The High Order Interaction Solutions Comprising Lump Solitons for the (2+1)-dimensional Caudrey-Dodd-Gibbon-Kotera-Sawada Equation

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Abstract. This paper deals with localized waves in the (2+1)-dimensional Caudrey-Dodd-Gibbon-Kotera-Sawada (CDGKS) equation in the incompressible fluid. Based on Hirota's bilinear method, N-soliton solutions related to CDGKS equation are constructed. For the case N = 5 and N = 6, the exact expression of multiple localized wave solutions comprising lump solitons are obtained by using the long wave limit method. A variety of interactions are illustrated analytically and graphically. The influence of parameters on propagation is analyzed and summarized. The results and phenomena obtained in this paper enrich the dynamic behavior of the evolution of nonlinear localized waves.

AMS subject classifications: 39A11, 39A30, 39A10

Key words: Bilinear operator; Long wave limit; Lump soliton; Periodic soliton; Interaction.

1. Introduction

Generally, it has always been a vital task to solve soliton equation based on the soliton theory. Except for numerical calculation and computer simulation, the mainstream research has been focused on finding the exact solution of the soliton equation. Seeking the exact solution of soliton equation possesses significant value from both theoretical and practical perspectives, which not only helps to further understand the essential properties and algebraic structure of the soliton equation, but also can explain related natural phenomenon reasonably. With the rapid development of soliton theory, many systematic methods have been proved effective, such as the inverse scattering method [1,2], Riemann-Hilbert problem [3], Darboux transformation [4], Bäcklund transformation [5], Hirota bilinear method [6,7], Wronskian technique [8], KP reductions [9], Painlevé analysis [10,11] and algebra-geometric method [12] etc. Among these methods, Hirota bilinear method uses the bilinear derivative as a tool and it is only related to the equation to be solved and independent on the spectral problem of the equation or the Lax pair. As a result, Hirota bilinear method is featured as intuitive and straightforward, which has become a common method to solve several multiple solition solutions of nonlinear evolution

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equations [13–19]. Many researchers have been working on various extensions and applications of Hirota bilinear methods, which further develops and broadens this methods. For instance, by using Hirota bilinear method, Ma et al. [20–22] studied lump solutions and interaction solutions to integrable equations.

In this paper, we will focus on the following (2+1)-dimensional Caudrey-Dodd-Gibbon-Kotera-Sawada (CDGKS) equation [23]:

$$36u_t + u_{5x} + 15(uu_{xx})_x + 45u^2u_x - 5u_{xxy} - 15uu_y - 15u_x\partial_x^{-1}u_y - 5\partial_x^{-1}u_{yy} = 0, \tag{1.1}$$

where $u = u(x, y, t)$ is a differentiable function with the scaled space variables x, y and time variable t, and the operator ∂_x^{-1} is the inverse operator of ∂_x. When $u_y = 0$, Eq. (1.1) reduces to the (1+1)-dimensional CDGKS equation. CDGKS equation was proposed by Konopelchenko and Dubrovsky [23], which is a higher-order generalization of the celebrated Korteweg-de Vries equation. It is widely applied in nonlinear sciences such as the conservative flow of Liouville equation, 2-dimentional gauge field theory of quantum gravity and theory of conformal field etc. [24, 25]. And it is one of the most important integrable equations in soliton theory for describing a large range of nonlinear dispersive physical phenomena.

Recently, CDGKS equation equation has attracted the attentions of many researchers, and delicate works have been conducted to solve the equation. By applying Painlevé expansion method and extended homoclinic test approach, Wang and Xian [26] obtained the homoclinic breather-wave solutions, periodic wave solutions and kink solitary wave solutions for Eq. (1.1). In [27], new non-traveling wave solutions of (2+1)-dimensional CDGKS equation were derived by combining the Lie point group method to proper nonlinear traveling wave method, and moreover, the localized structures were discussed. The other solutions to Eq. (1.1) including rational solutions and triangular periodic solutions, quasi-periodic solutions and novel periodic solitary wave have been derived by tanh method, Darboux transformation and Hirota bilinear method, respectively [28–33]. Results about (1+1)-dimensional CDGKS equation can be found in Refs. [34, 35].

Up to now, there are few results about different soliton interaction solutions of the (2+1)-dimensional CDGKS equation, such as the interaction between line soliton and periodic soliton, the interaction between line soliton and lump soliton and the interaction between periodic soliton and lump soliton. By using Hirota bilinear method, [36] investigated the interactions among different kinds of single solitary wave, such as line-line, line-lump, lump-lump, etc. Due to the lump soliton is the periodically infinite increment of periodic soliton, in other words, it is derived by taking the limit of periodic soliton. The interactions among soliton solutions became more complicated with high order of the solution, which will be discussed in detail in this paper.

2. N-soliton solution of (2+1)-dimensional CDGKS equation

Bilinear form of Eq. (1.1) has been obtained via the dependent variable transformation

$$u = 2(\ln g)_{xx}, \tag{2.1}$$

which could be written as

$$(5D_y(D_x^3 + D_y) - D_x(D_x^5 + 36D_t))(g \cdot g) = 0. \tag{2.2}$$

Based on the Hirota's bilinear theory, Eq. (2.2) has standard N-soliton solution in the form of [7,37]

$$g_N = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N} \mu_i \eta_i + \sum_{1\le i<j} \mu_i \mu_j \ln(A_{ij})\right) \tag{2.3}$$

where

$$\eta_i = a_i x + b_i y + c_i t + \eta_{0i}, \;\; c_i = -\frac{5a_i^3 b_i + 5b_i^2 - a_i^6}{36a_i},$$

$$A_{ij} = -\frac{(a_i-a_j)^6 - 5(a_i-a_j)^3(b_i-b_j) + 36(a_i-a_j)(c_i-c_j) - 5(b_i-b_j)^2}{(a_i+a_j)^6 - 5(a_i+a_j)^3(b_i+b_j) + 36(a_i+a_j)(c_i+c_j) - 5(b_i+b_j)^2},$$

with a_i, b_i, c_i and η_{0i} $(i = 1, 2, \ldots, N)$ any arbitrary constants, and $\sum_{\mu=0,1}$ summation total of taking over all possible combinations of $\eta_i, \eta_j = 0, 1 (i, j = 1, 2, 3, \ldots, N)$. Based on the work of [28, 38], the following theorem is proposed.

Theorem 1. Let $b_k = q_k a_k (k = 1, \cdots N), a_j = l_j \epsilon,\; \exp(\eta_j^0) = -1\ (j = 1, \cdots 2M),\ q_n = q^*_{n+M}\ (n = 1, \cdots M)('*'$ is conjugate), $a_{2M+l} = a^*_{2M+P+l},\ (l = 1, \cdots P)$ and $a_{2M+2P+h}\ \ (h = 1, \cdots Q)$ are real constants, when $\epsilon \to 0$, the N-soliton solution u of Eq. (2.1) with (2.3) can reduce to the interaction solutions of M-lump, P-breather and Q-line soliton, where $N = 2M + 2P + Q$, in which M, P, Q are nonnegative integers and express the numbers of lump, breather and line soliton, respectively.

3. The solutions comprising one lump soliton

3.1. The case of Theorem 1 with $M = 1, 2P + Q = 3$

To construct interaction solutions comprising one lump soliton satisfying the condition, the parameters in Eq.(2.3) need to satisfy the following conditions

$$b_i = a_i q_i\ (i = 1, 2, \cdots, 5), a_1 = l_1 \varepsilon, a_2 = l_2 \varepsilon, \eta_{01} = \eta^*_{02} = i\pi, \eta_{03} = \eta_{04} = \eta_{05} = 0,$$

and take the long wave limit as $\varepsilon \to 0$ in five-soliton solution, then we have

$$\begin{aligned} g =& (\varrho_1\varrho_2 + d_{12}) l_1 l_2 \varepsilon^2 + \sum_{j=3}^{(5)} (\varrho_1\varrho_2 + d_{2j}\varrho_1 + d_{1j}\varrho_2 + d_{12} + d_{1j}d_{2j}) \exp(\eta_j) l_1 l_2 \varepsilon^2 \\ &+ \sum_{3\le j<k}^{(5)} d_{jk}[\varrho_1\varrho_2 + (d_{2j} + d_{2k})\varrho_1 + (d_{1j} + d_{1k})\varrho_2 + d_{12} + (d_{1j} + d_{1k})(d_{2j} + d_{2k})] \\ &\cdot \exp(\eta_j + \eta_k) l_1 l_2 \varepsilon^2 + \prod_{3\le j<k}^{5} d_{jk}[\varrho_1\varrho_2 + \sum_{s=3}^{(5)} (d_{2s}\varrho_1 + d_{1s}\varrho_2) + d_{12} + \sum_{s=3}^{(5)} d_{1s} \sum_{s=3}^{(5)} d_{2s}] \\ &\cdot \exp(\sum_{s=3}^{(5)} \eta_s) l_1 l_2 \varepsilon^2 + O(\varepsilon^3), \end{aligned} \tag{3.1}$$

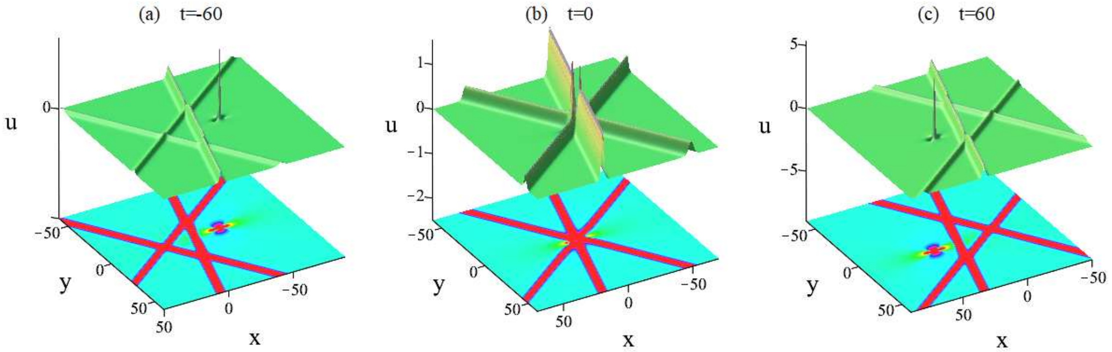

Figure 1: Three-dimensional plots and density plots of the interaction solution for Eq. (3.1) at different time with parameters: $q_1 = q_2^* = -\frac{1}{3} - 2i, q_3 = -\frac{3}{2}, q_4 = \frac{3}{4}, q_5 = -\frac{1}{3}, a_3 = -\frac{4}{5}, a_4 = \frac{4}{5}, a_5 = -\frac{3}{2}$.

with

$$\varrho_i = x + q_i y + \frac{5}{36} q_i^2 t \ \ (i = 1, 2), d_{12} = \frac{6(q_1 + q_2)}{(q_1 - q_2)^2}, \tag{3.2}$$

$$d_{sj} = -\frac{6a_j(a_j^2 - q_s - q_j)}{a_j^4 - (q_s + 2q_j)a_j^2 + (q_s - q_j)^2} \ \ (s = 1, 2, \ j = 3, 4, 5), \tag{3.3}$$

$$d_{sj} = \frac{M}{N}, \ \ (3 \le s < j \le 5), \tag{3.4}$$

where

$$M = a_s^4 - 3a_s^3 a_j + (4a_j^2 - 2q_s - q_j)a_s^2 - 3a_j(a_j^2 - q_s - q_j)a_s + a_j^4 - (q_s + 2q_j)a_j^2 + (q_s - q_j)^2,$$
$$N = a_s^4 + 3a_s^3 a_j + (4a_j^2 - 2q_s - q_j)a_s^2 + 3a_j(a_j^2 - q_s - q_j)a_s + a_j^4 - (q_s + 2q_j)a_j^2 + (q_s - q_j)^2.$$

Inserting Eqs. (3.1)-(3.4) into Eq. (2.1), the solution of Eq. (1.1) can be obtained.

(i) In the special case of $P = 0, Q = 3$.

If taking

$$q_1 = q_2^* = -\frac{1}{3} - 2i, q_3 = -\frac{3}{2}, q_4 = \frac{3}{4}, q_5 = -\frac{1}{3}, a_3 = -\frac{4}{5}, a_4 = \frac{4}{5}, a_5 = -\frac{3}{2},$$

the solution u given by Eq. (3.1) expresses the interaction among a lump soliton and three bell-shaped line solitons. Fig. 1 presents the interaction behavior between three bell-shaped line solitons and a lump soliton in Eq. (3.1) at different time. It can be observed that the lump spreads together with the three bell-shaped line solitons. During the interaction process, we can find that the shape and velocity of three bell-shaped line solitons and lump remain unchanged, which exhibit the characteristic of "elastic collision".

(ii) In the special case of $P = 1, Q = 1$.

If taking

$$q_1 = q_2^* = -\frac{1}{3} - 2i, q_3 = q_4^* = -\frac{1}{4} - \frac{1}{2}i, q_5 = 1, a_3 = a_4 = -\frac{1}{5}, a_5 = \frac{3}{4},$$

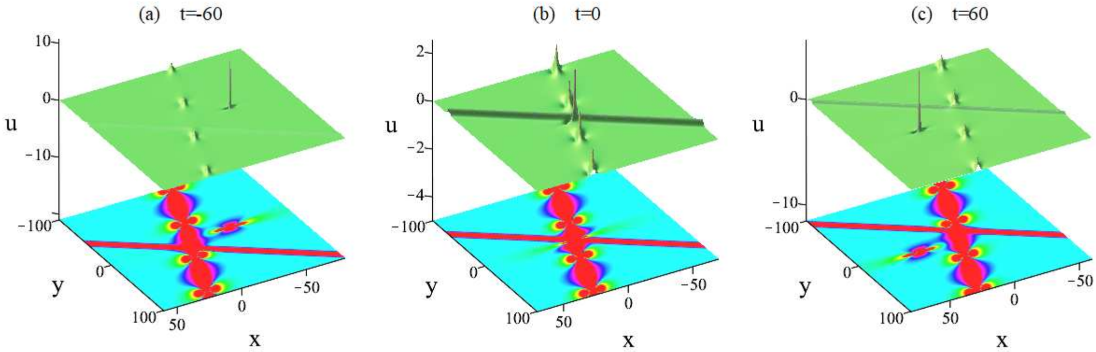

Figure 2: Three-dimensional plots and density plots of the interaction solution for Eq. (3.1) at different time with parameters: $q_1=q_2^*=-\frac{1}{3}-2i, q_3=q_4^*=-\frac{1}{4}-\frac{1}{2}i, q_5=1, a_3=a_4=-\frac{1}{5}, a_5=\frac{3}{4}$.

in Eq. (3.1), Fig. 2 presents the interaction behavior between one lump, one breather and one bell-shaped line soliton. The period of the breather is 20π along the y direction. It can be observed that the lump spreads together with the breather and line soliton. During the interaction process, the shape and velocity of the lump and line soliton remain unchanged, the period of the breather remain unchanged.

3.2. The case of Theorem 1 with $M=1, 2P+Q=4$

To construct interaction solutions comprising one lump soliton satisfying the condition, the parameters in Eq.(2.3) need to satisfy the following conditions

$$b_i=a_iq_i\ (i=1,2,\cdots,6), a_1=l_1\varepsilon, a_2=l_2\varepsilon, \eta_{01}=\eta_{02}^*=i\pi, \eta_{03}=\eta_{04}=\eta_{05}=\eta_{06}=0,$$

and take the long wave limit as $\varepsilon\to 0$ in the function g in Eq.(2.3), then

$$\begin{aligned}
g=&(\varrho_1\varrho_2+d_{12})l_1l_2\varepsilon^2+\sum_{j=3}^{(6)}(\varrho_1\varrho_2+d_{2j}\varrho_1+d_{1j}\varrho_2+d_{12}+d_{1j}d_{2j})\exp(\eta_j)l_1l_2\varepsilon^2\\
&+\sum_{3\le j<k}^{(6)}d_{jk}[\varrho_1\varrho_2+(d_{2j}+d_{2k})\varrho_1+(d_{1j}+d_{1k})\varrho_2+d_{12}+(d_{1j}+d_{1k})(d_{2j}+d_{2k})]\\
&\cdot\exp(\eta_j+\eta_k)l_1l_2\varepsilon^2+\sum_{3\le j<k<s}^{(6)}d_{jk}d_{js}d_{ks}[\varrho_1\varrho_2+(d_{2j}+d_{2k}+d_{2s})\varrho_1+(d_{1j}+d_{1k}\\
&+d_{1s})\varrho_2+d_{12}+(d_{1j}+d_{1k}+d_{1s})(d_{2j}+d_{2k}+d_{2s})]\exp(\eta_j+\eta_k+\eta_s)l_1l_2\varepsilon^2\\
&+\prod_{3\le j<k}^{6}d_{jk}[\varrho_1\varrho_2+\sum_{s=3}^{(6)}(d_{2s}\varrho_1+d_{1s}\varrho_2)+d_{12}+\sum_{s=3}^{(6)}d_{1s}\sum_{s=3}^{(6)}a_{2s}]\exp(\sum_{s=3}^{(6)}\eta_s)l_1l_2\varepsilon^2+O(\varepsilon^3),
\end{aligned}\tag{3.5}$$

where

$$\varrho_i=x+q_iy+\frac{5}{36}q_i^2t,\ \ (i=1,2),\tag{3.6}$$

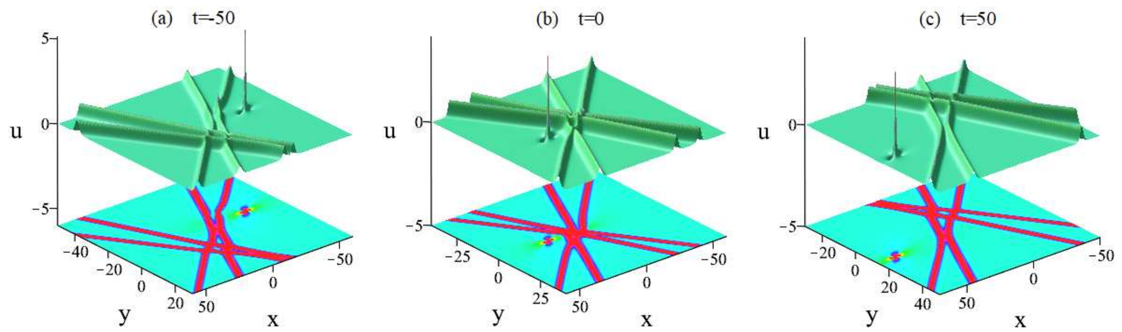

Figure 3: The interaction among one lump soliton and four line solitons with parameters: $q_1 = q_2^* = -1 - 2i, q_3 = -\frac{3}{2}, q_4 = -\frac{3}{4}, q_5 = 1, q_6 = \frac{2}{3}, a_3 = a_4 = 1, a_5 = a_6 = \frac{5}{4}$.

$$d_{12} = \frac{6(q_1 + q_2)}{(q_1 - q_1)^2}, \tag{3.7}$$

$$d_{sj} = -\frac{6a_j(a_j^2 - q_s - q_j)}{a_j^4 - (q_s + 2q_j)a_j^2 + (q_s - q_j)^2} \ \ (s = 1, 2, \ j = 3, 4, 5, 6), \tag{3.8}$$

$$d_{sj} = \frac{M}{N}, \ \ (3 \le s < j \le 6), \tag{3.9}$$

where

$$M = a_s^4 - 3a_s^3 a_j + (4a_j^2 - 2q_s - q_j)a_s^2 - 3a_j(a_j^2 - q_s - q_j)a_s + a_j^4 - (q_s + 2q_j)a_j^2 + (q_s - q_j)^2,$$
$$N = a_s^4 + 3a_s^3 a_j + (4a_j^2 - 2q_s - q_j)a_s^2 + 3a_j(a_j^2 - q_s - q_j)a_s + a_j^4 - (q_s + 2q_j)a_j^2 + (q_s - q_j)^2.$$

Inserting Eqs. (3.5)-(3.9) into Eq. (2.1), the solution of Eq. (1.1) can be obtained.

(i) In the special case of $P = 0, Q = 4$.

If taking

$$q_1 = q_2^* = -1 - 2i, q_3 = -\frac{3}{2}, q_4 = -\frac{3}{4}, q_5 = 1, q_6 = \frac{2}{3}, a_3 = a_4 = 1, a_5 = a_6 = \frac{5}{4},$$

the solutions u given by Eq. (3.5) express the elastic interaction between one lump and four bell-shaped solitons at different time as shown in Fig. 3.

(ii) In the special case of $P = 1, Q = 2$.

In Eq. (3.5), if taking

$$q_1 = q_2^* = -1 + 2i, q_3 = q_4^* = -\frac{4}{3}i, q_5 = -\frac{3}{4}, q_6 = \frac{3}{4}, a_3 = a_4 = \frac{1}{3}, a_5 = a_6 = 1,$$

the solution of Eq. (1.1) corresponds to the interaction behavior among one lump, one breather and two bell-shaped line solitons, as shown in Fig. 4.

(iii) In the special case of $P = 2, Q = 0$.

If taking

$$q_1 = q_2^* = -1 - 3i, q_3 = q_4^* = -1 - \frac{4}{3}i, q_5 = q_6^* = -\frac{1}{7} - \frac{1}{2}i, a_3 = a_4 = \frac{1}{8}, a_5 = a_6 = \frac{1}{5},$$

the solutions u given by Eq. (3.5) express the elastic interaction between one lump and two breather solitons at different time as shown in Fig. 5.

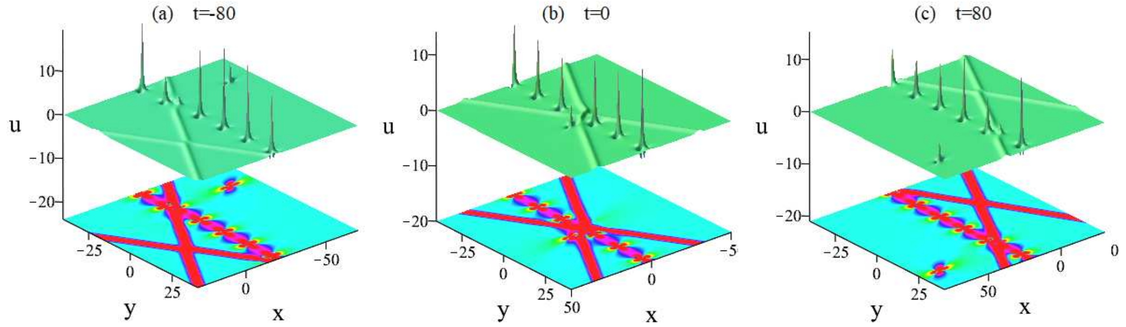

Figure 4: Three-dimensional plots and density plots of the interaction solution for Eq. (3.5) at different time with parameters: $q_1 = q_2^* = -1+2i, q_3 = q_4^* = -\frac{4}{3}i, q_5 = -\frac{3}{4}, q_6 = \frac{3}{4}, a_3 = a_4 = \frac{1}{3}, a_5 = a_6 = 1$.

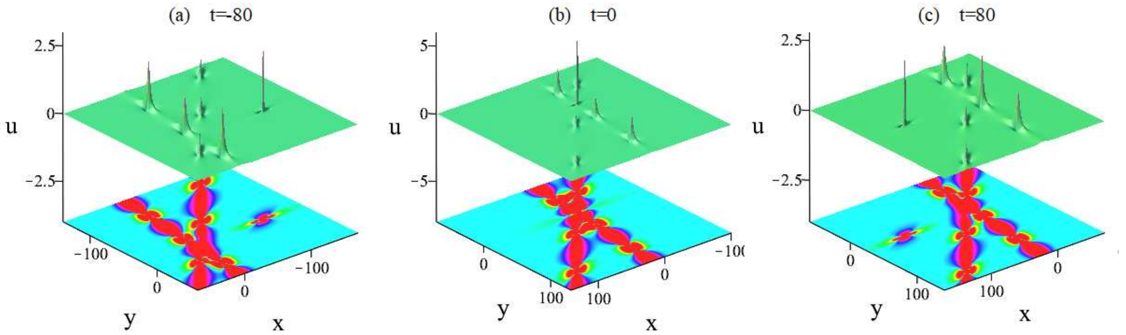

Figure 5: The interaction among one lump soliton and two breather solitons with parameters: $q_1 = q_2^* = -1-3i, q_3 = q_4^* = -1-\frac{4}{3}i, q_5 = q_6^* = -\frac{1}{7}-\frac{1}{2}i, a_3 = a_4 = \frac{1}{8}, a_5 = a_6 = \frac{1}{5}$.

4. The solutions comprising two lump solitons

4.1. The case of Theorem 1 with $M=2, P=0, Q=1$

To construct interaction solutions comprising two lump solitons satisfying the condition, the parameters in Eq.(2.3) need to satisfy the following conditions

$$b_i = a_i q_i, a_i = l_i \varepsilon \ (i=1,2,3,4), b_5 = a_5 q_5, \eta_{01} = \eta_{02}^* = \eta_{03} = \eta_{04}^* = i\pi, \eta_{05} = 0,$$

and take the long wave limit as $\varepsilon \to 0$ in five-soliton solution, we can obtain

$$\begin{aligned}
g =& (\prod_{j=1}^{4} \varrho_j + \sum_{1\le s<j}^{(4)} d_{sj} \prod_{k\neq s,j}^{4} \varrho_k + \sum_{\substack{1<j\neq k,\\ 1<s<k}}^{(4)} d_{1j} d_{sk}) l_1 l_2 l_3 l_4 \varepsilon^4 + \{\prod_{j=1}^{4} \varrho_j \\
&+ \sum_{j=1}^{(4)} d_{j5} \prod_{k\neq j}^{4} \varrho_k + \sum_{\substack{j<k,\\ s<m\neq j,k}}^{(4)} \varrho_j \varrho_k (d_{s5} d_{m5} + a_{sm}) + \sum_{\substack{j\neq k<s\le 5,\\ m\neq j\neq k\neq s,\\ m<w\le 5}}^{(4)} \varrho_j [(d_{ks} d_{mw} \\
&+ \prod_{n\neq j}^{4} d_{n5}] + \sum_{\substack{s<k,\\ 1<j\neq s}}^{(4)} d_{1j} a_{ks} + \sum_{\substack{s<m,\\ k<n\neq s,m}}^{(4)} d_{s5} d_{m5} d_{kn} + \prod_{i=1}^{4} d_{i5}\} \exp(\eta_5) \\
&\times l_1 l_2 l_3 l_4 \varepsilon^4 + O(\varepsilon^5), \qquad (4.1)
\end{aligned}$$

where

$$\varrho_i = x + q_i y + \frac{5}{36} q_i^2 t \ \ (i=1,2,3,4), \qquad (4.2)$$

$$d_{sj} = \frac{6(q_s+q_j)}{(q_s-q_j)^2} \ \ (1 \le s < j \le 4), \qquad (4.3)$$

and

$$d_{s5} = -\frac{6a_5(a_5^2 - q_s - q_5)}{a_5^4 - (q_s + 2q_5)a_5^2 + (q_s - q_5)^2} \ \ (s=1,2,3,4). \qquad (4.4)$$

Inserting Eqs. (4.1)-(4.4) into Eq. (2.1), the solution of Eq. (1.1) can be obtained.

If taking

$$q_1 = q_2^* = -\frac{1}{3} - 2i, q_3 = q_4^* = -\frac{1}{2} - i, q_5 = \frac{2}{5}, a_5 = \frac{3}{4},$$

the solutions u given by Eq. (4.1) express the elastic interaction between two lump solitons and one bell-shaped line soliton at different time as shown in Fig. 6. With the evolution of time, the two lump solitons move along the positive x-axis, and the line soliton moves along the negative x-axis. After elastic collision, the two lump solitons pass through the line soltion, and switch their positions.

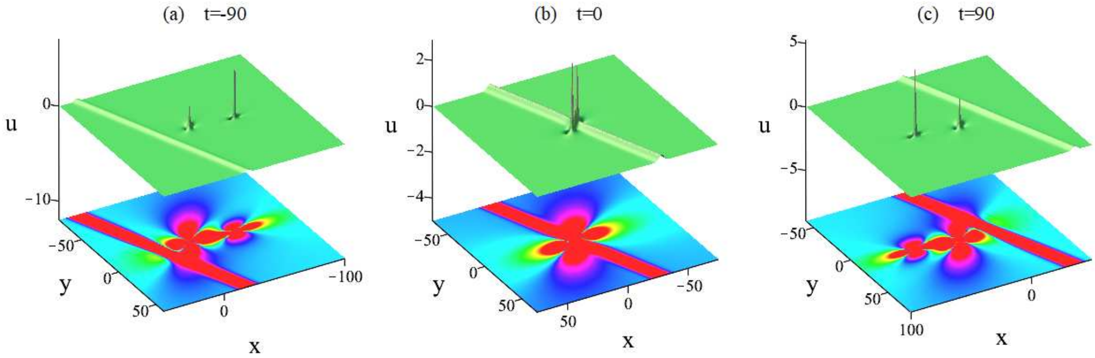

Figure 6: The interactions among two lump solitons and a line soliton at different time with parameters: $q_1 = q_2^* = -\frac{1}{3} - 2i, q_3 = q_4^* = -\frac{1}{2} - i, q_5 = \frac{2}{5}, a_5 = \frac{3}{4}$.

4.2. The case of Theorem 1 with $M = 2, 2P + Q = 2$

To construct interaction solutions comprising two lump solitons satisfying the condition, the parameters in Eq.(2.3) need to satisfy the following conditions

$$b_i = a_i q_i (i = 1, 2, \cdots, 6), a_k = l_k \varepsilon (k = 1, 2, 3, 4), \eta_{01} = \eta_{02}^* = \eta_{03} = \eta_{04}^* = i\pi, \eta_{05} = \eta_{06} = 0,$$

and take the long wave limit as $\varepsilon \to 0$ in six-soliton solution, we can obtain

$$\begin{aligned} g = &(\prod_{j=1}^{4} \varrho_j + \sum_{1\le s<j}^{(4)} d_{sj} \prod_{k\neq s,j}^{4} \varrho_k + \sum_{\substack{s<j\neq m,\\ s<k<m}}^{(4)} d_{sj} d_{km}) l_1 l_2 l_3 l_4 \varepsilon^4 + \sum_{w=5}^{(6)} \{ \prod_{j=1}^{4} \varrho_j \\ &+ \sum_{j=1}^{(4)} d_{jw} \prod_{k\neq j}^{4} \varrho_k + \sum_{\substack{j<k,\\ s<m\neq j,k}}^{(4)} \varrho_j \varrho_k (d_{sw} d_{mw} + d_{sm}) + \sum_{\substack{j\neq k<s,\\ m\neq j\neq k\neq s}}^{(4)} \varrho_j [(d_{ks} d_{mw} \\ &+ \prod_{n\neq j}^{4} d_{nw}] + \sum_{\substack{s<j\neq m,\\ s<k<m}}^{(4)} d_{sj} d_{km} + \sum_{\substack{s<m,\\ k<n\neq s,m}}^{(4)} d_{sw} d_{mw} d_{kn} + \prod_{n=1}^{4} d_{nw} \} \exp(\eta_w) \\ &\times l_1 l_2 l_3 l_4 \varepsilon^4 + d_{56} \{ \prod_{j=1}^{4} \varrho_j + \sum_{j=1}^{(4)} (d_{j5} + d_{j6}) \prod_{k\neq j}^{4} \varrho_k + \sum_{\substack{j<k,\\ s<m\neq j,k}}^{(4)} \varrho_j \varrho_k [(d_{s5} \\ &+ d_{s6})(d_{m5} + d_{m6}) + d_{sm}] + \sum_{\substack{j\neq k<s,\\ m\neq j\neq k\neq s}}^{(4)} \varrho_j [d_{ks}(d_{m5} + d_{m6}) + \prod_{n\neq j}^{4} (d_{n5} + d_{n6})] \\ &+ \sum_{\substack{s<j\neq m,\\ s<k<m}}^{(4)} d_{sj} d_{km} + \sum_{\substack{s<m,\\ k<n\neq s,m}}^{(4)} (d_{s5} + d_{s6})(d_{m5} + d_{m6}) d_{kn} + \prod_{n=1}^{4} (d_{n5} + d_{n6}) \} \\ &\times \exp(\eta_5 + \eta_6) l_1 l_2 l_3 l_4 \varepsilon^4 + O(\varepsilon^5), \end{aligned} \tag{4.5}$$

where

$$\varrho_i = x + q_i y + \frac{5}{36} q_i^2 t \quad (i = 1, 2, 3, 4), \tag{4.6}$$

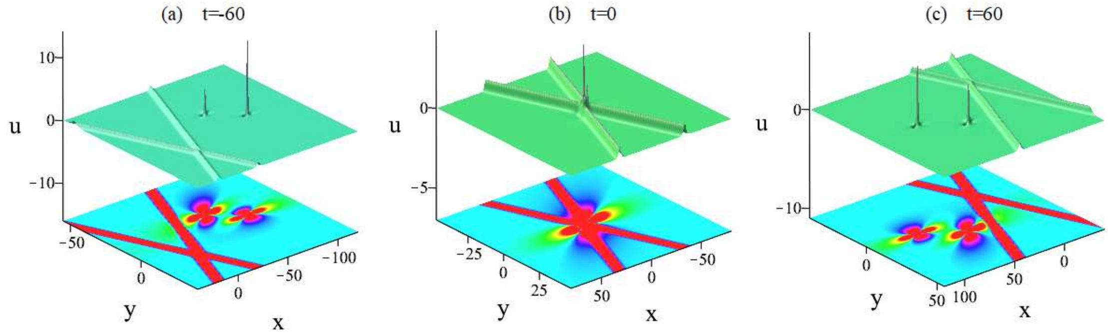

Figure 7: The interaction among two lump solitons and two line solitons with parameters: $q_1 = q_2^* = -1 - 2i, q_3 = q_4^* = -\frac{1}{4} - 3i, q_5 = -\frac{2}{3}, q_6 = \frac{2}{3}, a_5 = a_6 = -\frac{6}{5}$.

$$d_{sj} = \frac{6(q_s + q_j)}{(q_s - q_j)^2} \quad (1 \leq s < j \leq 4), \tag{4.7}$$

and

$$d_{sj} = -\frac{6a_j(a_j^2 - q_s - q_j)}{a_j^4 - (q_s + 2q_j)a_j^2 + (q_s - q_j)^2} \quad (s = 1, 2, 3, 4, \quad j = 5, 6), \tag{4.8}$$

$$d_{56} = \frac{M}{N}, \tag{4.9}$$

where

$$M = a_5^4 - 3a_5^3 a_6 + (4a_6^2 - 2q_5 - q_6)a_5^2 - 3a_6(a_6^2 - q_5 - q_6)a_5 + a_6^4 - (q_5 + 2q_6)a_6^2 + (q_5 - q_6)^2,$$
$$N = a_5^4 + 3a_5^3 a_6 + (4a_6^2 - 2q_5 - q_6)a_5^2 + 3a_6(a_6^2 - q_5 - q_6)a_5 + a_6^4 - (q_5 + 2q_6)a_6^2 + (q_5 - q_6)^2.$$

Inserting Eqs. (4.5)-(4.9) into Eq. (2.1), the solution of Eq. (1.1) can be obtained.

(i) In the special case of $P = 0, Q = 2$.

If taking

$$q_1 = q_2^* = -1 - 2i, q_3 = q_4^* = -\frac{1}{4} - 3i, q_5 = -\frac{2}{3}, q_6 = \frac{2}{3}, a_5 = a_6 = -\frac{6}{5},$$

the solutions u given by Eq. (4.5) express the elastic interaction between two lump and two bell-shaped line solitons at different time as shown in Fig. 7.

(ii) In the special case of $P = 1, Q = 0$.

If taking

$$q_1 = q_2^* = -3 - 3i, q_3 = q_4^* = -1 - 2i, q_5 = q_6^* = -1 - i, a_5 = a_6 = -\frac{1}{4},$$

the solutions u given by Eq. (4.5) express the elastic interaction between two lump solitons and one breather soliton, the period of the breather is 8π along the y direction, as shown in Fig. 8.

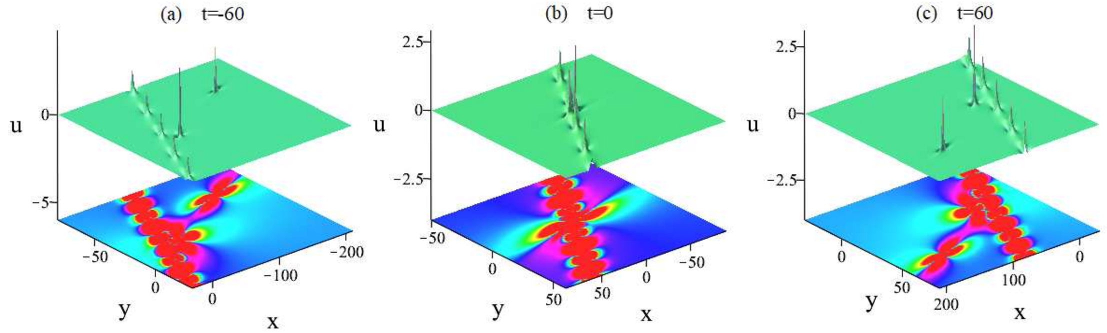

Figure 8: The elastic interaction between two lump solitons and one periodic soliton at different time by choosing parameters as: $q_1 = q_2^* = -3-3i, q_3 = q_4^* = -1-2i, q_5 = q_6^* = -1-i, a_5 = a_6 = -\frac{1}{4}$.

5. The interactions solutions among three lumps

In the special case of **Theorem 1** with $M = 3, P+Q = 0$, we obtain the interaction solution among three lumps. About the pure lumps solution, there have the following result [6]:

Corollary 1. In (2.3), setting $N = 2M$, $b_k = q_k a_k (k = 1, \cdots N), a_j = l_j \epsilon$, $\exp(\eta_j^0) = -1$ $(j = 1, \cdots 2M)$, $q_n = q_{n+M}^*$ $(n = 1, \cdots M)$, when $\epsilon \to 0$, the N-soliton solution of Eq. (1.1) can reduce to the interaction solutions of M-lump [39, 40]. The expression can be obtained by (2.1) with

$$\begin{aligned} g_{2M} &= \prod_{j=1}^{2M} \varrho_j + \frac{1}{2} \sum_{s,j}^{(2M)} d_{sj} \prod_{l \neq s,j}^{2M} \varrho_l + \frac{1}{2!2^2} \sum_{s,j,k,m}^{(2M)} d_{sj} d_{km} \prod_{l \neq s,j,k,m} \varrho_l + \cdots \\ &+ \frac{1}{M!2^M} \sum_{s,j,k,m}^{(2M)} d_{sj} \overbrace{d_{rl} \cdots d_{wn}}^{M} \prod_{p \neq s,j,r,l,\cdots,w,n}^{2M} \varrho_p + \cdots, \end{aligned} \tag{5.1}$$

where ϱ_s and d_{sj} meet following requirements,

$$\varrho_i = x + q_i y + \frac{5}{36} q_i^2 t, \ \ (s = 1, 2, \cdots, 2M), \tag{5.2}$$

and

$$d_{sj} = \frac{6(q_s + q_j)}{(q_s - q_j)^2} \ \ (1 \le s < j \le 2M), \tag{5.3}$$

where j, s are positive integers, m is arbitrary complex constant. When $M = 3$, the solution of Eq.(1.1) corresponds to interaction among three lump solitons.

In the followings, the large time asymptotic behaviors of the three lumps solution are analyzed. Fixing the modulus of a phase function, e.g. $|\varrho_1|^2$ = constant, considering the limit of $t \to \pm\infty$, $\varrho_2, \varrho_2^*, \varrho_3, \varrho_3^* = O(t)$ and $\varrho_2 \varrho_2^* = O(t^2)$, $\varrho_3 \varrho_3^* = O(t^2)$, function g has the following asymptotic states

$$g \sim |\varrho_1|^2 |\varrho_2|^2 |\varrho_3|^2 + d_{14} |\varrho_2|^2 |\varrho_3|^2. \tag{5.4}$$

Considering the properties of bilinear transformation of CDGKS equation, the function g in Eq. (5.4) can be farther equivalent to

$$g \sim |\varrho_1|^2 + d_{14}. \tag{5.5}$$

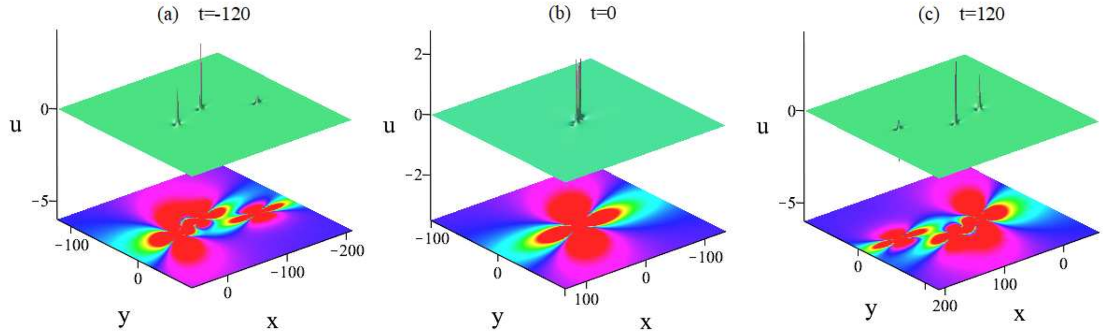

Figure 9: The elastic interaction among three lump solitons at different time by choosing parameters as: $q_1 = q_2^* = -1-2i, q_3 = q_4^* = -\frac{1}{4} - 3i, q_5 = q_6^* = -\frac{2}{3} - \frac{6}{5}i$.

If $\left|\varrho_2\right|^2$ = constant or $\left|\varrho_3\right|^2$ = constant, similar conclusion to Eq. (5.5) can be obtained. Thus, in the limit of $t \to \pm\infty$, the three lumps solution tends to become three single lump with different velocity and their phase function are $\varrho_1 = x + q_1 y + \frac{5}{36}q_1^2 t, \varrho_2 = x + q_2 y + \frac{5}{36}q_2^2 t, \varrho_3 = x + q_3 y + \frac{5}{36}q_3^2 t$, respectively. According to expression of phase function, each single lump has no phase shift, in other word, there is no phase shift for these three lumps during collision process.

In Eq. (5.1), if taking

$$q_1 = q_2^* = -1-2i, q_3 = q_4^* = -\frac{1}{4} - 3i, q_5 = q_6^* = -\frac{2}{3} - \frac{6}{5}i,$$

we can obtain the elastic interactions among three lump solitons at different time as shown in Fig. 9. It can be observed that the three lumps form a triangle structure before the collision. When $t = 0$, these three lumps merge into single one. After the collision, the three lumps separate from each other and maintain a triangle structure. The interaction among the lumps is elastic, indicating that these three lumps remain their shapes, amplitudes and velocity both before and after the interactions.

6. Conclusions

In this paper, we have studied the exact expression of multiple localized wave solutions comprising lump solitons and interaction structures from five-soliton and six-soliton solutions of the CDGKS equation via Hirota bilinear method. Some mathematical features to obtain localized waves and their interactions from the five- and six-soliton solutions were illustrated. By choosing appropriate parameters and using long wave limit method on the five-soliton and six-soliton solutions, some novel results and interaction phenomena have been found including the elastic interactions among one lump and three bell-shaped solitons (see Fig. 1), one lump and one periodic breather and one bell-shaped soliton (see Fig. 2), one lump and four bell-shaped solitons (see Fig. 3), one lump and one periodic breather and two bell-shaped solitons (see Fig. 4), one lump and two periodic breathers (see Fig. 5), two lumps and one bell-shaped soliton (see Fig. 6), two lumps and two bell-shaped solitons (see Fig. 7), two lumps and one periodic breather (see Fig. 8), and three lumps (see Figs. 9). The relevant interaction evolution processes and dynamic characteristics are presented and analyzed. The results presented in this paper might be helpful for understanding some physical phenomena of the propagation of nonlin-

Table 1: The localized wave interaction structures of N-soliton solution

M-lump	Interaction structures of localized waves	Parameters
$M=1$	$M=1,P=0,Q=3.$ one lump + three LSs	$b_i=a_iq_i\ (i=1,\cdots,5)$, $a_1=l_1\varepsilon, a_2=l_2\varepsilon, a_3=\delta_1,$ $a_4=\delta_2, a_5=\delta_3, q_1=q_2^*=\alpha_1+i\beta_1, q_3=\vartheta_1, q_4=\vartheta_2,$ $q_5=\vartheta_3, \eta_{01}=\eta_{02}^*=i\pi, \varepsilon\to 0$
	$M=1,P=1,Q=1.$ one lump + one PB + one LS	$b_i=a_iq_i\ (i=1,\cdots,5)$, $a_1=l_1\varepsilon, a_2=l_2\varepsilon,$ $a_3=a_4=\delta_4, a_5=\delta_5, q_1=q_2^*=\alpha_2+i\beta_2,$ $q_3=q_4^*=\alpha_3+i\beta_3, q_5=\vartheta_4, \eta_{01}=\eta_{02}^*=i\pi, \varepsilon\to 0$
	$M=1,P=0,Q=4.$ one lump + four LSs	$b_i=a_iq_i\ (i=1,\cdots,6)$, $a_1=l_1\varepsilon$, $a_2=l_2\varepsilon,$ $q_1=q_2^*=\tau_1+i\nu_1, q_3=\kappa_1, q_4=\kappa_2,$ $q_5=\kappa_3, q_6=\kappa_4, a_3=a_4=\varsigma_1, a_5=a_6=\varsigma_2,$ $\eta_{01}=\eta_{02}^*=i\pi, \varepsilon\to 0$
	$M=1,P=1,Q=2.$ one lump + one PB +two LSs	$b_i=a_iq_i\ (i=1,\cdots,6)$, $a_1=l_1\varepsilon$, $a_2=l_2\varepsilon,$ $a_3=a_4=\varsigma_3, a_5=a_6=\varsigma_4,$ $q_1=q_2^*=\tau_2+i\nu_2, q_3=q_4^*=\tau_3+i\nu_3, q_5=\kappa_5,$ $q_6=\kappa_6, \eta_{01}=\eta_{02}^*=i\pi, \varepsilon\to 0$
	$M=1,P=2,Q=0.$ one lump + two PBs	$b_i=a_iq_i\ (i=1,\cdots,6)$, $a_1=l_1\varepsilon$, $a_2=l_2\varepsilon,$ $a_3=a_4=\varsigma_5, a_5=a_6=\varsigma_6, q_1=q_2^*$ $=\tau_4+i\nu_4, q_3=q_4^*=\tau_5+i\nu_5, q_5=q_6^*=\tau_6+i\nu_6,$ $\eta_{01}=\eta_{02}^*=i\pi, \varepsilon\to 0$
$M=2$	$M=2,P=0,Q=1.$ two lumps + one LS	$b_i=a_iq_i$, $a_i=l_i\varepsilon\ (i=1,\cdots,4), b_5=a_5q_5,$ $q_1=q_2^*=\alpha_4+i\beta_4, q_3=q_4^*=\alpha_5+i\beta_5, q_5=\vartheta_5,$ $a_5=\delta_6, \eta_{01}=\eta_{02}^*=\eta_{03}=\eta_{04}^*=i\pi,\ \varepsilon\to 0$
	$M=2,P=0,Q=2.$ two lumps+ two LSs	$b_i=a_iq_i\ (i=1,\cdots,6)$, $a_1=l_1\varepsilon$, $a_2=l_2\varepsilon,$ $a_3=l_3\varepsilon, a_4=l_4\varepsilon, a_5=a_6=\varsigma_7,$ $q_1=q_2^*=\omega_1+i\iota_1, q_3=q_4^*=\omega_2+i\iota_2, q_5=\kappa_7, q_6=$ $\kappa_8,\ \eta_{01}=\eta_{02}^*=\eta_{03}=\eta_{04}^*=i\pi,\ \varepsilon\to 0$
	$M=2,P=1,Q=0.$ two lumps + one PB	$b_i=a_iq_i\ (i=1,\cdots,6)$, $a_1=l_1\varepsilon, a_2=l_2\varepsilon,$ $a_3=l_3\varepsilon, a_4=l_4\varepsilon, a_5=a_6=\varsigma_8,$ $q_1=q_2^*=\omega_3+i\iota_3, q_3=q_4^*=\omega_4+i\iota_4, q_5=q_6^*=$ $\omega_5+i\iota_5,\ \eta_{01}=\eta_{02}^*=\eta_{03}=\eta_{04}^*=i\pi,\ \varepsilon\to 0$
$M=3$	$M=3,P=0,Q=0.$ three lumps	$b_i=a_iq_i, a_i=l_i\varepsilon, (i=1,\cdots,6), q_1=q_2^*$ $=\omega_6+i\iota_6, q_3=q_4^*=\omega_7+i\iota_7, q_5=q_6^*=\omega_8+i\iota_8,$ $\eta_{01}=\eta_{02}^*=\eta_{03}=\eta_{04}^*=\eta_{05}=\eta_{06}^*=i\pi, \varepsilon\to 0$

Note: LS=Line soliton, PB= Periodic breather. Here, $\delta_s, \alpha_j, \beta_j, \vartheta_j, \tau_s, \nu_s, \kappa_l, \varsigma_l, \omega_l, \iota_l$ $(s=1,\cdots,6, j=1,\cdots,5, l=1,\cdots,8)$, are nonzero real constants.

ear localized waves.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Nos. 11905013, 12071042 and 11772063), the Scientific Research Common Program of Beijing Municipal Commission of Education under Grant (No. KM201911232011), and the Beijing Natural Science Foundation under Grant(No. 1202006).

Conflicts of Interest

This work does not have any conflicts of interest.

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