A parametric (b, θ)-metric space and some fixed point theorems

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Abstract. In this paper, motivated by Kamran et al. [Mathematics, 5(19), (2017), 7 pages] we introduce the notion of parametric (b, θ)-metric space as an extended form of parametric b-metric space, which was introduced by Hussain et al.[J. Nonlinear Sci. Appl., 8 (2015), 719–739] and improve the results of Alsulami et al.[Abstract and Applied Analysis, (2014), Article ID 187031, 10 pages] and others in such space. We also apply our result to establish an existence of solution of integral equation.

Mathematics Subject Classification (2010). Primary 47H10; Secondary 54H25.

Keywords. Altering distance function, comparison function, partial order, parametric α-orbital admissible, parametric (b, θ)-metric space, integral equation.

1. Introduction and Preliminaries

The concept of metric space has been generalized from one to different directions by many researchers. Bakhtin [7] (also, Czerwik [13]) introduced the concept of b-metric space as a generalization of metric space. In 2014, Hussain et al. [15] introduced the notion of parametric metric space as a generalization of metric space and proved some fixed point theorems in such spaces. In 2015, Hussain et al. [16] introduced the concept of parametric b-metric space as a generalization of parametric metric space and investigated the existence of fixed points under various contractive conditions in such space. On the other hand, recently Kamran et al. [17] introduced the concept of extended b-metric space as a generalization of b-metric space and proved Banach fixed point theorem in this space.

In 2003, Ran and Reurings [27] proved an analogue of Banach contraction principle for continuous monotone mapping in metric space endowed with partial order and studied the existence of solutions of linear and nonlinear matrix equations. After that Nieto and Rodríguez-López [21, 22] generalized the results of Ran and Reurings [27] by relaxing the conditions of continuity as well as monotonicity and presented some applications to first-order ordinary differential equations with periodic boundary conditions (for more results on fixed point in ordered set, we refer to [1, 3, 12, 15, 16, 21, 22, 27, 28] and references therein).

Definition 1.1. [7, 13] Let X be a nonempty set and $s \geq 1$ a given real number. A function $d_b : X \times X \to [0, +\infty)$ is called a b-metric on X, if for all $x, y, z \in X$, following conditions hold:
(d_b1) $d_b(x, y) = 0$, if and only $x = y$;
(d_b2) $d_b(x, y) = d_b(y, x)$;
(d_b3) $d_b(x, y) \leq s[d_b(x, y) + d_b(y, z)]$.
Then the pair (X, d_b) is called a b-metric space.

Every metric is b-metric for $s = 1$ but, the converse does not hold in general. Hence the class of b-metric spaces is effectively larger than that of metric spaces. The following example shows that b-metric space need not be metric space.

Example 1.2. [2] Let (X, d) a metric space and $d_b(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then, (X, d_b) is a b-metric space with $s = 2^{p-1}$. However, (X, d_b) is not a metric space.

Note that the distance function d_b used in b-metric spaces is not continuous in general (see [11, 14]). For more examples and fixed point results in b-metric spaces we refer to [2, 8, 11, 13, 14, 24] and references therein).

Definition 1.3. [17] Let X be a nonempty set and $\theta : X \times X \to [1, +\infty)$. A function $d_\theta : X \times X \to [0, +\infty)$ is called an extended b-metric if for all $x, y, z \in X$, the following conditions hold:
$(d_\theta 1)$ $d_\theta(x, y) = 0$ if and only if $x = y$;
$(d_\theta 2)$ $d_\theta(x, y) = d_\theta(y, x)$;
$(d_\theta 3)$ $d_\theta(x, y) \leq \theta(x, y)[d_\theta(x, z) + d_\theta(z, y)]$.
Then the pair (X, d_θ) is called an extended b-metric space.

Note that if $\theta(x, y) = s > 1$, for all $x, y \in X$, then extended b-metric space becomes a b-metric space. Therefore, every metric space is a b-metric space and b-metric space is an extended b-metric space, but the converse need not be true in general. For more results on extended b-metric spaces and references, we refer to [4, 5, 6, 17, 25].

Example 1.4. Let $X = \mathbb{R}$. Define $\theta : X \times X \to [1, +\infty)$ and $d_\theta : X \times X \to [0, +\infty)$ as: $\theta(x, y) = 1 + |x| + |y|$, for all $x, y \in X$ and

$$d_\theta(x, y) = \begin{cases} |x| + |y|, & x, y \in X, x \neq y; \\ 0, & x = y. \end{cases}$$

Then, (X, d_θ) is an extended b-metric space.

Definition 1.5. [15] Let X be a nonempty set and $\mathcal{P} : X^2 \times (0,+\infty) \to [0,+\infty)$ be a function. We say $\mathcal{P}$ is a parametric metric on X if for all $x, y, z \in X$ the following conditions hold:
($\mathcal{P}1$) $\mathcal{P}(x,y,t) = 0$, if and only if $x = y$, for all $t > 0$;
($\mathcal{P}2$) $\mathcal{P}(x,y,t) = \mathcal{P}(y,x,t)$, for all $t > 0$;
($\mathcal{P}3$) $\mathcal{P}(x,y,t) \leq \mathcal{P}(x,z,t) + \mathcal{P}(z,y,t)$, for all $t > 0$.
Then the pair $(X,\mathcal{P})$ is called a parametric metric space.

Example 1.6. [15] Let X be a set of all functions $f : (0,+\infty) \to \mathbb{R}$. Define $\mathcal{P} : X^2 \times (0,+\infty) \to [0,+\infty)$ as $\mathcal{P}(f,g,t) = |f(t) - g(t)|$, for all $f, g \in X$, $t > 0$. Then $(X,\mathcal{P})$ is a parametric metric space.

Example 1.7. [15] Let $X = [0,+\infty)$ and $\mathcal{P} : X^2 \times (0,+\infty) \to [0,+\infty)$ be defined by

$$\mathcal{P}(x,y,t) = \begin{cases} t\max\{x,y\}, & x \neq y; \\ 0, & x = y, \end{cases}$$

for all $x, y \in X$ and for all $t > 0$. Then $(X,\mathcal{P})$ is a parametric metric space.

Let $(X,\mathcal{P})$ be a parametric metric space. Let $a \in X$ and $r > 0$, then $B(a,r) = \{x \in X : \mathcal{P}(a,x,t) < r, \text{ for all } t > 0\}$ is called an open ball of radius $r > 0$ centred at $a \in X$.

Remark 1.8. If $(X,\mathcal{P})$ be a parametric metric space, then parametric metric $\mathcal{P}$ is a continuous function.

Definition 1.9. [16] Let X be a nonempty set and $s \geq 1$. A function $\mathcal{P}_b : X^2 \times (0,+\infty) \to [0,+\infty)$ is said to be a parametric b-metric if for all $x, y, z \in X$, the following conditions hold:
($\mathcal{P}_b1$) $\mathcal{P}_b(x,y,t) = 0$, if and only if $x = y$, for all $t > 0$;
($\mathcal{P}_b2$) $\mathcal{P}_b(x,y,t) = \mathcal{P}_b(y,x,t)$, for all $t > 0$;
($\mathcal{P}_b3$) $\mathcal{P}_b(x,y,t) \leq s[\mathcal{P}_b(x,z,t) + \mathcal{P}_b(z,y,t)]$, for all $t > 0$.
The pair $(X,\mathcal{P}_b)$ is called a parametric b-metric space.

Note that if $s = 1$ then, parametric b-metric space becomes a parametric space. Therefore, every parametric metric space is a parametric b-metric space, but the converse need not be true in general. In general, a parametric b-metric function $\mathcal{P}_b$ for $s > 1$ is not continuous in its variables (Example 1.7 of Hussain et al. [16]).

Example 1.10. [16] Let $X = [0,+\infty)$ and define $\mathcal{P}_b(x,y,t) = t|x-y|^p$, for all $x, y \in X$ and for all $t > 0$. Then $\mathcal{P}_b$ is a parametric b-metric with constant $s = 2^p$, where $p \geq 1$.

Let X be a non-empty set and $\alpha : X \times X \to \mathbb{R}$ be a mapping.

Definition 1.11. [28] A mapping $T : X \to X$ is an α-admissible, if $x, y \in X$, $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$.

Definition 1.12. [20] An α- admissible mapping T is said to be an α^*-admissible, if for all $x, x^* \in Fix(T) \neq \emptyset$, $\alpha(x,x^*) \geq 1$.

Remark 1.13. We denote $Fix(T) = \{x \in X | Tx = x\}$.

Definition 1.14. [18] A mapping $T : X \to X$ is a triangular α-admissible if
(T_1) T is an α-admissible;
(T_2) $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$ imply $\alpha(x, z) \geq 1$.

Definition 1.15. [3] A mapping $T : X \to X$ is a weak triangular α-admissible if
(T_1) T is an α-admissible;
(T_2) $\alpha(x, Tx) \geq 1$ implies $\alpha(x, T^2x) \geq 1$.

The following hypothesis is also used in Alsulami et al. [3] for the existence of uniqueness of fixed point.
Condition (B): For $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$.

Definition 1.16. [26] A mapping $T : X \to X$ said to be an α-orbital admissible if $x \in X$, $\alpha(x, Tx) \geq 1$ implies $\alpha(Tx, T^2x) \geq 1$.

Note that every α-admissible mapping is an α-orbital admissible mapping (for more details we refer to [26]).

Definition 1.17. An α-orbital admissible mapping T is said to be an α^*-orbital admissible, if for all $x, x^* \in Fix(T) \neq \emptyset$, $\alpha(x, x^*) \geq 1$.

Definition 1.18. [23] A continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an altering distance if it is non-decreasing and $\varphi(r) = 0$ if and only if $r = 0$. Denote by Φ the set of all altering distance functions.

Example 1.19. Let $\varphi_i : \mathbb{R}^+ \to \mathbb{R}^+$, where $i = 1, 2$ be defined by:
(i) $\varphi_1(r) = e^{\alpha r} + \beta r - 1$;
(ii) $\varphi_2(r) = \alpha r^2 + \ln(\beta r + 1)$,
where $\alpha, \beta > 0$.

Clearly, φ_1, φ_2 are altering distance functions (for more examples on altering distance function, we refer to Sintunavarat [29]).

Lemma 1.20. [19] *Suppose* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is nondecreasing. Then, for every* $r > 0$, $\lim\limits_{n\to+\infty} \psi^n(r) = 0$ *implies* $\psi(r) < r$, *where* ψ^n *denotes the nth-iterate of* ψ.

Definition 1.21. [8, 9] A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a comparison function, if it is monotonically increasing and $\psi^n(r) \to 0$ as $n \to +\infty$, for all $r > 0$ and Ψ denotes the set of all comparison functions.

Example 1.22. [10] Let $\psi_i : \mathbb{R}^+ \to \mathbb{R}^+, i = 1, 2, 3,$ be defined by
(ψ_1) $\psi_1(r) = \alpha r$, where $0 \leq \alpha < 1$;
(ψ_2) $\psi_2(r) = \frac{t}{1+r}$;
(ψ_3) $\psi_3(r) = \beta\gamma(r)$, where $\psi_3(r)$ is monotonically increasing, $0 \leq \beta < 1$ and $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\gamma^n(r) \to 0$ as $n \to +\infty$.

It is easy to see that $\psi_i, i = 1, 2, 3$ are comparison functions. Note that if ψ is comparison function, then $\psi(r) < r$, for all $r > 0$ and $\psi(0) = 0$. We denote Ψ, the set of all comparison functions.

In the next section, we introduce the concept of parametric (b, θ)-metric space and prove some fixed point theorems in such space. We also extend the our result to parametric (b, θ)-metric space endowed with partial order and apply it to prove the existence of solution of integral equation.

2. Main Results

Motivated by Kamran et al. [17], we introduce the notion of parametric (b, θ)-metric space and investigate some examples. We also introduce the notion of α-orbital admissible [26] mapping in this space.

Definition 2.1. Let X be a nonempty set and $\theta : X \times X \times (0, +\infty) \to [1, +\infty)$. A function $\mathcal{P}_\theta : X \times X \times (0, +\infty) \to [0, +\infty)$ is said to be parametric (b, θ)-metric if for all $x, y, z \in X$ and for all $t > 0$, the following conditions hold:
$(\mathcal{P}_\theta 1)$ $\mathcal{P}_\theta(x, y, t) = 0$, if and only if $x = y$;
$(\mathcal{P}_\theta 2)$ $\mathcal{P}_\theta(x, y, t) = \mathcal{P}_\theta(y, x, t)$;
$(\mathcal{P}_\theta 3)$ $\mathcal{P}_\theta(x, y, t) \leq \theta(x, y, t)[\mathcal{P}_\theta(x, z, t) + \mathcal{P}_\theta(z, y, t)]$.
The pair $(X, \mathcal{P}_\theta)$ is called a parametric (b, θ)-metric space.

If $\theta(x, y, t) = s \geq 1$, for all $t > 0$, then parametric (b, θ)-metric space becomes a parametric b-metric space. Therefore, every parametric metric space is a parametric b-metric and parametric b-metric space is a parametric (b, θ)-metric space, but the converse may not be true in general. In the following, we discuss some examples of parametric (b, θ)-metric spaces.

Example 2.2. Let $X = \mathbb{R}$ and $\mathcal{P}_\theta : X^2 \times (0, +\infty) \to [0, +\infty)$ be defined by $\mathcal{P}_\theta(x, y, t) = t(x - y)^2$, where $\theta(x, y, t) = 2 + t(|x| + |y|)$, for all $x, y \in X$ and for all $t > 0$. Then, $(X, \mathcal{P}_\theta)$ is a parametric (b, θ)-metric space.

Example 2.3. Let $X = \mathbb{R}$ and $\theta : X^2 \times (0, +\infty) \to [1, +\infty)$ be defined by $\theta(x, y, t) = 1 + t(|x| + |y|)$, for all $x, y \in X$ and for all $t > 0$. Let $\mathcal{P}_\theta : X^2 \times (0, +\infty) \to [0, +\infty)$ be given by

$$\mathcal{P}_\theta(x, y, t) = \begin{cases} t(|x|^p + |y|^p), & x, y \in X, x \neq y; \\ 0, & x = y, \end{cases}$$

where, $p \geq 1$. Then, $(X, \mathcal{P}_\theta)$ is a parametric (b, θ)-metric space.

Example 2.4. Let $X = [0, 1]$ and $\theta : X^2 \times (0, +\infty) \to [1, +\infty)$ be a function given by $\theta(x, y, t) = \frac{1+x+y+t}{x+y}$, where $x + y > 0$ and $\theta(0, 0, t) = 1$, for all $t > 0$. Define $\mathcal{P}_\theta : X^2 \times (0, +\infty) \to [0, +\infty)$ as

$$\begin{aligned} \mathcal{P}_\theta(x, y, t) &= \frac{t}{xy}, \quad \text{for } x, y \in (0, 1], x \neq y, t > 0; \\ \mathcal{P}_\theta(x, y, t) &= 0, \quad \text{for } x = y, t > 0; \\ \mathcal{P}_\theta(x, 0, t) &= \mathcal{P}_\theta(0, x, t) = \frac{t}{x}, \quad \text{for } x \in (0, 1], t > 0. \end{aligned}$$

Then $(X, \mathcal{P}_\theta)$ is a parametric (b,θ)-metric space.

Example 2.5. Consider the set $X = l_p(\mathbb{R})$ with $0 < p < 1$, where

$$l_p(\mathbb{R}) = \Big\{ \{x_i\} \subseteq \mathbb{R} | \sum_{i=1}^{\infty} |x_i|^p < \infty \Big\}.$$

Define $\theta : X^2 \times (0, +\infty) \to [1, +\infty)$ and $\mathcal{P}_\theta : X^2 \times (0, +\infty) \to [0, +\infty)$ by

$$\theta(x, y, t) = 2^{\frac{1}{p}} + t(|x| + |y|)$$

and

$$\mathcal{P}_\theta(x, y, t) = \Big(\sum_{i=1}^{\infty} |\eta(t)(x_i - y_i)|^p \Big)^{\frac{1}{p}},$$

where $0 < \eta(t) < \infty$, for all $x = \{x_i\}$, $y = \{y_i\} \in X$, for all $t > 0$. Then $(X, \mathcal{P}_\theta)$ is a parametric (b,θ)-metric with $\theta(x, y, t) > 1$.

Example 2.6. The space $L_p[0,1]$ of all real functions $x(\rho)$, $\rho \in [0,1]$ such that $\int_0^1 |x(\rho)|^p d\rho < 1$, where $0 < p < 1$. Define

$$\mathcal{P}_\theta(x, y, t) = \Big(\int_0^1 \Big| \frac{x(\rho) - y(\rho)}{1+t} \Big|^p d\rho \Big)^{\frac{1}{p}}$$

and

$$\theta(x, y, t) = 2^{\frac{1}{p}} + \frac{1+t}{x+y}, \text{ where } x + y > 0 \text{ and } \theta(0, 0, t) = 1$$

for all $x, y \in L_p[0,1]$ and for all $t > 0$. Then $\mathcal{P}_\theta$ is a parametric (b,θ)-metric on $L_p[0,1]$.

Let $(X, \mathcal{P}_\theta)$ be a parametric (b,θ)-metric space, where $\mathcal{P}_\theta$ is a continuous parametric (b,θ)-metric and let $a \in X$ and $r > 0$, we write

$$\mathcal{B}(a, r) = \{x \in X | \mathcal{P}_\theta(a, x, t) < r, \text{ for all } t > 0\}.$$

Then, $\mathcal{B}(a, r)$ is called an open ball of radius $r > 0$ centred at a. Let $\{x_n\}$ be a sequence in X, then a point $x \in X$ is called a limit of the sequence $\{x_n\}$ if $\lim_{n\to+\infty} \mathcal{P}_\theta(x_n, x, t) = 0$, for all $t > 0$ and we say that the sequence $\{x_n\}$ is convergent to $x \in X$ and denotes it as $x_n \to x$ an $n \to +\infty$.

The concepts of Cauchy sequence and completeness in a parametric (b,θ)-metric space can be formulated analogous to the case of parametric b-metric space.

Definition 2.7. Let $(X, \mathcal{P}_\theta)$ be a parametric (b,θ)-metric space and $\{x_n\}$ be a sequence in X. We say that
(a) $\{x_n\}$ in X is said to be a Cauchy sequence in X if for all $t > 0$, $\mathcal{P}_\theta(x_m, x_n, t) \to 0$ as $m, n \to +\infty$;
(b) $(X, \mathcal{P}_\theta)$ is said to be a complete if and only if every Cauchy sequence in X is convergent.

Definition 2.8. Let $(X,\mathcal{P}_\theta)$ be a parametric (b,θ)-metric space and $T: X \to X$ be a mapping, then we say that T is a continuous at $x \in X$, if for any sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to +\infty$, we have $Tx_n \to Tx$ as $n \to +\infty$.

Recall that, in general a parametric b-metric function $\mathcal{P}_b$ with $s > 1$ is not continuous in its variables, so $\mathcal{P}_\theta$ is also not continuous in general.

Though out the following sections, we assume that a parametric (b,θ)-metric $\mathcal{P}_\theta$ is a continuous function.

Definition 2.9. Let X be a non-empty set and $\alpha : X^2 \times (0,+\infty) \to \mathbb{R}$ be a mapping. A mapping $T : X \to X$ is said to be a parametric α-orbital admissible if $x \in X$, $\alpha(x,Tx,t) \geq 1$ implies $\alpha(Tx,T^2x,t) \geq 1$, for all $t > 0$.

In addition, T is said to be a parametric α^*-orbital admissible, if for all $x,y \in Fix(T) \neq \emptyset$, $\alpha(x,y,t) \geq 1$, for all $t > 0$.

Example 2.10. Let $X = [0,+\infty)$. Define $\alpha : X^2 \times (0,+\infty) \to \mathbb{R}$ and $T: X \to X$ as:

$$\alpha(x,y,t) = \begin{cases} 1, & \text{if } x \geq y; \\ 0, & \text{otherwise,} \end{cases}$$

for all $t > 0$ and $Tx = \ln(1+x)$, for all $x \in X$. Then, T is a parametric α-orbital admissible.

Example 2.11. Let $X = [0,+\infty)$ and $T : X \to X$ be a mapping defined by $Tx = \frac{x^2}{2}$, for all $x \in X$. Define $\alpha : X^2 \times (0,+\infty) \to \mathbb{R}$ as

$$\alpha(x,y,t) = \begin{cases} 1, & \text{if } x,y \in [0,2]; \\ 0, & \text{otherwise,} \end{cases}$$

for all $t > 0$. Note that $Fix(T) = \{0,2\}$. Then T is a parametric α-orbital admissible and parametric α^*-orbital admissible as well.

Example 2.12. Let $X = [0,\infty)$. Define $T : X \to X$ and $\alpha : X^2 \times (0,\infty) \to \mathbb{R}$ as $Tx = \sqrt{\left|\frac{x(x^2+11)-6}{6}\right|}$, for all $x \in X$ and for all $t > 0$,

$$\alpha(x,y,t) = \begin{cases} 1, & \text{if} x,y \in [0,2]; \\ 0, & \text{otherwise.} \end{cases}$$

Then T is a parametric α-orbital admissible but not a parametric α^*-orbital admissible as $Fix(T) = \{1,2,3\}$.

Lemma 2.13. *Let $\{x_n\}$ be any sequence in a parametric (b,θ)-metric space $(X,\mathcal{P}_\theta)$. If there exist two functions $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$0 < \varphi\Big(\mathcal{P}_\theta(x_n,x_{n+1},t)\Big) \leq \psi\Big(\mathcal{P}_\theta(x_{n-1},x_n,t)\Big)$$

$$\textit{and} \quad \lim_{n,m\to+\infty} \frac{\theta(x_n,x_m,t)\psi^n\Big(\mathcal{P}_\theta(x_0,x_1,t)\Big)}{\psi^{n-1}\Big(\mathcal{P}_\theta(x_0,x_1,t)\Big)} < 1$$

for any $m > n \geq 1$ and for all $t > 0$, then $\{x_n\}$ is a Cauchy sequence in X.

Proof. For all $t > 0$, we obtain

$$0 < \varphi\Big(\mathcal{P}_\theta(x_n, x_{n+1}, t)\Big) \leq \psi\Big(\mathcal{P}_\theta(x_{n-1}, x_n, t)\Big) \leq \ldots \leq \psi^n\Big(\mathcal{P}_\theta(x_0, x_1, t)\Big).$$

Letting $n \to +\infty$, we obtain

$$\lim_{n\to+\infty} \varphi\Big(\mathcal{P}_\theta(x_n, x_{n+1}, t)\Big) = 0 \text{ if and only if } \lim_{n\to+\infty} \mathcal{P}_\theta(x_n, x_{n+1}, t) = 0.$$

Setting $\theta_i = \theta(x_i, x_{i+p}, t)$ for each $i \in \mathbb{N}$, $p \geq 1$ and $\mathcal{P}_\theta(x_0, x_1, t) = \omega$, we obtain

$$\begin{aligned}
\mathcal{P}_\theta(x_n, x_{n+p}, t) \leq& \theta(x_n, x_{n+p}, t)\Big[\mathcal{P}_\theta(x_n, x_{n+1}, t) + \mathcal{P}_\theta(x_{n+1}, x_{n+p}, t)\Big] \\
\leq& \theta(x_n, x_{n+p}, t)\mathcal{P}_\theta(x_n, x_{n+1}, t) + \theta(x_n, x_{n+p}, t)\mathcal{P}_\theta(x_{n+1}, x_{n+p}, t) \\
\leq& \theta(x_n, x_{n+p}, t)\mathcal{P}_\theta(x_n, x_{n+1}, t) + \\
& \theta(x_n, x_{n+p}, t)\theta(x_{n+1}, x_{n+p})\mathcal{P}_\theta(x_{n+1}, x_{n+2}, t) + \ldots + \\
& \theta(x_n, x_{n+p}, t)\theta(x_{n+1}, x_{n+p}, t)\ldots \\
& \theta(x_{n+p-1}, x_{n+p}, t)\mathcal{P}_\theta(x_{n+p-1}, x_{n+p}, t) \\
\leq& \theta_n\psi^n\Big(\mathcal{P}_\theta(x_0, x_1, t)\Big) + \theta_n\theta_{n+1}\psi^{n+1}\Big(\mathcal{P}_\theta(x_0, x_1, t)\Big) + \ldots \\
& + \theta_n\theta_{n+1}\ldots\theta_{n+p-1}\psi^{n+p-1}\Big(\mathcal{P}_\theta(x_0, x_1, t)\Big) \\
=& \theta_n\psi^n(\omega) + \theta_n\theta_{n+1}\psi^{n+1}(\omega) + \ldots + \theta_n\theta_{n+1}\ldots\theta_{n+p-1}\psi^{n+p-1}(\omega) \\
=& \sum_{i=n}^{n+p-1} \psi^i(\omega) \prod_{j=n}^{i} \theta_j.
\end{aligned}$$

Multiplying $\prod_{i=1}^{n-1} \theta_i$ on the right side of the above inequality, we obtain

$$\begin{aligned}
\mathcal{P}_\theta(x_n, x_{n+p}, t) \leq& \sum_{i=n}^{n+p-1} \psi^i(\omega) \prod_{j=1}^{i} \theta_j \\
=& \sum_{i=1}^{n+p-1} \psi^i(\omega) \prod_{j=1}^{i} \theta_j - \sum_{i=1}^{n-1} \psi^i(\omega) \prod_{j=1}^{i} \theta_j.
\end{aligned}$$

Since,

$$\lim_{i\to+\infty} \frac{\theta(i, i+p, t)\psi^i\Big(\mathcal{P}_\theta(x_0, x_1, t)\Big)}{\psi^{i-1}\Big(\mathcal{P}_\theta(x_0, x_1, t)\Big)} = \lim_{i\to+\infty} \frac{\theta_i\psi^i(\omega)}{\psi^{i-1}(\omega)} < 1.$$

Therefore, by Ratio test the series $\sum_{i=1}^{\infty} \psi^i(\omega) \prod_{j=1}^{i} \theta_j$ converges.

Let $S = \sum_{i=1}^{\infty} \psi^i(\omega) \prod_{j=1}^{i} \theta_j$ and $S_n = \sum_{i=1}^{n} \psi^i(\omega) \prod_{j=1}^{i} \theta_j$, the sequence of partial sum. Consequently, we obtain

$$\mathcal{P}_\theta(x_n, x_{n+p}, t) \leq \Big[S_{n+p-1} - S_{n-1}\Big].$$

for any $n \in \mathbb{N}$ and $p \geq 1$. Letting limit as $n \to +\infty$, we obtain $\{x_n\}$ is Cauchy sequence in X. ∎

Setting $\varphi(\xi) = \xi$ and $\psi(\xi) = k\xi$, where $\xi \in \mathbb{R}^+$, $k \in [0,1)$, then we obtain the following lemma.

Lemma 2.14. *Let $\{x_n\}$ be any sequence in a parametric (b,θ)-metric space $(X,\mathcal{P}_\theta)$ such that*

$$0 < \mathcal{P}_\theta(x_n, x_{n+1}, t) \leq k\mathcal{P}_\theta(x_{n-1}, x_n, t) \tag{2.1}$$

and

$$\lim_{n,m\to+\infty} \theta(x_n, x_m, t) < \frac{1}{k}, \tag{2.2}$$

where $k \in [0,1)$, for any $m > n \geq 1$ and for all $t > 0$, then $\{x_n\}$ is a Cauchy sequence in X.

Theorem 2.15. *Let $(X,\mathcal{P}_\theta)$ be a complete parametric (b,θ)-metric space and $T : X \to X$ be a continuous self mapping on X. Assume that there exist $\alpha : X^2 \times (0,\infty) \to \mathbb{R}$, $\varphi \in \Phi$ and $\psi \in \Psi$ such that $\varphi(r) > \psi(r)$, for $r > 0$ satisfying*

$$\alpha(x,y,t)\varphi\Big(\mathcal{P}_\theta(Tx,Ty,t)\Big) \leq \psi\Big(\mathcal{M}(x,y,t)\Big) \tag{2.3}$$

where

$$\mathcal{M}(x,y,t) = \max\Big\{\mathcal{P}_\theta(x,y,t), \mathcal{P}_\theta(x,Tx,t), \mathcal{P}_\theta(y,Ty,t), \frac{\mathcal{P}_\theta(x,Ty,t) + \mathcal{P}_\theta(y,Tx,t)}{2\theta(x,y,t)}\Big\},$$

for all $x,y \in X$ and for all $t > 0$. If
(i) T is a parametric α-orbital admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0,t) \geq 1$, for all $t > 0$;
(iii) $\lim_{n,m\to+\infty} \dfrac{\theta(x_n,x_m,t)\psi^n\Big(\mathcal{P}_\theta(x_0,x_1,t)\Big)}{\psi^{n-1}\Big(\mathcal{P}_\theta(x_0,x_1,t)\Big)} < 1$,
where $x_n = T^n x_0$, $m > n \geq 1$, for all $t > 0$.
Then T possesses a fixed point $\zeta \in X$. Moreover the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to $\zeta \in X$.

Proof. By given assumption, there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0,t) \geq 1$, for all $t > 0$. Define a sequence $\{x_n\}$ in X such that $x_n = T^n x_0$, for all $n \in \mathbb{N}$. If $x_{k-1} = x_k = Tx_{k-1}$, for some $k \in \mathbb{N}$, then we have $x_{k-1} = Tx_{k-1}$. Without lost of generality we assume that $x_{n-1} \neq x_n$, for all $n \in \mathbb{N}$.

Since T is a parametric α-orbital admissible,

$$\alpha(x_0,x_1,t) = \alpha(x_0,Tx_0,t) \geq 1$$

implies

$$\alpha(x_1,x_2,t) = \alpha(Tx_0,T^2x_0,t) \geq 1,$$

for each $t > 0$. Similarly,

$$\alpha(x_1,x_2,t) = \alpha(Tx_0,T^2x_0,t) \geq 1$$

implies

$$\alpha(x_2,x_3,t) = \alpha(T^2x_0,T^3x_0,t) \geq 1,$$

for all $t > 0$. Continuing in this way, we obtain inductively that $\alpha(x_{n-1}, x_n, t) \geq 1$, where $n \in \mathbb{N}$, for all $t > 0$.

Taking $x = x_{n-1}$ and $y = x_n$ for all $t > 0$, we obtain

$$\begin{aligned}\varphi\Big(\mathcal{P}_\theta(x_n, x_{n+1}, t)\Big) =&\varphi\Big(\mathcal{P}_\theta(Tx_{n-1}, Tx_n, t)\Big)\\ \leq&\alpha(x_{n-1}, x_n, t)\varphi\Big(\mathcal{P}_\theta(Tx_{n-1}, Tx_n, t)\Big)\\ \leq&\psi\Big(\mathcal{M}(x_{n-1}, x_n, t)\Big),\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}(x_{n-1}, x_n, t) =&\max\Big\{\mathcal{P}_\theta(x_{n-1}, x_n, t), \mathcal{P}_\theta(x_{n-1}, Tx_{n-1}, t), \mathcal{P}_\theta(x_n, Tx_n, t),\\ &\frac{\mathcal{P}_\theta(x_{n-1}, Tx_n, t) + \mathcal{P}_\theta(x_n, Tx_{n-1}, t)}{2\theta(x_{n-1}, x_n, t)}\Big\}\\ =&\max\Big\{\mathcal{P}_\theta(x_{n-1}, x_n, t), \mathcal{P}_\theta(x_{n-1}, x_n, t), \mathcal{P}_\theta(x_n, x_{n+1}, t),\\ &\frac{\mathcal{P}_\theta(x_{n-1}, x_{n+1}, t) + \mathcal{P}_\theta(x_n, x_n, t)}{2\theta(x_{n-1}, x_n, t)}\Big\}\\ =&\max\Big\{\mathcal{P}_\theta(x_{n-1}, x_n, t), \mathcal{P}_\theta(x_n, x_{n+1}, t), \frac{\mathcal{P}_\theta(x_{n-1}, x_{n+1}, t)}{2\theta(x_{n-1}, x_n, t)}\Big\}\end{aligned}$$

For the refinement of the inequality, we shall consider the following cases:
Case (i): If $\mathcal{M}(x_{n-1}, x_n, t) = \mathcal{P}_\theta(x_{n-1}, x_n, t)$, we obtain

$$\varphi\Big(\mathcal{P}_\theta(x_n, x_{n+1}, t)\Big) \leq \psi\Big(\mathcal{P}_\theta(x_{n-1}, x_n, t)\Big), \text{ for all } t > 0.$$

Case (ii): If $\mathcal{M}(x_{n-1}, x_n, t) = \mathcal{P}_\theta(x_n, x_{n+1}, t)$, we obtain

$$\varphi\Big(\mathcal{P}_\theta(x_n, x_{n+1}, t)\Big) \leq \psi\Big(\mathcal{P}_\theta(x_n, x_{n+1}, t)\Big) < \varphi\Big(\mathcal{P}_\theta(x_n, x_{n+1}, t)\Big),$$

for all $t > 0$, which is a contradiction.
Case (iii): If $\mathcal{M}(x_{n-1}, x_n, t) = \frac{\mathcal{P}_\theta(x_{n-1}, x_{n+1}, t)}{2\theta(x_{n-1}, x_n, t)}$, we obtain

$$\begin{aligned}\varphi\Big(\mathcal{P}_\theta(x_n, x_{n+1}, t)\Big) \leq&\psi\Big(\frac{\mathcal{P}_\theta(x_{n-1}, x_{n+1}, t)}{2\theta(x_{n-1}, x_n, t)}\Big)\\ \leq&\psi\Big(\frac{1}{2}\Big\{\mathcal{P}_\theta(x_{n-1}, x_n, t) + \mathcal{P}_\theta(x_n, x_{n+1}, t)\Big\}\Big), \text{ for all } t > 0.\end{aligned}$$

If $\max\{\mathcal{P}_\theta(x_{n-1}, x_n, t), \mathcal{P}_\theta(x_n, x_{n+1}, t)\} = \mathcal{P}_\theta(x_n, x_{n+1}, t)$ that is $\mathcal{P}_\theta(x_{n-1}, x_n, t) < \mathcal{P}_\theta(x_n, x_{n+1}, t)$, then

$$\varphi\Big(\mathcal{P}_\theta(x_n, x_{n+1}, t)\Big) \leq \psi\Big(\mathcal{P}_\theta(x_n, x_{n+1}, t)\Big)$$

which is again a contradiction to the fact that $\varphi(r) > \psi(r)$, for $r > 0$ and for all $t > 0$. Therefore, Case (iii) holds whenever $\mathcal{P}_\theta(x_n, x_{n+1}, t) < \mathcal{P}_\theta(x_{n-1}, x_n, t)$, for all $t > 0$.

Consequently, we obtain

$$0 < \varphi\Big(\mathcal{P}_\theta(x_n, x_{n+1}, t)\Big) \leq\psi\Big(\mathcal{P}_\theta(x_{n-1}, x_n, t)\Big), \text{ for all } t > 0.$$

Since,

$$\lim_{n,m\to+\infty} \frac{\theta(x_n,\, x_m,\, t)\psi^n\Big(\mathcal{P}_\theta(x_0,\, x_1,\, t)\Big)}{\psi^{n-1}\Big(\mathcal{P}_\theta(x_0,\, x_1,\, t)\Big)} < 1, \text{ for all } t > 0.$$

It follows from Lemma 2.13, we obtain that $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete parametric (b, θ)-metric space, there is $\zeta \in X$ such that $x_n \to \zeta$ as $n \to +\infty$ i.e., $\lim_{n\to+\infty} \mathcal{P}_\theta(x_n, \zeta, t) = 0$, for all $t > 0$. We suppose that T is continuous on X, then $Tx_n \to T\zeta$ as $n \to +\infty$ but, $Tx_n = x_{n+1} \to \zeta$ as $n \to +\infty$. Therefore, $T\zeta = \zeta$. ∎

Example 2.16. Let $X = [0, +\infty)$ and $\mathcal{P}_\theta : X^2 \times (0, +\infty) \to [0, +\infty)$ be a parametric (b, θ)-metric equip with

$$\mathcal{P}_\theta(x, y, t) = \begin{cases} t(x^2 + y^2), & x, y \in X, x \neq y; \\ 0, & x = y, \end{cases}$$

where $\theta(x, y, t) = 1 + t(x + y)$, for all $x, y \in X$ and for all $t > 0$.

Consider $T : X \to X$ be a continuous mapping defined by

$$Tx = \begin{cases} \frac{3x}{4}, & 0 \leq x \leq 1; \\ 2x - \frac{5}{4}, & x > 1. \end{cases}$$

Define $\alpha : X^2 \times (0, +\infty) \to \mathbb{R}$ as: for all $t > 0$,

$$\alpha(x, y, t) = \begin{cases} 1, & x, y \in [0, 1]; \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, for $x \in [0, 1]$, $\alpha(x, Tx, t) \geq 1$ and $\alpha(Tx, T^2x, t) \geq 1$, for all $t > 0$. Therefore T is a parametric α-orbital admissible. Also, we define $\psi(r) = kr$ and $\varphi(r) = r$, where $k = \frac{9}{16}$, then $\varphi(r) > \psi(r)$, for all $r > 0$. In fact for all $x, y \in X$, we obtain

$$\begin{aligned} \alpha(x, y, t)\varphi\Big(\mathcal{P}_\theta(Tx, Ty, t)\Big) =& \frac{9}{16} t(x^2 + y^2) = k\mathcal{P}_\theta(x, y, t) \\ \leq& \psi\Big(\mathcal{M}(x, y, t)\Big), \text{ for all } t > 0. \end{aligned}$$

Also, there exists $x_0 \in X$ such that $\alpha(x_0, x_1, t) = \alpha(x_0, Tx_0, t) \geq 1$ implies $\alpha(x_1, x_2, t) = \alpha(Tx_0, T^2x_0, t) \geq 1$, for all $t > 0$. We obtain inductively that $\alpha(x_n, x_{n+1}, t) = \alpha(T^n x_0, T^{n+1}x_0, t) \geq 1$, for all $t > 0$, where $x_n = T^n x_0 = (\frac{3}{4})^n x_0$. In fact $x_n = T^n x_0 \to 0$ as $n \to +\infty$ and

$$\lim_{n,m\to+\infty} \theta(T^n x_0, T^m x_0, t) = 1 < \frac{1}{k}.$$

Thus all the conditions of Theorem 2.15 are satisfied and hence, T possesses a fixed point. Note that $Fix(T) = \{0, \frac{5}{4}\}$.

In the following theorem, we omit the continuity assumption of T.

Theorem 2.17. *Let $(X, \mathcal{P}_\theta)$ be a complete parametric (b, θ)-metric space and $T : X \to X$ be a self mapping on X. Assume that there exist $\alpha : X^2 \times (0, \infty) \to \mathbb{R}$, $\varphi \in \Phi$ and $\psi \in \Psi$ such that $\varphi(r) > \psi(r)$, $r > 0$ satisfying*

$$\alpha(x, y, t)\varphi\Big(\mathcal{P}_\theta(Tx, Ty, t)\Big) \leq \psi\Big(\mathcal{M}(x, y, t)\Big),$$

where

$$\mathcal{M}(x, y, t) = \max\Big\{\mathcal{P}_\theta(x, y, t), \mathcal{P}_\theta(x, Tx, t), \mathcal{P}_\theta(y, Ty, t), \frac{\mathcal{P}_\theta(x, Ty, t) + \mathcal{P}_\theta(y, Tx, t)}{2\theta(x, y, t)}\Big\},$$

for all $x, y \in X$ and $t > 0$. If
(i) T is a parametric α-orbital admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$, for all $t > 0$;
(iii) $\lim_{n,m\to+\infty} \frac{\theta(x_n, x_m, t)\psi^n\left(\mathcal{P}_\theta(x_0, x_1, t)\right)}{\psi^{n-1}\left(\mathcal{P}_\theta(x_0, x_1, t)\right)} < 1$,
where $x_n = T^n x_0$, $m > n \geq 1$, for all $t > 0$;
(iv) $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ and $x_n \to \zeta \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, \zeta, t) \geq 1$, for all $t > 0$, where $n_k \geq n_0 \geq 1$. Then T possesses a fixed point in X.

Proof. Following the proof of Theorem 2.15, we obtain that $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete parametric (b, θ)-metric space, there exists $\zeta \in X$ such that $x_n \to \zeta$ an $n \to +\infty$. From (iv) we obtain $\alpha(x_{n_k}, \zeta, t) \geq 1$, $n_k \geq n_0$, for all $t > 0$. Taking $x = x_{n_k}$ and $y = \zeta$ for all $t > 0$, we obtain

$$\begin{aligned}
\varphi\Big(\mathcal{P}_\theta(x_{n_k+1}, T\zeta, t)\Big) =& \varphi\Big(\mathcal{P}_\theta(Tx_{n_k}, T\zeta, t)\Big) \\
=& \alpha(x_{n_k}, \zeta, t)\varphi\Big(\mathcal{P}_\theta(Tx_{n_k}, T\zeta, t)\Big) \\
\leq& \psi\Big(\mathcal{M}(x_{n_k}, \zeta, t)\Big) \\
<& \varphi\Big(\mathcal{M}(x_{n_k}, \zeta, t)\Big),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}(x_{n_k}, \zeta, t) = \max\Big\{&\mathcal{P}_\theta(x_{n_k}, \zeta, t), \mathcal{P}_\theta(x_{n_k}, Tx_{n_k}, t), \mathcal{P}_\theta(\zeta, T\zeta, t), \\
&\frac{\mathcal{P}_\theta(x_{n_k}, T\zeta, t) + \mathcal{P}_\theta(\zeta, Tx_{n_k}, t)}{2\theta(x_{n_k}, \zeta, t)}\Big\} \\
= \max\Big\{&\mathcal{P}_\theta(x_{n_k}, \zeta, t), \mathcal{P}_\theta(x_{n_k}, x_{n_k+1}, t), \mathcal{P}_\theta(\zeta, T\zeta, t), \\
&\frac{\mathcal{P}_\theta(x_{n_k}, T\zeta, t) + \mathcal{P}_\theta(\zeta, x_{n_k+1}, t)}{2\theta(x_{n_k}, \zeta, t)}\Big\}.
\end{aligned}$$

Letting $k \to +\infty$ and continuity of φ, we obtain

$$\varphi\Big(\mathcal{P}_\theta(\zeta, T\zeta, t)\Big) < \varphi\Big(\lim_{n_k\to+\infty} \mathcal{M}(x_{n_k}, \zeta, t)\Big) = \varphi\Big(\mathcal{P}_\theta(\zeta, T\zeta, t)\Big),$$

which is a contradiction. Therefore, we conclude that $\mathcal{P}_\theta(\zeta, T\zeta, t) = 0$ and hence, $T\zeta = \zeta$. ∎

Theorem 2.18. *In addition to the hypothesis of Theorem* 2.15 *(resp. Theorem* 2.17*), suppose the mapping $T : X \to X$ is a parametric α^*-orbital admissible. Then T possesses a unique fixed point $\xi \in X$. Moreover the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to $\xi \in X$.*

Proof. By Theorem 2.15 and Theorem 2.17, T possesses a fixed point in X i.e. $Fix(T) \neq \emptyset$. Since T is a parametric α^*-orbital admissible, then

$$\alpha(\zeta, \zeta^*, t) = \alpha(T\zeta, T\zeta^*, t) \geq 1,$$

for all $\zeta, \zeta^* \in Fix(T)$ and for all $t > 0$.
Suppose that $\zeta \neq \zeta^*$, for all $t > 0$, we obtain

$$\begin{aligned} 0 < \varphi\Big(\mathcal{P}_\theta(\zeta, \zeta^*, t)\Big) =& \varphi\Big(\mathcal{P}_\theta(T\zeta, T\zeta^*, t)\Big) \\ \leq& \alpha(\zeta, \zeta^*, t)\varphi\Big(\mathcal{P}_\theta(T\zeta, T\zeta^*, t)\Big) \\ \leq& \psi\Big(\mathcal{M}(\zeta, \zeta^*, t)\Big) \\ =& \psi\Big(\mathcal{P}_\theta(\zeta, \zeta^*, t)\Big) \\ <& \varphi\Big(\mathcal{P}_\theta(\zeta, \zeta^*, t)\Big), \end{aligned}$$

This is a contradiction and hence, T possesses a unique fixed point in X. ∎

Corollary 2.19. Let $(X, \mathcal{P}_\theta)$ be a complete parametric (b,θ)-metric space and $T : X \to X$ be a continuous self mapping satisfying

$$\mathcal{P}_\theta(Tx, Ty, t) \leq k\mathcal{M}(x, y, t)$$

where $\mathcal{M}(x, y, t) = \max\left\{\mathcal{P}_\theta(x, y, t), \frac{\mathcal{P}_\theta(x,Tx,t)+\mathcal{P}_\theta(y,Ty,t)}{2}, \frac{\mathcal{P}_\theta(x,Ty,t)+\mathcal{P}_\theta(y,Tx,t)}{2\theta(x,y,t)}\right\}$, for all $x, y \in X$ and for all $t > 0$. Moreover, if for any $x_0 \in X$,

$$\lim_{n,m\to\infty} \theta(x_n, x_m, t) < \frac{1}{k},$$

where $x_n = T^n x_0$ and $0 \leq k < 1$, for all $t > 0$. Then T possesses a unique fixed point $\zeta \in X$.

Remark 2.20. (i) In Example 2.16, T is a parametric α-orbital admissible and $Fix(T) = \{0, \frac{5}{4}\}$, but $\alpha(\frac{5}{4}, T\frac{5}{4}, t) = \alpha(\frac{5}{4}, \frac{5}{4}, t) = 0$, for all $t > 0$. This shows that T is not parametric α^*-orbital admissible. In this case, Theorem 2.18 is not application in Example 2.16.
(ii) In Example 2.16, taking $x = \frac{1}{2}$ and $y = 2$, then

$$\mathcal{P}_\theta(Tx, Ty, t) = \mathcal{P}_\theta(T\frac{1}{2}, T2, t) = \frac{493t}{64} > \frac{17t}{4} = \mathcal{P}_\theta(\frac{1}{2}, 2, t),$$

for all $t > 0$. This shows that Corollary 2.19 is not application in Example 2.16.

In the following, we give a theorem which is the direct consequences of Theorems 2.15, 2.17 and 2.18 in metric space.

Theorem 2.21. *Let (X, d) be a complete metric space and $T : X \to X$ be a self mapping on X. Assume that there exist $\alpha : X \times X \to \mathbb{R}$, $\varphi \in \Phi$ and $\psi \in \Psi$ such that $\varphi(r) > \psi(r)$, for $r > 0$ satisfying*

$$\alpha(x,y)\varphi\Big(d(Tx,Ty)\Big) \le \psi\Big(\mathcal{M}(x,y)\Big), \text{ for all } x, y \in X,$$

where $\mathcal{M}(x,y) = \max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\Big\}$. If
(i) T is an α-orbital admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;
(iii) $\lim_{n\to+\infty} \dfrac{\psi^n\Big(d(x_0,x_1)\Big)}{\psi^{n-1}\Big(d(x_0,x_1)\Big)} < 1$, where $x_n = T^n x_0$;
(iv) T is continuous, or $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ and $x_n \to \zeta \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, \zeta) \ge 1$, where $n_k \ge n_0 \ge 1$.
Then T possesses a fixed point $\zeta \in X$. Moreover the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to $\zeta \in X$.

Theorem 2.22. *In addition to the hypothesis of Theorem 2.21, suppose the mapping $T : X \to X$ is an α^*-orbital admissible. Then, T possesses a unique fixed point $\zeta \in X$. Moreover the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to $\zeta \in X$.*

Recall that if $(X, \preceq)$ be a partial ordered set and $T : X \to X$, we say that T is monotone non-decreasing, if $x, y \in X$, $x \preceq y$ implies $Tx \preceq Ty$.

Theorem 2.23. *Let $(X, \preceq)$ be a partial ordered set and suppose that there exists a parametric (b,θ)-metric $\mathcal{P}_\theta$ such that $(X, \mathcal{P}_\theta)$ be a complete parametric (b,θ)-metric space. Let $T : X \to X$ be a monotone non-decreasing self mapping w.r.t. $\preceq$ such that there exist $\varphi \in \Phi$ and $\psi \in \Psi$, $\varphi(r) > \psi(r)$, for $r > 0$ satisfying*

$$\varphi\Big(\mathcal{P}_\theta(Tx,Ty,t)\Big) \le \psi\Big(\mathcal{M}(x,y,t)\Big)$$

where

$$\mathcal{M}(x,y,t) = \max\Big\{\mathcal{P}_\theta(x,y,t), \mathcal{P}_\theta(x,Tx,t), \mathcal{P}_\theta(y,Ty,t), \frac{\mathcal{P}_\theta(x,Ty,t) + \mathcal{P}_\theta(y,Tx,t)}{2\theta(x,y,t)}\Big\},$$

for all $x, y \in X$ with $x \preceq y$. If
(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
(ii) $\lim_{n,m\to+\infty} \dfrac{\theta(x_n,x_m,t)\psi^n\Big(\mathcal{P}_\theta(x_0,x_1,t)\Big)}{\psi^{n-1}\Big(\mathcal{P}_\theta(x_0,x_1,t)\Big)} < 1$, where $x_n = T^n x_0$, for all $t > 0$;
(iii)(a) T is continuous, or (b) $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to \zeta$ as $x_n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq \zeta$, where $n_k \ge n_0$.

Then $Fix(T) \neq \phi$. Further, if every pair of elements $\zeta, \zeta^* \in Fix(T)$ is comparable, then $Fix(T)$ is a singleton.

Proof. Define a mapping $\alpha : X^2 \times (0, +\infty) \to [0, +\infty)$ as:

$$\alpha(x, y, t) = \begin{cases} 1, & x \preceq y \text{ or } y \preceq x; \\ 0, & \text{otherwise}, \end{cases}$$

for all $t > 0$. Then, we obtain

$$\alpha(x, y, t)\varphi\Big(\mathcal{P}_\theta(Tx, Ty, t)\Big) \leq \psi\Big(\mathcal{M}(x, y, t)\Big)$$

for all $x, y \in X$ with $x \preceq y$ and for all $t > 0$.
Since, T is monotone non-decreasing mapping w.r.t. $\preceq$, so T is a parametric α-orbital admissible. Indeed, if $x \in X$ such that $\alpha(x, Tx, t) \geq 1$, for all $t > 0$, then $x \preceq Tx$, or $Tx \preceq x$. Since, T is monotone non-decreasing mapping w.r.t $\preceq$, we have $Tx \preceq T^2x$, or $T^2x \preceq Tx$, which in turn gives $\alpha(Tx, T^2x, t) \geq 1$, for all $t > 0$.

On the other hand, from (ii) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then $\alpha(x_0, Tx_0, t) \geq 1$, for all $t > 0$.

From $(iii)(a)$ if T is continuous, then all the hypothesis of Theorem 2.15 are satisfied. Again, from $(iii)(b)$ suppose that $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to \zeta$ as $x_n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq \zeta$, $n_k \geq n_0$, which in turn gives $(x_{n_k}, \zeta, t) \geq 1$, for all $t > 0$. Thus, all the hypothesis of Theorem 2.17 are satisfied.

Consequently, T possess a fixed point in X i.e., $Fix(T) \neq \phi$. Further, Assume that every pair of elements $\zeta, \zeta^* \in Fix(T)$ are comparable, then $\zeta \preceq \zeta^*$, or $\zeta^* \preceq \zeta$ which is turn gives $\alpha(\zeta, \zeta^*, t) \geq 1$, for all $t > 0$. Therefore, T is a parametric α^*-orbital admissible. Thus all the hypothesis of Theorem 2.18 are satisfied and hence $Fix(T)$ is a singleton. ∎

3. Application

Let $X = \mathcal{C}([0, T], \mathbb{R})$ be a set of all real valued continuous functions on $[0, T]$ and define a parametric (b, θ)-metric $\mathcal{P}_\theta : X \times X \times (0, +\infty) \to [0, +\infty)$ as

$$\mathcal{P}_\theta(x, y, t) = \sup_{\sigma \in [0,T]} \Big\{ t|x(\sigma) - y(\sigma)|^2 \Big\},$$

with $\theta(x, y, t) = 2 + t(x + y)$, for all $x, y \in X$ and for all $t > 0$.

Then (X, d_θ) is a complete parametric (b, θ)-metric space. Let $\preceq$ be a partial order on X defined by $x \preceq y$ if and only if $x(\sigma) \preceq y(\sigma)$ for all $\sigma \in [0, T]$.

Consider an integral equation

$$x(r) = \eta(r) + \int_0^T K(r, s) f\Big(s, x(s)\Big) ds \tag{3.1}$$

with the following assumption that:
(H_1) : $f : [0, T] \times \mathbb{R} \to \mathbb{R}$, $\eta : [0, T] \to \mathbb{R}$, and $K : [0, T] \times [0, T] \to [0, +\infty)$

are continuous functions;
(H_2) : $\sup_{r\in[0,T]}\left(\int_0^T K^2(r,s)ds\right)^{\frac{1}{2}} < \frac{\sqrt{k}}{T}$, where $k = \frac{1}{2^2}$;
(H_3) :

$$0 \leq \Big(f(s,x(s)) - f(s,y(s))\Big) \leq \Big(\max\Big\{|x(s)-y(s)|^2, |x(s)-Tx(s)|^2, |y(s)-Ty(s)|^2, \frac{|x(s)-Ty(s)|^2 + |y(s)-Tx(s)|^2}{2\theta(x(s),y(s),t)}\Big\}\Big)^{\frac{1}{2}}$$

for all $x, y \in X$, $x \preceq y$, $s \in [0,T]$ and for all $t > 0$, where

$$Tx(r) = \eta(r) + \int_0^T K(r,s)f\Big(s,x(s)\Big)ds, \ r \in [0,T] \text{ and for all } x \in X;$$

(H_4) : there exists $x_0 \in X$ such that

$$x_0(r) \leq \eta(r) + \int_0^T K(r,s)f\Big(s,x_0(s)\Big)ds;$$

(H_5) : $\lim_{n,m\to\infty}\theta(x_n,x_m,t) < \frac{1}{k}$, where $x_n = T^n x_0$, $m > n \geq n_0 \in \mathbb{N}$ and for all $t > 0$.

We have the following theorem for the existence of solution integral equation.

Theorem 3.1. *Suppose that $(H_1) - (H_5)$ are hold. Then the integral equation (3.1) has a solution in X.*

Proof. Suppose $T : X \to X$ be a continuous mapping defined by

$$Tx(r) = \eta(r) + \int_0^T K(r,s)f\Big(s,x(s)\Big)ds, \ r \in [0,T] \text{ and for all } x \in X.$$

First we show that T is non-decreasing mapping with respect to $\preceq$. For this, let $x \preceq y$, then by (H_3), we have

$$0 \leq \Big(f(s,x(s)) - f(s,y(s))\Big), \ \text{for all } s \in [0,T].$$

Also we have

$$Ty - Tx = \int_0^T K(r,s)\Big[f\Big(s,y(s)\Big) - f\Big(s,x(s)\Big)\Big]ds \geq 0, \ \text{for all } r \in [0,T].$$

Then $Tx \preceq Ty$ i.e., T is monotone non-decreasing mapping with respect to $\preceq$. On the other hand by (H_2), (H_3) and for all $t>0$, we have

$$
\begin{aligned}
\mathcal{P}_\theta(Tx,Ty,t) = &\sup_{r\in[0,T]} t|Tx(r)-Ty(r)|^2\\
\leq &t\Big(\sup_{r\in[0,T]}\int_0^T K(r,s)\Big[f\Big(s,x(s)\Big)-f\Big(s,y(s)\Big)\Big]ds\Big)^2\\
\leq &t\sup_{r\in[0,T]}\Big[\Big(\int_0^T K^2(r,s)ds\Big)^{\frac{1}{2}}\Big(\int_0^T\Big[f\Big(s,x(s)\Big)-f\Big(s,y(s)\Big)\Big]^2ds\Big)^{\frac{1}{2}}\Big]^2\\
\leq &\frac{kt}{T^2}\max\Big\{|x-y|^2,|x-Tx|^2,|y-Ty|^2,\\
&\frac{|x-Ty|^2+|y-Tx|^2}{2\theta(x,y,t)}\Big\}\Big(\int_0^T ds\Big)^2\\
= &k\max\Big\{\mathcal{P}(x,y,t),\mathcal{P}(x,Tx,t),\mathcal{P}(y,Ty,t),\frac{\mathcal{P}(x,Ty,t)+\mathcal{P}(y,Tx,t)}{2\theta(x,y,t)}\Big\}\\
= &k\mathcal{M}(x,y,t).
\end{aligned}
$$

From (H_4), there exists $x_0\in X$ such that $x_0\in X$ such that $x_0\preceq Tx_0$. Thus all the conditions of Theorem 2.23 are satisfied setting with the condition that $\varphi(r)=r$ and $\psi(r)=kr$, where $k\in(0,1]$ and hence, the integral equation (3.1) has a solution in X. Further, the uniqueness of the solution is obtained if every pair of elements $v,v'\in Fix(T)\subseteq X$ is comparable. ∎

Open Problem 3.2. Can the condition (2.2) be replaced by a weaker condition

$$\lim_{n,m\to+\infty}\theta(x_n,x_m,t)<+\infty? \tag{3.2}$$

Conclusion 3.3. (*i*) In this paper, we used two types of control functions $\varphi\in\Phi$ and $\psi\in\Psi$ such that $\varphi(r)>\psi(r)$, $r>0$, where Φ is the set of altering distance functions and Ψ, the set of comparison functions, but $\psi\in\Psi$ is not necessarily a continuous function.
(*ii*) Taking $\mathcal{N}(x,y)=\max\Big\{d(x,y),\frac{d(x,Tx)+d(y,Ty)}{2},\frac{d(x,Ty)+d(y,Tx)}{2}\Big\}$, then $\mathcal{N}(x,y)\leq\mathcal{M}(x,y)$, for all $x,y\in X$. Indeed Theorem 2.21 improves Theorem 8 (resp. Theorem 9, 10 and 11) of Alsulami et al. [3], in which the condition of continuity in control function ψ is replaced by comparison function, and weak triangular α-admissible mapping is also replaced by an α-orbital admissible of mapping T.
(*iii*) If condition (B) [3] is added to the hypothesis of Theorem 10 (resp. Theorem 11) of Alsulami et al. [3], then fixed point of T is unique, but condition (B) is not sufficient for the uniqueness of fixed point for the mappings involved in Theorem 8 (resp. Theorem 9) of Alsulami et al. [3]. However, this drawback is overcome by using α^*-orbital admissible mapping T.

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