Uniqueness for multidimensional kernel determination problems from a parabolic integro-differential equation

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Abstract. We study two problems of determining the kernel of the integral terms in a parabolic integro-differential equation. In the first problem the kernel depends on time t and $x = (x_1, ..., x_n)$ spatial variables in the multidimensional integro-differential equation of heat conduction. In the second problem the kernel it is determined from one dimensional integro-differential heat equation with a time-variable coefficient of thermal conductivity. In both cases it is supposed that the initial condition for this equation depends on a parameter $y = (y_1, ..., y_n)$ and the additional condition is given with respect to a solution of direct problem on the hyperplanes $x = y$. It is shown that if the unknown kernel has the form $k(x,t) = \sum_{i=o}^{N} a_i(x)b_i(t)$, then it can be uniquely determined.

Keywords: parabolic equation, Cauchy problem, integral equation, linearly independence, uniqueness.

1.Introduction. Formulation of problem

Integro–differential equations play an important role in the mathematical modeling of many fields: physical, biological phenomena, engineering sciences and others fields of the natural sciences where it is necessary to take into account the effect of a prehistory (or memory) of process. Constitutive relations in the linear non-homogeneous diffusion and wave propagation processes with memory contain time - and space-dependent memory kernel with convolution type integrals. Usually, they are obtained from experiences. For many cases, in the practise these kernels are unknown functions and thus the inverse problems are arose on determining of these functions from the observable information about the solutions of the corresponding equations. Problems of identification of memory kernels in parabolic and hyperbolic equations have been intensively studied starting at the end of the last century [1]-[4].

Often, in cases of equations describing the propagation of electrodynamic and elastic waves with integral convolution terms are reduced to one second-order hyperbolic integro–differential equation. One- and multidimensional problems of recovering the kernel of convolution integral in these equations were investigated in [5]-[24] (see, also references therein). The numerical solutions for kernel determination problems from integro–differential equations were considered in the works [25]-[27]. Inverse problems to determine time- and space-dependent kernels in parabolic integro-differential equations with several additional conditions have been studied by many authors [28]-[33]. In these papers there were proved existence, uniqueness and stability theorems. In the works [34]-[39] the authors discussed the linear inverse source and nonlinear inverse coefficient problems for parabolic integro-differential equations. Here also has been applied a numerical approach for solving such problems. It should be noted that nowadays there are few publications where the problems of determining multidimensional memory would be studied.

In the present paper we study inverse problems to determine a time and spatially varying kernel $k(x,t)$, $x \in \mathbb{R}^n$, $t > 0$ in a parabolic integro–differential equations governing the heat flow in materials with memory.

Consider Cauchy problem for the n-dimensional parabolic integro–differential equation with a time-variable coefficient of thermal conductivity

$$\frac{\partial u}{\partial t} - c(t)\triangle_x u = \int_0^t k(x,t-\tau)u(x,y,\tau)d\tau,$$
$$x = (x_1,...,x_n) \in \mathbb{R}^n,\ t \in (0,\ T], \tag{1.1}$$

$$u(x,y,0) = \varphi(x,y), \tag{1.2}$$

where $c(t)$ is an enough smooth positive function, $\triangle_x$ is Laplacian on the variables $x = (x_1,...,x_n)$, $y = (y_1,...,y_n) \in \mathbb{R}^n$ is a parameter of problem, T is a fixed positive number.

In this paper we investigate the following problems:

Inverse problem 1: when $c(t) = 1$, find a kernel $k(x,t)$ of the integral term in (1.1) if a solution to the Cauchy problem (1.1) and (1.2) is known on $x = y$ for all $y \in \mathbb{R}^n$ and $t \in [0,T]$:

$$u(y,y,t) = \psi(y,t), \quad \psi(y,0) = \varphi(y,y). \tag{1.3}$$

Inverse problem 2: when $n = 1$, find a kernel $k(x,t)$ of the integral term in (1.1) if a solution to the Cauchy problem (1.1) and (1.2) is known and it is given by (1.3).

Among the works which are close to the inverse problems 1 and 2 we note [31]-[33]. In [31] there was proven the uniqueness theorem for solution of kernel determination problem for one-dimensional heat conduction equation. The papers [32], [33] deal with the inverse problems of determining the kernel depending on a time variable t and $(n-1)$-dimensional spatial variable $x' = (x_1,...,x_{n-1})$. While the main part of the considered integro-differential equation is n-dimensional heat conduction operator and the integral term has a convolution type form with respect to unknown functions: the solutions of direct and inverse problems. It should be here noted that the kernel $k(t,x)$ in (1.1) depends on all variables like, and the solution $u(x,t)$ of the direct problem (1.1) and (1.2).

Let $B^m(Q)$ be the class of m times continuously differentiable with respect to all variables and bounded together with all derivatives up to the order of m in the domain Q functions. When $m = 0$, $B^0(Q) =: B(Q)$ and this is usual space of continuous and bounded functions.

In this paper we assume that the function $k(x,t)$ with derivatives $k_{x_i x_j}$, $i,j = 1,2,...,n$, k_t belongs to $B(D_T)$, $D_T := \{(x,t) : x \in \mathbb{R}^n, 0 \le t \le T\}$ for any fixed $T > 0$, and the function $\varphi(x,y)$ is in $B^4(\mathbb{R}^n \times \mathbb{R}^n)$.

Besides, let the function $k(x,t)$ have the separable form, i.e. it can be expressed as the sum of a finite number N of terms, each of which is the product of a function of x only and a function of t only:

$$k(x,t)=\sum_{i=0}^{N}a_i(x)b_i(t),\ \ a_i(x)\in B^2(\mathbb{R}^n),\ \ b_i(t)\in C^1(\mathbb{R}), \tag{1.4}$$

where $C^1(\mathbb{R})$ is the class of continuously differentiable in $\mathbb{R}$ functions. The functions $a_i(x)$ can be assumed to be linearly independent, otherwise the number of terms in relation (1.4) can be reduced.

2. Inverse problem 1

The main result of this section is the following uniqueness theorem for the inverse problem 1:

Theorem 2.1. *Suppose that the all assumptions about function $\varphi(x,y)$ in Section 1 are fulfilled. Besides, the function $\psi(y,t)$ together with derivatives ψ_t, ψ_{tt} and $\psi_{ty_iy_i}$, $i=1,...,n$ belong to the class $B\left(D_T\right)$ for any fixed $T>0$ and*

$$\inf_{y\in\mathbb{R}^n}\mid\psi(y,t)\mid=\mu(t)\geq\mu_0>0, \tag{2.1}$$

where μ_0 is a known numbers, then any function $k(x,t)$ having the form (1.4) is uniquely determined by the information (1.3) in domain D_T,

2.1. Auxiliary problem

In this paper we are not dwell on issues related to the existence theorem of the inverse problem 1. We note only natural necessary conditions which must satisfy the function $\psi(y,t)$. They are the second equalities of (1.3), (2.3) and

$$\Delta_y\psi_t(y,0)-\psi_{tt}(y,0)=\Delta_x\Delta_y\varphi(y,y)+2\sum_{i=1}^{n}\Delta_x\varphi_{x_iy_i}-k(y,0)\varphi(y,y),$$

which follows from the equalities (2.8) and (2.9).

First of all we write the problem (1.1)-(1.3) with respect to the functions u_t, k. It follows from (1.1)-(1.3) the problem for these functions:

$$(u_t)_t-\Delta_x u_t=k(x,t)\varphi(x,y)+\int_0^t k(x,\tau)u_t(x,y,t-\tau)d\tau, \tag{2.2}$$

$$u_t(x,y,0)=\Delta_x\varphi(x,y), \tag{2.3}$$

$$u_t(y,y,t)=\psi_t(y,t),\quad \psi_t(y,0)=\Delta_x\varphi(y,y),\ \ y\in\mathbb{R}^n. \tag{2.4}$$

Here, the initial condition (2.3) was obtained from equality (1.1) by setting $t=0$.

Further introduce the notations

$$\omega_i:=u_{ty_i},\quad i=1,2,...,n,\quad v=2div_x\omega+div_y\omega.$$

Here and below, a variable in index of operators *div*, *grad* indicates that they apply in this variable.

Differentiating (2.2) and (2.3) with respect to y_i, we get the Cauchy problem for the determining of functions $\omega_i(x,y,t)$

$$(\omega_i)_t - \Delta_x(\omega_i) =$$

$$= k(x,t)\varphi_{y_i}(x,y) + \int_0^t k(x,\tau)\omega_i(x,y,t-\tau)d\tau,\ x\in\mathbb{R}^n,\ t\in(0,\ T], \tag{2.5}$$

$$\omega_i(x,y,0) = \Delta_x\varphi_{y_i}(x,y), \quad i=1,2,...,n. \tag{2.6}$$

Applying by differential operators $2\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_i}$ to equation (2.5) alternately and summing the results with respect to i from $i=1$ to $i=n$, taking into account the above introduced notation, we obtain the equation for function $v(x,y,t)$

$$v_t - \Delta_x v =$$

$$= k(x,t)\left[2\sum_{i=1}^n \varphi_{x_iy_i} + \Delta_y\varphi\right](x,y) + \int_0^t v(x,\tau)v(x,y,t-\tau)d\tau +$$

$$+2grad_x k(x,t)\cdot grad_y\varphi(x,y) + 2\int_0^t grad_x k(x,\tau)\cdot\omega(x,y,t-\tau)d\tau, \tag{2.7}$$

where $a\cdot b$ means the scalar product of vectors a and b. From (2.6) in this way, we get the initial condition

$$v(x,y,0) = 2\sum_{i=1}^n \Delta_x\varphi_{x_iy_i}(x,y) + \Delta_x\Delta_y\varphi(x,y). \tag{2.8}$$

It follows from (2.4) the relations

$$\omega_i(y,y,t) = \frac{\partial}{\partial y_i}u_t(y,y,t) = (u_{tx_i}+u_{ty_i})(y,y,t) = \psi_{ty_i}(y,t),$$

$$\omega_{iy_i}(y,y,t) = \frac{\partial^2}{\partial y_i^2}u_t(y,y,t) =$$

$$= (u_{tx_ix_i} + 2u_{tx_iy_i} + u_{ty_iy_i})(y,y,t) = \psi_{ty_iy_i}(y,t), \quad i=1,2,...,n.$$

$$div_y\omega(y,y,t) = \left(\Delta_x u_t + 2\sum_{i=1}^n u_{tx_iy_i} + \Delta_y u_t\right)(y,y,t) = \Delta_y\psi_t(y,t).$$

In view of the last equalities and (2.2), we note that the condition (1.3) in the term of function v can be written in the form

$$v(y,y,t) = \Delta_y\psi_t(y,t) - \psi_{tt}(y,t) + k(y,t)\varphi(y,y) + \int_0^t k(y,\tau)\psi_t(y,t-\tau)d\tau. \tag{2.9}$$

Note that at the found from (2.5) and (2.6) $\omega_i,\ i=1,2,...,n$, the function v can be determined from the problem (2.7) and (2.8).

We present two lemmata, which will be needed in future use.

Lemma 2.1. *For a solution $p(x,t)$ of problem*

$$p_t - \triangle_x p = \int_0^t h(x,\tau)p(x,t-\tau)d\tau + f(x,t), \quad p|_{t=0} = \lambda(x), \quad x \in \mathbb{R}^n, t > 0 \qquad (2.10)$$

in the domain D_T takes place the estimate

$$|p(x,t)| \leq \Phi e^{T\|h\|_T t} + \int_0^t F(\tau)e^{T\|h\|_T (t-\tau)}d\tau, \qquad (2.11)$$

where

$$\|h\|_T := \max_{0\leq t\leq T} \sup_{x\in\mathbb{R}^n} |h(x,t)|, \, \Phi := \sup_{x\in\mathbb{R}^n} |\lambda(x)|, \quad F(t) := \sup_{x\in\mathbb{R}^n} |f(x,t)|.$$

For **proof** of this lemma, we note that the solution of Cauchy problem (2.10) satisfies the integral equation

$$p(x,t) = \frac{1}{\left(2\sqrt{\pi t}\right)^n} \int_{x\in\mathbb{R}^n} e^{\frac{-|x-\xi|^2}{4t}} \lambda(\xi)d\xi +$$

$$+\frac{1}{(2\sqrt{\pi})^n} \int_0^t \int_{x\in\mathbb{R}^n} \frac{e^{\frac{-|x-\xi|^2}{4(t-\tau)}}}{\left(\sqrt{t-\tau}\right)^n} f(\xi,\tau)d\xi d\tau + \frac{1}{(2\sqrt{\pi})^n} \int_0^t \int_{x\in\mathbb{R}^n} \frac{e^{\frac{-|x-\xi|^2}{4(t-\tau)}}}{\left(\sqrt{t-\tau}\right)^n} \times$$

$$\times \int_0^\tau h(\xi,\tau-\alpha)p(\xi,\alpha)d\alpha d\xi d\tau.$$

Using the standard method for estimating integrals, we have

$$U(t) \leq \Phi + \int_0^t F(\tau)d\tau + \|h\|_T T \int_0^t U(\tau)d\tau,$$

where $U(t) := \sup_{x\in\mathbb{R}^n} |p(x,t)|$. From here, based on Gronwall's inequality follows (2.10).

Lemma 2.2. [40]. *Let $k(x,t)$ has the form (1.4) and $K(t) := \sup\limits_{x\in\mathbb{R}^n} |k(x,t)|$. Then there exists a constant K_0 (generally speaking, different for each function k) so that the inequality*

$$|k_{x_i}(x,t)| \leq K_0 K(t), \; i = 1, ..., n \qquad (2.12)$$

is true.

The **proof** of this lemma is based on the assumption that the system of functions a_i, $i = 1, 2, ..., N$ can be considered linearly independent in $\mathbb{R}^n$ (otherwise, one can rearrange the terms in (1.4), leaving only a linearly independent system of functions a_i). In fact, then there is $\beta > 0$ so that $\sup\limits_{x\in\mathbb{R}^n} \left|\sum_{j=1}^N c_j a_j(x)\right| \geq \beta$, if $\sum_{j=1}^N |c_j| = 1$. In view of this, we have

$$\sup_{x\in\mathbb{R}^n} |k(x,t)| = \sup_{x\in\mathbb{R}^n} \left|\sum_{j=1}^N a_j(x) \frac{b_j(t)}{\sum_{l=1}^N |b_l(t)|}\right| \sum_{l=1}^N |b_l(t)| \geq \beta \sum_{l=1}^N |b_l(t)|.$$

At the same time, it follows from (1.4)

$$|k_{x_i}(x,t)| \leq \max_{1\leq j\leq N} \sup_{x\in\mathbb{R}^n} |a_{jx_i}(x)| \sum_{j=1}^{N} |b_j(t)| \, .$$

Matching the last two inequalities, we find

$$|k_{x_i}(x,t)| \leq K_0 \sup_{x\in\mathbb{R}^n} |k(x,t)| \leq K_0 K(t),$$

where

$$K_0 := \frac{1}{\beta} \max_{1\leq j\leq N} \sup_{x\in\mathbb{R}^n} |a_{jx_i}(x)| \, .$$

2.2. Proof of main result

For proof of the main result, we suppose that there are two solutions k_1 and k_2 of problem (1.1)-(1.3) and denote the corresponding to these functions solutions of Cauchy problem (1.1), (1.2) by u_1 and u_2, respectively. Introduce functions $\omega^{(1)} = \left(\omega_1^{(1)},...,\omega_n^{(1)}\right)$, $\omega^{(2)} = \left(\omega_1^{(2)},...\omega_n^{(2)}\right)$, v_1, v_2, similarly to functions $\omega = (\omega_1,...,\omega_n)$, v. We also denote

$$\widetilde{k} = k_1 - k_2,\ \widetilde{\omega} = \omega^{(1)} - \omega^{(2)},\ \widetilde{\omega_i} = \omega_i^{(1)} - \omega_i^{(2)},\ i = 1,...,n,\ \widetilde{v} = v_1 - v_2.$$

Then for $\widetilde{\omega_i}$, $\widetilde{v}$, from the equations (2.5)-(2.8) we find

$$\widetilde{\omega}_{it} - \Delta\widetilde{\omega}_i = \int_0^t k_1(x,\tau)\widetilde{\omega}_i(x,y,t-\tau)d\tau +$$

$$+\widetilde{k}(x,t)\varphi_{y_i}(x,y) + \int_0^t \widetilde{k}(x,\tau)\omega_i^{(2)}(x,y,t-\tau)d\tau, \tag{3.1}$$

$$\widetilde{\omega}_i|_{t=0} = 0, \quad i = 1,...,n, \tag{3.2}$$

$$\widetilde{v}_t - \triangle_x\widetilde{v} = \int_0^t k_1(x,\tau)\widetilde{v}(x,y,t-\tau)d\tau +$$

$$+\widetilde{k}(x,t)\left[2\sum_{i=1}^{n}\varphi_{x_iy_i} + \Delta_y\varphi(x,y)\right] + 2grad_x\widetilde{k}(x,t)\cdot grad_y\varphi(x,y) +$$

$$+\int_0^t \widetilde{k}(x,\tau)v_2(x,y,t-\tau)d\tau + 2\int_0^t grad_x\widetilde{k}(x,\tau)\cdot\omega^{(2)}(x,y,t-\tau)d\tau +$$

$$+2\int_0^t grad_x k_1(x,\tau)\cdot\widetilde{\omega}(x,y,t-\tau)d\tau, \tag{3.3}$$

$$\widetilde{v}|_{t=0} = 0. \tag{3.4}$$

It follows from (2.9) the equality

$$\widetilde{v}|_{x=y} = \widetilde{k}(y,t)\varphi(y,y) + \int_0^t \widetilde{k}(y,\tau)\psi_t(y,t-\tau)d\tau. \tag{3.5}$$

The equations (3.1)-(3.5) present the homogenous system of equations with respect to unknown functions $\widetilde{\omega}_i,\ i=1,...,n,\ \widetilde{v}$ and $\widetilde{k}$. It is required to proof that this system has only trivial solution in the domain D_T. To show this fact we need the estimates of functions $\widetilde{\omega},\ i=1,...,n,\ \widetilde{v}$ through $\widetilde{k}$.

In what follows we use the following notations for norms of the known functions depending on different variables:

$\|z_1\| := \sup\limits_{(x,y)\in\mathbb{R}^{2n}} |z_1(x,y)|$-for functions depending on (x,y);

$\|z_2\|_T := \sup\limits_{(x,t)\in D_T} |z_2(x,t)|$-for functions depending on (x,t);

$\|z_3\|^T := \sup\limits_{(x,y)\in\mathbb{R}^{2n},t\in(0,T]} |z_3(x,y,t)|$-for functions depending on (x,y,t) and notations for norms of the unknown functions

$$\|\widetilde{\omega}\|^T := \max_{1\le i\le n}\max_{0\le t\le T}\sup_{(x,y)\in\mathbb{R}^{2n}} |\widetilde{\omega}_i(x,y,t)|\,, \quad \|\widetilde{v}\|^T := \max_{0\le t\le T}\sup_{(x,y)\in\mathbb{R}^{2n}} |\widetilde{v}(x,y,t)|\,.$$

Using lemma 2.2 for $\widetilde{\omega}_i$ from (3.1) and (3.2), we obtain the estimate

$$|\widetilde{\omega}_i(x,y,t)| \le \int_0^t \left[\|\varphi_{y_i}\|\widetilde{K}(\tau) + \|\omega_i^2\|^T\int_0^\tau \widetilde{K}(\alpha)d\alpha\right] e^{T\|k_1\|_T(t-\tau)}d\tau \le$$

$$\le \left(\|\varphi_{y_i}\| + T\|\omega_i^2\|_T\right)\int_0^t \widetilde{K}(\tau)e^{T\|k_1\|_T(t-\tau)}d\tau, \quad i=1,...,n. \tag{3.6}$$

Similarly, from (3.3) and (3.4) we have the estimate for $\widetilde{v}$:

$$|\widetilde{v}(x,y,t)| \le \left[2n\max_{1\le i\le n}\|\varphi_{x_iy_i}\| + \|\Delta_y\varphi\|\right]\int_0^t \widetilde{K}(\tau)e^{T\|k_1\|_T(t-\tau)}d\tau +$$

$$+2\int_0^t \sup_{x\in\mathbb{R}^n}\left|grad_x\widetilde{k}(x,t)\,|grad_y\varphi(x,y)|\right| e^{T\|k_1\|_T(t-\tau)}d\tau +$$

$$+\int_0^t\int_0^\tau \sup_{(x,y)\in\mathbb{R}^{2n}}\Big|\widetilde{k}(x,\alpha)v_2(x,y,\tau-\alpha)d\alpha + 2grad_x\widetilde{k}(x,\alpha)\cdot\omega^{(2)}(x,y,\tau-\alpha)d\alpha +$$

$$+2grad_x k(x,\alpha)\cdot\widetilde{\omega}(x,y,\tau-\alpha)d\alpha\Big| e^{T\|k_1\|_T(t-\tau)}d\tau, \tag{3.7}$$

In accordance with the assumption of theorem, since functions $k_1,\ k_2$, are representable in the form (1.4), then $\widetilde{k}$ is also representable in this form. In view of lemma 2.2, for function $k(x,t)$ the inequality (2.12) holds. Therefore for the function $\widetilde{k}$ is valid the inequality with constant K_{00} :

$$\widetilde{k}_{x_i} \le K_{00}\widetilde{K}(t).$$

Taking into account this inequality for $\widetilde{k}$ and (3.6) for function $\widetilde{\omega}$ we rewrite the estimate (3.7) as follows

$$|\widetilde{v}(x,y,t)| \leq N\left(T,n,K_0,K_{00}\right)\int_0^t \widetilde{k}(\tau)d\tau, \tag{3.8}$$

where

$$N(T,n,K_0,K_{00}) := \Big[\|\Delta\varphi\| + 2n\max_{1\leq i\leq n}\|\varphi_{x_iy_i}\| + T\|v_2\|^T + 2T\|\omega^2\|^T +$$

$$+2nK_0\max_{1\leq i\leq n}\|\varphi_{y_i}\| + 2nK_0\left(\max_{1\leq i\leq n}\|\varphi_{y_i}\| + T\|\omega^2\|^T)T^2\right)e^{\|k_1\|_T T^2}\Big]e^{\|k_1\|_T T^2}.$$

From the equality (3,5), in view of (2.1) and (3.8), we have

$$\widetilde{K}(t) \leq \frac{1}{\mu_0}\left|\widetilde{v}|_{x=y} - \int_0^t \widetilde{k}(y,\tau)\psi_t(y,t-\tau)d\tau\right| \leq$$

$$\leq \frac{1}{\mu_0}\left[N(T,n,K_0,K_{00}) + \psi_0\right]\int_0^t \widetilde{K}(\tau)d\tau,\ \ \psi_0 = \max_{t\in[0,T]}\sup_{y\in\mathbb{R}^n}|\psi_t(y,t)|.$$

It follows from this inequality that $\widetilde{k} \equiv 0$, i.e. $k_1(x,t) = k_2(x,t)$ for $(x,t) \in D_T$, and the theorem is proven.

3. Inverse problem 2

The main result of this section is the following theorem of uniqueness for inverse problem 2.

Theorem 3.1. *Assume that* $c(t) \in C\,[0,T]$ $0 < c_0 \leq c(t) \leq c_1 \leq 1$ *and* $\varphi(x,y) \in B^4(\mathbb{R}^2)$. *Moreover, let the function* $\psi(y,t)$, *together with the derivatives* ψ_t, ψ_{tt}, ψ_{tyy} *belongs to the class* $B(D_T)$ *for any finite* $T > 0$, $c(0)(\varphi_{xxyy}(y,y) + 2\varphi_{xxxy}(y,y)) - \frac{1}{c(0)}k(y,0)\varphi(y,y) = \psi_{tyy}(y,0) - \frac{1}{c(0)}\psi_{tt}(y,0) + \frac{c'(0)}{c^2(0)}\psi_t(y,0)$ *and*

$$\inf_{(x,y)\in\mathbb{R}^2}|\varphi(x,y)| \geq \beta_0 > 0,$$

where c_i, $i = 1,2$, β_0 *are known number. Then, the function* $k(x,t)$ *representable in the form (1.4) is uniquely determined in the domain* $D(T)$.

Proof. In order to prove the theorem by differentiating the original equation (1.1) and condition (1.2), we obtain additional relations for auxiliary functions $u_y := \vartheta$:

$$\vartheta_t - c(t)\vartheta_{xx} = \int\limits_0^t k(x,t-\tau)\vartheta(x,y,\tau)d\tau, \tag{4.1}$$

$$\vartheta|_{t=0} = \varphi_y(x,y). \tag{4.2}$$

From (4.1)-(4.2), we get new problem by entering $\vartheta_t := \vartheta^{(1)}$:

$$\vartheta_t^{(1)} - c(t)\vartheta_{xx}^{(1)} = c'(t)\vartheta_{xx} + \int\limits_0^t k(x,\tau)\vartheta^{(1)}(x,y,t-\tau)d\tau + k(x,t)\varphi_y(x,y) \tag{4.3}$$

$$\vartheta^{(1)}\Big|_{t=0} = c(0)\varphi_{xxy}(x,y), \tag{4.4}$$

where

$$\vartheta_{xx} = \frac{1}{c(t)}\vartheta^{(1)} - \frac{1}{c(t)}\int_0^t k(x,t-\tau)\vartheta(x,y,\tau)d\tau.$$

Besides, entering with (4.3), (4.4) the new function $\omega := 2\vartheta_x^{(1)} + \vartheta_y^{(1)}$ we obtain the following problem:

$$\omega_t - c(t)\omega_{xx} = (\ln c(t))'\omega + k(x,t)(2\varphi_{xy}(x,y) + \varphi_{yy}(x,y)) + 2k_x(x,t)\varphi_y(x,y) +$$

$$+\int_0^t k(x,\tau)\omega(x,y,t-\tau)d\tau - (\ln c(t))'\int_0^t k(x,t-\tau)\omega(x,y,\tau)d\tau -$$

$$-(\ln c(t))'\int_0^t k_x(x,t-\tau)\vartheta(x,y,\tau)d\tau + \int_0^t k_x(x,t-\tau)\vartheta^{(1)}(x,y,\tau)d\tau$$

$$\omega\Big|_{t=0} = c(0)\left[\varphi_{xxyy}(x,y) + 2\varphi_{xxxy}(x,y)\right].$$

For the function ω, differentiating (1.3) first with respect to t and then twice with respect to y, we obtain the following conditions

$$(u_{xxt} + 2u_{xyt} + u_{yyt})\Big|_{x=y} = \psi_{tyy}(y,t),$$

$$\omega\Big|_{x=y} = \psi_{tyy}(y,t) - \frac{1}{c(t)}\psi_{tt}(y,t) + \frac{c'(t)}{c^2(t)}\psi_t(y,t) + \frac{c'(t)}{c^2(t)}\int_0^t k(y,t-\tau)\psi(y,\tau)d\tau -$$

$$+\frac{1}{c(t)}k(y,t)\varphi(y,y) + \frac{1}{c(t)}\int_0^t k(y,t-\tau)\psi_t(y,\tau)d\tau.$$

The further proof of Theorem 3.1 is completely analogous to the proof of Theorem 2.1. In this case, it is necessary to use the formula

$$p(x,t) = \int_{\mathbb{R}} \varphi(\xi)G(x-\xi;\theta(t))d\xi + \int_0^{\theta(t)} \frac{d\tau}{c(\theta^{-1}(\tau))}\times$$

$$\times\int_{\mathbb{R}} F(\xi,\theta^{-1}(\tau))G(x-\xi;\theta(t)-\tau)d\xi,$$

which provides the solution of the following Cauchy problem for the heat equation with time-variable coefficient of thermal conductivity:

$$p_t - c(t)p_{xx} = F(x,t),\ \ x\in\mathbb{R}, t>0,$$

$$p(x,0)=\varphi(x),\ x\in\mathbb{R}.$$

In (4.8) $\theta(t)=\int\limits_0^t c(\tau)d\tau$ and $\theta^{-1}(t)$ is the inverse function to $\theta(t)$; $G(x-\xi;\theta(t)-\tau)=\frac{1}{(2\sqrt{\pi(\theta(t)-\tau)})^n}e^{\frac{-|x-\xi|^2}{4(\theta(t)-\tau)}}$, $\xi=(\xi_1,...,\xi_n)$, $\xi'=(\xi_1,...,\xi_{n-1})$, $d\xi=d\xi_1...d\xi_n$, $|x|^2=x_1^2+...+x_n^2$.

For unknown functions ϑ and ϑ' in (4.5) the integral equations are derived from Cauchy problems (4.1), (4.2) and (4.3), (4.4), respectively. Carrying out similar estimates as in Section 2.2 completes the proof of Theorem 3.1.

1 Conclusion

In this article, we proved the uniqueness theorems for the definition of the convolution kernel in a parabolic integro-differential equation describing thermal processes with memory. In contrast to the results obtained in [30], [32], [33], here the kernel depends on all variables x and t. The study of the existence of solutions to inverse problems 1 and 2 is difficult and remains an open question.

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