int_0^t \bar{S}_m^{(k)}(s)ds. \tag{3.15}$$

Similarly

$$\begin{aligned}
\left\langle \hat{F}_m(t), -\Delta\ddot{u}_m^{(k)}(t)\right\rangle &= \left\langle F_m(t), -\Delta\ddot{u}_m^{(k)}(t)\right\rangle - \left\langle \mu_{1m}(t), -\Delta\ddot{u}_{mx}^{(k)}(t)\right\rangle \\
&\quad + \int_0^t g(t-s)\left\langle \bar{\mu}_{1m}(s), -\Delta\ddot{u}_{mx}^{(k)}(t)\right\rangle ds \\
&= \left\langle F_{mx}(t), \ddot{u}_{mx}^{(k)}(t)\right\rangle - \frac{d}{dt}\left\langle \mu_{1mx}(t), \Delta\dot{u}_m^{(k)}(t)\right\rangle + \left\langle \mu'_{1mx}(t), \Delta\dot{u}_m^{(k)}(t)\right\rangle \\
&\quad + \frac{d}{dt}\int_0^t g(t-s)\left\langle \bar{\mu}_{1mx}(s), \Delta\dot{u}_m^{(k)}(t)\right\rangle ds - g(0)\left\langle \bar{\mu}_{1mx}(t), \Delta\dot{u}_m^{(k)}(t)\right\rangle \\
&\quad - \int_0^t g'(t-s)\left\langle \bar{\mu}_{1mx}(s), \Delta\dot{u}_m^{(k)}(t)\right\rangle ds,
\end{aligned}$$

thus

$$
\begin{aligned}
J_{14} &= 2\int_0^t \left\langle \hat{F}_m(s), -\Delta\ddot{u}_m^{(k)}(s)\right\rangle ds \\
&= 2\left\langle \mu_{1mx}(0), \Delta\tilde{u}_{1k}\right\rangle + 2\int_0^t \left\langle F_{mx}(s), \ddot{u}_{mx}^{(k)}(s)\right\rangle ds - 2\left\langle \mu_{1mx}(t), \Delta\dot{u}_m^{(k)}(t)\right\rangle \\
&\quad +2\int_0^t \left\langle \mu_{1mx}'(s), \Delta\dot{u}_m^{(k)}(s)\right\rangle ds + 2\int_0^t g(t-s)\left\langle \bar{\mu}_{1mx}(s), \Delta\dot{u}_m^{(k)}(t)\right\rangle ds \\
&\quad -2g(0)\int_0^t \left\langle \bar{\mu}_{1mx}(s), \Delta\dot{u}_m^{(k)}(s)\right\rangle ds - 2\int_0^t d\tau\int_0^\tau g'(\tau-s)\left\langle \bar{\mu}_{1mx}(s), \Delta\dot{u}_m^{(k)}(\tau)\right\rangle ds \\
&= 2\left\langle \mu_{1mx}(0), \Delta\tilde{u}_{1k}\right\rangle + q_1 + \cdots + q_6.
\end{aligned}
$$

In order to estimate $q_1, \cdots, q_6$, we use the following inequalities

$$
\begin{aligned}
\|F_{mx}(s)\| &\leq (1+4M)\tilde{K}_M(f), \\
\|\mu_{1mx}(t)\| &\leq \|\mu_{1mx}(0)\| + \int_0^t \left\|\mu_{1mx}'(s)\right\| ds \leq \|\mu_{1mx}(0)\| + T(1+3M+M^2)K_M(\mu).
\end{aligned}
$$

Then

$$
\begin{aligned}
q_1 &= 2\int_0^t \left\langle F_{mx}(s), \ddot{u}_{mx}^{(k)}(s)\right\rangle ds \\
&\leq \frac{1}{\beta_*}T(1+4M)^2\tilde{K}_M^2(f) + \beta_*\bar{S}_m^{(k)}(t) = T\bar{q}_1(M) + \beta_*\bar{S}_m^{(k)}(t), \\
q_2 &= -2\left\langle \mu_{1mx}(t), \Delta\dot{u}_m^{(k)}(t)\right\rangle \\
&\leq \beta_*\bar{S}_m^{(k)}(t) + \frac{2}{\beta_*}\|\mu_{1mx}(0)\|^2 + \frac{2T}{\beta_*}T(1+3M+M^2)^2K_M^2(\mu) \\
&= \beta_*\bar{S}_m^{(k)}(t) + \frac{2}{\beta_*}\|\mu_{1mx}(0)\|^2 + T\bar{q}_2(M), \\
q_3 &= 2\int_0^t \left\langle \mu_{1mx}'(s), \Delta\dot{u}_m^{(k)}(s)\right\rangle ds \\
&\leq T(1+3M+M^2)^2K_M^2(\mu) + \int_0^t \bar{S}_m^{(k)}(s)ds = T\bar{q}_3(M) + \int_0^t \bar{S}_m^{(k)}(s)ds,
\end{aligned}
$$

$$
\begin{aligned}
q_4 &= 2\int_0^t g(t-s)\left\langle \bar{\mu}_{1mx}(s), \Delta\dot{u}_m^{(k)}(t)\right\rangle ds \\
&\leq \beta_*\bar{S}_m^{(k)}(t) + \frac{1}{\beta_*}T(1+M)^2K_M^2(\bar{\mu})\|g\|_{L^2(0,T^*)}^2 = \beta_*\bar{S}_m^{(k)}(t) + T\bar{q}_4(M), \\
q_5 &= -2g(0)\int_0^t \left\langle \bar{\mu}_{1mx}(s), \Delta\dot{u}_m^{(k)}(s)\right\rangle ds \\
&\leq T|g(0)|(1+M)^2K_M^2(\bar{\mu}) + \int_0^t \bar{S}_m^{(k)}(s)ds = T\bar{q}_5(M) + \int_0^t \bar{S}_m^{(k)}(s)ds, \\
q_6 &= -2\int_0^t d\tau\int_0^\tau g'(\tau-s)\left\langle \bar{\mu}_{1mx}(s), \Delta\dot{u}_m^{(k)}(\tau)\right\rangle ds \\
&\leq \int_0^t \bar{S}_m^{(k)}(\tau)d\tau + TT^*(1+M)^2K_M^2(\bar{\mu})\left\|g'\right\|_{L^2(0,T^*)}^2 = \int_0^t \bar{S}_m^{(k)}(\tau)d\tau + T\bar{q}_6(M).
\end{aligned}
$$

Thus, J_{14} is estimated by

$$
J_{14} \leq 2\left\langle \mu_{1mx}(0), \Delta\tilde{u}_{1k}\right\rangle + \frac{2}{\beta_*}\|\mu_{1mx}(0)\|^2 + TC_{14}(M) + 3\beta_*\bar{S}_m^{(k)}(t) + 3\int_0^t \bar{S}_m^{(k)}(s)ds, \tag{3.16}
$$

where $C_{14}(M) = \sum\limits_{j=1}^{6} \bar{q}_j(M)$.

Combining (3.12)–(3.16), it implies from (3.10) and (3.11) that

$$\bar{S}_m^{(k)}(t) \leq \bar{S}_{0m}^{(k)} + TD_1(M) + D_2(M) \int_0^t \bar{S}_m^{(k)}(s)ds, \tag{3.17}$$

where

$$\begin{aligned}
\bar{S}_{0m}^{(k)} &= \frac{2}{\bar{\mu}_*} S_m^{(k)}(0) + \frac{4}{\bar{\mu}_*} \left[\langle \mu_{3mx}(0)\tilde{u}_{0kx}, \Delta\tilde{u}_{0k} \rangle + \langle \mu_{1mx}(0), \Delta\tilde{u}_{1k} \rangle \right] \\
&\quad + \frac{4}{\bar{\mu}_*} \left\langle \frac{\partial}{\partial x} \left(\mu_{3m}(0)\tilde{u}_{0kx} \right), \Delta\tilde{u}_{1k} \right\rangle \\
&\quad + \frac{2}{\bar{\mu}_*} \|\tilde{u}_{0kx}\|^2 + \frac{4}{\bar{\mu}_*} \left(\|\mu_{3mx}(0)\tilde{u}_{0kx}\|^2 + \|\Delta\tilde{u}_{0k}\|^2 \right) \\
&\quad + \frac{40}{\bar{\mu}_*^2} \left(\left\| \frac{\partial}{\partial x} \left(\mu_{3m}(0)\tilde{u}_{0kx} \right) \right\|^2 + \|\mu_{1mx}(0)\|^2 \right), \\
S_m^{(k)}(0) &= \|\tilde{u}_{1k}\|^2 + \|\tilde{u}_{1kx}\|^2 + \left\| \sqrt{\mu_{3m}(0)}\tilde{u}_{0kx} \right\|^2 + \left\| \sqrt{\mu_{3m}(0)}\Delta\tilde{u}_{0k} \right\|^2 + \lambda \|\Delta\tilde{u}_{1k}\|^2, \\
D_1(M) &= \frac{2}{\bar{\mu}_*} \left[C_{12}(M) + C_{13}(M) + C_{14}(M) \right], \\
D_2(M) &= \frac{2}{\bar{\mu}_*} \left(5 + \sum\nolimits_{j=1}^{11} C_j(M) \right).
\end{aligned} \tag{3.18}$$

On the other hand, $\mu_{1mx}(x,0) = D_1^2\mu(x,0,\tilde{u}_0(x)) + D_3D_1\mu(x,0,\tilde{u}_0(x))\tilde{u}_{0x}(x)$, $\mu_{3m}(x,0) = D_3\mu(x,0,\tilde{u}_0(x))$, $\mu_{3mx}(x,0) = D_1D_3\mu(x,0,\tilde{u}_0(x)) + D_3^2\mu(x,0,\tilde{u}_0(x))\tilde{u}_{0x}(x)$, are independent of m, so it implies from $(3.18)_{1,2}$ that $S_m^{(k)}(0)$ and $\bar{S}_{0m}^{(k)}$ are also independent of m.

The convergences given by (3.18) show that there exists a positive constant M independent of k and m such that

$$\bar{S}_{0m}^{(k)} \leq \frac{M^2}{2}, \text{ for all } m,\ k \in \mathbb{N}. \tag{3.19}$$

The local existence is obtained by choosing T small enough as in the following lemma.

Lemma 3.4. *Suppose that there exists a positive constant M satisfying (3.19). For any $T \in (0, T^*]$, put*

$$k_T = 3\sqrt{D_1^*(M,T)} \exp\left(TD_2^*(M) \right), \tag{3.20}$$

where

$$\begin{aligned}
D_1^*(M,T) &= \frac{2}{\bar{\mu}_*} \left(\sigma + 2\sqrt{T}\tilde{K}_M(f) \right)^2 + \frac{12T}{\bar{\mu}_*^2} (1+M)^2 \left(K_M^2(\mu) + K_M^2(\bar{\mu}) \|g\|_{L^2(0,T^*)}^2 \right) \\
&\quad + \frac{T}{\bar{\mu}_*} (1+M)^2 \left(|g(0)| + \sqrt{T^*} \|g'\|_{L^2(0,T^*)} \right) K_M(\bar{\mu}), \\
D_2^*(M) &= \frac{2}{\bar{\mu}_*} \left[1 + (1+M)K_M(\mu) + 4\left(|g(0)| + \sqrt{T^*} \|g'\|_{L^2(0,T^*)} \right) K_M(\bar{\mu}) \right] \\
&\quad + \frac{12}{\bar{\mu}_*^2} K_M^2(\bar{\mu}) \|g'\|_{L^2(0,T^*)}.
\end{aligned}$$

Then, T can be chosen small enough such that

$$\begin{cases} \left(\dfrac{M^2}{2} + TD_1(M) \right) e^{TD_2(M)} \leq M^2, \\ k_T < 1. \end{cases} \tag{3.21}$$

Proof. By the assumption $0<\sigma<\dfrac{\sqrt{\bar{\mu}_*}}{3\sqrt{2}}$, it is easy to get that

$$\lim_{T\to 0_+} k_T=\lim_{T\to 0_+} 3\sqrt{D_1^*(M,T)}\exp\left(TD_2^*(M)\right)=3\sqrt{\frac{2}{\bar{\mu}_*}}\sigma<1,$$

and

$$\lim_{T\to 0_+}\left(\frac{M^2}{2}+TD_1(M)\right)e^{TD_2(M)}=\frac{M^2}{2}<M^2. \ \square$$

It follows from (3.17) and (3.21) that

$$\bar{S}_m^{(k)}(t)\leq M^2 e^{-TD_2(M)}+D_2(M)\int_0^t \bar{S}_m^{(k)}(s)ds.$$

By using Gronwall's Lemma, we deduce from the above inequality that

$$\bar{S}_m^{(k)}(t)\leq M^2 e^{-TD_2(M)}e^{tD_2(M)}\leq M^2,$$

for all $t\in[0,T]$, for all $m,\ k\in\mathbb{N}$. Therefore, we have

$$u_m^{(k)}\in W_1(M,T),\ \text{for all } m \text{ and } k\in\mathbb{N}. \tag{3.22}$$

Step 3. Limiting process. By (3.22), there exists a subsequence of $\{u_m^{(k)}\}$ with the same symbol, such that

$$\begin{cases} u_m^{(k)}\to u_m & \text{in} \quad L^\infty(0,T;H^2\cap H_0^1) \text{ weak*},\\ \dot{u}_m^{(k)}\to u_m' & \text{in} \quad L^\infty(0,T;H^2\cap H_0^1) \text{ weak*},\\ \ddot{u}_m^{(k)}\to u_m'' & \text{in} \quad L^2(0,T;H_0^1) \text{ weak},\\ u_m\in W(M,T). \end{cases} \tag{3.23}$$

Passing to limit in (3.8) and (3.9), we have u_m satisfying (3.6) and (3.7) in $L^2(0,T)$.

On the other hand, we deduce from $(3.6)_1$ and $(3.23)_4$ that

$$\begin{aligned} u_m'' &= \lambda u_{mxx}'+\frac{\partial}{\partial x}\left(\mu_{1m}(t)+\mu_{3m}(t)u_{mx}(t)\right)-\int_0^t g\left(t-s\right)\frac{\partial}{\partial x}\left(\bar{\mu}_{1m}(s)+\bar{\mu}_{3m}(s)u_{mx}(s)\right)ds+F_m\\ &\equiv \tilde{F}_m\in L^\infty(0,T;L^2). \end{aligned}$$

Thus, $u_m\in W_1(M,T)$. Theorem 3.1 is proved. $\square$

By using Theorem 3.1 and the compact imbedding theorems, we shall prove the existence and uniqueness of weak local solutions to Prob. (1.1). We first introduce the Banach space (see Lions [25]) as follows

$$W_1(T)=\left\{u\in C^0([0,T];H_0^1)\cap C^1([0,T];L^2): u'\in L^2(0,T;H_0^1)\right\},$$

with respect to the norm

$$\|u\|_{W_1(T)}=\|u\|_{C^0([0,T];H_0^1)}+\left\|u'\right\|_{C^0(0,T;L^2)}+\left\|u'\right\|_{L^2(0,T;H_0^1)}.$$

Then we have the following theorem.

Theorem 3.5. *Suppose that the assumptions $(H_1)-(H_4)$ hold. Then the recurrent sequence $\{u_m\}$ defined by (3.8)-(3.9) strongly converges to u in $W_1(T)$. Furthermore, u is a unique weak solution of Prob. (1.1) and $u\in W_1(M,T)$. On the other hand, the following estimation is valid*

$$\|u_m-u\|_{W_1(T)}\leq C_T k_T^m, \ \textit{for all } m\in\mathbb{N},$$

where $k_T\in[0,1)$ is defined as in (3.20) and C_T is a constant depending only on T, f, g, μ, $\bar{\mu}$, $\tilde{u}_0$, $\tilde{u}_1$.

Proof of Theorem 3.5. First, we prove the local existence of Prob. (1.1). We begin by proving that $\{u_m\}$ (in Theorem 3.1) is a Cauchy sequence in $W_1(T)$. Let $w_m = u_{m+1} - u_m$. Then w_m satisfies the variational problem

$$\begin{cases} \langle w_m''(t), v\rangle + \lambda\langle w_{mx}'(t), v_x\rangle + B_m(t,v) \\ \qquad\qquad = \int_0^t g(t-s)\bar{B}_m(s,v)ds + \langle F_{m+1}(t) - F_m(t), v\rangle, \ \forall v \in H_0^1, \\ w_m(0) = w_m'(0) = 0, \end{cases} \tag{3.24}$$

where

$$\begin{aligned} B_m(t,v) &= a_{m+1}(t; u_{m+1}(t), v) - a_m(t; u_m(t), v), \\ \bar{B}_m(t,v) &= \bar{a}_{m+1}(t; u_{m+1}(t), v) - \bar{a}_m(t; u_m(t), v), \ v \in H_0^1. \end{aligned}$$

Taking $v = w_m'(t)$ in $(3.24)_1$ and then integrating in t, we get

$$\begin{aligned} \bar{\mu}_* \bar{S}_m(t) \ &\leq \ 2\int_0^t \left\langle F_{m+1}(s) - F_m(s), w_m'(s)\right\rangle ds + \int_0^t ds \int_0^1 \mu_{3m+1}'(x,s) w_{mx}^2(x,s)dx \qquad (3.25) \\ &\quad -2\int_0^t \left\langle \left[\mu_{3m+1}(s) - \mu_{3m}(s)\right] u_{mx}(s) + \mu_{1m+1}(s) - \mu_{1m}(s), w_{mx}'(s)\right\rangle ds \\ &\quad +2\int_0^t g(t-s)\bar{B}_m(s, w_m(t))ds - 2g(0)\int_0^t \bar{B}_m(s, w_m(s))ds \\ &\quad -2\int_0^t d\tau \int_0^\tau g'(\tau - s)\bar{B}_m(s, w_m(\tau))ds = \sum\nolimits_{j=1}^6 \bar{I}_j, \end{aligned}$$

where $\bar{\mu}_* = \min\{1, \mu_*, \lambda\}$ and

$$\bar{S}_m(t) = \left\|w_m'(t)\right\|^2 + \left\|w_{mx}(t)\right\|^2 + \int_0^t \left\|w_{mx}'(s)\right\|^2 ds. \tag{3.26}$$

Next, the integrals on right-hand side of (3.25) are estimated as follows.

By the following inequalities

$$\begin{aligned} \left\|F_{m+1}(t) - F_m(t)\right\| \ &\leq \ 2\tilde{K}_M(f)\left[\left\|\nabla w_{m-1}(t)\right\| + \left\|w_{m-1}'(t)\right\|\right] + \sigma\left\|\nabla w_{m-1}'(t)\right\| \\ &\leq \ 2\tilde{K}_M(f)\left\|w_{m-1}\right\|_{W_1(T)} + \sigma\left\|\nabla w_{m-1}'(t)\right\|, \\ \left(\int_0^t \left\|F_{m+1}(s) - F_m(s)\right\|^2 ds\right)^{1/2} \ &\leq \ 2\sqrt{T}\tilde{K}_M(f)\left\|w_{m-1}\right\|_{W_1(T)} + \sigma\left(\int_0^t \left\|\nabla w_{m-1}'(s)\right\|^2 ds\right)^{1/2} \\ &\leq \ (2\sqrt{T}\tilde{K}_M(f) + \sigma)\left\|w_{m-1}\right\|_{W_1(T)}, \\ \left|\mu_{im+1}(x,t) - \mu_{im}(x,t)\right| \ &\leq \ K_M(\mu)\left|w_{m-1}(x,t)\right| \leq K_M(\mu)\left\|w_{m-1}\right\|_{W_1(T)}, \ i = 1, 3, \end{aligned}$$

the terms $\bar{I}_1$, $\bar{I}_2$, $\bar{I}_3$ are estimated by

$$\begin{aligned} \bar{I}_1 \ &= \ 2\int_0^t \left\langle F_{m+1}(s) - F_m(s), w_m'(s)\right\rangle ds \qquad (3.27) \\ &\leq \ (2\sqrt{T}\tilde{K}_M(f) + \sigma)\left\|w_{m-1}\right\|_{W_1(T)}^2 + \int_0^t \bar{S}_m(s)ds, \\ \bar{I}_2 \ &= \ \int_0^t ds \int_0^1 \mu_{3m+1}'(x,s) w_{mx}^2(x,s)dx \leq (1+M)K_M(\mu)\int_0^t \bar{S}_m(s)ds, \\ \bar{I}_3 \ &= \ -2\int_0^t \left\langle \left[\mu_{3m+1}(s) - \mu_{3m}(s)\right] u_{mx}(s) + \mu_{1m+1}(s) - \mu_{1m}(s), w_{mx}'(s)\right\rangle ds \\ &\leq \ 2(1+M)K_M(\mu)\left\|w_{m-1}\right\|_{W_1(T)} \int_0^t \left\|w_{mx}'(s)\right\| ds \\ &\leq \ \frac{\bar{\mu}_*}{6}\bar{S}_m(t) + \frac{6}{\bar{\mu}_*}T(1+M)^2 K_M^2(\mu)\left\|w_{m-1}\right\|_{W_1(T)}^2. \end{aligned}$$

For the integral $\bar{I}_4$, $\bar{I}_5$, $\bar{I}_6$, we note that

$$\begin{aligned}\bar{B}_m(s,w_m(t)) &= \bar{a}_{m+1}(s;u_{m+1}(s),w_m(t)) - \bar{a}_m(s;u_m(s),w_m(t)) \\ &= \left\langle \bar{\mu}_{3m+1}(s) w_{mx}(s), w_{mx}(t)\right\rangle \\ &\quad + \left\langle \left[\bar{\mu}_{3m+1}(s)-\bar{\mu}_{3m}(s)\right] u_{mx}(s) + \bar{\mu}_{1m+1}(s) - \bar{\mu}_{1m}(s), w_{mx}(t)\right\rangle,\end{aligned}$$

hence

$$\left|\bar{B}_m(s,w_m(t))\right| \leq K_M(\bar{\mu}) \left[\sqrt{\bar{S}_m(s)} + (1+M)\left\|w_{m-1}\right\|_{W_1(T)}\right]\sqrt{\bar{S}_m(t)}.$$

Then

$$\begin{aligned}\bar{I}_4 &= 2\int_0^t g(t-s)\bar{B}_m(s,w_m(t))ds \qquad (3.28)\\ &\leq 2K_M(\bar{\mu})\int_0^t |g(t-s)|\sqrt{\bar{S}_m(s)}\sqrt{\bar{S}_m(t)}ds \\ &\quad +2(1+M)K_M(\bar{\mu})\left\|w_{m-1}\right\|_{W_1(T)}\int_0^t |g(t-s)|\sqrt{\bar{S}_m(t)}ds \\ &\leq \frac{\bar{\mu}_*}{3}\bar{S}_m(t) + \frac{6}{\bar{\mu}_*}T(1+M)^2K_M^2(\bar{\mu})\left\|g\right\|^2_{L^2(0,T^*)}\left\|w_{m-1}\right\|^2_{W_1(T)} \\ &\quad +\frac{6}{\bar{\mu}_*}K_M^2(\bar{\mu})\left\|g\right\|^2_{L^2(0,T^*)}\int_0^t \bar{S}_m(s)ds, \\ \bar{I}_5 &= -2g(0)\int_0^t \bar{B}_m(s,w_m(s))ds \\ &\leq 2\left|g(0)\right|K_M(\bar{\mu})\int_0^t\left[\bar{S}_m(s) + (1+M)\left\|w_{m-1}\right\|_{W_1(T)}\sqrt{\bar{S}_m(s)}\right]ds \\ &= 4\left|g(0)\right|K_M(\bar{\mu})\int_0^t \bar{S}_m(s)ds + \frac{1}{2}T\left|g(0)\right|K_M(\bar{\mu})(1+M)^2\left\|w_{m-1}\right\|^2_{W_1(T)}, \\ \bar{I}_6 &= -2\int_0^t d\tau\int_0^\tau g'(\tau-s)\bar{B}_m(s,w_m(\tau))ds \leq 2K_M(\bar{\mu})\int_0^t d\tau\int_0^\tau \left|g'(\tau-s)\right|\sqrt{\bar{S}_m(s)}\sqrt{\bar{S}_m(\tau)}ds \\ &\quad +2(1+M)K_M(\bar{\mu})\left\|w_{m-1}\right\|_{W_1(T)}\int_0^t d\tau\int_0^\tau \left|g'(\tau-s)\right|\sqrt{\bar{S}_m(\tau)}ds \\ &\leq 4K_M(\bar{\mu})\sqrt{T^*}\left\|g'\right\|_{L^2(0,T^*)}\int_0^t \bar{S}_m(s)ds + \frac{1}{2}T(1+M)^2K_M(\bar{\mu})\sqrt{T^*}\left\|g'\right\|_{L^2(0,T^*)}\left\|w_{m-1}\right\|^2_{W_1(T)}.\end{aligned}$$

Combining the estimations (3.27) and (3.28), we deduce from (3.26) that

$$\bar{S}_m(t) \leq D_1^*(M,T)\left\|w_{m-1}\right\|^2_{W_1(T)} + 2D_2^*(M)\int_0^t \bar{S}_m(s)ds,$$

where $D_1^*(M,T)$, $D_2^*(M)$ are defined as in Lemma 3.4.

Using Gronwall's lemma, we get from (3.26) that

$$\bar{S}_m(t) \leq TD_1^*(M)\left\|w_{m-1}\right\|^2_{W_1(T)}\exp\left(2TD_2^*(M)\right), \qquad (3.29)$$

hence, it leads to

$$\left\|w_m\right\|_{W_1(T)} \leq k_T\left\|w_{m-1}\right\|_{W_1(T)}, \ \forall m\in\mathbb{N},$$

where the constant $k_T\in[0,1)$ is defined as in (3.20), which implies that

$$\left\|u_{m+p}-u_m\right\|_{W_1(T)} \leq \frac{M}{1-k_T}k_T^m, \ \forall m, \ p\in\mathbb{N}.$$

It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \longrightarrow u \text{ strongly in } W_1(T). \tag{3.30}$$

Note that $u_m \in W(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in} \quad L^\infty(0,T;H^2 \cap H_0^1) \text{ weak*}, \\ u'_{m_j} \to u' & \text{in} \quad L^\infty(0,T;H^2 \cap H_0^1) \text{ weak*}, \\ u''_{m_j} \to u'' & \text{in} \quad L^2(0,T;H_0^1) \text{ weak}, \\ u \in W(M,T). \end{cases} \tag{3.31}$$

Since

$$\|F_m(t) - F[u](t)\|_{L^2(Q_T)} \leq (\sigma + 2\tilde{K}_M(f)) \|u_{m-1} - u\|_{W_1(T)}, \tag{3.32}$$

hence, by (3.30) and (3.32), we have

$$F_m \longrightarrow f[u] \text{ strongly in } L^2(Q_T). \tag{3.33}$$

On the other hand, using the equality

$$\begin{aligned} a_m(t;u_m(t),v) - a(t;u(t),v) &= \langle \mu_{3m}(t)u_{mx}(t) - \mu_3[u](t)u_x(t) + \mu_{1m}(t) - \mu_1[u](t), v_x\rangle \\ &= \langle \mu_{3m}(t)\left[u_{mx}(t) - u_x(t)\right] + \left[\mu_{3m}(t) - \mu_3[u](t)\right] u_x(t), v_x\rangle \\ &\quad + \langle \mu_{1m}(t) - \mu_1[u](t), v_x\rangle, \end{aligned}$$

and the inequality

$$\left|\mu_{im+1}(x,t) - \mu_{im}(x,t)\right| \leq K_M(\mu)\left|w_{m-1}(x,t)\right| \leq K_M(\mu)\|w_{m-1}\|_{W_1(T)}, \; i = 1,3,$$

we get

$$|a_m(t;u_m(t),v) - a(t;u(t),v)| \leq K_M(\mu)\left[\|u_m - u\|_{W_1(T)} + (1+M)\|u_{m-1} - u\|_{W_1(T)}\right]\|v_x\|.$$

Hence

$$a_m(t;u_m(t),v) \longrightarrow a(t;u(t),v) \text{ in } L^\infty(0,T) \text{ weak*, for all } v \in H_0^1. \tag{3.34}$$

Similarly

$$\int_0^t g(t-s)\bar{a}_m(s;u_m(s),v)ds \longrightarrow \int_0^t g(t-s)\bar{a}_m(s;u_m(s),v)ds, \tag{3.35}$$

in $L^\infty(0,T)$ weak*, for all $v \in H_0^1$.

Passing to limit in (3.8) and (3.9) as $m = m_j \to \infty$, it implies from (3.33), (3.34) and (3.35) that there exists $u \in W(M,T)$ satisfying (3.1), (3.2).

On the other hand, we derive from (3.1) and $(3.31)_4$ that

$$\begin{aligned} u'' &= \lambda u'_{xx} + \frac{\partial^2}{\partial x^2}\left(\mu(t,u(t))\right) - \int_0^t g(t-s)\frac{\partial^2}{\partial x^2}\left(\bar{\mu}(s,u(s))\right)ds + f[u] \\ &\equiv \tilde{f} \in L^\infty(0,T;L^2). \end{aligned}$$

Thus $u \in W_1(M,T)$.The proof of existence is completed.

Finally, we need to prove the uniqueness of solutions. Let u_1, $u_2 \in W_1(M,T)$ be two weak solutions of Prob. (1.1). Then $u = u_1 - u_2$ satisfies the variational problem

$$\begin{cases} \langle u''(t),v\rangle + \lambda\langle u'_x(t),v_x\rangle + B(t,v) \\ \qquad\qquad = \displaystyle\int_0^t g(t-s)\bar{B}(s,v)ds + \left\langle \bar{F}_1(t) - \bar{F}_2(t), v\right\rangle, \; \forall v \in H_0^1, \\ u(0) = u'(0) = 0, \end{cases} \tag{3.36}$$

where

$$\begin{aligned}
B(t,v) &= a(t;u_1(t),v) - a(t;u_2(t),v) \\
&= \langle \mu_3[u_1](t)u_x(t) + [\mu_3[u_1](t) - \mu_3[u_2](t)]\, u_{2x}(t), v_x\rangle + \langle \mu_1[u_1](t) - \mu_1[u_2](t), v_x\rangle, \\
\bar{B}(t,v) &= \bar{a}(t;u_1(t),v) - \bar{a}(t;u_2(t),v) \\
&= \langle \bar{\mu}_3[u_1](t)u_x(t) + [\bar{\mu}_3[u_1](t) - \bar{\mu}_3[u_2](t)]\, u_{2x}(t), v_x\rangle + \langle \bar{\mu}_1[u_1](t) - \bar{\mu}_1[u_2](t), v_x\rangle,\ v \in H_0^1, \\
\mu_i[u](x,t) &= D_i\mu(x,t,u(x,t)),\ \bar{\mu}_i[u](x,t) = D_i\bar{\mu}(x,t,u(x,t)),\ i = 1,3, \\
\bar{F}_j(t) &= f[u_j](t),\ j = 1,2.
\end{aligned}$$

Taking $v = u'(t)$ in $(3.36)_1$ and integrating in time from 0 to t, we get

$$\begin{aligned}
\bar{\mu}_* \bar{Z}(t) \le & \int_0^t ds \int_0^1 \mu_3'[u_1](x,s) u_x^2(x,s)dx \\
& -2\int_0^t \left\langle [\mu_3[u_1](s) - \mu_3[u_2](s)]\, u_{2x}(s), u_x'(s)\right\rangle ds - 2\int_0^t \left\langle \mu_1[u_1](s) - \mu_1[u_2](s), u_x'(s)\right\rangle ds \\
& +2\int_0^t g(t-s)\bar{B}(s,u(t))ds - 2g(0)\int_0^t \bar{B}(s,u(s))ds \\
& -2\int_0^t d\tau \int_0^\tau g'(\tau - s)\bar{B}(s,u(\tau))ds + 2\int_0^t \left\langle \bar{F}_1(s) - \bar{F}_2(s), u'(s)\right\rangle ds,
\end{aligned} \tag{3.37}$$

where

$$\bar{Z}(t) = \left\|u'(t)\right\|^2 + \|u_x(t)\|^2 + \int_0^t \left\|u_x'(s)\right\|^2 ds. \tag{3.38}$$

Through similar calculations in Theorem 3.1, we obtain from (3.37), (3.38) that

$$\left(\bar{\mu}_* - 2\sigma^2 - 2\gamma\right) \bar{Z}(t) \le \eta(M,\gamma)\int_0^t \bar{Z}(s)ds, \tag{3.39}$$

for all $\gamma > 0$, where

$$\begin{aligned}
\eta(M,\gamma) = \; & 1 + 16\tilde{K}_M^2(f) + (1+M)K_M(\mu) \\
& +2(2+M)K_M(\bar{\mu})\left(|g(0)| + \sqrt{T^*}\left\|g'\right\|_{L^2(0,T^*)}\right) \\
& +\frac{1}{\gamma}\left[(1+M)^2K_M^2(\mu) + (2+M)^2K_M^2(\bar{\mu})\,\|g\|_{L^2(0,T^*)}^2\right].
\end{aligned}$$

Since $0 < \sigma < \dfrac{\sqrt{\bar{\mu}_*}}{3\sqrt{2}}$, it follows that $\bar{\mu}_* - 2\sigma^2 > 0$. Then, by choosing $\gamma > 0$ such that $\bar{\mu}_* - 2\sigma^2 - 2\gamma > 0$ and using Gronwall lemma, we deduce from (3.39) that $\bar{Z}(t) \equiv 0$, i.e., $u = u_1 - u_2 = 0$.

Therefore, uniqueness is proved. The proof of Theorem 3.5 is done. $\square$

4 Continuous dependence

In this section, we assume that $\lambda > 0$ and $\tilde{u}_0,\ \tilde{u}_1 \in H^2 \cap H_0^1$. By Theorem 3.5, Prob. (1.1) admits a unique solution u depending on the datum $\mu,\ \bar{\mu},\ f,\ g$

$$u = u(\mu, \bar{\mu}, f, g),$$

where $\mu,\ \bar{\mu},\ f,\ g$ satisfy the assumptions $(H_2) - (H_4)$.

First, we note that if the datum $(\mu, \bar{\mu}, f, g)$, $(\mu_j, \bar{\mu}_j, f_j, g_j)$ satisfy $(H_2) - (H_4)$ and in addition, the following condition is fulfilled

$$\begin{aligned} d_1(\mu_j, \mu) &\equiv \sup_{M>0} \max_{|\beta|\leq 3} \left\| D^\beta \mu_j - D^\beta \mu \right\|_{C^0(A_M)} \to 0, \\ d_1(\bar{\mu}_j, \bar{\mu}) &\equiv \sup_{M>0} \max_{|\beta|\leq 3} \left\| D^\beta \bar{\mu}_j - D^\beta \bar{\mu} \right\|_{C^0(A_M)} \to 0, \\ \tilde{d}(f_j, f) &\equiv \sup_{M>0} \max_{|\alpha|\leq 1} \| D^\alpha f_j - D^\alpha f \|_{C^0(\tilde{A}_M)} \to 0, \\ & \qquad \|g_j - g\|_{H^1(0,T^*)} \to 0, \end{aligned} \tag{4.1}$$

as $j \to \infty$, then there exists $j_0 \in \mathbb{N}$ (independent of M) such that

$$\begin{cases} \left\| D^\beta \mu_j \right\|_{C^0(A_M)} \leq 1 + \left\| D^\beta \mu \right\|_{C^0(A_M)}, \forall \beta \in \mathbb{Z}_+^3,\ |\beta| \leq 3,\ \forall M > 0,\ \forall j \geq j_0, \\ \left\| D^\beta \bar{\mu}_j \right\|_{C^0(A_M)} \leq 1 + \left\| D^\beta \bar{\mu} \right\|_{C^0(A_M)}, \forall \beta \in \mathbb{Z}_+^3,\ |\beta| \leq 3,\ \forall M > 0,\ \forall j \geq j_0, \\ \| D^\alpha f_j \|_{C^0(\tilde{A}_M)} \leq 1 + \| D^\alpha f \|_{C^0(\tilde{A}_M)}, \forall \alpha \in \mathbb{Z}_+^6,\ |\alpha| \leq 1,\ \forall M > 0,\ \forall j \geq j_0, \\ \|g_j\|_{H^1(0,T^*)} \leq 1 + \|g\|_{H^1(0,T^*)}, \forall j \geq j_0. \end{cases}$$

By setting the constants $K_M(\mu)$, $K_M(\bar{\mu})$, $\tilde{K}_M(f)$ and (H_3), we deduce from the above estimation that

$$\begin{cases} K_M(\mu_j) \leq 1 + K_M(\mu),\ \forall M > 0,\ \forall j \geq j_0, \\ K_M(\bar{\mu}_j) \leq 1 + K_M(\bar{\mu}),\ \forall M > 0,\ \forall j \geq j_0, \\ \tilde{K}_M(f_j) \leq 1 + \tilde{K}_M(f),\ \forall M > 0,\ \forall j \geq j_0, \\ \|g_j\|_{H^1(0,T^*)} \leq 1 + \|g\|_{H^1(0,T^*)}, \forall j \geq j_0. \end{cases}$$

Therefore, the Galerkin approximation sequence $\{u_m^{(k)}\}$ corresponding to $(\mu, \bar{\mu}, f, g) = (\mu_j, \bar{\mu}_j, f_j, g_j)$, $j \geq j_0$ also satisfies the priori estimates as in Theorem 3.1 and

$$u_m^{(k)} \in W_1(M, T), \text{ for all } m \text{ and } k \in \mathbb{N},$$

where M, T are constants independent of j. Indeed, in the process, we can choose the positive constants M and T as in (3.19) and (3.21) with replacing $K_M(\mu)$, $K_M(\bar{\mu})$, $\tilde{K}_M(f)$, $|g(0)|$, $|\mu_{1mx}(0)|$, $|\mu_{3mx}(0)|$ by $1 + K_M(\mu)$, $1 + K_M(\bar{\mu})$, $1 + \tilde{K}_M(f)$, $1 + |g(0)|$, $1 + |\mu_{1mx}(0)|$, $1 + |\mu_{3mx}(0)|$, respectively.

Hence, the limitation u_j of $\{u_m^{(k)}\}$, as $k \to +\infty$ and $m \to +\infty$ later, is the unique weak solution of Prob. (1.1) corresponding to $(\mu, \bar{\mu}, f) = (\mu_j, \bar{\mu}_j, f_j)$, $j \geq j_0$ satisfying

$$u_j \in W_1(M, T), \text{ for all } j \geq j_0.$$

Moreover, by the same argument used in Theorem 3.5, we can prove that the limitation u of $\{u_j\}$ as $j \to +\infty$, is the unique weak solution of Prob. (1.1) and $u \in W_1(M, T)$.

Consequently, we have the following theorem.

Theorem 4.1. *For any $\lambda > 0$, $\tilde{u}_0$, $\tilde{u}_1 \in H^2 \cap H_0^1$, suppose that $(H_2) - (H_4)$ and the condition (4.1) hold. Then, there exists a positive constant T such that the solution of Prob. (1.1) is continuous dependence on the datum μ, $\bar{\mu}$, f, g, i.e., if $(\mu, \bar{\mu}, f, g)$ and $(\mu_j, \bar{\mu}_j, f_j, g_j)$ satisfy $(H_2) - (H_4)$ and (4.1), then*

$$u_j = u(\mu_j, \bar{\mu}_j, f_j, g_j) \longrightarrow u \text{ strongly in } W_1(T), \text{ as } j \to \infty.$$

Moreover, we have the estimation

$$\|u_j - u\|_{W_1(T)} \leq C_T \left(d_1(\mu_j, \mu) + d_1(\bar{\mu}_j, \bar{\mu}) + \tilde{d}(f_j, f) + \|g_j - g\|_{H^1(0,T^*)} \right), \ \forall j \geq j_0,$$

where C_T is a constant only depending on T, f, g, μ, $\bar{\mu}$, $\tilde{u}_0$ and $\tilde{u}_1$.

Proof of Theorem 4.1. Setting

$$\begin{aligned}
\tilde{g}_j &= g_j - g, \\
\tilde{F}_j(x,t) &= f_j[u_j](x,t) - f[u](x,t), \\
f_j[u_j](x,t) &= f_j(x,t,u_j(x,t),u_j'(x,t),u_{jx}(x,t),u_{jx}'(x,t)), \\
f[u](x,t) &= f(x,t,u(x,t),u'(x,t),u_x(x,t),u_x'(x,t)),
\end{aligned}$$

then $w_j = u_j - u$, satisfies the variational problem

$$\begin{cases}
\langle w_j''(t),v\rangle + \lambda\langle w_{jx}'(t),v_x\rangle + a_j(t;u_j(t),v) - a(t;u(t),v) \\
\qquad\qquad = \displaystyle\int_0^t [g_j(t-s)\bar{a}_j(s;u_j(s),v) - g(t-s)\bar{a}(s;u(s),v)]\,ds \\
\qquad\qquad + \left\langle \tilde{F}_j(t),v\right\rangle, \ \forall v \in H_0^1, \\
w_j(0) = w_j'(0) = 0,
\end{cases} \tag{4.2}$$

where

$$\begin{aligned}
a_j(t;u_j(t),v) &= \left\langle D_3\mu_j(t,u_j(t))u_{jx}(t),v_x\right\rangle + \left\langle D_1\mu_j(t,u_j(t)),v_x\right\rangle, \\
a(t;u(t),v) &= \left\langle D_3\mu(t,u(t))u_x(t),v_x\right\rangle + \left\langle D_1\mu(t,u(t)),v_x\right\rangle, \\
\bar{a}_j(t;u_j(t),v) &= \left\langle D_3\bar{\mu}_j(t,u_j(t))u_{jx}(t),v_x\right\rangle + \left\langle D_1\bar{\mu}_j(t,u_j(t)),v_x\right\rangle, \\
\bar{a}(t;u(t),v) &= \left\langle D_3\bar{\mu}(t,u(t))u_x(t),v_x\right\rangle + \left\langle D_1\bar{\mu}(t,u(t)),v_x\right\rangle.
\end{aligned}$$

On the other hand, by the following equalities

$$\begin{aligned}
& a_j(t;u_j(t),v) - a(t;u(t),v) \\
=\ & \left\langle D_3\mu_j(t,u_j(t))w_{jx}(t),v_x\right\rangle + \left\langle \left[D_3\mu_j(t,u_j(t)) - D_3\mu(t,u(t))\right]u_x(t),v_x\right\rangle \\
& + \left\langle D_1\mu_j(t,u_j(t)) - D_1\mu(t,u(t)),v_x\right\rangle, \\
& \bar{a}_j(s;u_j(s),v) - \bar{a}(s;u(s),v) \\
=\ & \left\langle D_3\bar{\mu}_j(s,u_j(s))w_{jx}(s),v_x\right\rangle + \left\langle \left[D_3\bar{\mu}_j(s,u_j(s)) - D_3\bar{\mu}(s,u(s))\right]u_x(s),v_x\right\rangle \\
& + \left\langle D_1\bar{\mu}_j(s,u_j(s)) - D_1\bar{\mu}(s,u(s)),v_x\right\rangle, \\
& g_j(t-s)\bar{a}_j(s;u_j(s),v) - g(t-s)\bar{a}(s;u(s),v) \\
=\ & [g_j(t-s) - g(t-s)]\,\bar{a}_j(s;u_j(s),v) + g(t-s)\left[\bar{a}_j(s;u_j(s),v) - \bar{a}(s;u(s),v)\right] \\
=\ & [g_j(t-s) - g(t-s)]\left[\left\langle D_3\bar{\mu}_j(s,u_j(s))u_{jx}(s),v_x\right\rangle + \left\langle D_1\bar{\mu}_j(s,u_j(s)),v_x\right\rangle\right] \\
& + g(t-s)\left\langle D_3\bar{\mu}_j(s,u_j(s))w_{jx}(s),v_x\right\rangle \\
& + g(t-s)\left\langle \left[D_3\bar{\mu}_j(s,u_j(s)) - D_3\bar{\mu}(s,u(s))\right]u_x(s),v_x\right\rangle \\
& + g(t-s)\left\langle D_1\bar{\mu}_j(s,u_j(s)) - D_1\bar{\mu}(s,u(s)),v_x\right\rangle,
\end{aligned}$$

we rewrite (4.2) by

$$\begin{cases}
\langle w_j''(t),v\rangle + \lambda\langle w_{jx}'(t),v_x\rangle + \left\langle D_3\mu_j(t,u_j(t))w_{jx}(t),v_x\right\rangle \\
\quad = \displaystyle\int_0^t g(t-s)\left\langle D_3\bar{\mu}_j(s,u_j(s))w_{jx}(s),v_x\right\rangle ds \\
\quad + \displaystyle\int_0^t [g_j(t-s) - g(t-s)]\left[\left\langle D_3\bar{\mu}_j(s,u_j(s))u_{jx}(s),v_x\right\rangle + \left\langle D_1\bar{\mu}_j(s,u_j(s)),v_x\right\rangle\right] ds \\
\quad + \displaystyle\int_0^t g(t-s)\left\langle \left[D_3\bar{\mu}_j(s,u_j(s)) - D_3\bar{\mu}(s,u(s))\right]u_x(s),v_x\right\rangle ds \\
\quad + \displaystyle\int_0^t g(t-s)\left\langle D_1\bar{\mu}_j(s,u_j(s)) - D_1\bar{\mu}(s,u(s)),v_x\right\rangle ds \\
\quad - \left\langle \left[D_3\mu_j(t,u_j(t)) - D_3\mu(t,u(t))\right]u_x(t),v_x\right\rangle - \left\langle D_1\mu_j(t,u_j(t)) - D_1\mu(t,u(t)),v_x\right\rangle \\
\quad + \left\langle \tilde{F}_j(t),v\right\rangle, \ \forall v \in H_0^1, \\
w_j(0) = w_j'(0) = 0.
\end{cases} \tag{4.3}$$

Taking $v = w_j'(t)$ in $(4.4)_1$ and then integrating in t, we get

$$\begin{aligned}
\bar{\mu}_* \bar{S}_j(t) &\leq \int_0^t ds \int_0^1 \frac{\partial}{\partial s}\left[D_3\mu_j(x,s,u_j(x,s))\right] w_{jx}^2(x,s)dx \\
&\quad +2\int_0^t d\tau \int_0^\tau g(\tau-s)\left\langle D_3\bar{\mu}_j(s,u_j(s))w_{jx}(s), w_{jx}'(\tau)\right\rangle ds \\
&\quad +2\int_0^t d\tau \int_0^\tau \left[g_j(\tau-s)-g(\tau-s)\right]\left\langle D_1\bar{\mu}_j(s,u_j(s)) + D_3\bar{\mu}_j(s,u_j(s))u_{jx}(s), w_{jx}'(\tau)\right\rangle ds \\
&\quad +2\int_0^t d\tau \int_0^\tau g(\tau-s)\left\langle D_1\bar{\mu}_j(s,u_j(s)) - D_1\bar{\mu}(s,u(s)), w_{jx}'(\tau)\right\rangle ds \\
&\quad +2\int_0^t d\tau \int_0^\tau g(\tau-s)\left\langle \left[D_3\bar{\mu}_j(s,u_j(s)) - D_3\bar{\mu}(s,u(s))\right]u_x(s), w_{jx}'(\tau)\right\rangle ds \\
&\quad -2\int_0^t \left\langle \left[D_3\mu_j(s,u_j(s)) - D_3\mu(s,u(s))\right]u_x(s), w_{jx}'(s)\right\rangle ds \\
&\quad -2\int_0^t \left\langle D_1\mu_j(s,u_j(s)) - D_1\mu(s,u(s)), w_{jx}'(s)\right\rangle ds + 2\int_0^t \left\langle \tilde{F}_j(s), w_j'(s)\right\rangle ds \\
&= \sum_{j=1}^{8} I_j,
\end{aligned} \tag{4.4}$$

where $\bar{\mu}_* = \min\{1, \lambda, \mu_*\}$ and

$$\bar{S}_j(t) = \left\|w_j'(t)\right\|^2 + \|w_{jx}(t)\|^2 + \int_0^t \left\|w_{jx}'(s)\right\|^2 ds.$$

We estimate the terms I_j on the right-hand side of (4.4) as follows.

Estimate of I_1. By the estimation

$$\begin{aligned}
\left|\frac{\partial}{\partial s}\left[D_3\mu_j(x,s,u_j(x,s))\right]\right| &= \left|D_2D_3\mu_j(x,s,u_j(x,s)) + D_3^2\mu_j(x,s,u_j(x,s))u_j'(x,s)\right| \\
&\leq K_M(\mu_j)\left(1+\left|u_j'(x,s)\right|\right) \leq \left(1+K_M(\mu)\right)(1+M),
\end{aligned}$$

we have

$$\begin{aligned}
I_1 &= \int_0^t ds \int_0^1 \frac{\partial}{\partial s}\left[D_3\mu_j(x,s,u_j(x,s))\right] w_{jx}^2(x,s)dx \\
&\leq \left(1+K_M(\mu)\right)(1+M)\int_0^t \|w_{jx}(s)\|^2 ds \leq \left(1+K_M(\mu)\right)(1+M)\int_0^t \bar{S}_j(s)ds.
\end{aligned} \tag{4.5}$$

Estimate of I_2. By the estimation

$$\left|D_3\bar{\mu}_j(x,s,u_j(x,s))\right| \leq K_M(\bar{\mu}_j) \leq 1 + K_M(\bar{\mu}),$$

we obtain

$$\begin{aligned}
I_2 &= 2\int_0^t d\tau \int_0^\tau g(\tau-s)\left\langle D_3\bar{\mu}_j(s,u_j(s))w_{jx}(s), w_{jx}'(\tau)\right\rangle ds \\
&\leq 2\left(1+K_M(\bar{\mu})\right)\int_0^t \left\|w_{jx}'(\tau)\right\| d\tau \int_0^\tau |g(\tau-s)|\,\|w_{jx}(s)\| ds \\
&\leq 2\left(1+K_M(\bar{\mu})\right)\sqrt{\bar{S}_j(t)}\sqrt{T^*}\,\|g\|_{L^2(0,T^*)}\left(\int_0^t \|w_{jx}(s)\|^2 ds\right)^{1/2} \\
&\leq \beta\bar{S}_j(t) + \frac{1}{\beta}\left(1+K_M(\bar{\mu})\right)^2 T^*\|g\|_{L^2(0,T^*)}^2 \int_0^t \|w_{jx}(s)\|^2 ds \\
&\leq \beta\bar{S}_j(t) + \frac{1}{\beta}\left(1+K_M(\bar{\mu})\right)^2 T^*\|g\|_{L^2(0,T^*)}^2 \int_0^t \bar{S}_j(s)ds.
\end{aligned} \tag{4.6}$$

Estimate of I_3. Note that

$$\left\| D_1\bar{\mu}_j(s,u_j(s)) + D_3\bar{\mu}_j(s,u_j(s))u_{jx}(s)\right\| \le K_M(\bar{\mu}_j)(1+\|u_{jx}(s)\|) \le (1+K_M(\bar{\mu}))\,(1+M),$$

hence

$$\begin{aligned} I_3 &= 2\int_0^t d\tau \int_0^\tau [g_j(\tau-s)-g(\tau-s)] \left\langle D_1\bar{\mu}_j(s,u_j(s)) + D_3\bar{\mu}_j(s,u_j(s))u_{jx}(s), w'_{jx}(\tau)\right\rangle ds \quad (4.7)\\ &\le 2\,(1+K_M(\bar{\mu}))\,(1+M)\int_0^t \left\|w'_{jx}(\tau)\right\| d\tau \int_0^\tau |g_j(\tau-s)-g(\tau-s)|\,ds \\ &\le 2\,(1+K_M(\bar{\mu}))\,(1+M)\sqrt{\bar{S}_j(t)}\sqrt{T^*}\,\|g_j-g\|_{L^2(0,T^*)} \\ &\le \beta\bar{S}_j(t) + \frac{1}{\beta}\,(1+K_M(\bar{\mu}))^2\,(1+M)^2T^*\,\|g_j-g\|^2_{L^2(0,T^*)}\,. \end{aligned}$$

Estimate of I_4. Using the estimation

$$\begin{aligned} &\left|D_1\bar{\mu}_j(x,s,u_j(x,s)) - D_1\bar{\mu}(x,s,u(x,s))\right| \\ \le\ &\left|D_1\bar{\mu}_j(x,s,u_j(x,s)) - D_1\bar{\mu}(x,s,u_j(x,s))\right| \\ &+ \left|D_1\bar{\mu}(x,s,u_j(x,s)) - D_1\bar{\mu}(x,s,u(x,s))\right| \\ \le\ &\sup_{(x,t,y)\in A_M}\left|D_1\bar{\mu}_j(x,s,y) - D_1\bar{\mu}(x,s,y)\right| + K_M(\bar{\mu})\,|u_j(x,s)-u(x,s)| \\ \le\ &d_1(\bar{\mu}_j,\bar{\mu}) + K_M(\bar{\mu})\sqrt{\bar{S}_j(s)}, \end{aligned}$$

we get

$$\begin{aligned} I_4 &= 2\int_0^t d\tau\int_0^\tau g(\tau-s)\left\langle D_1\bar{\mu}_j(s,u_j(s)) - D_1\bar{\mu}(s,u(s)), w'_{jx}(\tau)\right\rangle ds \quad (4.8)\\ &\le 2\int_0^t d\tau\int_0^\tau |g(\tau-s)|\left\|D_1\bar{\mu}_j(s,u_j(s)) - D_1\bar{\mu}(s,u(s))\right\|\left\|w'_{jx}(\tau)\right\| ds \\ &\le 2\sqrt{\bar{S}_j(t)}\sqrt{T^*}\,\|g\|_{L^2(0,T^*)}\left(\int_0^t \left\|D_1\bar{\mu}_j(s,u_j(s)) - D_1\bar{\mu}(s,u(s))\right\|^2 ds\right)^{1/2} \\ &\le \beta\bar{S}_j(t) + \frac{1}{\beta}T^*\,\|g\|^2_{L^2(0,T^*)}\int_0^t \left\|D_1\bar{\mu}_j(s,u_j(s)) - D_1\bar{\mu}(s,u(s))\right\|^2 ds \\ &\le \beta\bar{S}_j(t) + \frac{2}{\beta}T^*\,\|g\|^2_{L^2(0,T^*)}\int_0^t \left[d_1^2(\bar{\mu}_j,\bar{\mu}) + K_M^2(\bar{\mu})\bar{S}_j(s)\right] ds \\ &\le \beta\bar{S}_j(t) + \frac{2}{\beta}\left(T^*\,\|g\|_{L^2(0,T^*)}\right)^2 d_1^2(\bar{\mu}_j,\bar{\mu}) + \frac{2}{\beta}T^*\left(\|g\|_{L^2(0,T^*)}\,K_M(\bar{\mu})\right)^2\int_0^t \bar{S}_j(s)ds. \end{aligned}$$

Estimate of I_5. By the following inequality

$$\begin{aligned} &\left|D_1\bar{\mu}_j(x,s,u_j(x,s)) - D_1\bar{\mu}(x,s,u(x,s))\right| \\ \le\ &\left|D_1\bar{\mu}_j(x,s,u_j(x,s)) - D_1\bar{\mu}(x,s,u_j(x,s))\right| \\ &+ \left|D_1\bar{\mu}(x,s,u_j(x,s)) - D_1\bar{\mu}(x,s,u(x,s))\right| \\ \le\ &\sup_{(x,t,y)\in A_M}\left|D_1\bar{\mu}_j(x,s,y) - D_1\bar{\mu}(x,s,y)\right| + K_M(\bar{\mu})\,|u_j(x,s)-u(x,s)| \\ \le\ &d_1(\bar{\mu}_j,\bar{\mu}) + K_M(\bar{\mu})\sqrt{\bar{S}_j(s)}, \end{aligned}$$

we obtain

$$\left\|\left[D_3\bar{\mu}_j(s,u_j(s)) - D_3\bar{\mu}(s,u(s))\right]u_x(s)\right\| \le M\left(d_1(\bar{\mu}_j,\bar{\mu}) + K_M(\bar{\mu})\sqrt{\bar{S}_j(s)}\right).$$

Hence

$$\begin{aligned} I_5 &= 2\int_0^t d\tau \int_0^\tau g(\tau - s) \left\langle \left[D_3\bar{\mu}_j(s, u_j(s)) - D_3\bar{\mu}(s, u(s)) \right] u_x(s), w'_{jx}(\tau) \right\rangle ds \qquad (4.9)\\ &\leq 2\int_0^t d\tau \int_0^\tau |g(\tau - s)| \left\| \left[D_3\bar{\mu}_j(s, u_j(s)) - D_3\bar{\mu}(s, u(s)) \right] u_x(s) \right\| \left\| w'_{jx}(\tau) \right\| ds \\ &\leq \beta \bar{S}_j(t) + \frac{2}{\beta} \left(T^* M \|g\|_{L^2(0,T^*)} \right)^2 d_1^2(\bar{\mu}_j, \bar{\mu}) + \frac{2}{\beta} T^* \left(M \|g\|_{L^2(0,T^*)} K_M(\bar{\mu}) \right)^2 \int_0^t \bar{S}_j(s) ds. \end{aligned}$$

Estimate of I_6. Similarly, we verify that

$$\left\| \left[D_3\mu_j(s, u_j(s)) - D_3\mu(s, u(s)) \right] u_x(s) \right\| \leq M \left(d_1(\mu_j, \mu) + K_M(\mu) \sqrt{\bar{S}_j(s)} \right),$$

so

$$\begin{aligned} I_6 &= -2\int_0^t \left\langle \left[D_3\mu_j(s, u_j(s)) - D_3\mu(s, u(s)) \right] u_x(s), w'_{jx}(s) \right\rangle ds \qquad (4.10)\\ &\leq \beta \int_0^t \left\| w'_{jx}(s) \right\|^2 ds + \frac{1}{\beta} \int_0^t \left\| \left[D_3\mu_j(s, u_j(s)) - D_3\mu(s, u(s)) \right] u_x(s) \right\|^2 ds \\ &\leq \beta \bar{S}_j(t) + \frac{2}{\beta} T^* M^2 d_1^2(\mu_j, \mu) + \frac{2}{\beta} M^2 K_M^2(\mu) \int_0^t \bar{S}_j(s) ds. \end{aligned}$$

Estimate of I_7. Repeating the estimation similarly to I_6, we obtain

$$\begin{aligned} &\left| D_1\mu_j(x, s, u_j(x, s)) - D_1\mu(x, s, u(x, s)) \right| \\ \leq\ &\left| D_1\mu_j(x, s, u_j(x, s)) - D_1\mu(x, s, u_j(x, s)) \right| + \left| D_1\mu(x, s, u_j(x, s)) - D_1\mu(x, s, u(x, s)) \right| \\ \leq\ &\sup_{(x,t,y) \in A_M} \left| D_1\mu_j(x, s, y) - D_1\mu(x, s, y) \right| + K_M(\mu) \left| u_j(x, s) - u(x, s) \right| \\ \leq\ &d_1(\mu_j, \mu) + K_M(\mu) \sqrt{\bar{S}_j(s)}, \end{aligned}$$

so it follows

$$\begin{aligned} I_7 &= -2\int_0^t \left\langle D_1\mu_j(s, u_j(s)) - D_1\mu(s, u(s)), w'_{jx}(s) \right\rangle ds \qquad (4.11)\\ &\leq \beta \int_0^t \left\| w'_{jx}(s) \right\|^2 ds + \frac{1}{\beta} \int_0^t \left\| D_1\mu_j(s, u_j(s)) - D_1\mu(s, u(s)) \right\|^2 ds \\ &\leq \beta \bar{S}_j(t) + \frac{2}{\beta} \int_0^t \left(d_1^2(\mu_j, \mu) + K_M^2(\mu) \bar{S}_j(s) \right) ds \\ &\leq \beta \bar{S}_j(t) + \frac{2}{\beta} T^* d_1^2(\mu_j, \mu) + \frac{2}{\beta} K_M^2(\mu) \int_0^t \bar{S}_j(s) ds. \end{aligned}$$

Estimate of I_8. We note that

$$\begin{aligned} \tilde{F}_j(t) &= \tilde{F}_j(x, t) = F_j(x, t) - F(x, t) \\ &= f_j[u_j](x, t) - f[u_j](x, t) + f[u_j](x, t) - f[u](x, t). \end{aligned}$$

Since

$$\begin{aligned} &\left| f_j[u_j](x, t) - f[u_j](x, t) \right| \\ =\ &\left| f_j(x, t, u_j(x, t), u'_j(x, t), \nabla u_j(x, t), \nabla u'_j(x, t)) - f(x, t, u_j(x, t), u'_j(x, t), \nabla u_j(x, t), \nabla u'_j(x, t)) \right| \\ \leq\ &\left\| f_j - f \right\|_{C^0(\tilde{A}_M)} \leq \tilde{d}(f_j, f), \end{aligned}$$

it follows

$$\begin{aligned}\|f[u_j](t)-f[u](t)\| &\leq \tilde{K}_M(f)\left[\|w_j(t)\|+\left\|w_j'(t)\right\|+\|w_{jx}(t)\|\right]+\tilde{K}_M(f)\left\|w_{jx}'(t)\right\| \\ &\leq 2\tilde{K}_M(f)\left[\left\|w_j'(t)\right\|+\|w_{jx}(t)\|\right]+\tilde{K}_M(f)\left\|w_{jx}'(t)\right\| \\ &\leq 2\sqrt{2}\tilde{K}_M(f)\sqrt{\bar{S}_j(t)}+\tilde{K}_M(f)\left\|w_{jx}'(t)\right\|.\end{aligned}$$

Then

$$\begin{aligned}\left\|\tilde{F}_j(t)\right\| &\leq \|f_j[u_j](t)-f[u_j](t)\|+\|f[u_j](t)-f[u](t)\| \\ &\leq \tilde{d}(f_j,f)+2\sqrt{2}\tilde{K}_M(f)\sqrt{\bar{S}_j(t)}+\tilde{K}_M(f)\left\|w_{jx}'(t)\right\|.\end{aligned}$$

Hence

$$\begin{aligned}I_8 &= 2\int_0^t\left\langle\tilde{F}_j(s),w_j'(s)\right\rangle ds\leq 2\int_0^t\left\|\tilde{F}_j(s)\right\|\left\|w_j'(s)\right\|ds \qquad (4.12)\\ &\leq 2\int_0^t\left[\tilde{d}(f_j,f)+2\sqrt{2}\tilde{K}_M(f)\sqrt{\bar{S}_j(s)}+\tilde{K}_M(f)\left\|w_{jx}'(s)\right\|\right]\sqrt{\bar{S}_j(s)}ds \\ &\leq T^*\tilde{d}^2(f_j,f)+\int_0^t\bar{S}_j(s)ds+4\sqrt{2}\tilde{K}_M(f)\int_0^t\bar{S}_j(s)ds+\beta\int_0^t\left\|w_{jx}'(s)\right\|^2ds+\frac{1}{\beta}\tilde{K}_M^2(f)\int_0^t\bar{S}_j(s)ds \\ &\leq T^*\tilde{d}^2(f_j,f)+\int_0^t\bar{S}_j(s)ds+4\sqrt{2}\tilde{K}_M(f)\int_0^t\bar{S}_j(s)ds+\beta\bar{S}_j(t)+\frac{1}{\beta}\tilde{K}_M^2(f)\int_0^t\bar{S}_j(s)ds \\ &= T^*\tilde{d}^2(f_j,f)+\beta\bar{S}_j(t)+\left[1+4\sqrt{2}\tilde{K}_M(f)+\frac{1}{\beta}\tilde{K}_M^2(f)\right]\int_0^t\bar{S}_j(s)ds.\end{aligned}$$

Finally, by choosing $\beta=\frac{\bar{\mu}_*}{14}$, we get from (4.5)-(4.10) that

$$\bar{S}_j(t)\leq R_j(M)+D_M\int_0^t\bar{S}_j(s)ds,$$

where

$$\begin{aligned}R_j(M) &= \frac{2}{\bar{\mu}_*}T^*\tilde{d}^2(f_j,f)+\frac{28}{\bar{\mu}_*^2}\left(1+K_M(\bar{\mu})\right)^2(1+M)^2T^*\left\|g_j-g\right\|_{L^2(0,T^*)}^2 \\ &\quad+\frac{56}{\bar{\mu}_*^2}T^*\left(1+M^2\right)\left[d_1^2(\mu_j,\mu)+\left(T^*\|g\|_{L^2(0,T^*)}\right)^2d_1^2(\bar{\mu}_j,\bar{\mu})\right], \\ D_M &= \frac{2}{\bar{\mu}_*}\left(1+4\sqrt{2}\tilde{K}_M(f)+(1+K_M(\mu))(1+M)+\frac{14}{\bar{\mu}_*}\tilde{K}_M^2(f)\right) \\ &\quad+2\frac{28}{\bar{\mu}_*^2}\left(1+K_M(\bar{\mu})\right)^2T^*\|g\|_{L^2(0,T^*)}^2 \\ &\quad+\frac{56}{\bar{\mu}_*^2}\left(1+M^2\right)\left(K_M^2(\mu)+T^*\|g\|_{L^2(0,T^*)}^2K_M^2(\bar{\mu})\right).\end{aligned}$$

Using Gronwall's lemma, we have

$$\bar{S}_j(t)\leq R_j(M)\exp(TD_M).$$

This derive that

$$\begin{aligned}\|u_j-u\|_{W_1(T)} &\leq 3\sqrt{\exp(TD_M)R_j(M)} \\ &\leq C_T\left(d_1(\mu_j,\mu)+d_1(\bar{\mu}_j,\bar{\mu})+\tilde{d}(f_j,f)+\|g_j-g\|_{H^1(0,T^*)}\right),\ \forall j\geq j_0,\end{aligned}$$

where

$$
\begin{aligned}
C_T &= 3\sqrt{\exp(TD_M)}\max\left\{\sqrt{\frac{2}{\bar{\mu}_*}T^*},\ \frac{2\sqrt{7T^*}}{\bar{\mu}_*}\left(1+K_M(\bar{\mu})\right)(1+M),\right. \\
&\qquad \left.\frac{2\sqrt{14}}{\bar{\mu}_*}\sqrt{T^*\left(1+M^2\right)},\ \frac{2\sqrt{14}}{\bar{\mu}_*}\sqrt{T^*\left(1+M^2\right)}T^*\left\|g\right\|_{L^2(0,T^*)}\right\}.
\end{aligned}
$$

Theorem 4.1 is proved. □

Remark 4.2. We give here an example, in which the condition (4.1) is satisfied.

(i) Considering $\{f_j\}$ defined by

$$
f_j(x,t,y_1,\cdots,y_4) = f(x,t,y_1,\cdots,y_4) + \frac{x^2t^2y_1^2}{j\left(1+y_1^2\right)},\ (x,t,y_1,\cdots,y_4)\in[0,1]\times[0,T^*]\times\mathbb{R}^4,
$$

where $f\in C^1([0,1]\times[0,T^*]\times\mathbb{R}^4)$ satisfies (H_4).

It is easy to check that $f_j\in C^1([0,1]\times[0,T^*]\times\mathbb{R}^4)$ also satisfies (H_4) and

$$
\tilde{d}(f_j,f)\equiv\sup_{M>0}\left(\max_{|\alpha|\le 1}\left\|D^\alpha f_j - D^\alpha f\right\|_{C^0(\tilde{A}_M)}\right)\to 0.
$$

Indeed, for all $(x,t,y_1,\cdots,y_4)\in\tilde{A}_M = [0,1]\times[0,T^*]\times[-M,M]^2\times[-\sqrt{2}M,\sqrt{2}M]^2$, we can estimate that

$$
\left\|f_j - f\right\|_{C^0(\tilde{A}_M)} \le \frac{\left(T^*\right)^2M^2}{j\left(1+M^2\right)} \le \frac{\left(T^*\right)^2}{j},\ \forall M>0.
$$

Similarly, we have

$$
\begin{aligned}
\left\|D_1f_j - D_1f\right\|_{C^0(\tilde{A}_M)} &\le \frac{2\left(T^*\right)^2M^2}{j\left(1+M^2\right)} \le \frac{2\left(T^*\right)^2}{j},\ \forall M>0, \\
\left\|D_2f_j - D_2f\right\|_{C^0(\tilde{A}_M)} &\le \frac{2T^*M^2}{j\left(1+M^2\right)} \le \frac{2T^*}{j},\ \forall M>0, \\
\left\|D_3f_j - D_3f\right\|_{C^0(\tilde{A}_M)} &\le \frac{\left(T^*\right)^2}{j},\ \forall M>0, \\
\left\|D_if_j - D_if\right\|_{C^0(\tilde{A}_M)} &= 0,\ i=4,5,6,
\end{aligned}
$$

then

$$
\tilde{d}(f_j,f)\equiv\sup_{M>0}\max_{|\alpha|\le 1}\left\|D^\alpha f_j - D^\alpha f\right\|_{C^0(\tilde{A}_M)} \le \frac{2}{j}\max\left\{\left(T^*\right)^2, T^*\right\}\to 0,\ \text{as } j\to\infty.
$$

(ii) Considering $\{\mu_j\}$ defined by

$$
\mu_j(x,t,y) = \mu(x,t,y) + \frac{xty^2}{j\left(1+y^2\right)},\ (x,t,y)\in[0,1]\times[0,T^*]\times\mathbb{R},
$$

where $\mu\in C^3([0,1]\times[0,T^*]\times\mathbb{R})$ satisfies (H_2).

It is easy to check that $\mu_j\in C^3([0,1]\times[0,T^*]\times\mathbb{R})$ also satisfies (H_2) and

$$
d_1(\mu_j,\mu)\equiv\sup_{M>0}\left(\max_{|\beta|\le 3}\left\|D^\beta\mu_j - D^\beta\mu\right\|_{C^0(A_M)}\right)\le\frac{1}{j}\max\left\{5, 18T^*\right\}\to 0,\ \text{as } j\to\infty.
$$

(iii) It is similar to give $\{\bar{\mu}_j\}$ and $\{g_j\}$, we omit here. □

5 Global existence and general decay

In this section, we investigate the general decay of solutions to Prob. (1.1) in the specific case $\mu = \mu(t,u)$, $\bar{\mu}(u) = u$, $f = -\lambda_1 u_t + f(u) - \frac{1}{2}D_2^2\mu(t,u)u_x^2 + F(x,t)$. Precisely, we shall consider the following problem

$$\begin{cases} u_{tt} + \lambda_1 u_t - \lambda u_{txx} - \dfrac{\partial^2}{\partial x^2}\mu(t,u(x,t)) + \displaystyle\int_0^t g(t-s)u_{xx}(x,s))ds \\ \qquad\qquad\qquad = f(u) - \dfrac{1}{2}D_2^2\mu(t,u)u_x^2 + F(x,t),\ 0<x<1,\ 0<t<T, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{5.1}$$

where $\lambda > 0$, $\lambda_1 > 0$ are given constants and μ, g, f, F, $\tilde{u}_0$, $\tilde{u}_1$ are given functions satisfying the following assumptions.

We first note that, by Theorem 3.5, under the assumptions corresponding to this special case, Prob. (5.1) has a unique local weak solution u such that

$$\begin{aligned} u &\in C([0,T];H^2\cap H_0^1)\cap C^1([0,T];H_0^1)\cap L^\infty(0,T;H^2\cap H_0^1), \\ u' &\in C([0,T];H_0^1)\cap L^\infty(0,T;H^2\cap H_0^1), \\ u'' &\in L^2(0,T;H_0^1)\cap L^\infty(0,T;L^2), \end{aligned}$$

for T chosen small enough. Furthermore, using the standard arguments of density, we can propose the assumptions to get the local existence and uniqueness of a weak solution for Prob. (5.1) with less smoothness as follows.

$(\bar{H}_1)$ $(\tilde{u}_0, \tilde{u}_1) \in H_0^1 \times L^2$;

$(\hat{H}_2)$ $\mu \in C^3(\mathbb{R}_+ \times \mathbb{R})$ and there exists the positive constant μ_* such that

(i) $D_2\mu(t,z) \geq \mu_* > 0$, for all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$,

(ii) $D_1D_2\mu(t,z) \leq 0$, for all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$;

$(\bar{H}_3)$ $g \in C^1(\mathbb{R}_+)$;

$(\bar{H}_4)$ $f \in C^1(\mathbb{R})$, such that $f(0) = 0$ and $yf(y) > 0$, for all $y \in \mathbb{R}$;

$(\hat{H}_5)$ $F \in L^2((0,1)\times\mathbb{R}_+)$.

We then obtain the following theorem.

Theorem 5.1. *Let* $(\bar{H}_1)$, $(\hat{H}_2)$, $(\bar{H}_3)$, $(\bar{H}_4)$, $(\hat{H}_5)$ *hold. Then, there exist* $T > 0$ *and a unique solution of Prob.* (5.1) *such that*

$$u \in C^0([0,T];H_0^1)\cap C^1([0,T];L^2),\ u' \in L^2(0,T;H_0^1). \tag{5.2}$$

We now prove the existence of global solution and the energy of the solution decays as $t \to +\infty$. For this purpose, we strengthen the following assumptions.

$(\bar{H}_1)$ $(\tilde{u}_0, \tilde{u}_1) \in H_0^1 \times L^2$;

(H_2^d) $\mu \in C^3(\mathbb{R}_+ \times \mathbb{R})$ and there exist the positive constants μ_*, μ_{1*}, μ_{2*} such that

(i) $D_2\mu(t,z) \geq \mu_* > 0$, for all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$,

(ii) $D_1D_2\mu(t,z) \leq 0$, for all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$,

(iii) $\frac{1}{2}zD_2^2\mu(t,z) + D_2\mu(t,z) \geq \mu_{1*} > 0$, for all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$,

(iv) $zD_2^2\mu(t,z) \geq -\mu_{2*}$, for all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$;

(H_3^d) $g \in C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ such that

(i) $L_* = \mu_* - \bar{g}(\infty) > 0$,

(ii) there exists a function $\xi \in C^1(\mathbb{R}_+)$ such that

$\xi'(t) \leq 0 < \xi(t)$, for all $t \geq 0$, $\int_0^\infty \xi(s)\,ds = \infty$, and

$g'(t) \leq -\xi(t)g(t) < 0$, for all $t \geq 0$, where $\bar{g}(t) = \int_0^t g(s)\,ds$, $\bar{g}(\infty) = \int_0^\infty g(s)\,ds$;

(H_4^d) $f \in C^1(\mathbb{R})$, $f(0) = 0$, $yf(y) > 0$, for all $y \in \mathbb{R}$ and there exist the constants α, β, d_2, $\bar{d}_2 > 0$, with $\alpha > 2$, $\beta > 2$, such that

(i) $yf(y) \le d_2 \int_0^y f(z)dz$, for all $y \in \mathbb{R}$,

(ii) $\int_0^y f(z)dz \le \bar{d}_2\left(|y|^\alpha + |y|^\beta\right)$, for all $y \in \mathbb{R}$;

(H_5^d) $F \in L^\infty(\mathbb{R}_+; L^2) \cap L^1(\mathbb{R}_+; L^2)$, and there exist two positive constants C_0, γ_0 such that $\|F(t)\|^2 \le C_0 \exp(-\gamma_0 t)$, for all $t \ge 0$;

(H_6^d) $p > \max\{2, d_2\}$, $\mu_* > \frac{p}{2d_2}\mu_{2*} + \left(1 + \frac{p}{d_2}\right)\bar{g}(\infty)$.

We next prove that if $\int_0^1 D_2\mu(0, \tilde{u}_0(x))\tilde{u}_{0x}^2(x)dx - p\int_0^1 dx \int_0^{\tilde{u}_0(x)} f(z)dz > 0$, and if the initial energy and $\|F(t)\|$ are small enough, then the solution is globally extended in time and its energy decays to zero, as t tends to infinity. To achieve this goal, we first construct the Lyapunov functional in the form

$$\mathcal{L}(t) = E(t) + \delta\psi(t), \tag{5.3}$$

where δ is a positive constant suitably chosen and

$$E(t) = \frac{1}{2}\left\|u'(t)\right\|^2 + \left(\frac{1}{2} - \frac{1}{p}\right)[(g * u)(t) + N(u)] + \frac{1}{p}I(t), \tag{5.4}$$

$$\psi(t) = \langle u'(t), u(t)\rangle + \frac{\lambda_1}{2}\|u(t)\|^2 + \frac{\lambda}{2}\|u_x(t)\|^2, \tag{5.5}$$

with $(g * u)(t) = \int_0^t g(t-s)\|u_x(t) - u_x(s)\|^2\,ds$ and

$$\begin{aligned} I(t) &= (g * u)(t) + N(u) - p\int_0^1 \mathcal{F}(u(x,t))dx, \qquad (5.6)\\ N(u) &= \int_0^1 D_2\mu(t, u(x,t))u_x^2(x,t)dx - \bar{g}(t)\|u_x(t)\|^2, \\ \mathcal{F}(y) &= \int_0^y f(z)dz. \end{aligned}$$

Lemma 5.2. *If* $(\bar{H}_1)$, $(H_2^d) - (H_6^d)$ *hold and* u *is the solution of* (5.1), *then the energy functional* $E(t)$ *satisfies*

$$\begin{aligned} \text{(i)}\;\; E'(t) &\le \frac{1}{2}\|F(t)\| + \frac{1}{2}\|F(t)\|\left\|u'(t)\right\|^2, \qquad (5.7)\\ \text{(ii)}\;\; E'(t) &\le -\left(\lambda_1 - \frac{\varepsilon_1}{2}\right)\left\|u'(t)\right\|^2 - \lambda\left\|u_x'(t)\right\|^2 - \frac{1}{2}\xi(t)(g * u)(t) + \frac{1}{2\varepsilon_1}\|F(t)\|^2, \end{aligned}$$

for all $\varepsilon_1 > 0$.

Proof. Multiplying (5.1) by $u'(x,t)$ and integrating over $[0,1]$, we get

$$\begin{aligned} E'(t) &= -\lambda_1\left\|u'(t)\right\|^2 - \lambda\left\|u_x'(t)\right\|^2 + \frac{1}{2}\left(g' * u\right)(t) - \frac{1}{2}g(t)\|u_x(t)\|^2 \qquad (5.8)\\ &\quad + \frac{1}{2}\int_0^1 D_1 D_2\mu(t, u(x,t))u_x^2(x,t)dx + \left\langle F(t), u'(t)\right\rangle. \end{aligned}$$

Using the assumptions (H_2^d), (H_3^d), (H_5^d), we obtain

$$\begin{aligned} \frac{1}{2}\int_0^1 D_1D_2\mu(t,u(x,t))u_x^2(x,t)dx &\leq 0, \\ \frac{1}{2}\left(g'*u\right)(t) &\leq -\frac{1}{2}\xi(t)\left(g*u\right)(t), \end{aligned} \tag{5.9}$$

so

$$E'(t) \leq \langle F(t), u'(t)\rangle \leq \frac{1}{2}\|F(t)\| + \frac{1}{2}\|F(t)\|\left\|u'(t)\right\|^2.$$

This assures (5.7)-(i).

By applying Cauchy-Schwartz inequality, we have

$$\langle F(t), u'(t)\rangle \leq \frac{1}{2\varepsilon_1}\|F_1(t)\|^2 + \frac{\varepsilon_1}{2}\left\|u'(t)\right\|^2, \text{ for all } \varepsilon_1 > 0. \tag{5.10}$$

Then, by using (5.8), (5.9) and (5.10), it is easy to see (5.7)-(ii) holds. Lemma 5.2 is proved. □

Lemma 5.3. *If* $(H_2^d)-(H_6^d)$ *hold and* $(\tilde{u}_0, \tilde{u}_1) \in H_0^1 \times L^2$ *such that* $I(0) > 0$ *and*

$$\eta^* \equiv \mu_* - \bar{g}(\infty) - p\bar{d}_2\left(R_*^{\alpha-2} + R_*^{\beta-2}\right) > \frac{p}{2d_2}\mu_{2*} + \frac{p}{d_2}\bar{g}(\infty), \tag{5.11}$$

where $R_* = \left(\frac{2pE_*}{(p-2)L_*}\right)^{1/2}$, $E_* = \left(E(0) + \frac{1}{2}\rho_1\right)\exp(\rho_1)$, $\rho_1 = \int_0^\infty \|F(t)\|\,dt$, $L_* = \mu_* - \bar{g}(\infty) > 0$.

Then $I(t) \geq 0$, *for all* $t \geq 0$.

We note more that the condition (5.11) holds if $\bar{g}(\infty)$, E_* are chosen small enough and $\mu_* > 0$ is suitably large.

Proof. By the continuity of $I(t)$ and $I(0) > 0$, there exists $\tilde{T} > 0$ such that

$$I(t) = I(u(t)) > 0, \ \forall t \in [0, \tilde{T}].$$

From (5.4) and (5.6), we get

$$\begin{aligned} E(t) &\geq \frac{1}{2}\left\|u'(t)\right\|^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\left[(g*u)(t) + N(u)\right] \\ &\geq \frac{1}{2}\left\|u'(t)\right\|^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\left[(g*u)(t) + L_*\|u_x(t)\|^2\right] \\ &\geq \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{p-2}{2p}(g*u)(t) + \frac{(p-2)L_*}{2p}\|u_x(t)\|^2, \ \forall t \in [0, \tilde{T}]. \end{aligned} \tag{5.12}$$

Combining (5.7)-(i) and (5.12) and using Gronwall's inequality, we obtain

$$\begin{aligned} &\|u_x(t)\|^2 + \frac{p}{(p-2)L_*}\left\|u'(t)\right\|^2 + \frac{1}{L_*}(g*u)(t) \\ \leq\ &\frac{2pE(t)}{(p-2)L_*} \leq \frac{2pE_*}{(p-2)L_*} \equiv R_*^2, \ \forall t \in [0, \tilde{T}]. \end{aligned} \tag{5.13}$$

Then, it follows from (H_4^d)-(ii) and (5.13) that

$$\begin{aligned} p\int_0^1 \mathcal{F}(u(x,t))\,dx &\leq p\bar{d}_2\left(\|u(t)\|_{L^\alpha}^\alpha + \|u(t)\|_{L^\beta}^\beta\right) \\ &\leq p\bar{d}_2\left(\|u_x(t)\|^\alpha + \|v_x(t)\|^\beta\right) \leq p\bar{d}_2\left(R_*^{\alpha-2} + R_*^{\beta-2}\right)\|u_x(t)\|^2. \end{aligned}$$

Thus

$$I(t) \geq (g*u)(t) + \eta^*\|u_x(t)\|^2 \geq 0, \ \forall t \in [0, \tilde{T}], \tag{5.14}$$

where the positive constant η^* is defined as in (5.11).

Next, we prove that $I(t) > 0$, $\forall t \geq 0$. Put $T_\infty = \sup\left\{\tilde{T} > 0 : I(t) > 0,\ \forall t \in [0, \tilde{T}]\right\}$, we have to show that $T_\infty = +\infty$. Indeed, if $T_\infty < +\infty$ then, by the continuity of $I(t)$, we have $I(T_\infty) \geq 0$.

In case of $I(T_\infty) > 0$, by the same arguments as above, we can reduce that there exists $\tilde{T}_\infty > T_\infty$ such that $I(t) > 0$, $\forall t \in [0, \tilde{T}_\infty]$. This is contrary to the definition of T_∞.

In case of $I(T_\infty) = 0$, it implies from (5.14) that

$$0 = I(T_\infty) \geq (g * u)(T_\infty) + \eta^* \|u_x(T_\infty)\|^2 \geq 0.$$

Therefore

$$\|u(T_\infty)\| = (g * u)(T_\infty) = 0.$$

Due to the function $s \longmapsto g(T_\infty - s)\|u_x(T_\infty) - u_x(s)\|^2$ is continuous on $[0, T_\infty]$ and $g(T_\infty - s) > 0$, $\forall s \in [0, T_\infty]$, we have

$$(g * u)(T_\infty) = \int_0^{T_\infty} g(T_\infty - s)\|u_x(s)\|^2 ds = 0,$$

it follows that $\|u_x(s)\|^2 = 0$, $\forall s \in [0, T_\infty]$. Thus, $u(0) = 0$. This is contrary to $I(0) > 0$.

Consequently, $T_\infty = +\infty$, i.e. $I(t) > 0$, $\forall t \geq 0$. Lemma 5.3 is proved. $\square$

It is clear to see that Lemmas 5.2, 5.3 assure a global existence of the solution for Prob. (5.1).

Next, we put

$$E_1(t) = \|u'(t)\|^2 + \|u_x(t)\|^2 + N(u) + (g * u)(t) + I(t). \tag{5.15}$$

In order to discuss general decay, we need more the following lemmas.

Lemma 5.4. *If the assumptions of Lemma 5.3 hold, there exist the positive constants β_1, $\bar{\beta}_1$, β_2, $\bar{\beta}_2$ such that*

$$\begin{array}{rrcl} \text{(i)} & \beta_1 E_1(t) & \leq & \mathcal{L}(t) \leq \beta_2 E_1(t),\ \textit{for all } t \geq 0, \\ \text{(ii)} & \bar{\beta}_1 E_1(t) & \leq & E(t) \leq \bar{\beta}_2 E_1(t),\ \textit{for all } t \geq 0, \end{array} \tag{5.16}$$

for δ is small enough.

Proof. It is easy to see that

$$\begin{aligned} \mathcal{L}(t) &= \frac{1}{2}\|u'(t)\|^2 + \frac{p-2}{2p}(g * u)(t) + \frac{p-2}{2p}N(u) + \frac{1}{p}I(t) \\ &\quad + \delta\langle u'(t), u(t)\rangle + \frac{\delta}{2}\left(\lambda_1\|u(t)\|^2 + \lambda\|u_x(t)\|^2\right). \end{aligned}$$

Using Cauchy-Schwartz inequality, we get the estimations

$$\begin{aligned} |\langle u'(t), u(t)\rangle| &\leq \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|u_x(t)\|^2, \\ N(u) &\geq L_*\|u_x(t)\|^2. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{L}(t) &\geq \frac{1}{2}\|u'(t)\|^2 + \frac{p-2}{2p}(g * u)(t) + \frac{p-2}{2p}N(u) + \frac{1}{p}I(t) - \delta\left(\frac{\|u'(t)\|^2 + \|u_x(t)\|^2}{2}\right) \\ &\geq \frac{1}{2}\|u'(t)\|^2 + \frac{p-2}{2p}(g * u)(t) + \frac{p-2}{2p}\left[\varepsilon_* N(u) + (1-\varepsilon_*)L_*\|u_x(t)\|^2\right] + \frac{1}{p}I(t) \\ &\quad -\delta\left(\frac{\|u'(t)\|^2 + \|u_x(t)\|^2}{2}\right) \\ &= \frac{1-\delta}{2}\|u'(t)\|^2 + \frac{p-2}{2p}(g * u)(t) + \frac{(p-2)\varepsilon_*}{2p}N(u) + \left[\frac{(p-2)(1-\varepsilon_*)L_*}{2p} - \frac{\delta}{2}\right]\|u_x(t)\|^2 \\ &\quad + \frac{1}{p}I(t) \geq \beta_1 E_1(t), \end{aligned}$$

where

$$0<\varepsilon_*<1,\ \beta_1=\min\left\{\frac{1-\delta}{2},\ \frac{(p-2)\varepsilon_*}{2p},\ \frac{1}{p},\ \left[\frac{(p-2)(1-\varepsilon_*)L_*}{2p}-\frac{\delta}{2}\right]\right\},$$

and $\delta>0$ are chosen small enough such that

$$0<\delta<\min\left\{1;\ \frac{(p-2)(1-\varepsilon_*)L_*}{p}\right\}.$$

On the other hand

$$\begin{aligned}\mathcal{L}(t) &\le \frac{1}{2}\left\|u'(t)\right\|^2+\frac{p-2}{2p}(g*u)(t)+\frac{p-2}{2p}N(u)+\frac{1}{p}I(t)\\ &\quad+\delta\left(\frac{\|u'(t)\|^2+\|u_x(t)\|^2}{2}\right)+\frac{\delta}{2}\left(\lambda_1\|u_x(t)\|^2+\lambda\|u_x(t)\|^2\right)\\ &\le \frac{1+\delta}{2}\left\|u'(t)\right\|^2+\frac{p-2}{2p}(g*u)(t)+\frac{p-2}{2p}N(u)+\frac{1}{p}I(t)\\ &\quad+\frac{\delta}{2}(1+\lambda_1+\lambda)\|u_x(t)\|^2\ \le\beta_2E_1(t),\end{aligned}$$

where $\beta_2=\max\left\{\frac{1+\delta}{2},\ \frac{p-2}{2p},\ \frac{\delta}{2}(1+\lambda+\lambda_1)\right\}$. Thus, the estimation (5.16)-(i) holds. Similarly, we verify that (5.16)-(ii) also holds.

Lemma 5.4 is proved completely. □

Lemma 5.5. *If the assumptions of Lemma 5.3 hold, then the functional $\psi(t)$ defined by* (5.5) *satisfies the following estimation*

$$\begin{aligned}\psi'(t) &\le \left\|u'(t)\right\|^2+\frac{1}{2\varepsilon_2}\|F(t)\|^2+\left(\frac{d_2}{p}+\frac{1}{2\varepsilon_2}\right)(g*u)(t)\\ &\quad-\frac{\delta_1 d_2}{p}I(t)-\left(1-\frac{d_2}{p}-\delta_*\right)N(u)\\ &\quad-\left[\frac{d_2}{p}(1-\delta_1)\eta^*+\delta_*\mu_{1*}-\frac{1}{2}(1-\delta_*)\mu_{2*}-\frac{\varepsilon_2}{2}-\left(1+\frac{\varepsilon_2}{2}\right)\bar{g}(\infty)\right]\|u_x(t)\|^2,\end{aligned}\tag{5.17}$$

for all $\varepsilon_2>0$, δ_*, $\delta_1\in(0,1)$.

Proof. Multiplying $(5.1)_1$ by $u(x,t)$ and integrating over $[0,1]$, we obtain

$$\begin{aligned}\psi'(t) &= \left\|u'(t)\right\|^2-\frac{1}{2}\langle D_2^2\mu(t,u(t))u_x^2(t),u(t)\rangle-\langle D_2\mu(t,u(t))u_x(t),u_x(t)\rangle+\langle F(t),u(t)\rangle\\ &\quad+\int_0^t g(t-s)\langle u_x(s),u_x(t)\rangle ds+\langle f(u(t)),u(t)\rangle\\ &= \left\|u'(t)\right\|^2+\langle F(t),u(t)\rangle+\int_0^t g(t-s)\langle u_x(s),u_x(t)\rangle ds\\ &\quad-\delta_*\int_0^1\left[\frac{1}{2}u(x,t)D_2^2\mu(t,u(x,t))+D_2\mu(t,u(x,t))\right]u_x^2(x,t)dx\\ &\quad-(1-\delta_*)\int_0^1\left[\frac{1}{2}u(x,t)D_2^2\mu(t,u(x,t))+D_2\mu(t,u(x,t))\right]u_x^2(x,t)dx+\langle f(u(t)),u(t)\rangle.\end{aligned}$$

Using Cauchy-Schwartz inequality, we have

$$\begin{aligned}\langle F(t),u(t)\rangle &\le \frac{\varepsilon_2}{2}\|u_x(t)\|^2+\frac{1}{2\varepsilon_2}\|F(t)\|^2,\\ \int_0^t g(t-s)\langle u_x(s),u_x(t)\rangle ds &\le \left(1+\frac{\varepsilon_2}{2}\right)\bar{g}(t)\|u_x(t)\|^2+\frac{1}{2\varepsilon_2}(g*u)(t),\\ I(t) &\ge \eta^*\|u_x(t)\|^2,\end{aligned}\tag{5.18}$$

for all $\varepsilon_2 > 0$.

By asumption (H_2^d)-(iii) and (H_2^d)-(iv), we get

$$-\delta_* \int_0^1 \left[\frac{1}{2}u(x,t)D_2^2\mu(t,u(x,t)) + D_2\mu(t,u(x,t))\right] u_x^2(x,t)dx \leq -\delta_*\mu_{1*} \left\|u_x(t)\right\|^2, \tag{5.19}$$

$$\begin{aligned} & -(1-\delta_*) \int_0^1 \left[\frac{1}{2}u(x,t)D_2^2\mu(t,u(x,t)) + D_2\mu(t,u(x,t))\right] u_x^2(x,t)dx \\ = \; & -\frac{1}{2}(1-\delta_*) \int_0^1 u(x,t)D_2^2\mu(t,u(x,t))u_x^2(x,t) - (1-\delta_*) \int_0^1 D_2\mu(t,u(x,t))u_x^2(x,t)dx \\ = \; & -\frac{1}{2}(1-\delta_*) \int_0^1 u(x,t)D_2^2\mu(t,u(x,t))u_x^2(x,t) - (1-\delta_*) \left[N(u) + \bar{g}(t)\left\|u_x(t)\right\|^2\right] \\ \leq \; & \frac{1}{2}(1-\delta_*)\mu_{2*}\left\|u_x(t)\right\|^2 - (1-\delta_*)N(u). \end{aligned} \tag{5.20}$$

On the other hand, by asumption (H_4^d)-(i) and definition of $I(t)$ given by (5.6), we obtain

$$\begin{aligned} \langle f(u(t)), u(t)\rangle & \leq d_2\int_0^1 \mathcal{F}\left(u\left(x,t\right)\right)dx \\ & = \frac{d_2}{p}\left[\left(g*u\right)(t) + N(u) - \delta_1 I(t) - (1-\delta_1)I(t)\right] \\ & \leq \frac{d_2}{p}\left[\left(g*u\right)(t) + N(u) - \delta_1 I(t) - (1-\delta_1)\eta^*\left\|u_x(t)\right\|^2\right]. \end{aligned} \tag{5.21}$$

Then, it follows from (5.18)-(5.21) that the inequality (5.17) is valid.

Lemma 5.5 is proved completely. □

Using Lemmas 5.2 - 5.5, we state and prove our main result in this section as follows.

Theorem 5.6. *If $(H_2^d) - (H_6^d)$ hold and $(\tilde{u}_0, \tilde{u}_1) \in H_0^1 \times L^2$ satisfy $I(0) > 0$ and (5.11). Then, there exist positive constants $\bar{C}$, $\bar{\gamma}$ such that*

$$\left\|u'(t)\right\|^2 + \left\|u_x(t)\right\|^2 \leq \bar{C}\exp\left(-\bar{\gamma}\int_0^t \xi\left(s\right)ds\right), \text{ for all } t \geq 0. \tag{5.22}$$

Proof of Theorem 5.6.

First, due to the definition of $\mathcal{L}(t)$ and the inequalities (5.7)-(ii), (5.17), we deduce that

$$\begin{aligned} \mathcal{L}'(t) \leq & -\left(\lambda_1 - \frac{\varepsilon_1}{2} - \delta\right)\left\|u'(t)\right\|^2 - \frac{1}{2}\xi(t)\left(g*u\right)(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\left\|F\left(t\right)\right\|^2 \\ & + \delta\left(\frac{d_2}{p} + \frac{1}{2\varepsilon_2}\right)\left(g*u\right)(t) - \frac{\delta\delta_1 d_2}{p}I(t) - \delta\theta_1 N(u) - \delta\theta_2\left\|u_x(t)\right\|^2. \end{aligned} \tag{5.23}$$

where

$$\begin{aligned} \theta_1 & = \theta_1\left(\delta_*\right) = 1 - \frac{d_2}{p} - \delta_*, \\ \theta_2 & = \theta_2\left(\delta_*, \delta_1, \varepsilon_2\right) = \frac{d_2}{p}(1-\delta_1)\eta^* + \delta_*\mu_{1*} - \frac{1}{2}(1-\delta_*)\mu_{2*} - \frac{\varepsilon_2}{2} - \left(1 + \frac{\varepsilon_2}{2}\right)\bar{g}\left(\infty\right). \end{aligned}$$

Clearly

$$\begin{aligned} \lim_{\delta_*\to 0_+}\theta_1\left(\delta_*\right) & = 1 - \frac{d_2}{p} > 0, \\ \lim_{\delta_*\to 0_+,\; \delta_1\to 0_+,\; \varepsilon_2\to 0_+}\theta_2\left(\delta_*, \delta_1, \varepsilon_2\right) & = \frac{d_2}{p}\eta^* - \frac{1}{2}\mu_{2*} - \bar{g}\left(\infty\right) > 0. \end{aligned}$$

Then, we can choose δ_*, $\delta_1 \in (0,1)$ and $\varepsilon_2 > 0$ small enough such that

$$\theta_1 = \theta_1(\delta_*) > 0, \ \theta_2 = \theta_2(\delta_*, \delta_1, \varepsilon_2) > 0.$$

Moreover, we also choose $\varepsilon_1 > 0$, $\delta > 0$ small enough and satisfying

$$\bar{\theta}_1 = \lambda_1 - \frac{\varepsilon_1}{2} - \delta > 0, \ 0 < \delta < \min\left\{1; \ \frac{(p-2)(1-\varepsilon_*)L_*}{p}\right\}.$$

Putting

$$\bar{\theta}_* = \min\left\{\bar{\theta}_1, \delta\theta_1, \delta\theta_2, \frac{\delta\delta_1 d_2}{p}\right\}, \ \bar{\theta}_3 = \delta\left(\frac{d_2}{p} + \frac{1}{2\varepsilon_2}\right), \tag{5.24}$$

we get from (5.23) and (5.24) that

$$\begin{aligned} \mathcal{L}'(t) &\leq -\bar{\theta}_*\left[\left\|u'(t)\right\|^2 + \|u_x(t)\|^2 + I(t) + N(u)\right] \\ &\quad +\bar{\theta}_3(g*u)(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\|F(t)\|^2 \\ &= -\bar{\theta}_* E_1(t) + \left(\bar{\theta}_* + \bar{\theta}_3\right)(g*u)(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\|F(t)\|^2. \end{aligned} \tag{5.25}$$

Combining (5.7)-(ii) and (5.25), we obtain

$$\begin{aligned} \xi(t)\mathcal{L}'(t) &\leq -\bar{\theta}_*\xi(t)E_1(t) + \left(\bar{\theta}_* + \bar{\theta}_3\right)\xi(t)(g*u)(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\xi(0)\|F(t)\|^2 \\ &\leq -\bar{\theta}_*\xi(t)E_1(t) + 2\left(\bar{\theta}_* + \bar{\theta}_3\right)\left[-E'(t) + \frac{1}{2\varepsilon_1}\|F(t)\|^2\right] + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\xi(0)\|F(t)\|^2 \\ &= -\bar{\theta}_*\xi(t)E_1(t) - 2\left(\bar{\theta}_* + \bar{\theta}_3\right)E'(t) + \left[\frac{\bar{\theta}_* + \bar{\theta}_3}{\varepsilon_1} + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\right]\xi(0)\|F(t)\|^2 \\ &\leq -\bar{\theta}_*\xi(t)E_1(t) - 2\left(\bar{\theta}_* + \bar{\theta}_3\right)E'(t) + \bar{C}_0 e^{-\gamma_0 t}, \end{aligned} \tag{5.26}$$

where $\bar{C}_0 = \left[\frac{\bar{\theta}_* + \bar{\theta}_3}{\varepsilon_1} + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\right]\xi(0)C_0$.

Setting the functional

$$L(t) = \xi(t)\mathcal{L}(t) + 2\left(\bar{\theta}_* + \bar{\theta}_3\right)E(t),$$

then we have

$$L(t) \leq \left[\xi(0)\beta_2 + 2\left(\bar{\theta}_* + \bar{\theta}_3\right)\bar{\beta}_2\right]E_1(t) \equiv \hat{\beta}_2 E_1(t),$$

and

$$\begin{aligned} L'(t) &= \xi'(t)\mathcal{L}(t) + \xi(t)\mathcal{L}'(t) + 2\left(\bar{\theta}_* + \bar{\theta}_3\right)E'(t) \\ &\leq -\bar{\theta}_*\xi(t)E_1(t) + \bar{C}_0 e^{-\gamma_0 t} \leq -\frac{\bar{\theta}_*}{\hat{\beta}_2}\xi(t)L(t) + \bar{C}_0 e^{-\gamma_0 t}. \end{aligned}$$

By choosing $0 < \bar{\gamma} < \min\left\{\frac{\bar{\theta}_*}{\hat{\beta}_2}, \frac{\gamma_0}{\xi(0)}\right\}$, we get

$$L'(t) + \bar{\gamma}\xi(t)L(t) \leq \bar{C}_0 e^{-\gamma_0 t}.$$

Integrating the above inequality, we deduce

$$L(t) \leq \left(L(0) + \frac{\bar{C}_0}{\gamma_0 - \bar{\gamma}\xi(0)}\right)\exp\left(-\bar{\gamma}\int_0^t \xi(\tau)d\tau\right). \tag{5.27}$$

On the other hand

$$
\begin{aligned}
L(t) &= \xi(t)\mathcal{L}(t) + 2\left(\bar{\theta}_* + \bar{\theta}_3\right) E(t) \geq 2\left(\bar{\theta}_* + \bar{\theta}_3\right) \bar{\beta}_1 E_1(t) \\
&\geq 2\left(\bar{\theta}_* + \bar{\theta}_3\right) \bar{\beta}_1 \left(\left\|u'(t)\right\|^2 + \|u_x(t)\|^2\right).
\end{aligned} \tag{5.28}
$$

Then, by (5.27) and (5.28), we get (5.22). Theorem 5.6 is proved completely. □

Remark 5.7. We also give here an example, in which μ satisfies the assumption (H_2^d). We shall consider the function

$$
\mu(t,z) = \mu_* z + \bar{\mu}_* e^{-t} |z|^{k-1} z,
$$

where $\mu_* > 0$, $\bar{\mu}_* > 0$, $k > 3$ are constants. By the direct computations, we have

$$
\begin{aligned}
D_2\mu(t,z) &= \mu_* + k\bar{\mu}_* e^{-t} |z|^{k-1} \geq \mu_* > 0; \\
D_1 D_2\mu(t,z) &= -k\bar{\mu}_* e^{-t} |z|^{k-1} \leq 0; \\
z D_2^2\mu(t,z) &= k(k-1)\bar{\mu}_* e^{-t} |z|^{k-1} \geq 0 > -\mu_{2*}; \\
\frac{1}{2} z D_2^2\mu(t,z) + D_2\mu(t,z) &= \frac{1}{2}(k-1)\left[D_2\mu(t,z) - \mu_*\right] + D_2\mu(t,z) \\
&\geq D_2\mu(t,z) \geq \mu_* = \mu_{1*} > 0.
\end{aligned}
$$

This claims that (H_2^d) holds. □

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