Solving inverse Sturm-Liouville problem featuring a constant delay by Chebyshev interpolation method

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Abstract

The inverse nodal problem for Sturm-Liouville operator with a constant delay has been investigated in the present paper. To do so, we have computed the nodal points and nodal lengths. Therefore, we have tried Chebyshev interpolation method to obtain the numerical solution of inverse nodal problem. Following that, a number of numerical examples have been given. The numerical calculations in the present paper have been conducted via pc applying some programs encoded in Matlab software.

Keywords: Inverse problem (IP); Sturm-Liouville (SL) operator; Constant delay; Nodes; Chebyshev interpolation.
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1 Introduction

The research area known as Inverse problem (IP) explores the inversion of models or data. It is a mathematical framework which is utilized to achieve information regarding physical object or system with regard to observed measurements.

Once a solution is found for such a problem, it can be used to find out information relating to physical parameter that can not be observed directly. Hence, such problems (IPs) are considered significant and extensively investigated in science and mathematics.

A large number of applications can be attributed to IPs, a few of which are medical imaging, geophysics, computer vision, astronomy, and non-destructive testing.

IPs can be dealt with via two different approaches, that is, either to survey inverse eigenvalue problem or to investigate inverse nodal problem [1–3]. It was McLaughlin [4] who for the first time solved inverse nodal problem regarding the Sturm-Liouville problems (SLP). It was proved by McLaughlin that the potential function of the SLP could be achieved via certain dense subset of the nodes for the eigenfunction relating to a constant.

The definition of inverse nodal problem is to find potential function and some constants in boundary conditions from nodal point(zeros of eigenfunction). A number of authors considered the inverse nodal problems (see [4–13]).

In various real-world processes, the future behavior of the system depends not only on its present state and rate of change of the state (corresponding to the values of the function and its derivatives at the current point), but also on its states in the past. Such processes are described by functional differential equations with delay and reflected in physics, biology and especially in engineering and control theory (see the monographs [14, 15]).

Many authors studied these problems with the constant delay (for example see [16–20]). Rashed computed the approximate solutions of integral equations (IEs) and integral-differential equations (IDEs) by Chebyshev interpolation approach in [21, 22].

For SLPs, we have three types of problems: direct problems, isospectral problems, and inverse problems (IPs). In direct problems, the eigenvalues, eigenfunctions, and some properties of the problem are estimated from the known coefficients. Different numerical methods for solving direct problem are applied in [23, 24]. In isospectral problems, for a given problem, we want to obtain different problems of the same form, which have the same eigenvalues of the initial problem. Isospectral SLPs are studied in [25–27]. The third type of problems related to the SLPs are IPs. The inverse spectral SLP can be regarded from three aspects: existence, uniqueness, and reconstruction of the coefficients with specific properties of eigenvalues and eigenfunctions, (see [28–33] and the references therein).

In [4, 5, 8], the authors solved the inverse nodal SLPs without the constant delay by using Chebyshev polynomials. In this study, we have tried to communicate between the inverse nodal problem including a delay constant and the integral equation and apply Chebyshev interpolation method to solve the inverse nodal problem with a delay constant.

We consider SLP $L_i = L_i(q, h), i = 0, 1$ with a constant delay of the form

$$-y''(x) + q(x)y(x - a) = \lambda\, y(x), \qquad 0 < x < 1, \tag{1.1}$$

with initial conditions:

$$y'(0) - h\, y(0) = y^{(i)}(1) = 0, \tag{1.2}$$

in which

- $a \in (0, 1)$,
- h is a real number,
- $q(x) = 0$ for $x \in (0, a]$,
- λ is the spectral parameter,
- $q(x)$ is a real function and integrable on $(a, 1)$.

In Section 2, the asymptotic form of nodes and nodal lengths are calculated. In Section 3, we approximate the solution of inverse nodal problem by Chebyshev interpolathon method and present some examples to show the numerical results in Section 4.

2 Preliminaries

In the present part, we present a description of the asymptotic form of eigenvalues, nodes and the length of nodes of the boundary value problems L_i, $i = 0, 1$.

Let $N \in \mathbb{N}$ be such that $aN < 1 \leq a(N+1)$, i.e. $a \in [\frac{1}{N+1}, \frac{1}{N})$. Let $\varphi(x, \lambda)$ be solution of the Eq. (1.1) under the initial conditions $\varphi(0, \lambda) = 1$, $\varphi'(0, \lambda) = h$, then the result can be written as (see [16])

$$\varphi(x, \lambda) = \varphi_0(x, \lambda) + \varphi_1(x, \lambda) + ... + \varphi_N(x, \lambda),$$

where

$$\begin{cases} \varphi_0(x, \lambda) = cos\rho x + h\frac{sin\rho x}{\rho}, & x \geq 0, \\ \varphi_k(x, \lambda) = \int_{ka}^{x} \frac{sin\rho(x-t)}{\rho} q(t)\varphi_{k-1}(t-a, \lambda)dt, & x \geq ka, \end{cases}$$

and $\varphi_k(x, \lambda) = 0$ for $x \leq ka$. Therefore, for $\rho \to \infty$, the following estimate hold:

$$\varphi(x, \lambda) = \frac{\sin \rho(x-a)}{2\rho} \int_a^x q(t)dt + \frac{\sin \rho x}{\rho} h + \cos \rho x + o\left(\frac{1}{\rho}\right). \tag{2.1}$$

Then, for $i = 0, 1$

$$\Delta_i(\lambda) := (-\rho)^i \cos(\frac{i\pi}{2} - \rho) + h\frac{\sin(\frac{i\pi}{2} + \rho)}{\rho^{1-i}} + \frac{\sin(\frac{i\pi}{2} + \rho)(1-a)}{2\rho^{1-i}} \int_a^1 q(t)dt + o\left(\frac{1}{\rho^{1-i}}\right).$$

Let the eigenvalues of the boundary value problems L_i, $i = 0, 1$ to be $\lambda_{0i} < \lambda_{1i} < ... \to \infty$ and $0 < x_1^{ni} < ... < x_j^{ni} < 1$, $j = \overline{1, n-1}$ be the nodes of n-th eigenfunction.

Lemma 2.1. [16] The eigenvalues $\{\lambda_{ni}\}_{n\geq 0}$, $i = 0, 1$ for $n \to \infty$ in the boundary value problems L_i, $i = 0, 1$ are formulated in the form of

$$\rho_{ni} := \sqrt{\lambda_{ni}} = (n + \frac{1-i}{2-i})\pi + \frac{h}{n\pi} + \frac{cos(n + \frac{1-i}{2-i})\pi a}{2n\pi} \int_a^1 q(t)dt + o\left(\frac{1}{n}\right), \quad i = 0, 1. \tag{2.2}$$

Theorem 2.2. Suppose the equation (1.1) be considered regarding the initial conditions

$$y(0, \lambda) = 1, \quad y'(0, \lambda) = h.$$

Therefore, the nodes of boundary value problems L_i, $i = 0, 1$ and the length of nodes are formulated in the following form

$$x_j^{ni} = \frac{(j - \frac{1}{2})\pi}{\rho_{ni}} + \frac{h}{\rho_{ni}^2} + \frac{cos\rho_{ni}a}{2\rho_{ni}^2} \int_a^{x_j^{ni}} q(t)dt + o\left(\frac{1}{n^2}\right), \quad i = 0, 1, \quad j = 0, 1, ..., n-1, \tag{2.3}$$

$$l_j^{ni} = \frac{\pi}{\rho_{ni}} + \frac{cos\rho_{ni}a}{2\rho_{ni}^2} \int_{x_j^{ni}}^{x_{j+1}^{ni}} q(t)dt + o\left(\frac{1}{n^2}\right), \quad i = 0, 1, \quad j = 0, 1, ..., n-1. \tag{2.4}$$

Proof. We have from (2.2)

$$\varphi(x,\lambda) = \cos\rho x + h\frac{\sin\rho x}{\rho} + \frac{\sin\rho(x-a)}{2\rho}\int_a^x q(t)dt + o\left(\frac{1}{\rho}\right).$$

Now, we set $\varphi(x_j^{ni},\lambda_{ni}) = 0$, because the nodes $\{x_j^{ni}\}$, $i = 0,1$, $n > 1$, $j = \overline{1, n-1}$, are the zeroes of $n-$th eigenfunction. Thus,

$$\cos\rho_{ni}x_j^{ni} + h\frac{\sin\rho_{ni}x_j^{ni}}{\rho_{ni}} + \frac{\sin\rho_{ni}(x_j^{ni}-a)}{2\rho_{ni}}\int_a^{x_j^{ni}} q(t)dt + o\left(\frac{1}{\rho_{ni}}\right) = 0.$$

It can be written as

$$\cot\rho_{ni}x_j^{ni} = -\frac{h}{\rho_{ni}} - [\cos\rho_{ni}a - \cot\rho_{ni}x_j^{ni}\sin\rho_{ni}a]\frac{1}{2\rho_{ni}}\int_a^{x_j^{ni}} q(t)dt + o\left(\frac{1}{\rho_{ni}}\right).$$

Then,

$$\cot\rho_{ni}x_j^{ni}[1 + o\left(\frac{1}{\rho_{ni}}\right)] = -\frac{h}{\rho_{ni}} - \frac{cos\rho_{ni}a}{2\rho_{ni}}\int_a^{x_j^{ni}} q(t)dt + o\left(\frac{1}{\rho_{ni}}\right).$$

We apply Taylor's expansion as $n \to \infty$ for the *arccot* and gain

$$x_j^{ni} = \frac{(j-\frac{1}{2})\pi}{\rho_{ni}} + \frac{h}{\rho_{ni}^2} + \frac{cos\rho_{ni}a}{2\rho_{ni}^2}\int_a^{x_j^{ni}} q(t)dt + o\left(\frac{1}{n^2}\right).$$

So, the length of nodes are

$$l_j^{ni} = x_{j+1}^{ni} - x_j^{ni};$$

therefore,

$$l_j^{ni} = \frac{\pi}{\rho_{ni}} + \frac{cos\rho_{ni}a}{2\rho_{ni}^2}\int_{x_j^{ni}}^{x_{j+1}^{ni}} q(t)dt + o\left(\frac{1}{\rho_{ni}^2}\right).$$

□

3 Main results

Inverse Problem (IP). Presenting the nodes $\{x_j^{ni}\}$, $i = 0,1$, $n > 1$, $j = \overline{1, n-1}$, $x_j^{ni} > a$ construct of the potential function $q(x)$.

Since the nodes $\{x_j^{ni}\}$, $i = 0,1$, $n > 1$, $j = \overline{1, n-1}$, $x_j^{ni} > a$ are the zeroes of n-th eigenfunction $\varphi(x,\lambda_{ni})$, the result will be

$$\varphi(x_j^{ni},\lambda_{ni}) = 0, \quad n > 1, \quad j = \overline{1, n-1}, \quad x_j^{ni} > a, \quad i = 0,1.$$

Using (2.1), we have

$$\int_a^{x_j^{ni}} \sin\rho_{ni}(x_j^{ni}-a)q(t)dt \cong -2\rho_{ni}\cos\rho_{ni}x_j^{ni} - 2h\sin\rho_{ni}x_j^{ni}. \tag{3.1}$$

Equation (3.1) is the first kind of Fredholm type integral equation, where the nodes $\{x_j^{ni}\}$, $i = 0, 1$, $n > 1$, $j = \overline{1, n-1}$, $x_j^{ni} > a$ are known input data and the potential function q is an unknown function.

As a result, for the purpose of finding the solution to IP, all we need to do is to compute the solution of Fredholm integral equation (3.1). One way to find the solution to integral equations and integral differential equations is the application of Chebyshev polynomials as the basic function in order to approximate the unknown function and to turn the integral equation into the system of linear equation. Then, this method can be used to solve the integral equation (3.1).

The first kind of Chebyshev polynomials $T_k(t)$ on the interval $[-1, 1]$ are described by a relation which has a recursive relation:

$$T_0(t) = 1, \quad T_1(t) = t,$$

$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t).$$

With the application of Chebyshev interpolation technique for the function $q(t)$, it can be show that [21, 22]

$$q(t) \cong \sum_{m=0}^{N} q_m l_{m,N}(t), \quad t \in (a, 1), \tag{3.2}$$

where

$$l_{m,N}(t) = \frac{2\delta_m}{N}\sum_{k=0}^{N}{}'' T_k(\frac{-2}{a-1}t + \frac{a+1}{a-1})\cos(\frac{km\pi}{N}),$$

$$\delta_m = \begin{cases} 0.5 & m = 0, N, \\ 1 & 0 < m < N, \end{cases}$$

and the functions $T_k(t)$, $k = \overline{0, N}$ are the first kind of Chebyshev polynomials and the numbers q_m, $m = \overline{0, N}$ exist the values of function $q(t)$ in the points $t_m = (a + 1 - (a-1)\cos(\frac{m\pi}{N}))/2$ which are the extrema of $T_N(\frac{-2}{a-1}t + \frac{a+1}{a-1})$. Meanwhile, $\sum''$ is the aggregate amount of all terms, but not the first and last two sentences in such a way that the aggregate amount of half of the sentences taken into account.

Replacing Eq. (3.2) at Eq. (3.1), we have

$$\sum_{m=0}^{N} q_m \frac{2\delta_m}{N}\sum_{k=0}^{N}{}'' \int_a^{x_j^{ni}} \sin\rho_{ni}(x_j^{ni}-a)T_k(\frac{-2}{a-1}t + \frac{a+1}{a-1})\cos(\frac{km\pi}{N})dt$$

$$\cong -2\rho_{ni}\cos\rho_{ni}x_j^{ni} - 2h\sin\rho_{ni}x_j^{ni}. \tag{3.3}$$

Denote $g(x_j^{ni}) = -2\rho_{ni}\cos\rho_{ni}x_j^{ni} - 2h\sin\rho_{ni}x_j^{ni}$, $n > 1$, $j = \overline{1, n-1}$, $x_j^{ni} > a$, $i = 0, 1$ and

$$I_k(x_j^{ni}) = \int_a^{x_j^{ni}} \sin\rho_{ni}(x_j^{ni} - a)T_k(\frac{-2}{a-1}t + \frac{a+1}{a-1})dt,$$

$$R(t_m, x_j^{ni}) = \frac{2\delta_m}{N}\sum_{k=0}^{N}{}'' I_k(x_j^{ni})\cos(\frac{km\pi}{N}).$$

Using the above formulas, we have

$$\sum_{m=0}^{N} R(t_m, x_j^{ni})q_m = g(x_j^{ni}), \quad n > 1, \quad j = \overline{1, n-1}, \quad i = 0, 1, \quad x_j^{ni} > a.$$

Consequently, one can compute the solution of IP by employing the algorithm below.

Algorithm *1.* An Algorithm to obtain the approximate solution of inverse problem via Chebyshev interpolation method.

Let the nodes $\{x_j^{ni}\}$, $i = 0, 1$, $n > 1$, $j = \overline{1, n-1}$, $x_j^{ni} > a$.

Step 1. Choose N.

Step 2. Determine the coefficients q_m, $m = \overline{0, N}$ by utilizing the next linear system:

$$A\hat{q} \cong B,$$

in which $\hat{q} = [q_0 \quad q_1 \quad ... \quad q_N]^T$,

$$A = [R(t_m, x_j^{ni})], \quad i = 0, 1, \quad j = \overline{1, n-1}, \quad n = N+2, \quad x_j^{ni} > a, \quad m = 0, 1, ..., N.$$

Moreover

$$B = [g(x_j^{ni})], \quad i = 0, 1, \quad j = \overline{1, n-1}, \quad n = N+2, \quad x_j^{ni} > a.$$

4 Numerical examples

Matlab software program has been used to find out the figures of exact and approximate solutions of IP.

Test example 4.1. Let the potential function $q(x) = \sin(3\pi x)$, $i = 1$, $a = 0.4$ and $h = -1$ be given. We get the nodes x_j^{n1}, $n = 23$, $j = \overline{1, 22}$ of the boundary value problem L_1 by using the formulas (2.2) and (2.3) shown in Table 1. Now, with the assumption that q is the known function and the eigenvalues and nodes that are computed in Table 1 are the given data in the IPs. The numerical values of the potential function q can be achieved via the application of the above mentioned algorithm in IP and the given data in Table 1 and by computing the approximate solution of IP through replacing the obtained numerical values with (3.3). In Fig. 1, we can observe the exact solution and the numerical approximation that are computed with $N = 21$.

Table 1: The nodes x_j^{n1} in Example 4.1.

j	1	2	3	4	5	6
x_j^{n1}	0.43170	0.47717	0.52264	0.58810	0.61357	0.65904
j	7	8	9	10	11	12
x_j^{n1}	0.70450	0.74996	0.79542	0.84088	0.88634	0.93181

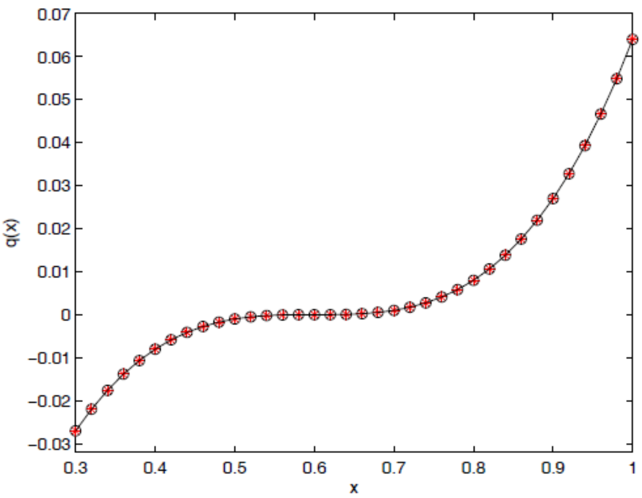

Fig. 1. Approximate and exact solutions of IP in Example 4.1: ($* * *$) for the exact solution and (o o o) for the approximate solution with $N = 21$.

Test example 4.2. Let the potential function $q(x) = (x - 0.6)^3$, $i = 0$, $a = 0.3$ and $h = 0.2$ be given. We calculate the nodes x_j^{n0}, $n = 23$, $j = \overline{1, 22}$ of the boundary value problem L_0 by using the formulas (2.2) and (2.3) seen in Table 2. Now, we suppose that q is the unknown function and

Table 2: The nodes x_j^{n0} in Example 4.2.

j	1	2	3	4	5	6	7	8
x_j^{n0}	0.31917	0.36173	0.40428	0.44683	0.48938	0.53193	0.57448	0.61704
j	9	10	11	12	13	14	15	
x_j^{n0}	0.65959	0.70214	0.74469	0.78724	0.82979	0.87234	0.91490	

the nodes obtained in Table 2 are the input data in the IPs. The exact solution and the numerical approximations calculated with $N = 21$ are shown in Fig. 2.

5 Conclusion

We have explored the inverse nodal SLP featuring a delay constant, and then Chebyshev interpolation approach has been employed to achieve the approximate solution of the assumed problem. In other words, it can be said that the above mentioned method is an approximate way to find the

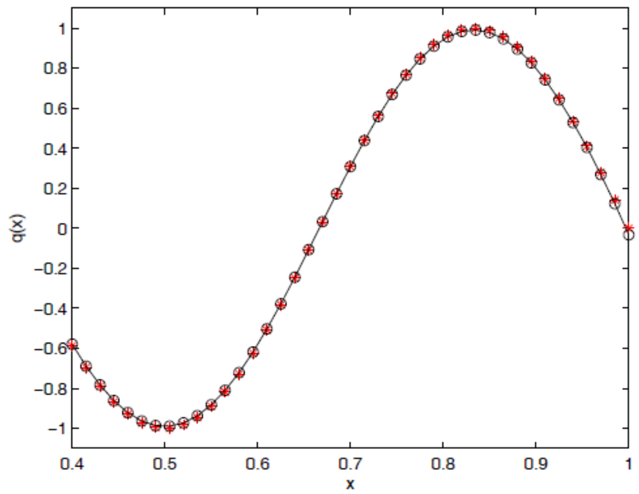

Fig. 2. Approximate and exact solutions of IP in Example 4.2: $(* * *)$ for the exact solution and (o o o) for the approximate solution with $N = 21$.

solution for the inverse nodal problem with a delay constant via the application of a limited number of nodal points.

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