INVERSE PROBLEM FOR HILL EQUATION WITH JUMP CONDITIONS

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ABSTRACT. In this paper, the inverse nodal problem is discussed for discontinuous periodic (or anti-periodic) Sturm-Liouville problem. Such type problems are different from regular problems because of discontinuity in the boundary conditions. Firstly, results of Sturm-Liouville problem including jump condition are present. Then, by deferring the zeros of eigenfunctions, inverse problem is solved as we desired. The method is based on considering a translation so that the periodic (or anti-periodic) problem is reduced to a Dirichlet problem as in Cheng and Law's paper [5]. But, our problem including also discontinuous conditions. In addition to all of these, although there are so many results on this subject, we combine both periodic conditions and discontinuity.

1. INTRODUCTION

Let us consider the Sturm-Liouville operator in Liouville normal form

$$Sy = -y'' + q(x)y = \lambda y, \tag{1.1}$$

where $q \in L^1[0,\pi]$ and λ is spectral parameter. The forward periodic problem means to find the eigenvalues of the following two problems:

$$Sy = \lambda y, \tag{1.2}$$

$$y(0) = y(\pi), \qquad y'(0) = y'(\pi) \tag{1.3}$$

and

$$Sy = \lambda y, \tag{1.4}$$

$$y(0) = -y(\pi), \qquad y'(0) = -y'(\pi). \tag{1.5}$$

The eigenfunctions of (1.2),(1.3) are periodic, i.e., $y(x+\pi) = y(x)$ and those of (1.4),(1.5) are anti-periodic, i.e., $y(x+\pi) = -y(x)$. We will indicate the

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eigenvalues of periodic(or anti-periodic) problems as λ_n ($\tilde{\lambda}_n$), respectively. Also,we will use the term *periodic spectrum* for all periodic and anti-periodic eigenvalues.

The term "Hill equation" is a suitable abbreviation defining the class of Sturm-Liouville equations with real and periodic coefficients. Hill equation has many applications in engineering, mechanics and astronomy [2], [6],[7],[15],[17].

Let us consider the periodic Sturm-Liouville problem with discontinuity conditions inside the interval $(0, \pi)$

$$Sy = \lambda y, \qquad 0 < x < \pi, \tag{1.6}$$

$$y(0) = y(\pi), \qquad y'(0) = y'(\pi), \tag{1.7}$$

$$y\left(\frac{\pi}{2}+0\right) = \alpha y\left(\frac{\pi}{2}-0\right), \quad y'\left(\frac{\pi}{2}+0\right) = \frac{1}{\alpha}y'\left(\frac{\pi}{2}-0\right), \tag{1.8}$$

where $\alpha > 0$ is real number.

Solution of discontinuous Sturm-Liouville problems appears in applied mathematics, mechanics and other branches. For instance, one of them is torsional modes of the earth [12],[29]. For these type problems, inverse problems where studied widely by many authors [21],[25],[26],[27],[28],[29],[30].

Inverse nodal problem means to look for the potential function q by position of zeros of eigenfunctions (nodal points). This problem was firstly solved by Mclaughlin [18]. Later the uniqueness problem was defined for the potential function q with general boundary conditions [31].The reconstruction and stability of the potential have been given by many authors [3],[4],[8],[10],[9],[11],[13],[14],[16],[19][22],[23][24]. Especially, the reconstruction formula for the potential function q was given as

$$q(x) = \lim_{n\to\infty} 2\lambda_n \left(\sum_{j=0}^{n-1} \frac{\sqrt{\lambda_n} l_j^{(n)} x_j^{(n)}}{\pi} - 1 \right),$$

where λ_n is the nth eigenvalue as the jth nodal point of the nth eigenfunction and $l_n^{(j)} = x_n^{(j+1)} - x_n^{(j)}$ as the nodal length and χ_A is characteristic function [16].

In this paper, our aim is to study inverse nodal problem for Hill equation with discontinuous conditions inside $[0, \pi]$. Firstly, we should give eigenvalues, nodal lengths and reconstruction formula for the potential discontinuous Sturm-Liouville equation with Dirichlet boundary conditions. We have obtained these results by Prüfer substitution [1] which is different from the other technichs. It seems that this method was firstly applied for discontinuous

Sturm-Liouville problem [20]. *Note that, we will abbreviate Discontinuous Hill Equation as DHE.*

Firstly we need a translation to obtain the results. The periodic (or anti-periodic) spectrum dos not change upon any translation of q for discontinuous Hill equation. Then, when we translate the problem by a distance equal to first nodal position x_1, so that the periodic (or anti-periodic) problem becomes a discontinuous Dirichlet problem. This idea is inspired by the work [16].

Let x_n^j ($\tilde{x}_n^j$) be periodic (anti-periodic) nodal points and λ_n^j ($\tilde{\lambda}_n^j$) be the spectrums of DHE. Denote $z_n^j = x_n^j - x_n^1$, $l_n^j = z_n^{j+1} - z_n^j$. Then z_n^j are nodal points of discontinuous Sturm-Liouville problem with Dirichlet conditions and the asymptotic formula of z_n^j follows that of the Dirichlet case.

2. Asymptotics of Eigenvalue and Nodal Parameters of DHE

In this part, we will give the estimates of DHE. We are aware of that after a translation x by the firstly nodal point x_1, DHE becomes

$$y''(x+x_1) + [\lambda - q(x+x_1)]y(x+x_1) = 0,$$

$$y(x_1) = y(\pi + x_1) = 0,$$

$$y\left(\left(\frac{\pi}{2}\right)_{+0} + x_1\right) = \alpha y\left(\left(\frac{\pi}{2}\right)_{-0} + x_1\right), \quad y'\left(\left(\frac{\pi}{2}\right)_{+0} + x_1\right) = \frac{1}{\alpha} y'\left(\left(\frac{\pi}{2}\right)_{-0} + x_1\right).$$

Taking $y(x+x_1) = Y(x)$, $q(x+x_1) = Q(x)$, the above problem will reconsidered

$$-Y''(x) + Q(x)Y(x) = \lambda Y(x), \tag{2.1}$$

$$Y(0) = Y(\pi) = 0, \tag{2.2}$$

$$Y\left(\frac{\pi}{2}+0\right) = \alpha Y\left(\frac{\pi}{2}-0\right), \quad Y'\left(\frac{\pi}{2}+0\right) = \frac{1}{\alpha} Y'\left(\frac{\pi}{2}-0\right), \tag{2.3}$$

which is Discontinuous Dirichlet Problem (DDP). Note that jth nodal point of (2.1)-(2.3) will be

$$z_j = x_j - x_1. \tag{2.4}$$

Theorem 2.1. *Let q be real-valued integrable on $(0, \pi)$ and $s_m = \lambda_m(q) = \lambda_{2n}(Q)$. Also, denote $\left[\frac{m+1}{2}\right] = n$. Then, we get* *(i) for periodic case;*

$$s_m(q) = 4n + \frac{1}{8n\pi}\left[\int_0^{\pi} q(s+x_m^1)ds + (-1)^m\left(\int_0^{\pi} q(s+x_m^1)ds - 2\int_0^{\frac{\pi}{2}} q(s+x_m^1)ds\right)\right] + O\left(\frac{1}{m^2}\right), \tag{2.5}$$

$$x_m^j = \frac{j\pi}{2n} + \frac{1}{32n^2}\int_0^{x_m^j} q(s)ds + O\left(\frac{1}{m^3}\right), \tag{2.6}$$

$$l_{2n}^j = L_m^j = \frac{2\pi}{s_m} + \frac{1}{2s_m^2}\int_{x_m^j}^{x_m^{j+1}} q(s)ds + o\left(\frac{1}{m^2}\right) \tag{2.7}$$

in case of $x_m^j \in \left(x_m^1, \frac{\pi}{2} + x_m^1\right)$.

$$x_m^j = \frac{j\pi}{2n} - \frac{j}{4n^2}\gamma_n + \frac{1}{32n^2}\int_0^{x_m^j} q(s)ds - \frac{1}{32n^2}\int_0^{\frac{\pi}{2}} q(s+x_m^1)ds + O\left(\frac{1}{m^3}\right), \tag{2.8}$$

$$l_{2n}^j = L_m^j = \frac{2\pi}{s_m} + \frac{\gamma_{m+1}}{s_m} + \frac{1}{2s_m^2}\int_{x_m^j}^{x_m^{j+1}} q(s)ds + o\left(\frac{1}{m^2}\right). \tag{2.9}$$

in case of $x_m^j \in \left(x_m^1 + \frac{\pi}{2}, x_m^1 + \pi\right)$.

(ii) for anti-periodic case;

$$\tilde{s}_m(q) = 2(2n-1) + \frac{1}{4(2n-1)\pi}\left[\int_0^{\pi} q(s+\tilde{x}_m^1)ds + (-1)^m\left(\int_0^{\pi} q(s+\tilde{x}_m^1)ds - 2\int_0^{\frac{\pi}{2}} q(s+\tilde{x}_m^1)ds\right)\right] + O\left(\frac{1}{m^2}\right),$$

$$\tilde{x}_m^j = \frac{j\pi}{2n-1} + \frac{1}{8(2n-1)^2}\int\limits_0^{\tilde{x}_m^j} q(s)ds + O\left(\frac{1}{m^3}\right),$$

$$\tilde{L}_m^j = \frac{2\pi}{\tilde{s}_m} + \frac{1}{2\tilde{s}_m^2}\int\limits_{\tilde{x}_m^j}^{\tilde{x}_m^{j+1}} q(s)ds + o\left(\frac{1}{m^2}\right).$$

in case of $\tilde{x}_m^j \in \left(\tilde{x}_m^1, \frac{\pi}{2} + \tilde{x}_m^1\right)$.

$$\tilde{s}_m = 2(2n-1) + \frac{1}{2(2n-1)\pi^2}\left[\int\limits_0^{\pi} q(s+\tilde{x}_m^1)ds + (-1)^{2n-2}\left(\int\limits_0^{\pi} q(s+\tilde{x}_m^1)ds \right.\right.$$
$$\left.\left. -2\int\limits_0^{\frac{\pi}{2}} q(s+\tilde{x}_m^1)ds\right)\right] + O\left(\frac{1}{m^2}\right),$$

$$\tilde{x}_m^j = \frac{j\pi}{2n-1} - \frac{j}{(2n-1)^2} + \frac{1}{8(2n-1)^2}\int\limits_0^{\tilde{x}_m^j} q(s)ds - \frac{1}{8(2n-1)^2}\int\limits_0^{\frac{\pi}{2}} q(s+x_m^1)ds + O\left(\frac{1}{m^3}\right),$$

$$\tilde{L}_m^j = \frac{2\pi}{\tilde{s}_m} + \frac{\gamma_m}{\tilde{s}_m} + \frac{1}{2\tilde{s}_m^2}\int\limits_{\tilde{x}_m^j}^{\tilde{x}_m^{j+1}} q(s)ds + o\left(\frac{1}{m^2}\right).$$

in case of $\tilde{x}_m^j \in \left(\tilde{x}_m^1 + \frac{\pi}{2}, \tilde{x}_m^1 + \pi\right)$.

Proof. Proof consists of only the periodic case. The anti-periodic case is similar. Let us consider asymptotic formulas of the eigenvalues, nodal points and nodal lengths for the Dirichlet discontinuous problem as [20]:

$$\lambda_n(Q) = 2n + \frac{1}{4n\pi}\left(\zeta_1 + (-1)^{n-1}\zeta_2 + \frac{8\gamma_n^2}{\pi}\right) + O\left(\frac{1}{n^2}\right),$$

where

$$\zeta_1 = \int_0^{\pi} Q(s)ds, \qquad \zeta_2 = \int_0^{\pi} Q(s)ds - 2\int_0^{\frac{\pi}{2}} Q(s)ds,$$

$$\gamma_n = \begin{cases} Arcsin\left(\frac{1}{\sqrt{1+\alpha^2}}\right), & \text{if n is even,} \\ \\ -Arcsin\left(\frac{|\alpha|}{\sqrt{1+\alpha^2}}\right), & \text{if n is odd.} \end{cases}$$

$$z_n^j = \frac{j\pi}{n} + \frac{1}{8n^2}\int_0^{z_n^j} Q(s)ds + O\left(\frac{1}{n^3}\right),$$

$$l_n^j = \frac{2\pi}{\lambda_n} + \frac{1}{2\lambda_n^2}\int_{z_n^j}^{z_n^{j+1}} Q(s)ds + o\left(\frac{1}{\lambda_n^2}\right), \quad z_n^j \in \left(0, \frac{\pi}{2}\right),$$

and

$$z_n^j = \frac{j\pi}{n} - \frac{j\gamma_n}{n^2} + \frac{1}{8n^2}\int_0^{z_n^j} Q(s)ds - \frac{1}{8n^2}\int_0^{\frac{\pi}{2}} Q(s)ds + O\left(\frac{1}{n^3}\right),$$

$$l_n^j = \frac{2\pi}{\lambda_n} + \frac{\gamma_n}{\lambda_n} + \frac{1}{2\lambda_n^2}\int_{z_n^j}^{z_n^{j+1}} Q(s)ds + o\left(\frac{1}{\lambda_n^2}\right), \quad z_n^j \in \left(\frac{\pi}{2}, \pi\right).$$

However, we can obtain the results of theorem 2.1 with Prüfer substitution, we will use the method used in [16]. For this, we replace the nodal points of Dirichlet problem by the first node x_1, then we obtain desired results for the periodic problem. Let $m = 2n - 1$ and by choosing $s_m = \lambda_m(q) = \lambda_{2n}(Q)$ and $Q(s) = q\left(s + x_m^1\right)$.

Now, we will chose $z = s + x_m^1$ in above asymptotic formulas, then (2.7) and (2.9) hold. Thus, eigenvalues and nodal points given in (2.5), (2.6), (2.8) are obtained, respectively. This completes the proof. □

3. Reconstruction of the Potential Function

Theorem 3.1. *Given the nodal set x_m^j. For $x \in \left(x_m^1, x_m^1 + \frac{\pi}{2}\right)$ the potential function $q \in L^1[0, \pi]$ can be reconstructed for the discontinuous periodic* *problem (1.6)-(1.8)*

$$q(x) = \lim_{m\to\infty} 2s_m^2 \left(\frac{s_m L_m^j}{2\pi} - 1 \right) \tag{3.1}$$

for $j = j_m(x) = max\,\{j | x_m^j < x\}$.

Proof. From (2.7), we have

$$\frac{s_m L_m^j}{2\pi} - 1 = \frac{1}{4\pi s_m} \int_{x_m^j}^{x_m^{j+1}} q(s)ds + o\left(\frac{1}{m}\right)$$

and appliying the mean-value problem, with fixed m and for each j, there exists $t \in (x_m^j, x_m^{j+1})$ such that

$$\int_{x_m^j}^{x_m^{j+1}} q(s)ds = L_m^j q(t).$$

Then we have

$$q(x) = \frac{4\pi s_m}{L_m^j} \left(\frac{s_m L_m^j}{2\pi} - 1 \right), \qquad x < x_m^1 + \frac{\pi}{2}.$$

Furthermore $L_m^j = \frac{2\pi}{s_m} + o\left(\frac{1}{m^2}\right)$ Then, for $m \to \infty$ we complete the proof. □

Theorem 3.2. *Given the subset* X_m^j *of the nodal points which is dense on* $(0,\pi)$*. Then, for* $x \in \left(x_m^1 + \frac{\pi}{2}, x_m^1 + \pi\right)$*, the potential function can be reconstructed as*

$$q(x) = \lim_{m\to\infty} 2s_m^2 \left(\frac{s_m L_m^j}{2\pi} - \frac{\gamma_{m+1}}{2\pi} - 1 \right).$$

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Proof. By (2.9), we have

$$\frac{s_m L_m^j}{2\pi} - \frac{\gamma_{m+1}}{2\pi} - 1 = \frac{1}{4\pi s_m} \int_{x_m^j}^{x_m^{j+1}} q(s)ds + o\left(\frac{1}{m}\right)$$

or

$$q(x) = \frac{4\pi s_m}{L_m^j} (\frac{s_m L_m^j}{2\pi} - \frac{\gamma_{m+1}}{2\pi} - 1) + o\left(\frac{1}{m}\right)$$

Then, it is not difficult to complete the proof for $x \in \left(x_m^1 + \frac{\pi}{2}, x_m^1 + \pi\right)$. □

Theorem 3.3. *For the discontinuous anti-periodic problem, reconstruction of the potential function can be written*

$$q(x)=\lim_{m\to\infty}2\tilde{s}_m^2\left(\frac{\tilde{s}_m\tilde{L}_m^j}{2\pi}-1\right),$$

$j=j_m(x)$ *and* $x\in\left(\tilde{x}_m^1,\tilde{x}_m^1+\frac{\pi}{2}\right)$

$$q(x)=\lim_{m\to\infty}2\tilde{s}_m^2\left(\frac{\tilde{s}_m\tilde{L}_m^j}{2\pi}-\frac{\gamma_{m+1}}{2\pi}-1\right)$$

$j=j_m(x)$ *and* $x\in\left(\tilde{x}_m^1+\frac{\pi}{2},\tilde{x}_m^1+\pi\right)$,

where $\tilde{s}_m$, $\tilde{L}_m$ are eigenvalues and nodal lengths of the anti-periodic problem.

4. Conclusion

In the present paper we propose a formula for solving inverse nodal problem of Sturm–Liouville problem with periodic and anti periodic conditions. The method works well for any considered in this type problem even if the use of different type equations like Diffusion or Dirac equations. The main key in this method is Prüfer substitution used in second order differential equations.

5. Declaration of Competing Interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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