Almost exponential decay of Bénard convection problem without surface tension

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Abstract

We consider the dynamics of an Boussinesq approximation Bénard convection fluid evolving in a three-dimensional domain bounded below by a fixed flatten boundary and above by a free moving surface. The domain is horizontally periodic and the effect of the surface tension is neglected on the free surface. By developing a priori estimates for the model, we prove the global existence and almost exponential decay of solutions in the framework of high regularity.

Keywords: almost exponential decay; Bénard convection; high regularity.

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1 Introduction

1.1. Formulation of the problem in Eulerian problem

We consider the Bénard convection problem in a shallow horizontal layer of a fluid heated from below evolving in a moving domain

$$\Omega(t) = \{y \in \Sigma \times R| - b < y_3 < \eta(y_1, y_2, t)\}.$$

where we assume that $\Sigma = (L_1\mathbb{T})\times(L_2\mathbb{T})$ for $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $L_1, L_2 > 0$ periodicity lengths. The depth of the lower boundary $b > 0$ is assumed to be fixed constant, but the upper boundary is a free surface that is the graph of the unknown function $\eta : \Sigma \times \mathbb{R}_+ \to \mathbb{R}$. We will write $\Sigma(t) = \{y_3 = \eta(y_1, y_2, t)\}$ for the free surface of the fluid and $\Sigma_b = \{y_3 = -b\}$ for the fixed bottom surface of the fluid. Assuming the Boussinesq approximation, we obtain the basic hydrodynamic equations governing Bénard convection as follows:

$$\begin{gathered}
\partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} + \tfrac{1}{\rho_0} \nabla \tilde{p} = \mu \Delta \tilde{u} + g\alpha\tilde{\theta}\mathbf{e}_{y_3}, \quad in\ \Omega(t), \\
\mathrm{div}\tilde{u} = 0, \quad in\ \Omega(t), \\
\partial_t \tilde{\theta} + \tilde{u} \cdot \nabla \tilde{\theta} = \kappa \Delta \tilde{\theta}, \quad in\ \Omega(t),
\end{gathered}$$

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$$\tilde{u}|_{t=0} = \tilde{u}_0(y_1, y_2, y_3), \tilde{\theta}|_{t=0} = \tilde{\theta}_0(y_1, y_2, y_3),$$

Here, $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ is the velocity field of the fluid satisfying $\mathrm{div}\tilde{u} = 0$, $\tilde{p}$ the pressure, $g > 0$ the strength of gravity, $\mu > 0$ the kinematic viscosity, α the thermal expansion coefficient, e_{y_3} the unit upward vector, $\tilde{\theta}$ the temperature field of the fluid, κ the thermal diffusively coefficient, and ρ_0 is the density at the temperature T_0. Notice that, we have made the shift of actual pressure $\tilde{p} = \overline{\tilde{p}} + gy_3 - \tilde{p}_{atm}$ with the constant atmosphere pressure $\tilde{p}_{atm}$.

The boundary condition is

$$\begin{aligned} \partial_t \eta = \tilde{u}_3 - \tilde{u}_1 \partial_{y_1} \eta - \tilde{u}_2 \partial_{y_2} \eta, \quad on\ \Gamma(t), \\ (\tilde{p}I - \mu \mathbb{D}(\tilde{u}))n = g\eta n, \quad on\ \Gamma(t), \\ n \cdot \nabla \tilde{\theta} + Bi \tilde{\theta} = -1, \quad on\ \Gamma(t), \\ \tilde{u}|_{y_3=-1} = 0, \tilde{\theta}|_{y_3=-1} = 0, \quad on\ \Gamma_b, \end{aligned}$$

Here, I the 3×3 identity matrix, $\mathbb{D}(\tilde{u})_{ij} = \partial_i \tilde{u}_j + \partial_j \tilde{u}_i$ the symmetric gradient of $\tilde{u}$, $\mathcal{N}$ the upper normal vector of the free boundary $y_3 = \eta$, $n = \mathcal{N}/|\mathcal{N}|$ the unit upward vector of the free surface $y_3 = \eta$ where $\mathcal{N} = (-\partial_1\eta, -\partial_2\eta, 1)$ is the upward normal vector of the free surface $y_3 = \eta$ and $|\mathcal{N}| = \sqrt{(\partial_1\eta)^2 + (\partial_2\eta)^2 + 1}$, $Bi \geq 0$ the Biot number. Here $\mathrm{div}_\star, \nabla_\star$ denote the horizontal differential operator.(along with writing $x_\star = (x_1, x_2)$).

We will always assume the natural condition that there exists a positive number δ_0 such that $b + \eta_0 \geq \delta_0 > 0$ on $\Gamma(0)$, which means that the initial free surface is strictly separated from the bottom. And without loss of generality, we may assume that $\rho_0 = \mu = \kappa = \alpha = g = Bi = 1$, i.e., we will consider the equations

$$\begin{cases} \partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} + \nabla \tilde{p} - \Delta \tilde{u} - \tilde{\theta} e_{y_3} = 0, & in\ \Omega(t) \\ \mathrm{div}\tilde{u} = 0, & in\ \Omega(t) \\ \partial_t \tilde{\theta} + \tilde{u} \cdot \nabla \tilde{\theta} - \Delta \tilde{\theta} = 0, & in\ \Omega(t) \\ (\tilde{p}I - \mathbb{D}\tilde{u})n = \eta n, & on\ \Gamma(t), \\ \nabla \tilde{\theta} \cdot n + \tilde{\theta} = -1, & on\ \Gamma(t), \\ \partial_t \eta + \tilde{u}_1 \partial_1 \eta + \tilde{u}_2 \partial_2 \eta = \tilde{u}_3, & on\ \Gamma(t), \\ \tilde{u} = 0,\ \tilde{\theta} = 0, & on\ \Gamma_b, \\ \tilde{u}|_{t=0} = \tilde{u}_0, \tilde{\theta}|_{t=0} = \tilde{\theta}_0, \eta|_{t=0} = \eta_0, \end{cases} \tag{1.1}$$

We assume that the initial surface function η_0 satisfies the "zero average" condition

$$\frac{1}{L_1 L_2} \int_\Sigma \eta_0 = 0. \tag{1.2}$$

Notice that for sufficiently regular solutions to the periodic problem, the condition (1.2) persists in time since $\partial_t \eta = \tilde{u} \cdot \nu \sqrt{1 + |\nabla_\star \eta|^2}$:

$$\frac{d}{dt} \int_\Sigma \eta = \int_\Sigma \partial_t \eta = \int_{\Gamma(t)} \tilde{u} \cdot \nu = \int_{\Omega(t)} \mathrm{div}\tilde{u} = 0.$$

1.2. Reformulation of equations

In order to work in a fixed domain, we use a flattening transformation introduce by Beale [1], [2], also see [7],[8],[9]. We consider the fixed equilibrium domain

$$\Omega := \{x \in \Sigma \times \mathbb{R} | -b < x_3 < 0\},$$

for which we will write the coordinates as $x \in \Omega$. We will think of Σ as the upper boundary of Ω, and we will write $\Sigma_b = \{x_3 = -b\}$ for the lower boundary. We continue to view η as a function on $\Sigma \times R^+$. We then define

$$\overline{\eta} := \mathcal{P}\eta = \textit{ harmonic extension of } \eta \textit{ into the lower half space},$$

where $\mathcal{P}$ is as defined by

$$\mathcal{P}\eta(x) = \sum_{n \in (L_1^{-1}\mathbb{Z}) \times (L_2^{-1}\mathbb{Z})} e^{2\pi i n \cdot x_\star} e^{2\pi |n| x_3} \hat{\eta}(n),$$

where we have written

$$\hat{\eta}(n) = \int_\Sigma \eta(x_\star) \frac{e^{-2\pi i n \cdot x_\star}}{L_1 L_2} dx_\star,$$

The harmonic extension $\overline{\eta}$ allows us to flatten the coordinate domain via the mapping

$$\Omega \ni x \mapsto (x_1, x_2, x_3 + \overline{\eta}(x,t)(1 + x_3/b)) = \Phi(x,t) = (y_1, y_2, y_3) \in \Omega(t), \tag{1.3}$$

Note that $\Phi(\Sigma, t) = \{y_3 = \eta(y_1, y_2, t)\} = \Phi(t)$ and $\Phi(\cdot, t)|_{\Sigma_b} = Id_{\Sigma_b}$, i.e. Φ maps Σ to the free surface and keeps the lower surface fixed. We have

$$\nabla\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ A & B & J \end{pmatrix} \quad \textit{and} \quad \mathcal{A} := (\nabla\Phi^{-1})^T = \begin{pmatrix} 1 & 0 & -AK \\ 0 & 1 & -BK \\ 0 & 0 & K \end{pmatrix}$$

for

$$A = \partial_1 \overline{\eta} \widetilde{b} - (x_3 \overline{\eta} \partial_1 b)/b^2, \quad B = \partial_2 \overline{\eta} \widetilde{b} - (x_3 \overline{\eta} \partial_2 b)/b^2,$$
$$J = 1 + \overline{\eta}/b + \partial_3 \overline{\eta} \widetilde{b}, \quad K = J^{-1},$$
$$\widetilde{b} = (1 + x_3/b).$$

Here $J = det\nabla\Phi$ is the Jacobian of the coordinate transformation.

Now we define the transformed quantities as

$$u(t,x) := \tilde{u}(t, \Phi(t,x)), \quad p(t,x) := \tilde{p}(t, \Phi(t,x)), \quad \theta(t,x) := \tilde{\theta}(t, \Phi(t,x)).$$

In the new coordinates we rewrite (1.1) as

$$\begin{cases} \partial_t u - \partial_t \overline{\eta} \widetilde{b} K \partial_3 u + u \cdot \nabla_{\mathcal{A}} u + \nabla_{\mathcal{A}} p - \Delta_{\mathcal{A}} u - \theta \nabla_{\mathcal{A}} y_3 = 0, \quad \textit{in } \Omega \\ \operatorname{div}_{\mathcal{A}} u = 0, \quad \textit{in } \Omega \\ \partial_t \theta - \partial_t \overline{\eta} \widetilde{b} K \partial_3 \theta + u \cdot \nabla_{\mathcal{A}} \theta - \Delta_{\mathcal{A}} \theta = 0, \quad \textit{in } \Omega \\ (pI - \mathbb{D}_{\mathcal{A}} u)\mathcal{N} = \eta \mathcal{N}, \quad \textit{on } \Sigma, \\ \nabla_{\mathcal{A}} \theta \cdot \mathcal{N} + \theta |\mathcal{N}| = -|\mathcal{N}|, \quad \textit{on } \Sigma, \\ \partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3, \quad \textit{on } \Sigma, \\ u = 0, \ \theta = 0, \quad \textit{on } \Sigma_b, \end{cases} \tag{1.4}$$

Here we have written the differential operators $\nabla_{\mathcal{A}}$, $\mathrm{div}_{\mathcal{A}}$, and $\Delta_{\mathcal{A}}$ with their actions given by $(\nabla_{\mathcal{A}}f)_i := \mathcal{A}_{ij}\partial_j f$, $\mathrm{div}_{\mathcal{A}}X = \mathcal{A}_{ij}\partial_j X_i$, and $\Delta_{\mathcal{A}}f = \mathrm{div}_{\mathcal{A}}\nabla_{\mathcal{A}}f$ for approximate f and X; for $u\cdot\nabla_{\mathcal{A}}u$ we mean $(u\cdot\nabla_{\mathcal{A}}u)_i := u_j\mathcal{A}_{jk}\partial_k u_i$. We have also written $(\mathbb{D}_{\mathcal{A}}u)_{ij} = \sum_k(\mathcal{A}_{ik}\partial_k u_j + \mathcal{A}_{jk}\partial_k u_i)$. Also, $\mathcal{N} = (-\nabla_\star\eta, 1)$ denotes the non-unit normal on $\Gamma(t)$.

The Bénard convection problem was firstly observed from the experiments by Bénard [3]. Later on, Rayleigh [5] gives the linearized stability of the Bénard convection model in the fixed slab $\{0 < x_3 < 1\}$. For the viscous surface wave problem with surface tension case, the existence and decay of global in time solutions with free boundary surface was proved by T.Nishida, Y.Teramoto and H.Yoshihara[11]. For small Rayleigh and Marangoni numbers, T.Ioraha [10] proved the existence of exponentially decaying solutions in the class of small initial data. For these similar results can be seen in [12],[13],[14]. They all utilized the framework of [1], [2] in the Lagrangian coordinates. In T.Ioraha' result and our previous paper [6], we can see that the surface tension appears as the requirement that the solutions would be exponential decay. Now we will consider the case when the surface tension is absent, by using the flattening coordinate [7] and high regularity framework [9], we prove almost exponential decay of solutions for the Bénard convection problem.

The paper is organized as follows. In section 2 we define the energy and dissipations, we also state our main result. In section 3 we develop basic energy-dissipative estimates. In section 4 we provide the estimates for the nonlinearities. In section 5 we enhanced the estimates by the elliptic estimates. In section 6 we complete our a priori estimates and prove our main results.

In the following, some notation were introduced. When using space-time differential multi-indices, we will use $\mathbb{N}^{1+m} = \{\alpha = (\alpha_0, \alpha_1, ..., \alpha_m)\}$ to emphasize that the 0-index term is related to temporal derivatives. For just spatial derivatives we write $\mathbb{N}^m$. For $\alpha \in \mathbb{N}^{1+m}$ we write $\partial^\alpha = \partial_t^{\alpha_0}\partial_1^{\alpha_1}...\partial_m^{\alpha_m}$. We define the parabolic counting of such multi-indices by writing $|\alpha| = 2\alpha_0 + \alpha_1 + ... + \alpha_m$. We will also write $\nabla_* f$ for the horizontal gradient of f, that is $\nabla_* f = \partial_1 f e_1 + \partial_2 f e_2$, while ∇f will denote the usual full gradient.

For a given norm $\|\cdot\|$ and an integer $k \geq 0$, we introduce the following notation for sums of spatial derivatives:

$$\|\nabla_*^k f\|^2 := \sum_{\alpha\in\mathbb{N}^2,|\alpha|\leq k}\|\partial^\alpha f\|^2 \quad \mathit{and} \quad \|\nabla^k f\|^2 := \sum_{\alpha\in\mathbb{N}^3,|\alpha|\leq k}\|\partial^\alpha f\|^2 \tag{1.5}$$

The convention we adopt in this notation is that ∇_* refers to only horizontal spatial derivatives, while ∇ refers to full spatial derivatives. For space-time derivatives we ass bars to our notation:

$$\|\bar{\nabla}_*^k f\|^2 := \sum_{\alpha\in\mathbb{N}^{1+2},|\alpha|\leq k}\|\partial^\alpha f\|^2 \quad \mathit{and} \quad \|\bar{\nabla}^k f\|^2 := \sum_{\alpha\in\mathbb{N}^{1+3},|\alpha|\leq k}\|\partial^\alpha f\|^2 \tag{1.6}$$

Here the spaces H^s denote the usual L^2- based Sobolev spaces of order s. For simplicity, we will write $\|\cdot\|_s$ for $H^s(\Omega)$ norm and $\|\cdot\|_{\Sigma,s}$ for $H^s(\Sigma)$ norms.

2 Main results

In order to state our main results we first define the energy and dissipation functionals that we shall use in our analysis. We will consider energies and dissipates at both the $N+2$ and $2N$ levels. For any $N \geq 3$, we define the high energy via

$$\mathcal{E}_n := \sum_{j=0}^{n} (\|\partial_t^j u\|_{2n-2j}^2 + \|\partial_t^j \theta\|_{2n-2j}^2 + \|\partial_t^j \eta\|_{2n-2j}^2) + \sum_{j=0}^{n-1} \|\partial_t^j p\|_{2n-2j-1}^2. \tag{2.1}$$

and define the corresponding dissipation as

$$\begin{aligned} \mathcal{D}_n := \sum_{j=0}^{n} (\|\partial_t^j u\|_{2n-2j+1}^2 + \|\partial_t^j \theta\|_{2n-2j+1}^2) + \sum_{j=0}^{n-1} \|\partial_t^j p\|_{2n-2j}^2 \\ + \|\eta\|_{2n-1/2}^2 + \|\partial_t \eta\|_{2n-1/2}^2 + \sum_{j=2}^{n+1} \|\partial_t^j \eta\|_{2n-2j+5/2}^2. \end{aligned} \tag{2.2}$$

We write the high-order spatial derivatives of η as

$$\mathcal{F}_{2N} := \|\eta\|_{4N+1/2}^2. \tag{2.3}$$

and specialized term as

$$\mathcal{K} = \|\nabla u\|_{L^\infty}^2 + \|\nabla^2 u\|_{L^\infty}^2 + \sum_{i=1}^{2} \|D u_i\|_{H^2(\Sigma)}^2. \tag{2.4}$$

Finally, we define the total energy as

$$\mathcal{G}_{2N} := \sup_{0\leq r\leq t} \mathcal{E}_{2N}(r) + \int_0^t \mathcal{D}_{2N}(r) dr + \sup_{0\leq r\leq} (1+r)^{4N-8} \mathcal{E}_{N+2}(r) + \sup_{0\leq r\leq t} \frac{\mathcal{F}_{2N}(r)}{(1+r)}, \tag{2.5}$$

We now state our a priori estimates for solution to (1.4).

Theorem 2.1. *Suppose that (u, P, η, M) solves (1.4) on the temporal interval $[0, T]$. There exists a universal constant $0 < \delta_* < 1$(independent of T) such that if $\mathcal{G}_{2N}(T) \leq \delta_*$, then*

$$\mathcal{G}_{2N} \leq C(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)) \tag{2.6}$$

for all $t \in [0, T]$, where C is a universal constant.

In order to prove the existence of almost exponential decaying solutions, we couple a priori estimates with a local existence result. In [15] Zheng has been constructed local-in-time solutions of the form (1.4) without surface tension. We will simply state the result that one can prove by [15] in straightforward ways.

To state the local result we will need to define $\mathcal{H}_1 := \{u \in H^1(\Omega) | u|_{\Sigma_b} = 0\}$ and

$$\mathcal{X}_T = \{u \in L^2([0, T]; \mathcal{H}_1) | \mathrm{div}_{\mathcal{A}(t)} u(t) = 0 \ for\ a.e.\ t\}.$$

The compatibility conditions for the initial data are natural ones that would be satisfied for solutions in our framework. They are cumbersome to write, so we shall not record them here. We refer the reader to [15] for their precise definition.

Now we can state the local existence result.

Theorem 2.2. *Let $N \geq 3$ be an integer. Assume that $\eta_0 + b \geq \delta > 0$, and that the initial data (u_0, θ_0, η_0) satisfies the bounds $\|u_0\|^2_{H^{4N}} + \|\theta_0\|^2_{H^{4N}} + \|\eta_0\|^2_{H^{4N+1/2}} < \infty$ as well as the appropriate compatibility conditions. Let $\varepsilon > 0$, there exists a $\delta_0 = \delta_\varepsilon > 0$, then there exists a $\delta_0 = \delta_0(\varepsilon) > 0$ and a*

$$T_0 = C(\varepsilon)\min\{1, \frac{1}{\|\eta_0\|_{4N+1/2}}\} > 0 \tag{2.7}$$

so that if $0 < T \leq T_0$ and $\|u_0\|^2_{4N} + \|\theta_0\|^2_{4N} + \|\eta_0\|^2_{4N} \leq \delta_0$, then there exists a unique solution (u, p, θ, η) to (1.4) on the interval $[0, T]$ that achieves the initial data. The solution obeys the estimate

$$\begin{aligned} &\sup_{0\leq t\leq T} \mathcal{E}_{2N}(t) + \int_0^T \mathcal{D}_{2N}(t)dt + \int_0^T (\|\partial_t^{2N+1}u\|_{(\mathcal{X}_T)^*} + \|\partial_t^{2N+1}\theta\|_{(\mathcal{X}_T)^*} + \|\partial_t^{2N}p(t)\|_0^2)dt \\ &\leq C_1(\|u_0\|^2_{H^{4N}} + \|\theta_0\|^2_{H^{4N}} + T\|\eta_0\|^2_{H^{4N+1/2}}), \end{aligned} \tag{2.8}$$

and

$$\sum_{0\leq t\leq T} \mathcal{F}_{2N}(t) \leq C_1(\|u_0\|^2_{H^{4N}} + \|\theta_0\|^2_{H^{4N}} + (1+T)\|\eta_0\|^2_{H^{4N+1/2}}), \tag{2.9}$$

The solution is unique among functions that achieve the initial data and for which the left-hand side of is finite. Moreover, η is such that the mapping $\Phi(\cdot, t)$, defined by (1.3), is a C^1 diffeomorphism for each $t \in [0, T]$.

Coupled a priori estimates Theorem 2.1 with local existence Theorem 2.2, we may deduce a global existence and almost exponential decay result.

Theorem 2.3. *Suppose the initial data (u_0, θ_0, η_0) satisfying the compatibility conditions of Theorem 2.1, and assume that η_0 satisfy the zero average condition (1.2). Let $N \geq 3$ be an integer. There exists a $0 < \kappa = \kappa(N)$ so that if $\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) < \kappa$, then there exists a unique solution (u, p, η) on the interval $[0, \infty)$ that achieves the initial data. The solution obeys the estimate*

$$\mathcal{G}_{2N}(\infty) \leq C(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)) < C\kappa, \tag{2.10}$$

where $C > 0$ is a universal constant.

3 Energy-dissipation equations

In this section we show two forms of the energy-dissipation for solutions to (1.4). The one form is the geometric form which is ideal for estimating temporal derivatives. The other form is the perturbed linear form which is ideal for estimating horizonal spatial derivatives and for elliptic regularity.

3.1. Geometric form

In controlling the interaction between highest time derivative pressure and velocity, the perturbed linear form wound be failed. Thus, we adopt the geometric form which is a linear formulation of (1.4). We assume that u and η are given and that $\mathcal{A}, \mathcal{N}, \mathcal{J}$, etc. are given in terms of η as in (1.4).

Consider the following system for (v, q, ζ, h):

$$\begin{cases} \partial_t v - \partial_t \bar{\eta} \tilde{b} K \partial_3 v + u \cdot \nabla_{\mathcal{A}} v + \nabla_{\mathcal{A}} q - \Delta_{\mathcal{A}} v - \vartheta \nabla_{\mathcal{A}} y_3 = F^1, \quad in\ \Omega \\ \mathrm{div}_{\mathcal{A}} v = F^2, \quad in\ \Omega \\ \partial_t \vartheta - \partial_t \bar{\eta} \tilde{b} K \partial_3 \vartheta + u \cdot \nabla_{\mathcal{A}} \vartheta - \Delta_{\mathcal{A}} \vartheta = F^3, \quad in\ \Omega \\ (qI - \mathbb{D}_{\mathcal{A}} v)\mathcal{N} = \zeta \mathcal{N} + F^4, \quad on\ \Sigma, \\ \nabla_{\mathcal{A}} \vartheta \cdot \mathcal{N} + \vartheta |\mathcal{N}| = F^5, \quad on\ \Sigma, \\ \partial_t \zeta - v \cdot \mathcal{N} = F^6, \quad on\ \Sigma, \\ v = \vartheta = 0 \quad on\ \Sigma_b \end{cases} \tag{3.1}$$

We now record the energy-dissipation equality associated to solutions to (3.1)

Proposition 3.1. *Let u and η be given and solve* (1.4)*. If (v, q, ζ, ϑ) solve* (3.1) *then*

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega \frac{|v|^2}{2} J + \int_\Sigma \frac{|\zeta|^2}{2}\right) + \int_\Omega \frac{|\mathbb{D}_{\mathcal{A}} v|^2}{2} J = \int_\Omega (v \cdot F^1 + q \cdot F^2) J \\ &+ \int_\Sigma (-v \cdot F^4 + \zeta F^6) + \int_\Omega \vartheta \nabla_{\mathcal{A}} y_3 \cdot v J, \end{aligned} \tag{3.2}$$

and

$$\frac{d}{dt}\int_\Omega \frac{|\vartheta|^2}{2} J + \int_\Omega |\nabla_{\mathcal{A}} \vartheta|^2 J + \int_\Sigma |\vartheta|^2 |\mathcal{N}| = \int_\Omega \vartheta \cdot F^3 J + \int_\Sigma \vartheta \cdot F^5, \tag{3.3}$$

Proof. We take the product of the first equation in (3.1) with Jv and integrate over Ω to find that

$$I + II = III,$$

for

$$\begin{aligned} I &= \int_\Omega \partial_t v_i J v_i - \partial_t \bar{\eta} \tilde{b} \partial_3 v_i v_i + u_j \mathcal{A}_{jk} \partial_k v_i J v_i, \\ II &= \int_\Omega \mathcal{A}_{jk} \partial_k S_{ij}(v, q) J v_i, \quad III = \int_\Omega F^1 \cdot v J. \end{aligned}$$

A simple computation shows that

$$I = \frac{d}{dt}\int_\Omega \frac{|v|^2}{2} J.$$

To handle the term II we first integrate

$$\begin{aligned} II &= \int_\Omega -\mathcal{A}_{jk} S_{ij}(v, q) J \partial_k v_i + \int_\Sigma J \mathcal{A}_{j3} S_{ij}(v, q) v_i \\ &= \int_\Omega -q \mathcal{A}_{jk} \partial_k v_i J + J \frac{|\mathbb{D}_{\mathcal{A}} v|^2}{2} + \int_\Sigma S_{ij}(v, q) \mathcal{N}_j v_i \\ &= \int_\Omega -q J F^2 + J \frac{|\mathbb{D}_{\mathcal{A}} v|^2}{2} + \int_\Sigma \zeta \mathcal{N} \cdot v + F^4 \cdot v \end{aligned}$$

For the fourth equation in (3.1) we may compute

$$\int_\Sigma \zeta \mathcal{N} \cdot v = \int_\Sigma \zeta(\partial_t \zeta - F^6) = \frac{d}{dt}\int_\Sigma \frac{|\zeta|^2}{2} - \int_\Sigma \zeta F^6.$$

Similarly, multiplying the third equation in (3.1) with $J\vartheta$ and integrating over Ω, we have

$$IV + V = VI,$$

for

$$\begin{aligned}
IV &= \int_\Omega \partial_t \vartheta J \vartheta - \partial_t \bar{\eta} \tilde{b} \partial_3 \vartheta \vartheta + u_j \mathcal{A}_{jk} \partial_k \vartheta J \vartheta, \\
V &= -\int_\Omega \Delta_{\mathcal{A}} \vartheta J \vartheta, \quad VI = \int_\Omega F^3 \vartheta J.
\end{aligned}$$

A simple computation shows that

$$IV = \frac{d}{dt} \int_\Omega \frac{|\vartheta|^2}{2} J.$$

To handle the term V we first integrate

$$\begin{aligned}
V &= \int_\Omega |\nabla_{\mathcal{A}} \vartheta|^2 J - \int_\Sigma \nabla_{\mathcal{A}} \vartheta \cdot \mathcal{N} \vartheta \\
&= \int_\Omega |\nabla_{\mathcal{A}} \vartheta|^2 J + \int_\Sigma |\vartheta|^2 |\mathcal{N}| - F^5 \vartheta,
\end{aligned}$$

□

We will employ the form (3.1) to study the temporal derivative of solutions to (1.4). That is, we will employ ∂_α to (1.4) to deduce that $(v, q, \zeta, \vartheta) = (\partial_\alpha u, \partial_\alpha p, \partial_\alpha \eta, \partial_\alpha \theta)$ satisfy (3.1) for certain terms F^i for $\partial^\alpha = \alpha_t^{\alpha_0}$ with $\alpha_0 \leq 2N$. Below we record the form of these forcing terms $F^i, i = 1,2,3,4,5,6$ for this particular problem.

We have that $F^1 = \sum_{i=1}^5 F^{1,i}$, for

$$\begin{aligned}
F_i^{1,1} &:= \sum_{0<\beta<\alpha} C_{\alpha,\beta} \partial^\beta (\partial_t \bar{\eta} \tilde{b} K) \partial^{\alpha-\beta} \partial_3 u_i + \sum_{0<\beta\leq\alpha} C_{\alpha,\beta} \partial^{\alpha-\beta} \partial_t \bar{\eta} \partial^\beta (\tilde{b} K) \partial_3 u_i \\
F_i^{1,2} &:= -\sum_{0<\beta\leq\alpha} C_{\alpha,\beta} (\partial^\beta (u_j \mathcal{A}_{jk}) \partial^{\alpha-\beta} \partial_k u_i + \partial^\beta \mathcal{A}_{ik} \partial^{\alpha-\beta} \partial_k p) \\
F_i^{1,3} &:= \sum_{0<\beta\leq\alpha} C_{\alpha,\beta} \partial^\beta \mathcal{A}_{jl} \partial^{\alpha-\beta} \partial_l (\mathcal{A}_{im} \partial_m u_j + |\Updownarrow \partial_m u_i) \\
F_i^{1,4} &:= \sum_{0<\beta<\alpha} C_{\alpha,\beta} \mathcal{A}_{jk} \partial_k (\partial_\beta \mathcal{A}_{il} \partial_{\alpha-\beta} \partial_l u_j + \partial^\beta \mathcal{A}_{jl} \partial^{\alpha-\beta} \partial_l u_i) \\
F_i^{1,5} &:= \partial^\alpha \partial_t \bar{\eta} K \partial_3 u_i \quad \textit{and} \quad F_i^{1,6} := \mathcal{A}_{jk} \partial_k (\partial^\alpha \mathcal{A}_{il} \partial_l u_j + \partial^\alpha \mathcal{A}_{jl} \partial_l u_i) \\
F_i^{1,7} &:= \sum_{0<\beta\leq\alpha} C_{\alpha,\beta} \partial^{\alpha-\beta} \theta \partial^\beta (\mathcal{A}_{jl} \partial_l y_3),
\end{aligned} \tag{3.4}$$

$$F^{2,1} := -\sum_{0<\beta<\alpha} C_{\alpha,\beta} \partial^\beta \mathcal{A}_{ij} \partial^{\alpha-\beta} \partial_j u_i, \quad \textit{and} \quad F^{2,2} = -\partial^\alpha \mathcal{A}_{ij} \partial_j u_i. \tag{3.5}$$

$$\begin{aligned}
F^{3,1} &:= \sum_{0<\beta<\alpha} C_{\alpha,\beta} \partial^\beta (\partial_t \bar{\eta} \tilde{b} K) \partial^{\alpha-\beta} \partial_3 u_i + \sum_{0<\beta\leq\alpha} C_{\alpha,\beta} \partial^{\alpha-\beta} \partial_t \bar{\eta} \partial^\beta (\tilde{b} K) \partial_3 u_i \\
F^{3,2} &:= -\sum_{0<\beta\leq\alpha} C_{\alpha,\beta} \partial^\beta (u_j \mathcal{A}_{jk}) \partial^{\alpha-\beta} \partial_k \theta
\end{aligned}$$

$$
\begin{aligned}
F^{3,3} &:= \sum_{0<\beta\le\alpha} C_{\alpha,\beta}\partial^\beta\mathcal{A}_{jl}\partial^{\alpha-\beta}\partial_l\mathcal{A}_{jm}\partial_m\theta \\
F^{3,4} &:= \sum_{0<\beta<\alpha} C_{\alpha,\beta}\mathcal{A}_{jk}\partial_k\partial_\beta\mathcal{A}_{jl}\partial_{\alpha-\beta}\partial_l\theta \\
F^{3,5} &:= \partial^\alpha\partial_t\eta\bar{b}K\partial_3\theta \quad and \quad F^{3,6} := \mathcal{A}_{jk}\partial_k\partial^\alpha\mathcal{A}_{jl}\partial_l\theta.
\end{aligned} \tag{3.6}
$$

$F^4 = F^{4,1} + F^{4,2}$, where for $i = 1, 2, 3$ we have

$$
\begin{aligned}
F_i^{4,1} &:= -(\sum_{0<\beta\le\alpha} C_{\alpha,\beta}\partial^\beta D\eta(\partial^{\alpha-\beta}\eta - \partial^{\alpha-\beta}p) \\
F_i^{4,2} &:= \sum_{0<\beta\le\alpha} C_{\alpha,\beta}(\partial^\beta(\mathcal{N}_j\mathcal{A}_{im})\partial^{\alpha-\beta}\partial_m u_j + \partial^\beta(\mathcal{N}_j\mathcal{A}_{jm})\partial_{\alpha-\beta}\partial_m u_i),
\end{aligned} \tag{3.7}
$$

$$
F^5 := -\sum_{0<\beta\le\alpha} C_{\alpha,\beta}[\partial^\beta(\mathcal{A}_{jk}\cdot\mathcal{N}_j)\partial_k\partial^{\alpha-\beta}\theta - \partial^{\alpha-\beta}\theta\partial^\beta|\mathcal{N}|] - \partial^\alpha|\mathcal{N}|, \tag{3.8}
$$

$$
F^6 := -\sum_{0<\beta\le\alpha} C_{\alpha,\beta}\partial^\beta D\eta\cdot\partial^{\alpha-\beta}u. \tag{3.9}
$$

3.2. Perturbed Linear form

Next, we consider an alternate way of linearizing (1.4) that eliminates the $\mathcal{A}$ coefficients in favor for constant coefficients. This is advantageous for applying elliptic regularity results and is the context in which we will derive estimates horizontal spatial derivatives. We may rewrite (1.4) as

$$
\begin{cases}
\partial_t u + \nabla p - \Delta u - \theta e_3 = G^1, & in\ \Omega \\
\mathrm{div} u = G^2, & in\ \Omega \\
\partial_t\theta - \Delta\theta = G^3, & in\ \Omega \\
(pI - \mathbb{D}u - \eta I)e_3 = G^4, & on\ \Sigma, \\
\nabla\theta\cdot e_3 + \theta = G^5, & on\ \Sigma, \\
\partial_t\eta - u_3 = G^6, & on\ \Sigma,
\end{cases} \tag{3.10}
$$

Here we have written the nonlinear terms G^i for $i = 1, ..., 5$ as follows. We write $G^1 := G^{1,1} + G^{1,2} + G^{1,3} + G^{1,4} + G^{1,5} + G^{1,6}$, for

$$
\begin{aligned}
G_i^{1,1} &:= (\delta_{ij} - \mathcal{A}_{ij})\partial_j p \\
G_i^{1,2} &:= u_j\mathcal{A}_{jk}\partial_k u_i, \\
G_i^{1,3} &:= [K^2(1 + A^2 + B^2) - 1]\partial_{33}u_i - 2AK\partial_{13}u_i - 2BK\partial_{23}u_i, \\
G_i^{1,4} &:= [-K^3(1 + A^2 + B^2)\partial_3 J + AK^2(\partial_1 J + \partial_3 A) + BK^2(\partial_2 J + \partial_3 B)]\partial_3 u_i, \\
G_i^{1,5} &:= \partial_t\overline{\eta}(1 + x_3/b)K\partial_3 u_i, \\
G_i^{1,6} &:= \theta\nabla_{\mathcal{A}}y_3 - \theta e_3,
\end{aligned} \tag{3.11}
$$

$$
G^2 := AK\partial_3 u_1 + BK\partial_3 u_2 + (1 - K)\partial_3 u_3, \tag{3.12}
$$

$G^3 = G^{3,1} + G^{3,2} + G^{3,3}$, for

$$
G_i^{3,1} := u_j\mathcal{A}_{jk}\partial_k\theta,
$$

$$\begin{aligned} G_i^{3,2} &:= [K^2(1+A^2+B^2)-1]\partial_{33}\theta - 2AK\partial_{13}u_i - 2BK\partial_{23}\theta, \\ G_i^{3,3} &:= [-K^3(1+A^2+B^2)\partial_3 J + AK^2(\partial_1 J + \partial_3 A) + BK^2(\partial_2 J + \partial_3 B)]\partial_3\theta, \end{aligned} \tag{3.13}$$

$$\begin{aligned} G^4 := \partial_1\eta & \begin{pmatrix} p-\eta-2(\partial_1 u_1 - AK\partial_3 u_1) \\ -\partial_2 u_1 - \partial_1 u_2 + BK\partial_3 u_1 + AK\partial_3 u_2 \\ -\partial_1 u_3 - K\partial_3 u_1 + AK\partial_3 u_3 \end{pmatrix} \\ +\partial_2\eta & \begin{pmatrix} -\partial_2 u_1 - \partial_1 u_2 + BK\partial_3 u_1 + AK\partial_3 u_2 \\ p-\eta-2(\partial_2 u_2 - BK\partial_3 u_2) \\ -\partial_2 u_3 - K\partial_3 u_2 + BK\partial_3 u_3 \end{pmatrix} + \begin{pmatrix} (K-1)\partial_3 u_1 + AK\partial_3 u_3 \\ (K-1)\partial_3 u_2 + BK\partial_3 u_3 \\ 2(K-1)\partial_3 u_3 \end{pmatrix}, \end{aligned} \tag{3.14}$$

$$G^5 := -|\mathcal{N}| - (\nabla_{\mathcal{A}}\theta\cdot\mathcal{N} + \theta|\mathcal{N}|) + \nabla\theta\cdot e_3 + \theta, \tag{3.15}$$

$$G^6 := D\eta\cdot u, \tag{3.16}$$

Next we consider the energy-dissipation evolution equation for solutions to problem of the form (3.10).

Proposition 3.2. *Suppose* (v,q,ζ,ϑ) *solve*

$$\begin{cases} \partial_t v + \nabla q - \Delta v - \vartheta e_3 = \Phi^1, & in\ \Omega \\ \mathrm{div} v = \Phi^2, & in\ \Omega \\ \partial_t\vartheta - \Delta\vartheta = \Phi^3, & in\ \Omega \\ (qI - \mathbb{D}v - \zeta I)e_3 = \Phi^4, & on\ \Sigma, \\ \nabla\vartheta\cdot e_3 + \vartheta = \Phi^5, & on\ \Sigma, \\ \partial_t\zeta - v_3 = \Phi^6, & on\ \Sigma, \\ v = \theta = 0, & on\ \Sigma_b, \end{cases} \tag{3.17}$$

Then

$$\begin{aligned} &\frac{d}{dt}\Big(\int_\Omega \frac{|v|^2}{2} + \int_\Sigma \frac{|\zeta|^2}{2}\Big) + \int_\Omega \frac{|Dv|^2}{2} \\ &= \int_\Omega v\cdot\Phi^1 + q\cdot\Phi^2 + \int_\Sigma(-v\cdot\Phi^4 + \zeta\Phi^6) + \int_\Omega \vartheta e_3\cdot v, \end{aligned} \tag{3.18}$$

and

$$\frac{d}{dt}\int_\Omega \frac{|\vartheta|^2}{2} + \int_\Omega |\nabla\vartheta|^2 + \int_\Sigma |\vartheta|^2 = \int_\Omega \vartheta\cdot\Phi^3 + \int_\Sigma \vartheta\cdot\Phi^5, \tag{3.19}$$

Proof. From the first equation in (3.17) we compute

$$\partial_t v_i + \partial_i q - \Delta v_i - \vartheta e_3 - \partial_i\Phi^2 = \Phi_i^1 - \partial_i\Phi^2,$$

By the usual energy estimates we may compute

$$\frac{d}{dt}\int_\Omega \frac{|v|^2}{2} + \int_\Omega \frac{|Dv|^2}{2} + \underbrace{\int_\Sigma v_3\zeta}_{I} = \int_\Omega v\cdot\Phi^1 + q\Phi^2 - v\cdot\nabla\Phi^4 + \int_\Omega \vartheta e_3\cdot v,$$

We compute I by integrating by parts and using (3.17):

$$I = \int_\Sigma \zeta\partial_t\zeta - \zeta\Phi^6 = \frac{d}{dt}\int_\Sigma \frac{|\zeta|^2}{2} - \int_\Sigma \zeta\Phi^6,$$

Similarly, from the third equation in (3.17) and usual energy estimates, we compute

$$\frac{d}{dt}\int_\Omega |\vartheta|^2 + \int_\Omega |\nabla\vartheta|^2 + \int_\Sigma |\vartheta|^2 = \int_\Omega \Phi^3\vartheta + \int_\Sigma \vartheta\Phi^5,$$

□

4 Estimates of the nonlinearities

In this section, we record estimates for the nonlinearities that appear in (3.1) and (3.10). Throughout this section we will repeatedly use the estimates of Lemmas B.1 and B.2 in [16] to estimates $\bar{\eta}$, as well as Lemma B.3 in [16] to estimate various nonlinearities. Before doing these, we firstly give a lemma for moving the appearance of J and $\mathcal{A}$ factors.

4.1. Useful L^∞ estimates

Lemma 4.1. *There exists a universal $0 < \delta < 1$ so that if $\|\eta\|_{5/2}^2 \le \delta$, then the following hold.*
(1) We have the estimate

$$\|J-1\|_{L^\infty}^2 + \|A\|_{L^\infty}^2 + \|B\|_{L^\infty}^2 \le \frac{1}{2}, \quad \text{and} \quad \|K\|_{L^\infty}^2 + \|\mathcal{A}\|_{L^\infty}^2 \lesssim 1, \tag{4.1}$$

(2) The map Θ defined by (1.3) is a diffeomorphism.
(3) There exists a universal constant $C > 0$ such that for all $v \in H^1(\Omega)$ such that $v = 0$ on Σ_b we have

$$\int_\Omega |Dv|^2 \le \int_\Omega J|D_{\mathcal{A}}v|^2 + C\sqrt{\mathcal{E}}\int_\Omega |Dv|^2, \tag{4.2}$$

Proof. The proof of this lemma can be founded in [9] □

4.2. Nonlinearities in (3.1)

Our goal now is to estimate the nonlinear terms F^i for $i = 1, ..., 6$, as defined in (3.6)-(3.9). These estimates will be used principally to estimates the interaction terms on the right side of (3.2) and (3.3).

Theorem 4.2. *Let $\partial^\alpha = \partial_t^{\alpha_0}$ and let $F^1, ..., F^6$ de defined in* (3.6)-(3.9). *Then the following estimates hold. For $0 \le \alpha_0 \le 2N$, we have*

$$\|F^1\|_0^2 + \|\partial_t(JF^2)\|_0^2 + \|F^3\|_0^2 + \|F^4\|_0^2 + \|F^5\|_0^2 + \|F^6\|_0^2 \lesssim \mathcal{E}_{2N}\mathcal{D}_{2N}, \tag{4.3}$$

and

$$\|F^2\|_0^2 \le \mathcal{E}_{2N}^2, \tag{4.4}$$

For $0 \le \alpha_0 \le N+2$*, we have*

$$\|F^1\|_0^2 + \|\partial_t(JF^2)\|_0^2 + \|F^3\|_0^2 + \|F^4\|_0^2 + \|F^5\|_0^2 + +\|F^6\|_0^2 \lesssim \mathcal{E}_{2N}\mathcal{D}_{N+2}, \tag{4.5}$$

Also,

$$\|F^2\|_0^2 \le \mathcal{E}_{2N}\mathcal{E}_{N+2}, \tag{4.6}$$

Proof. The proof is similar to the estimates in Theorem 4.1-4.2 of [15].

□

4.3. Nonlinearities in (3.10)

Now we turn our attention to the nonlinear terms G^i for $i = 1, ..., 6$, as defined in (3.11)-(3.16).

Theorem 4.3. *Let* $G^1, ..., G^6$ *de defined in* (3.11)-(3.16)*. Then, the following estimates hold:*

$$\begin{aligned} &\|\bar\nabla^{4N-2}G^1\|_0^2 + \|\bar\nabla^{4N-2}G^2\|_0^2 + \|\bar\nabla^{4N-2}G^3\|_0^2 + \|\bar\nabla_*^{4N-2}G^4\|_{1/2}^2 \\ &+ \|\bar\nabla_*^{4N-2}G^5\|_{1/2}^2 + \|\bar\nabla_*^{4N-2}G^6\|_{1/2}^2 \lesssim \mathcal{E}_{2N}^{1+\theta}, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &\|\bar\nabla^{4N-2}G^1\|_0^2 + \|\bar\nabla^{4N-2}G^2\|_0^2 + \|\bar\nabla^{4N-2}G^3\|_0^2 + \|\bar\nabla_*^{4N-2}G^4\|_{1/2}^2 + \|\bar\nabla^{4N-2}G^5\|_{1/2}^2 \\ &+ \|\bar\nabla^{4N-2}G^6\|_{1/2}^2 + \|\bar\nabla^{4N-3}\partial_t G^1\|_0^2 + \|\bar\nabla^{4N-2}\partial_t G^2\|_0^2 + \|\bar\nabla^{4N-3}\partial_t G^3\|_0^2 \\ &+ \|\bar\nabla_*^{4N-3}\partial_t G^4\|_{1/2}^2 + \|\bar\nabla_*^{4N-3}\partial_t G^5\|_{1/2}^2 + \|\bar\nabla^{4N-2}\partial_t G^6\|_{1/2}^2 \lesssim \mathcal{E}_{2N}^{\theta}\mathcal{D}_{2N}, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} &\|\bar\nabla^{4N-1}G^1\|_0^2 + \|\bar\nabla^{4N-1}G^2\|_0^2 + \|\bar\nabla^{4N-1}G^3\|_0^2 + \|\bar\nabla_1^{4N-2}G^4\|_{1/2}^2 \\ &+ \|\bar\nabla^{4N-1}G^5\|_{1/2}^2 \lesssim \mathcal{E}^{\theta}\mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &\|\bar\nabla^{2(N+2)-2}G^1\|_0^2 + \|\bar\nabla^{2(N+2)-2}G^2\|_0^2 + \|\bar\nabla^{2(N+2)-2}G^3\|_0^2 + \|\bar\nabla_*^{2(N+2)-2}G^4\|_{1/2}^2 \\ &+ \|\bar\nabla_*^{2(N+2)-2}G^5\|_{1/2}^2 + \|\bar\nabla_*^{2(N+2)-2}G^6\|_{1/2}^2 \lesssim \mathcal{E}_{2N}^{\theta}\mathcal{E}_{N+2}, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} &\|\bar\nabla^{2(N+2)-1}G^1\|_0^2 + \|\bar\nabla^{2(N+2)-1}G^2\|_0^2 + \|\bar\nabla^{2(N+2)-1}G^3\|_0^2 + \|\bar\nabla_*^{2(N+2)-1}G^4\|_{1/2}^2 \\ &+ \|\bar\nabla_*^{2(N+2)-1}G^5\|_{1/2}^2 + \|\bar\nabla_*^{2(N+2)-1}G^5\|_{1/2}^2 + \|\bar\nabla_*^{2(N+2)-2}\partial_t G^6\|_{1/2}^2 \lesssim \mathcal{E}_{2N}^{\theta}\mathcal{D}_{N+2}, \end{aligned} \tag{4.11}$$

Proof. Here the term $G^1, ..., G^6$ terms are estimated similar as [16], so we omit it. □

5 A priori estimates

In this section we combine energy-dissipation estimates with various elliptic estimates and estimate the nonlinearities in order to deduce a system of a priori estimates.

5.1. Energy-dissipation estimates

In order to state our energy-dissipation estimates. First, we define the energy and dissipation involving only temporal derivatives by

$$\bar{\mathcal{E}}_n^0 := \sum_{j=0}^{n} (\|\sqrt{J}\partial_t^j u\|_0^2 + \|\sqrt{J}\partial_t^j \theta\|_0^2 + \|\partial_t^j \eta\|_0^2) \ \ and \ \ \bar{\mathcal{D}}_n^0 := \sum_{j=0}^{n} (\|\mathbb{D}\partial_t^j u\|_0^2 + \|\nabla\partial_t^j \theta\|_0^2). \tag{5.1}$$

Then define the horizontal energies and dissipations

$$\begin{aligned}\bar{\mathcal{E}}_n^+ :=& \|\bar{\nabla}_*^{2n-1} u\|_0^2 + \|\nabla_*\bar{\nabla}_*^{2n-1} u\|_0^2 + \|\bar{\nabla}_*^{2n-1} M\|_0^2 \\ &+ \|\nabla_*\bar{\nabla}_*^{2n-1} M\|_0^2 + \|\bar{\nabla}_*^{2n-1}\eta\|_0^2 + \|\nabla_*\bar{\nabla}_*^{2n-1}\eta\|_0^2,\end{aligned} \tag{5.2}$$

and

$$\bar{\mathcal{D}}_n^+ := \|\bar{\nabla}_*^{2n-1}\mathbb{D}(u)\|_0^2 + \|\nabla_*\bar{\nabla}_*^{2n-1}\mathbb{D}(u)\|_0^2 + \|\bar{\nabla}_*^{2n-1}\nabla\theta\|_0^2 + \|\nabla_*\bar{\nabla}_*^{2n-1}\nabla\theta\|_0^2 \tag{5.3}$$

We will also need to use the functional

$$\mathcal{H} := \int_\Omega \partial_t^{N+1} p F^2 J, \tag{5.4}$$

and

$$\bar{\mathcal{E}}_n = \bar{\mathcal{E}}_n^0 + \bar{\mathcal{E}}_n^+ \qquad and \qquad \bar{\mathcal{D}}_n = \bar{\mathcal{D}}_n^0 + \bar{\mathcal{D}}_n^+. \tag{5.5}$$

First, we present the temporal derivatives for solutions.

Theorem 5.1. *In the case $0 \le \alpha_0 \le 2N$, There exist a $\iota > 0$ so that*

$$\overline{\mathcal{E}}_{2N}^0(t) + \int_0^t \overline{\mathcal{D}}_{2N}^0 \lesssim \mathcal{E}_{2N}(0) + (\mathcal{E}_{2N}(t))^{3/2} + \int_0^t (\mathcal{E}_{2N})^\iota \mathcal{D}_{2N}. \tag{5.6}$$

In the case $0 \le \alpha_0 \le N+2$, we have

$$\frac{d}{dt}(\bar{\mathcal{E}}_{N+2}^0 - 2\mathcal{H}) + \bar{\mathcal{D}}_{N+2}^0 \lesssim \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{N+2}, \tag{5.7}$$

Proof. We write $\mathbb{N} = \{0, 1, 2, ...\}$ for the collection of non-negative integers. When using space-time differential multi-indices, we will write $\mathbb{N}^{1+m} = \{\alpha = (\alpha_0, \alpha_1, ..., \alpha_m)\}$ to emphasize that the 0-index term is related to temporal derivatives. For just spatial derivatives, we write $\mathbb{N}^m$. For $\alpha \in \mathbb{N}^{1+m}$, we write $\partial^\alpha = \partial_t^{\alpha_0}\partial_1^{\alpha_1}...\partial_m^{\alpha_m}$. We apply ∂^α to (1.4) to derive an equation for $(\partial^\alpha u, \partial^\alpha p, \partial^\alpha \eta, \partial^\alpha \theta)$. We will consider the form of this equation in different ways depending on α.

Suppose that $\partial^\alpha = \partial_t^{\alpha_0}$ with $0 \le \alpha_0 \le 2N$. Then $v = \partial_t^{\alpha_0} u, q = \partial_t^{\alpha_0} p, \zeta = \partial_t^{\alpha_0}\eta, \vartheta = \partial_t^{\alpha_0}\theta$ satisfy (3.1) with $F^1, ..., F^6$ as given in (3.6)-(3.9). According to Proposition 3.1 we then have that

$$\begin{aligned}&\frac{d}{dt}\Big(\int_\Omega \frac{|\partial_t^{\alpha_0} u|^2}{2} J + \int_\Sigma \frac{|\partial_t^{\alpha_0}\eta|^2}{2} + \int_\Omega \frac{|\partial_t^{\alpha_0}\theta|^2}{2} J\Big) + \int_\Omega \frac{|\mathbb{D}_{\mathcal{A}}\partial_t^{\alpha_0} u|^2}{2} J \\ &+ \int_\Omega |\nabla_{\mathcal{A}}\partial_t^{\alpha_0}\theta|^2 J + \int_\Sigma |\partial_t^{\alpha_0}\theta|^2 |\mathcal{N}| = \int_\Omega (\partial_t^{\alpha_0} u \cdot F^1 + \partial_t^{\alpha_0} p F^2 + \partial_t^{\alpha_0}\theta \cdot F^3) J \\ &+ \int_\Sigma (-\partial_t^{\alpha_0} u \cdot F^4 + \partial_t^{\alpha_0}\theta F^5 + \partial_t^{\alpha_0}\eta F^6) + \int_\Omega \partial_t^{\alpha_0}\theta \nabla_{\mathcal{A}} y_3 \cdot \partial_t^{\alpha_0} u J,\end{aligned} \tag{5.8}$$

We now estimate the in the right side hand of (5.8). We divide into two cases.

In the case $0 \le \alpha_0 \le 2N$, for the pressure term in the right side hand of (5.8), we then write

$$\int_0^t \partial_t^{2N} pJF^2 := -\int_0^t \int_\Omega \partial_t^{2N-1} p\partial_t(JF^2) + \int_\Omega (\partial_t^{2N-1} pJF^2)(t)$$
$$-\int_\Omega (\partial_t^{2N-1} pJF^2)(0).$$

It is easy to verify that

$$\int_\Omega (\partial_t^{2N-1} pF^2 J)(t) - \int_\Omega (\partial_t^{2N-1} pF^2 J)(0) \lesssim \mathcal{E}_{2N}(0) + (\mathcal{E}_{2N}(t))^{3/2},$$

Hence,

$$\int_0^T \int_\Omega \partial_t^{2N} pF^2 J \lesssim \mathcal{E}_{2N}(0) + (\mathcal{E}_{2N}(t))^{3/2} + \int_0^T \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}, \tag{5.9}$$

Next, we estimate all of the remaining terms on the right side of (5.7). In dealing with these terms, acoording to the estimates (4.3)-(4.4) in Theorem 4.2 we may bound

$$\int_\Omega (\partial_t^{\alpha_0} u \cdot F^1 + \partial_t^{\alpha_0} \theta \cdot F^3) J + \int_\Sigma (-\partial_t^{\alpha_0} u \cdot F^4 + \partial_t^{\alpha_0} \theta F^5 + \partial_t^{\alpha_0} \eta F^6)$$
$$+ \int_\Omega \theta \nabla_{\mathcal{A}} y_3 \cdot \partial_t^{\alpha_0} u J \lesssim \int_0^T \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}, \tag{5.10}$$

Furthermore, by lemma 4.1 we can deduce that

$$\int_\Omega \frac{|\mathbb{D}\partial_t^{\alpha_0} u|^2}{2} J \lesssim \int_\Omega \frac{|\mathbb{D}_{\mathcal{A}}\partial_t^{\alpha_0} u|^2}{2} J + \int_0^T \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}, \tag{5.11}$$

Therefore we complete the first claim of the Theorem 5.1.

On the other hand, in the case $0 \le \alpha_0 \le N+2$, as for the pressure term, we have the estimate

$$\int_\Omega \partial_t^{N+2} pF^2 J = \frac{d}{dt} \int_\Omega \partial_t^{N+1} pF^2 J - \int_\Omega \partial_t^{N+1} p\partial_t(F^2 J),$$

hence,

$$\int_\Omega \partial_t^{N+2} pF^2 J = \frac{d}{dt} \int_\Omega \partial_t^{N+1} pF^2 J + \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{N+2}, \tag{5.12}$$

Similarly, we may argue as in to show that we may bound

$$\int_\Omega (\partial_t^{\alpha_0} u \cdot F^1 + \partial_t^{\alpha_0} \theta \cdot F^3) J + \int_\Sigma (-\partial_t^{\alpha_0} u \cdot F^4 + \partial_t^{\alpha_0} \theta F^5 + \partial_t^{\alpha_0} \eta F^6)$$
$$+ \int_\Omega \partial_t^{\alpha_0} \theta \nabla_{\mathcal{A}} y_3 \cdot \partial_t^{\alpha_0} u J \lesssim \int_0^T \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{N+2}, \tag{5.13}$$

and

$$\int_\Omega \frac{|\mathbb{D}\partial_t^{\alpha_0} u|^2}{2} J \lesssim \int_\Omega \frac{|\mathbb{D}_{\mathcal{A}}\partial_t^{\alpha_0} u|^2}{2} J + \int_0^T \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{N+2}, \tag{5.14}$$

Then the second claim is completed. Thus the theorem 5.1 follows.

□

Theorem 5.2. *Let $\alpha \in \mathbb{N}^{1,2}$. In the case $0 \le \alpha_0 \le 2N-1$ and $|\alpha| \le 4N$, There exist a $\iota > 0$ so that*

$$\bar{\mathcal{E}}_{2N}^{+}(t) + \int_0^t \bar{\mathcal{D}}_{2N}^{+} \lesssim \bar{\mathcal{E}}_{2N}^{+}(0) + \int_0^t (\mathcal{E}_{2N})^{\iota}\mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}} \\ + \varepsilon\mathcal{D}_{2N} + \varepsilon^{-8N-1}\bar{\mathcal{D}}_{2N}^{0}. \tag{5.15}$$

In the case $0 \le \alpha_0 \le N+1$ and $|\alpha| \le 2(N+2)$, we have

$$\frac{d}{dt}\bar{\mathcal{E}}_{N+2}^{+} + \bar{\mathcal{D}}_{N+2}^{+} \lesssim \mathcal{E}_{2N}^{\iota}\mathcal{D}_{N+2} + \varepsilon\mathcal{D}_{N+2} + \varepsilon^{-4N-9}\bar{\mathcal{D}}_{N+2}^{0}. \tag{5.16}$$

Proof. Now we view (u, p, η, θ) in terms of (3.10), which then means that $(v, q, \zeta, \vartheta) = (\partial^\alpha u, \partial^\alpha p, \partial^\alpha \eta, \partial^\alpha \theta)$ satisfy (3.17) with $\Phi^i = \partial^\alpha G^i$ for $i = 1, ..., 5$, where the nonlinearities G^i are as defined in (3.11)-(3.16), then apply Proposition 3.2 to see that

$$\frac{d}{dt}\Big(\int_\Omega \frac{|\partial^\alpha u|^2}{2} + \int_\Sigma \frac{|\partial^\alpha \eta|^2}{2} + \int_\Omega \frac{|\partial^\alpha \theta|^2}{2}\Big) + \int_\Omega \frac{|\mathbb{D}\partial^\alpha u|^2}{2} + \int_\Omega |\nabla \partial^\alpha \theta|^2 + \int_\Sigma |\partial^\alpha \theta|^2 \\ = \int_\Omega (\partial^\alpha u \cdot \partial^\alpha G^1 + \partial^\alpha p \partial^\alpha G^2 + \partial^\alpha \theta \cdot \partial^\alpha G^3 + \partial^\alpha \theta e_3 \cdot \partial^\alpha u) \\ + \int_\Sigma (-\partial^\alpha u \cdot \partial^\alpha G^4 + \partial^\alpha \theta \cdot \partial^\alpha G^5) + \partial^\alpha \eta \partial^\alpha G^6, \tag{5.17}$$

We will estimate the right hand side of (5.17). In the case $0 \le \alpha_0 \le 2N-1$ and $|\alpha| \le 4N$. Assume initially that $|\alpha| \le 4N-1$, then by the estimate (4.7)-(4.11) in Theorem 4.3 we have

$$\int_\Omega (\partial^\alpha u \cdot \partial^\alpha G^1 + \partial^\alpha p \partial^\alpha G^2 + \partial^\alpha \theta \cdot \partial^\alpha G^3) \\ \lesssim \|\partial^\alpha u\|_0 \|\partial^\alpha G^1\|_0 + \|\partial^\alpha p\|_0 \|\partial^\alpha G^2\|_0 + \|\partial^\alpha \theta\|_0 \|\partial^\alpha G^3\|_0 \\ + \|\partial^\alpha \theta\|_0 \|\partial^\alpha u\|_0 \lesssim \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}^{\iota}\mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}}, \tag{5.18}$$

and

$$\int_\Omega \partial^\alpha \theta e_3 \cdot \partial^\alpha u \lesssim \|\partial^\alpha \theta\|_0 \|\partial^\alpha u\|_0 \\ \lesssim \|\nabla_0^{4N-2\alpha_0} \partial_t^{\alpha_0} u\|_0 \sqrt{\mathcal{D}_{2N}} \lesssim \sqrt{\mathcal{D}_{2N}} \|\partial_t^{\alpha_0} u\|_{4N-2\alpha_0} \tag{5.19}$$

We estimate the $4N - 2\alpha_0$ norm with standard Sobolev interpolation:

$$\|\partial_t^{\alpha_0} u\|_{4N-2\alpha_0} \lesssim \|\partial_t^{\alpha_0} u\|_0^{\chi} \|\partial_t^{\alpha_0} u\|_{4N-2\alpha_0+1}^{1-\chi} \lesssim (\bar{\mathcal{D}}_{2N}^{0})^{\chi/2} (\mathcal{D}_{2N})^{(1-\chi)/2}, \tag{5.20}$$

where $\chi = (4N - 2\alpha_0 + 1)^{-1} \in (0, 1)$. Then Young's inequality allows us to further bound

$$\sqrt{\mathcal{D}_{2N}} \|\partial_t^{\alpha_0} u\|_{4N-2\alpha_0} \lesssim \sqrt{\mathcal{D}_{2N}} (\bar{\mathcal{D}}_{2N}^{0})^{\chi/2} (\mathcal{D}_{2N})^{(1-\chi)/2} \\ \lesssim \varepsilon(1 - \frac{\chi}{2})\mathcal{D}_{2N} + \frac{\chi}{2}\varepsilon^{(\chi-2)/\chi}\bar{\mathcal{D}}_{2N}^{0} \lesssim \varepsilon\mathcal{D}_{2N} + \varepsilon^{-8N-1}\bar{\mathcal{D}}_{2N}^{0}. \tag{5.21}$$

where in the last inequality we have used the fact that $(2-\chi)/\chi = 8N - 4\alpha_0 + 1$ to find the largest power of $1/\varepsilon$ when $0 \le \alpha_0 \le 2N$.

Again by the estimate (4.7)-(4.11) in Theorem 4.3, together with the trace theorem, we have

$$\int_\Sigma (-\partial^\alpha u \cdot \partial^\alpha G^4 + \partial^\alpha \theta \cdot \partial^\alpha G^5) + \partial^\alpha \eta \partial^\alpha G^6 \\ \lesssim \|\partial^\alpha u\|_{\Sigma,0} \|\partial^\alpha G^4\|_0 + \|\partial^\alpha \theta\|_{\Sigma,0} \|\partial^\alpha G^5\|_0 \\ + \|\partial^\alpha \eta\|_0 \|\partial^\alpha G^6\|_0 \lesssim \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}^{\iota}\mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}}, \tag{5.22}$$

Now we assume that $|\alpha| = 4N$. Since $\alpha_0 \le 2N-1$, we have $\alpha_1+\alpha_2 \ge 2$, then we can integrate by parts on the horizontal directions. We write $\partial^\alpha = \partial^\beta \partial^\gamma$ so that $|\gamma| = 4N-1$. So by integrating by parts in the G^1 terms in (5.17) and using the estimates the estimate (4.7)-(4.11) in Theorem 4.3, we obtain

$$\begin{aligned} RHS\ of\ (5.10) &= \int_\Omega (-\partial^{\alpha+\beta} u \cdot \partial^\gamma G^1 + \partial^\alpha p \partial^\alpha G^2 + \partial^{\alpha+\beta}\theta \cdot \partial^\omega G^3 + \partial^\alpha \theta e_3 \cdot \partial^\alpha u) \\ &+ \int_\Sigma (-\partial^\alpha u \cdot \partial^\alpha G^4 + \partial^\alpha \theta \cdot \partial^\alpha G^5) \lesssim \|\partial^{\alpha+\beta} u\|_0 \|\partial^\gamma G^1\|_0 + \|\partial^\alpha p\|_0 \|\partial^\alpha G^2\|_0 \\ &+ \|\partial^\alpha \theta\|_0 \|\partial^\alpha G^3\|_0 + \|\partial^\alpha \theta\|_0 \|\partial^\alpha u\|_0 + \|\partial^\alpha u\|_{\Sigma,0} \|\partial^\alpha G^4\|_0 + \|\partial^\alpha \theta\|_0 \|\partial^\alpha G^5\|_0 \\ &\lesssim \sqrt{\mathcal{D}_{2N}} \sqrt{\mathcal{E}^\iota \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}}, \end{aligned} \tag{5.23}$$

For the G^6 term we split into two cases: $\alpha_0 \ge 1$ and $\alpha_0 = 0$. In the former case, we have $\|\partial^\alpha \eta\|_{1/2} \le \sqrt{\mathcal{D}_{2N}}$, and hence

$$\begin{aligned} &\int_\Sigma \partial^\alpha \eta \partial^\alpha G^6 \lesssim |\int_\Sigma \partial^{\alpha+\beta}\eta \partial^{\alpha-\beta} G^6| \lesssim \|\partial^{\alpha+\beta}\eta\|_{-1/2} \|\partial^{\alpha-\beta} G^6\|_{1/2} \\ &\lesssim \|\partial^{\alpha+\beta}\eta\|_{-1/2} \|\partial^{\alpha-\beta} G^6\|_{1/2} \lesssim \|\partial^\alpha \eta\|_{1/2} \|\partial^{\alpha-\beta} G^6\|_{1/2} \\ &\lesssim \sqrt{\mathcal{D}_{2N}} \sqrt{\mathcal{E}^\iota \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}}, \end{aligned} \tag{5.24}$$

In the latter case, ∂^α only involves spatial derivatives, we may use Lemma 5.1 of [9] to bound

$$\int_\Sigma \partial^\alpha \eta \partial^\alpha G^6 \lesssim \sqrt{\mathcal{D}_{2N}} \sqrt{\mathcal{E}^\iota \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}}, \tag{5.25}$$

Consequently, in light of (5.17)-(5.25), the first claim in the theorem follows. Similarly, we follow the steps of the first claim, we can conclude the second claim. Thus Theorem 5.2 follows. □

5.2. Enhanced energy estimates

From the energy-dissipative estimates of Theorem 5.1 and Theorem 5.2 we have control of $\bar{\mathcal{E}}_n$ and $\bar{\mathcal{D}}_n$. Our goal now is to show that these can be used to control $\mathcal{E}_n$ and $\mathcal{D}_n$ up to some error terms that we will be able to guarantee are small, for both $n = 2N$ and $n = N+2$. We begin with the energy estimate.

Theorem 5.3. *There exists a $\iota > 0$ so that*

$$\mathcal{E}_{2N} \lesssim \bar{\mathcal{E}}_{2N} + \mathcal{E}_{2N}^{1+\iota}, \tag{5.26}$$

and

$$\mathcal{E}_{N+2} \lesssim \bar{\mathcal{E}}_{N+2} + \mathcal{E}_{2N}^\iota \mathcal{E}_{N+2}, \tag{5.27}$$

Proof. We first let n denote either $2N$ or $N+2$ throughout the proof, and we compactly write

$$\mathcal{W}_n = \sum_{j=0}^{n-1} \|\partial_t^j G^1\|_{2n-2j-2}^2 + \|\partial_t^j G^2\|_{2n-2j-1}^2 + \|\partial_t^j G^4\|_{2n-2j-3/2}^2. \tag{5.28}$$

and

$$\mathcal{Z}_n = \sum_{j=0}^{n-1} \|\partial_t^j G^3\|_{2n-2j-2}^2 + \|\partial_t^j G^5\|_{2n-2j-3/2}^2. \tag{5.29}$$

According to the definitions of $\bar{\mathcal{E}}_n^0$ and $\bar{\mathcal{E}}_n^+$, we have

$$\|\partial_t^n u\|_0^2 + \|\partial_t^n \theta\|_0^2 + \sum_{j=0}^{n} \|\partial_t^j \eta\|_{2n-2j}^2 \lesssim \bar{\mathcal{E}}_n, \tag{5.30}$$

For estimating u and p we recall the standard Stokes estimates: for $r \geq 0$,

$$\|u\|_r + \|p\|_{r-1} \lesssim \|\phi\|_{r-2} + \|\psi\|_{r-1} + \|\alpha\|_{\Sigma, r-\frac{3}{2}}, \tag{5.31}$$

if

$$\begin{cases} -\Delta u + \nabla p = \phi \in H^{r-2}(\Omega) \\ \mathrm{div} v = \psi \in H^{r-1}(\Omega) \\ (pI - \mathbb{D}u)e_3 = \alpha \in H^{r-\frac{3}{2}}(\Sigma) \\ u|_{\Sigma_b} = 0, \end{cases}$$

Now we let $j = 0, ..., n-1$ and then apply ∂_t^j to the equations to find

$$\begin{cases} -\Delta \partial_t^j u + \nabla \partial_t^j p = -\partial_t^{j+1} u + \partial_t^j G^1 \\ \mathrm{div} \partial_t^j u = \partial_t^j G^2 \\ (\partial_t^j p I - \mathbb{D}(\partial_t^j u))e_3 = \partial_t^j \eta I e_3 - \partial_t^j G^4 \\ \partial_t^j u = 0, \quad on\ \Sigma_b, \end{cases} \tag{5.32}$$

and hence we may apply (5.11) and the estimate (4.4) of Theorem 4.3 to see that

$$\begin{aligned} \|\partial_t^j u\|_{2n-2j}^2 + \|\partial_t^j p\|_{2n-2j-1}^2 &\lesssim \|\partial_t^{j+1} u\|_{2n-2j-2}^2 + \|\partial_t^j G^1\|_{2n-2j-2}^2 + \|\partial_t^j G^2\|_{2n-2j-2}^2 \\ &\quad + \|\partial_t^j \eta\|_{2n-2j-3/2}^2 + \|G^4\|_{2n-2j-3/2}^2 \\ &\lesssim \|\partial_t^{j+1} u\|_{2n-2(j+1)}^2 + \bar{\mathcal{E}}_n + \mathcal{W}_n. \end{aligned} \tag{5.33}$$

Similarly, for estimating θ, we have

$$\begin{cases} -\Delta \partial_t^j \theta = -\partial_t \partial_t^j \theta + \partial_t^j G^3 \\ \nabla \partial_t^j \theta \cdot e_3 + \partial_t^j \theta = \partial_t^j G^5 \\ \partial_t^j \theta = 0, \quad on\ \Sigma_b, \end{cases}$$

and hence we deduce that

$$\begin{aligned} \|\partial_t^j \theta\|_{2n-2j}^2 &\lesssim \|\partial_t^{j+1} \theta\|_{2n-2j-2}^2 + \|\partial_t^j G^3\|_{2n-2j-2}^2 \\ &\quad + \|\partial_t^j \theta\|_{2n-2j-3/2}^2 + \|G^5\|_{2n-2j-3/2}^2 \\ &\lesssim \|\partial_t^{j+1} \theta\|_{2n-2(j+1)}^2 + \bar{\mathcal{E}}_n + \mathcal{Z}_n. \end{aligned} \tag{5.34}$$

We claim that

$$\mathcal{E}_n \lesssim \bar{\mathcal{E}}_n + \mathcal{W}_n + \mathcal{Z}_n. \tag{5.35}$$

We prove the claim by a finite induction based on the estimate. For $j = n-1$, we obtain

$$\begin{aligned} &\|\partial_t^{n-1} u\|_2^2 + \|\partial_t^{n-1} p\|_1^2 + \|\partial_t^{n-1} \theta\|_2^2 \\ &\lesssim \|\partial_t^n u\|_0^2 + \|\partial_t^n \theta\|_0^2 + \bar{\mathcal{E}}_n + \mathcal{W}_n + \mathcal{Z}_n \\ &\lesssim \bar{\mathcal{E}}_n + \mathcal{W}_n + \mathcal{Z}_n. \end{aligned} \tag{5.36}$$

Now suppose that the following holds for $1 \le l \le n-1$

$$\|\partial_t^{n-l} u\|_2^2 + \|\partial_t^{n-l} p\|_1^2 + \|\partial_t^{n-l}\theta\|_2^2 \lesssim \bar{\mathcal{E}}_n + \mathcal{W}_n + \mathcal{Z}_n. \tag{5.37}$$

We apply (5.31) with $j = n-(l+1)$ and use the induction hypothesis (5.37) to find

$$\begin{aligned} &\|\partial_t^{n-(l+1)} u\|_{2(l+1)}^2 + \|\partial_t^{n-(l+1)} p\|_{2(l+1)-1}^2 + \|\partial_t^{n-(l+1)}\theta\|_{2(l+1)}^2 \\ &\lesssim \|\partial_t^{n-l} u\|_{2l}^2 + \|\partial_t^{n-l}\theta\|_{2l}^2 + \bar{\mathcal{E}}_n + \mathcal{W}_n + \mathcal{Z}_n \\ &\lesssim \bar{\mathcal{E}}_n + \mathcal{W}_n + \mathcal{Z}_n. \end{aligned} \tag{5.38}$$

Hence by finite induction, the bound holds for all $l = 1, ..., n-1$. Summing (5.36) and (5.37) over $l = 1, ..., n-1$ and changing the index, we then have

$$\sum_{j=0}^{n-1} \|\partial_t^j u\|_{2n-2j}^2 + \|\partial_t^j p\|_{2n-2j-1}^2 + \|\partial_t^j\theta\|_{2n-2j}^2 \lesssim \bar{\mathcal{E}}_n + \mathcal{W}_n + \mathcal{Z}_n. \tag{5.39}$$

We then conclude the claim (5.35) by summing (5.31) and (5.39).

Finally, setting $n = 2N$ in (5.35), and using Theorem 4.3 to bound $\mathcal{W}_{2N} + \mathcal{Z}_n \lesssim (\mathcal{E}_{2N})^{1+\iota}$, we obtain (5.26); setting $n = N+2$ in (5.35), and using Theorem 4.3 to bound $\mathcal{W}_{N+2} + \mathcal{Z}_n \lesssim (\mathcal{E}_{2N})^{\iota}\mathcal{E}_{N+2}$, we obtain (5.27);

□

5.3. Enhanced dissipate estimates.

We now complete Theorem 5.2 by proving a corresponding result for the dissipation.

Theorem 5.4. *There exists a $\iota > 0$ so that*

$$\mathcal{D}_{2N} \lesssim \bar{\mathcal{D}}_{2N} + \mathcal{K}\mathcal{F}_{2N} + \mathcal{E}_{2N}^{1+\iota}, \tag{5.40}$$

and

$$\mathcal{D}_{N+2} \lesssim \bar{\mathcal{D}}_{N+2} + \mathcal{E}_{2N}^{\iota}\mathcal{D}_{N+2}, \tag{5.41}$$

Proof. We again let n denote either $2N$ or $N+2$ and compactly write

$$\begin{aligned} \mathcal{Y}_n =& \|\bar{\nabla}^{2n-1} G^1\|_0^2 + \|\bar{\nabla}^{2n-1} G^2\|_1^2 + \|\bar{\nabla}^{2n-1} G^3\|_0^2 + \|\bar{\nabla}_*^{2n-1} G^4\|_{1/2}^2 \\ &+ \|\bar{\nabla}_*^{2n-1} G^5\|_{1/2}^2 + \|\bar{\nabla}_*^{2n-1} G^6\|_{1/2}^2 + \|\bar{\nabla}_*^{2n-1}\partial_t G^6\|_{1/2}^2. \end{aligned} \tag{5.42}$$

First, by the definition of $\bar{\mathcal{D}}_n^0, \bar{\mathcal{D}}_n^+$ and Korn's inequality, we obtain

$$\|\bar{\nabla}_*^{2n-1} u\|_1^2 + \|\nabla_*\bar{\nabla}_*^{2n-1} u\|_1^2 + \|\bar{\nabla}_*^{2n-1}\theta\|_1^2 + \|\nabla_*\bar{\nabla}_*^{2n-1}\theta\|_1^2 \lesssim \bar{\mathcal{D}}_n^+, \tag{5.43}$$

and

$$\sum_{j=0}^{n} \|\partial_t^j u\|_1^2 + \sum_{j=0}^{n} \|\partial_t^j\theta\|_1^2 \lesssim \bar{\mathcal{D}}_n^0, \tag{5.44}$$

Summing these estimates (5.42) and (5.43), we find that

$$\|\bar{\nabla}_*^{2n} u\|_1^2 \lesssim \bar{\mathcal{D}}_n, \tag{5.45}$$

Notice that we have not yet derived an estimate of η in terms of dissipation, so we can not use previous boundary condition. Instead, we can apply the definition of Sobolev norm on T^2 and the trace theorem to see from (5.44) that

$$\begin{aligned}\|\partial_t^j u\|^2_{H^{2n-2j+1/2}(\Sigma)} &\lesssim \|\partial_t^j u\|^2_{L^2(\Sigma)} + \|\nabla_*^{2n-2j}\partial_t^j u\|^2_{H^{1/2}(\Sigma)}\\ &\lesssim \|\partial_t^j u\|^2_{H^1(\Omega)} + \|\nabla_*^{2n-2j}\partial_t^j u\|^2_{H^1(\Omega)} \lesssim \bar{\mathcal{D}}_n,\end{aligned} \tag{5.46}$$

Let $j = 0, ..., n-1$, and observe that $(\partial_t^j u, \partial_t^j p)$ solve the problem

$$\begin{cases} -\Delta\partial_t^j u + \nabla\partial_t^j p = -\partial_t^{j+1} u + \partial_t^j G^1 \quad in\ \Omega \\ \mathrm{div}\partial_t^j u = \partial_t^j G^2 \quad in\ \Omega \\ \partial_t^j u = \partial_t^j u \quad on\ \Sigma \\ \partial_t^j u = 0 \quad on\ \Sigma_b, \end{cases} \tag{5.47}$$

We apply the Stokes estimate with $r = 2n-2j+1$ to the problem (5.46) using the estimates and summing up, we find

$$\begin{aligned}&\|\partial_t^j u\|^2_{H^{2n-2j+1}} + \|\nabla\partial_t^j p\|^2_{H^{2n-2j-1}} \lesssim \|\partial_t^{j+1} u\|^2_{H^{2n-2j-1}}\\ &+ \|\partial_t^{j+1} G^1\|^2_{H^{2n-2j-1}} + \|\partial_t^j G^2\|^2_{H^{2n-2j}} + \|\partial_t^j u\|^2_{H^{2n-2j+1/2}(\Sigma)}\\ &\lesssim \|\partial_t^{j+1} u\|^2_{H^{2n-2j-1}} + \mathcal{Y}_n + \bar{\mathcal{D}}_n,\end{aligned} \tag{5.48}$$

Similarly, for the θ dissipative estimate, we directly use the Stokes elliptic estimates to following equations

$$\begin{cases} -\Delta\partial_t^j \theta = -\partial_t^{j+1}\theta + \partial_t^j G^3 \\ \nabla\partial_t^j\theta \cdot e_3 + \partial_t^j\theta = \partial_t^j G^5 \\ \partial_t^j\theta = 0, \quad on\ \Sigma_b, \end{cases}$$

and hence we derive that

$$\begin{aligned}&\|\partial_t^j \theta\|^2_{H^{2n-2j+1}} \lesssim \|\partial_t^{j+1}\theta\|^2_{H^{2n-2j-1}} + \|\partial_t^j G^3\|^2_{H^{2n-2j-1}}\\ &+ \|\partial_t^j \theta\|^2_{H^{2n-2j+1/2}(\Sigma)} + \|\partial_t^j G^5\|^2_{H^{2n-2j+1/2}(\Sigma)}\\ &\lesssim \|\partial_t^{j+1}\theta\|^2_{H^{2n-2j-1}} + \mathcal{Y}_n + \bar{\mathcal{D}}_n,\end{aligned} \tag{5.49}$$

We now claim that

$$\sum_{j=0}^{n} \|\partial_t^j u\|^2_{H^{2n-2j+1}} + \sum_{j=0}^{n-1} \|\partial_t^j \nabla p\|^2_{H^{2n-2j-1}} + \sum_{j=0}^{n} \|\partial_t^j \theta\|^2_{H^{2n-2j+1}} \lesssim \mathcal{Y}_n + \bar{\mathcal{D}}_n, \tag{5.50}$$

We prove this claim by a finite induction. For $j = n-1$, we obtain

$$\|\partial_t^{n-1} u\|^2_{H^3} + \|\nabla\partial_t^{n-1}\nabla p\|^2_{H^1} + \sum_{j=0}^{n} \|\partial_t^{n-1}\theta\|^2_{H^3} \lesssim \|\partial_t^n u\|_1^2 + \mathcal{Y}_n + \bar{\mathcal{D}}_n \lesssim \mathcal{Y}_n + \bar{\mathcal{D}}_n, \tag{5.51}$$

Now we suppose that the following holds for $1 \le l \le n-1$:

$$\|\partial_t^{n-l} u\|^2_{H^{2l+1}} + \|\nabla\partial_t^{n-l}\nabla p\|^2_{H^{2l-1}} + \sum_{j=0}^{n} \|\partial_t^{n-l}\theta\|^2_{H^{2l+1}} \lesssim \mathcal{Y}_n + \bar{\mathcal{D}}_n, \tag{5.52}$$

We apply with $j=n-(l+1)$ and use the induction hypothesis (5.52) to find

$$\begin{aligned}&\|\partial_t^{n-(l+1)}u\|^2_{H^{2(l+1)+1}}+\|\nabla\partial_t^{n-(l+1)}p\|^2_{H^{2(l+1)-1}}+\|\partial_t^{n-(l+1)}\theta\|^2_{H^{2(l+1)+1}}\\&\lesssim\|\partial_t^{n-l}u\|^2_{H^{2l+1}}+\|\partial_t^{n-l}\theta\|^2_{H^{2l+1}}+\mathcal{Y}_n+\bar{\mathcal{D}}_n\\&\lesssim\mathcal{Y}_n+\bar{\mathcal{D}}_n,\end{aligned}\tag{5.53}$$

Hence the bound (5.52) holds for all $l=1,...,n$. We then conclude the claim (5.50) by summing this over $l=1,...,n$, adding (5.44) and changing the index.

Now that we have obtained (5.50), we estimate the remaining parts in $\mathcal{D}_n$. We will turn to the boundary conditions in (3.10). First we derive estimates for η. For the term $\partial_t^j\eta$ for $j\geq 2$ we use the boundary condition

$$\partial_t\eta=u_3+G^6\quad on\ \ \Sigma,\tag{5.54}$$

Indeed, for $j=2,...,n+1$ we apply ∂_t^{j-1} to (5.54) to see that

$$\begin{aligned}\|\partial_t^j\eta\|^2_{2n-2j+5/2}&\lesssim\|\partial_t^{j-1}u_3\|^2_{H^{2n-2j+5/2}(\Sigma)}+\|\partial_t^{j-1}G^6\|^2_{H^{2n-2j+5/2}(\Sigma)}\\&\lesssim\|\partial_t^{j-1}u_3\|^2_{H^{2n-2(j-1)+1}}+\|\partial_t^{j-1}G^6\|^2_{H^{2n-2(j-1)+1/2}}\\&\lesssim\mathcal{Z}_n+\bar{\mathcal{D}}_n,\end{aligned}\tag{5.55}$$

For the term $\partial_t\eta$, we again use (5.54), (5.50) and (5.42) to find

$$\begin{aligned}\|\partial_t^j\eta\|^2_{2n-1/2}&\lesssim\|u_3\|^2_{H^{2n-1/2}(\Sigma)}+\|\partial_t^{j-1}G^6\|^2_{H^{2n-1/2}(\Sigma)}\\&\lesssim\|u_3\|^2_{H^{2n}}+\|\partial_t^{j-1}G^6\|^2_{H^{2n-1/2}}\\&\lesssim\mathcal{Z}_n+\bar{\mathcal{D}}_n,\end{aligned}\tag{5.56}$$

For the remaining η term, that is those without temporal derivatives, we use the boundary conditions

$$\eta=p-\partial_3u_3+G^4\tag{5.57}$$

Notice that at this point we do not have any bound on p on the boundary Σ, but we have bounded ∇p in Ω. Applying ∂_1,∂_2 to (5.56) and (5.57), respectively, by (5.50) and (5.42), we obtain

$$\begin{aligned}\|\nabla_*\eta\|^2_{2n-3/2}&\lesssim\|\nabla_*p\|^2_{H^{2n-3/2}(\Sigma)}+\|\nabla_*\partial_3u_3\|^2_{H^{2n-3/2}(\Sigma)}+\|\nabla_*G^6\|^2_{H^{2n-3/2}(\Sigma)}\\&\lesssim\|\nabla p\|^2_{H^{2n}}+\|u_3\|^2_{H^{2n+1}}+\|G^6\|^2_{H^{2n-1/2}}\\&\lesssim\mathcal{Y}_n+\bar{\mathcal{D}}_n,\end{aligned}\tag{5.58}$$

Since $\int_{T^2}\eta=0$, we may then use Poincare inequality on Σ to obtain from (5.58) that

$$\|\eta\|^2_{2n-1/2}\lesssim\|\eta\|^2_0+\|\nabla_*\eta\|^2_{2n-3/2}\lesssim\|\nabla_*\eta\|^2_{2n-3/2}\lesssim\mathcal{Y}_n+\bar{\mathcal{D}}_n,\tag{5.59}$$

Summing (5.55), (5.56) and (5.59), we complete the estimate for η:

$$\|\eta\|^2_{2n-1/2}+\|\partial_t\eta\|^2_{2n-1/2}+\sum_{j=0}^{n-1}\|\partial_t^j\eta\|^2_{2n-2j+5/2}\lesssim\mathcal{Y}_n+\bar{\mathcal{D}}_n,\tag{5.60}$$

It remains to bound $\|\partial_t^j p\|_0$. Applying $\partial_t^j, j=0,...,n-1$ to (5.47) and employing (5.50),(5.60) and (5.42), we find

$$\begin{aligned}\|\partial_t^j p\|_{L^2(\Sigma)}^2 &\lesssim \|\partial_t^j \eta\|_{L^2(\Sigma)}^2 + \|\partial_3\partial_t^j u_3\|_{L^2(\Sigma)}^2 + \|\partial_t^j G^6\|_0^2 \\ &\lesssim \|\partial_t^j \eta\|_{L^2(\Sigma)}^2 + \|\partial_t^j u_3\|_{H^2}^2 + \|\partial_t^j G^6\|_0^2 \lesssim \mathcal{Y}_n + \bar{\mathcal{D}}_n,\end{aligned} \tag{5.61}$$

This, and (5.44) and (5.42), and (5.60) allow us improve (5.50) to

$$\mathcal{D}_n \lesssim \mathcal{Z}_n + \bar{\mathcal{D}}_n, \tag{5.62}$$

Setting $n=2N$ in (5.62) and using the estimates (4.7)-(4.11) in Theorem 4.3 to estimate $\mathcal{Z}_{2N} \lesssim (\mathcal{E}_{2N})^\iota \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}$. On the other hand, we may set $n=N+2$ to bound $\mathcal{Z}_{N+2} \lesssim (\mathcal{E}_{2N})^\iota \mathcal{D}_{N+2}$.

□

6 Proof of main results

6.1. Estimates involving $\mathcal{F}_{2N}$ and $\mathcal{K}$

We first need to control $\mathcal{F}_{2N}$. This is achieved by the following proposition.

Proposition 6.1. *There exists a universal constant $0<\delta<1$ so that if $\mathcal{G}_{2N} \le \delta$, then*

$$\sup_{0\le r\le t} \mathcal{F}_{2N}(r) \lesssim \mathcal{F}_{2N}(0) + t\int_0^t \mathcal{D}_{2N}, \quad for\ all\ 0\le t\le T. \tag{6.1}$$

Proof. Based on the transport estimate on the kinematic boundary condition, we may show as in Lemma 7.1 of [9] that

$$\begin{aligned}&\sup_{0\le r\le t} \mathcal{F}_{2N}(r) \lesssim exp(C\int_0^t \sqrt{\mathcal{K}(r)}dr) \\ &\times [\mathcal{F}_{2N}(0) + t\int_0^t (1+\mathcal{E}_{2N}(r))\mathcal{D}_{2N}(r)dr + (\int_0^t \sqrt{\mathcal{K}(r)\mathcal{F}_{2N}(r)})^2].\end{aligned} \tag{6.2}$$

The Sobolev and trace embeddings allow us to estimate $\mathcal{K} \lesssim \mathcal{E}_{N+2}$, and hence

$$\int_0^t \sqrt{\mathcal{K}(r)}dr \lesssim \int_0^t \sqrt{\mathcal{E}_{N+2}(r)}dr \lesssim \sqrt{\delta}\int_0^t \frac{1}{(1+r)^{2N-4}}dr \lesssim \sqrt{\delta}. \tag{6.3}$$

Since $\delta \le 1$, this implies that for any constant $C>0$,

$$exp(C\int_0^t \sqrt{\mathcal{K}(r)}dr) \lesssim 1. \tag{6.4}$$

Then by (6.3) and (6.4), we deduce from (6.2) that

$$\begin{aligned}\sup_{0\le r\le t} \mathcal{F}_{2N}(r) &\lesssim \mathcal{F}_{2N}(0) + t\int_0^t \mathcal{D}_{2N}(r)dr + \sup_{0\le r\le t}\mathcal{F}_{2N}(r)(\int_0^t \sqrt{\mathcal{K}(r)}dr)^2 \\ &\lesssim \mathcal{F}_{2N}(0) + t\int_0^t \mathcal{D}_{2N}(r)dr + \delta \sup_{0\le r\le t}\mathcal{F}_{2N}(r)\end{aligned} \tag{6.5}$$

By taking δ small enough, we get (6.1) . □

This bound on $\mathcal{F}_{2N}$ allows us to estimate the integral of $\mathcal{K}\mathcal{F}_{2N}$ and $\sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}$ as in Corollary 7.3 of [9].

Corollary 6.2. *There exists* $0<\delta<1$ *so that if* $\mathcal{G}_{2N}\le\delta$, *then*

$$\int_0^t \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}} \lesssim \mathcal{F}_{2N}(0)+\sqrt{\delta}\int_0^t \mathcal{D}_{2N}(r)dr, \tag{6.6}$$

and

$$\int_0^t \mathcal{K}\mathcal{F}_{2N} \lesssim \delta\mathcal{F}_{2N}(0)+\delta\int_0^t \mathcal{D}_{2N}(r)dr, \tag{6.7}$$

Now we show the boundness of the high-order terms.

Proposition 6.3. *There exists* $0<\delta<1$ *so that if* $\mathcal{G}_{2N}\le\delta$, *then*

$$\sup_{0\le r\le t}\mathcal{E}_{2N}(r)+\int_0^t \mathcal{D}_{2N}+\sup_{0\le r\le t}\frac{\mathcal{F}_{2N}(r)}{(1+r)}\lesssim \mathcal{E}_{2N}(0)+\mathcal{F}_{2N}(0),\quad \textit{for all } 0\le t\le T. \tag{6.8}$$

Proof. Note first that since $\mathcal{E}_{2N}(t)\le\mathcal{G}_{2N}\le\delta$, by taking δ small, from Theorem 5.3 and Theorem 5.4 we know that

$$\mathcal{E}_{2N}\lesssim\bar{\mathcal{E}}_{2N}\lesssim\mathcal{E}_{N+2},\quad \textit{and}\quad \bar{\mathcal{D}}_{2N}\lesssim\mathcal{D}_{2N}\lesssim\bar{\mathcal{D}}_{2N}+\mathcal{K}\mathcal{F}_{2N}. \tag{6.9}$$

Now we multiplying (5.6) by a constant $1+C_*$ (with precise value to be choose later) and add this to (5.15) find that

$$\begin{aligned}
&\bar{\mathcal{E}}^{+}_{2N}(t)+\bar{\mathcal{E}}^{0}_{2N}(t)+\int_0^t(1+\beta)\bar{\mathcal{D}}^{0}_{2N}+\bar{\mathcal{D}}^{+}_{2N}\\
&\lesssim(1+\beta)\mathcal{E}_{2N}(0)+\bar{\mathcal{E}}^{+}_{2N}(0)+(1+\beta)(\mathcal{E}_{2N}(t))^{3/2}+\int_0^t(1+\beta)\mathcal{E}^{\iota}_{2N}\mathcal{D}_{2N}\\
&+\int_0^t\sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}+\varepsilon\mathcal{D}_{2N}+C(\varepsilon)\bar{\mathcal{D}}^{0}_{2N}.
\end{aligned} \tag{6.10}$$

Then we may improve

$$\begin{aligned}
\mathcal{E}_{2N}(t)+\int_0^t\mathcal{D}_{2N}\lesssim{}&(1+\beta)\mathcal{E}_{2N}(0)+(1+\beta)(\mathcal{E}_{2N}(t))^{1+\iota}+\int_0^t(2+\beta)\mathcal{E}^{\iota}_{2N}\mathcal{D}_{2N}\\
&+\int_0^t\sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}+\mathcal{K}\mathcal{F}_{2N}+\varepsilon\mathcal{D}_{2N}+C(\varepsilon)\bar{\mathcal{D}}^{0}_{2N}.
\end{aligned} \tag{6.11}$$

We plug the estimates (6.6) and (6.7) into (6.10), then take ε sufficiently small first, β sufficiently large second, and δ sufficiently small third; we may then conclude

$$\sup_{0\le r\le t}\mathcal{E}_{2N}(t)+\int_0^t\mathcal{D}_{2N}\lesssim\mathcal{E}_{2N}(0)+\mathcal{F}_{2N}(0), \tag{6.12}$$

□

It remains to show the decay estimates of $\mathcal{E}_{N+2}$.

Proposition 6.4. *There exists* $0 < \delta < 1$ *so that if* $\mathcal{G}_{2N} \leq \delta$, *then*

$$(1 + t^{4N-8})\mathcal{E}_{N+2}(t) \lesssim \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) \quad for\ all\ 0 \leq\leq T. \tag{6.13}$$

Proof. Since $\mathcal{E}_{2N}(t) \lesssim \mathcal{G}_{2N}(T)$, by taking δ small, from Theorem 5.3 and 5.4 we know that

$$\mathcal{E}_{N+2} \lesssim \bar{\mathcal{E}}_{N+2} \lesssim \mathcal{E}_{N+2}, \quad and \quad \bar{\mathcal{D}}_{N+2} \lesssim \mathcal{D}_{N+2} \lesssim \bar{\mathcal{D}}_{N+2}. \tag{6.14}$$

By these estimates and the smallness of δ, from Theorem 5.1 and 5.2 we may deduce that there exists an instantaneous energy, which is equivalent to $\mathcal{E}_{N+2}$, such that

$$\frac{d}{dt}\mathcal{E}_{N+2} + \mathcal{D}_{N+2} \lesssim 0. \tag{6.15}$$

On the other hand, based on the Sobolev interpolation inequality we can prove

$$\mathcal{E}_{N+2} \lesssim \mathcal{D}_{N+2}^{\iota}\mathcal{E}_{2N}^{1-\iota}, \quad where\ \iota = \frac{4N-8}{4N-7} \tag{6.16}$$

Now since we know that the boundness of high energy estimate Proposition 6.3, we get

$$\sup_{0\leq r\leq t} \mathcal{E}_{2N}(r) \lesssim \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) := \mathcal{M}_0, \quad where\ \iota = \frac{4N-8}{4N-7} \tag{6.17}$$

we obtain form (6.14) that

$$\mathcal{E}_{N+2} \lesssim \mathcal{M}^{1-\iota}\mathcal{D}_{N+2}^{\iota}. \tag{6.18}$$

Hence by (6.17) and (6.15), there exists some constant $C_1 > 0$ such that

$$\frac{d}{dt}\mathcal{E}_{N+2} + \frac{C_1}{\mathcal{M}_0^s}\mathcal{E}_{N+2}^{1+s} \lesssim 0, \quad where\ s = \frac{1}{\iota} - 1 = \frac{1}{4N-8}, \tag{6.19}$$

Solving this differential inequality directly, we obtain

$$\mathcal{E}_{N+2}(t) \lesssim \frac{\mathcal{M}_0}{(\mathcal{M}_0^s + sC_1(\mathcal{E}_{N+2}(0))^s t)^{1/s}}\mathcal{E}_{N+2}(0), \tag{6.20}$$

Using that $\mathcal{E}_{N+2}(0) \lesssim \mathcal{M}_0$ and the fact $1/s = 4n - 8 > 1$, we obtain that

$$\mathcal{E}_{N+2}(t) \lesssim \frac{\mathcal{M}_0}{1 + sC_1 t)^{1/s}} \lesssim \frac{\mathcal{M}_0}{1 + t)^{1/s}} \lesssim \frac{\mathcal{M}_0}{1 + t)^{4N-8}}. \tag{6.21}$$

This implies (6.13). □

Now we combine proposition to arrive at our ultimate a priori estimates for $\mathcal{G}_{2N}$.

Theorem 6.5. *There exists a universal* $0 < \delta < 1$ *so that if* $\mathcal{G}_{2N}(T) \leq \delta$, *then*

$$\mathcal{G}_{2N}(t) \leq C_2(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)) \quad for\ all\ 0 \leq t \leq T. \tag{6.22}$$

Proof. The conclusion follows directly from the definition of $\mathcal{G}_{2N}$ and Propositions. □

In order to combine the local existence result with the a priori estimates, we must be able to estimate $\mathcal{G}_{2N}$ in terms of the right and side. This is achieved by the following proposition 6.3 and 6.4.

Proposition 6.6. *There exists a universal constant $C_3 > 0$ so that the following hold. If $0 \leq T$, then we have the estimate*

$$\begin{aligned}\mathcal{G}_{2N}(t) \leq & \sup_{0\leq t\leq T} \mathcal{E}_{2N}(t) + \int_0^{T_2} \mathcal{D}_{2N}(t)dt \\ & + \sup_{0\leq t\leq T} \mathcal{F}_{2N}(t) + C_3(1+T)^{4N-8} \sup_{0\leq t\leq T} \mathcal{E}_{2N}(t).\end{aligned} \tag{6.23}$$

If $0 < T_1 \leq T_2$, then we have

$$\begin{aligned}\mathcal{G}_{2N}(T_2) \leq & C_3\mathcal{G}_{2N}(T_1) + \sup_{T_1\leq t\leq T_2} \mathcal{E}_{2N}(t) + \int_{T_1}^{T_2} \mathcal{D}_{2N}(t)dt \\ & + \frac{1}{(1+T_1)} \sup_{T_1\leq t\leq T_2} \mathcal{F}_{2N}(t) + C_3(T_2-T_1)^2(1+T)^{4N-8} \sup_{0\leq t\leq T} \mathcal{E}_{2N}(t).\end{aligned} \tag{6.24}$$

Proof. The proof is same as Proposition 9.1 of [9]. □

6.2. Global well-posedness

Now we turn to the completion of the proof of our main theorem.

Proof of Theorem 2.1. let $0 < \delta < 1$ and $C_2 > 0$ be the constants in Theorem 6.5, $C_1 > 0$ be the constant in () and $C_1 > 0$ be the constant in (2.9). By the local existence result, Theorem 2.2, for any $\varepsilon > 0$ there exists $\delta_0(\varepsilon) < 1$ and $0 < T < 1$ so that if $\kappa < \delta_0$, then there is a unique solution of (1.3) on [0,T] satisfying the estimates

$$\sup_{0\leq t\leq T} \mathcal{E}_{2N}(t) + \int_0^T \mathcal{D}_{2N}(t)dt + \int_0^T (\|\partial_t^{2N+1}u\|_{-1}^2 + \|\partial_t^{2N}p(t)\|_0^2)dt \leq \varepsilon \tag{6.25}$$

and

$$\sup_{0\leq t\leq T} \mathcal{F}_{2N}(t) \leq C_1\mathcal{F}_{2N}(0) + \varepsilon. \tag{6.26}$$

Hence if we choose $\varepsilon = \delta/(4 + C_3 2^{4N-7})$ and then choose $\kappa < \delta/(2C_1)$, we may use the estimate of Proposition to see that

$$\mathcal{G}_{2N}(T) \leq C_1\kappa + \varepsilon(2 + C_3 2^{4N-8}) < \delta. \tag{6.27}$$

Now we define

$$\begin{aligned}T_*(\kappa) = \sup\{ & for\ every\ choice\ of\ initial\ data\ satisfying\ the\ compatibility \\ & conditions\ and\ \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) < \kappa,\ there\ exists\ a\ unique \\ & solution\ of\ (1.3)\ on\ [0,T]\ satisfying\ \mathcal{G}_{2N}(T) \leq \delta\}.\end{aligned} \tag{6.28}$$

By the above analysis, $T_*(\kappa)$ is well-defined and satisfies $T_*(\kappa) > 0$ if κ is small enough, that is there is a $\kappa_1 > 0$ so that $T_* : (0,\kappa_1] \to (0,\infty]$. It is easy to verify that T_* is non-increasing on $(0,\kappa_1]$. Now we set

$$\varepsilon = \frac{\delta}{3}\min\{\frac{1}{2}, \frac{1}{C_3}\} \tag{6.29}$$

and then define $\kappa_0 \in (0,\kappa_1]$ by

$$\kappa_0 = \min\{\frac{\delta}{3C_2(C_3+2C_1)}, \frac{\delta_0(\varepsilon)}{C_2}, \kappa_1\} \tag{6.30}$$

We claim that $T_*(\kappa_0) = \infty$. Once the claim is established, the proof the theorem is complete since then $T_*(\kappa) = \infty$ for all $0 < \kappa < \kappa_0$.

We prove the claim by contradiction. Suppose that $T_*(\kappa_0) < \infty$. By the definition of $T_*(\kappa_0)$, for any $0 < T_1 < T_*(\kappa_0)$ and for any choice of data satisfying the compatibility conditions and the bound $\mathcal{E}_{2N}(0)+\mathcal{F}_{2N}(0) < \kappa_0$, there exists a unique solution of (1.3) on $[0, T_1]$ satisfying $\mathcal{G}_{2N}(T_1) \leq \delta$. Then by Theorem 6.5, we have

$$\mathcal{G}_{2N}(T_1) \leq C_2(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)) < C_2\kappa_0. \tag{6.31}$$

In particular (6.31), (6.30) imply

$$\mathcal{E}_{2N}(T_1) + \frac{\mathcal{F}_{2N}}{(1+T_1)} < C_2\kappa_0 \leq \delta_0(\varepsilon),\ \forall < T_1 < T_*(\kappa_0). \tag{6.32}$$

We can view $(u(T_1), p(T_1), \eta(T_1))$ as initial data for a new problem; they satisfy the compatibility conditions as initial data since they are already solutions on $[0, T_1]$. Since $\mathcal{E}_{2N} < \delta_0(\varepsilon)$, we may use Theorem to extend the solution to $[T_1, T_2]$, where T_2 is any time satisfying

$$0 < T_2 - T_1 \leq T_0 := C(\varepsilon)\min\{1, \mathcal{F}_{2N}(T_1)^{-1}\}. \tag{6.33}$$

By (6.31), we have

$$\bar{T} := C(\varepsilon)\min\{1, \frac{1}{\delta_0(1+T_*(\kappa_0))}\} \leq T_0. \tag{6.34}$$

Note that $\bar{T}$ depends on ε and $T_*(\kappa_0)$ but does depend on T_1. Let

$$\gamma = \min\{\bar{T}, T_\star(\kappa_0), \frac{1}{(1+2T_*(\kappa_0))^{2N-4}}\}, \tag{6.35}$$

and then choose $T_1 = T_*(\kappa_0) - \gamma/2$ and $T_2 = T_*(\kappa_0) + \gamma/2$. We have

$$0 < T_1 < T_*(\kappa_0) < T_2 < 2T_*(\kappa_0) \ \ \textit{and} \ \ 0 < \gamma = T_2 - T_1 \leq \bar{T} \leq T_0, \tag{6.36}$$

By the above argument, we have extended the solution to $[0, T_2]$ and on the extended time interval $[T_1, T_2]$ we have

$$\sup_{T_1 \leq t \leq T_2} \mathcal{E}_{2N}(t) + \int_{T_1}^{T_2} \mathcal{D}_{2N}(t)dt + \int_{T_1}^{T_2} (\|\partial_t^{2N+1}u\|_{-1}^2 + \|\partial_t^{2N}p(t)\|_0^2)dt \leq \varepsilon \tag{6.37}$$

and

$$\sup_{T_1 \leq t \leq T_2} \mathcal{F}_{2N}(t) \leq C_1\mathcal{F}_{2N}(0) + \varepsilon. \tag{6.38}$$

We now combine the estimates (6.37)-(6.38), (6.31)-(6.33) and (6.24) with the definitions (6.29), (6.30), and (6.37) to see that

$$\begin{aligned} \mathcal{G}_{2N}(T_2) &< C_2C_3\kappa_0 + \varepsilon + \frac{C_1C_2\kappa_0(1+T_1)+\varepsilon}{(1+T_1)} + \varepsilon C_3(T_2-T_1)^2(1+T_2)^{4N-8} \\ &\kappa_0C_2(C_3+C_1) + 2\varepsilon + \varepsilon C_3\gamma^2(1+2T_*(\kappa_0))^{4N-8} \\ &\frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta. \end{aligned} \tag{6.39}$$

Hence $\mathcal{G}_{2N} \leq \delta$, which contradicts the definition of $T_*(\kappa_0)$. Therefore, we have $T_*(\kappa_0) = \infty$. This proves the claim and complete the proof Theorem 2.1.

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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