manuscript No.
(will be inserted by the editor)

Self-adaptive methods for solving split problems of variational inclusion

Xiaojun Ma[1] · Zhifu Jia [2]

Received: date / Accepted: date

Abstract In this paper, we study the weak convergence of the algorithms for solving variational inclusion problems without using Lipschitz condition of the inverse strongly monotone operator in real Hilbert spaces. The algorithms are inspired by Tseng's modified forward-backward splitting method [4](SIAM J Control Optim **38**,431-446(2000))with a simple step size. The weak convergence theorems for our algorithms are established without any requirement of additionally resolvent operators and the prior knowledge of the bounded linear operator norm. Also, our methods are extended to solve the split feasible problem and split minimization problem. Finally, some numerical experiments are provided to demonstrate the efficiency of the proposed iterative method.

Keywords Variational inclusion problem, Resolvent operator, Splitting method, Bounded linear operator

Mathematics Subject Classification (2000) 47J20 · 49J40 · 35A15 · 47J25

1 Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Let $B : H \to 2^H$ be a set-valued mapping with its domain $\mathbb{D}(B) := \{x \in H : B(x) \neq \emptyset\}$. Suppose that B is named monotone, if for all $x, y \in H$,

$$\langle a - b, x - y\rangle \geq 0,\ \forall\ a \in Bx,\ b \in By.$$

Also, a monotone mapping B is known as *maximal* provided that the $graph(B) := \{(x, a) \in \mathbb{D}(B) \times Bx\}$ is not properly contained in the graph of any other monotone mapping. To our knowledge, a monotone mapping B is *maximal* if and only if for all $(x, a) \in H \times H$,

✉Xiaojun Ma
fzhuangmaxj@163.com
1 School of Mathematics and Statistics, Xidian University, Xi'an, 710126, Shaanxi, China

Zhifu Jia
jzflzbx@nuaa.edu.cn
2 State Key Laboratory of Mechanics Control of Mechanical Structures, Institute of Nano Science and Department of Mathematics, Nanjing University of Aeronautics and Astronautics, 210016 Nanjing, China

$$\langle a-b, x-y\rangle \geq 0,\ \forall\ (y,b) \in graph(B),$$

the resulted $a \in Bx$.

Let $B : H \to 2^H$ be a set-valued maximal monotone operator. The resolvent operator $J_\beta^B : H \to H$ associated with B is given by

$$J_\beta^B(x) = (I+\beta B)^{-1}(x),\ \forall x \in H,$$

where $\beta > 0$ and I is the identity operator on H.

In 2011, Moudafi [1] addressed the following split monotone variational inclusion problem ($SMVIP$):

$$\text{Find } \tilde{x} \in H_1 \text{ such that } 0 \in f_1(\tilde{x}) + B_1(\tilde{x}),\ \text{and } 0 \in f_2(A\tilde{x}) + B_2(A\tilde{x}), \tag{1}$$

where $H_1,\ H_2$ are real Hilbert spaces, $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ are two given operators, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings, and $A : H_1 \to H_2$ is a bounded and linear operator. As claimed in [1], the problem ($SMVIP$) (1) includes as special cases, split equilibrium problem, split zero problem, split minimization problem and relaxed split feasibility problem. These problems have been studied and have wide applications in real-life problems including intensity-modulated radiation therapy treatment planning [2] and sensor networks in computerized tomography and data compression [3]. If $f_1 = 0$ and $f_2 = 0$, then the problem(1) reduces to the split variational inclusion problem ($SVIP$), that is,

$$\text{Find } \tilde{x} \in H_1 \text{ such that } 0 \in B_1(\tilde{x}),\ \text{and } 0 \in B_2(A\tilde{x}), \tag{2}$$

and its solution set is denoted by

$$\Gamma = \{\tilde{x} \in H_1 : 0 \in B_1(\tilde{x}) \text{ and } 0 \in B_2(A\tilde{x})\}.$$

For a more general problem($SVIP$)(2), Byrne et al [5] proposed the following iterative algorithm which guaranteed the weak and strong convergence:

$$x_{n+1} = J_\lambda^{B_1}(x_n - \gamma A^*(I - J_\lambda^{B_2})Ax_n),\ \forall n \geq 1,\ \exists\lambda > 0,$$

where A^* is the adjoint operator of A, $\gamma \in (0, \frac{2}{L})$, $L = \|A\|^2$. In the last years, some well known scholars such as Byrne, Chuang and Kazmi and so on who made a contribution in the study of the $SVIP$, see [2,3,6,10, 17,19,23,28,?]. In recent years, inertial type algorithms have been proposed for solving $SVIP$ and its special cases, such as, inertial proximal algorithm[7,8,15], the inertial method [11,14], the split inertial proximal method [9,10,16], the hybrid inertial proximal algorithm [12] and its accelerated version[13]. Among them, Majee and Nahak [16] modified Chuang [12] by introducing the following weak convergence algorithm :

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = J_\beta^{B_1}(w_n - \lambda_n A^*(I - J_\beta^{B_2} Aw_n)); \\ d(w_n, \lambda_n) = w_n - y_n - \lambda_n[A^*(I - J_{\beta_n}^{B_2})Aw_n - A^*(I - J_{\beta_n}^{B_2})Ay_n]; \\ \alpha_n = \frac{\langle w_n - y_n, d(w_n, \lambda_n)\rangle}{\|d(w_n, \lambda_n)\|^2}; \\ x_{n+1} = w_n - \kappa\alpha_n d(w_n, \lambda_n), \kappa \in (0, 2), \end{cases}$$

where $\{\theta_n\}_{n\in\mathbb{N}} \subset [0, \theta) \subset [0, ((2-\kappa)/\kappa)/((2-\kappa)/\kappa + \phi + 1))$, $\kappa \in (0, 2)$, $\phi = \max\{1, (2-\kappa)/\kappa\}$, and $\{\lambda_n\} \subset [\lambda, \frac{\delta}{\|A\|^2}]$ satisfies $\lambda_n\|A^*(I - J_{\beta_n}^{B_2})Aw_n - A^*(I - J_{\beta_n}^{B_2})Ay_n\| \leq \delta\|w_n - y_n\|, 0 < \delta < 1$. Subsequently, Long et al.[30] introduced the following iterative steps:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = J_\beta^{B_1}(w_n - \lambda_n A^*(I - J_\beta^{B_2} Aw_n)); \\ x_{n+1} = (1 - \tilde{\beta}_n)x_n + \tilde{\beta}_n y_n, \end{cases}$$

where $\tilde{\beta}_n \in (0, 1)$, $\lambda_n \in (0, \frac{1}{\|A\|^2})$, $0 < \theta_n < \theta < 1$. Also, its weak convergence is obtained. All the algorithms mentioned above have a similar feature which is its computational shortcoming and it is clear that the step size λ_n is governed by the operator norm $\|A\|$,which is not often available. To avoid it, Kesornprom and Cholamjiak [15] introduced the following line search scheme

$$\lambda_n\|A^*(I - J_{\beta_n}^{B_2})Aw_n - A^*(I - J_{\beta_n}^{B_2})Ay_n\| \leq \delta\|w_n - y_n\|, 0 < \delta < 1.$$

where $\lambda_n = \sigma\rho^{m_n}$, $\sigma > 0$, $0 < \rho < 1$ and m_n is the smallest nonnegative integer. Based on Lemma 3.3 given in Takahashi[17], $A^*(I - J_{\beta_n}^{B_2})A$ is $\frac{1}{\|A\|^2}$-inverse strongly monotone. So, it is Lipschitzian with Lipschitz constant $\|A\|^2$. Under this condition, the Armijo-type rule above is well defined and so the convergence of the related algorithm is guaranteed. Also, it is obvious that this scheme often leads to additional computation costs. Soon, self-adaptive step size methods have been proposed (see, e.g., [18,19]), to cite a few. Very recently,Izuchukwu et al.[20] proposed A Relaxed Inertial Forward-Backward-Forward Algorithm for Solving Monotone inclusions and their step size is adaptively updated as follows :

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\delta\|x_n - y_n\|}{\|Bx_n - By_n\|}, \lambda_n\}, & \text{if } \|Bx_n - By_n\| \neq 0, \\ \lambda_n, & \text{otherwise}, \end{cases}$$

where $0 < \delta < 1$ and B is the inverse strongly monotone operator. Notice that the step size sequence given by Izuchukwu et al.[20] is non-increasing and of rather restriction.

Inspired by the work of Tseng [4], Kesornprom et al. [15], Majee and Nahak [16], Izuchukwu et al.[20] and Long et al.[26], in this paper, we propose self-adaptive methods with new variable step sizes for solving the SVIP so that no more running any line searches, and no prior information about the operator norm is required. Also, the designed step size is well defined and the convergence of the proposed iterative Algorithms are still obtained under the condition of the inverse strongly monotone operator $A^*(I - J_{\beta_n}^{B_2})A$.

The structure of this paper is the following: In Sect.2, we recall that some definitions and preliminaries will be used for the whole paper. In Sect.3, we introduce a new algorithm and analyze its convergence. In

Sect.4, a preliminary numerical example is provided to show our method performance. In Sect.5, we apply our algorithm to solve the split feasible problem and split minimization problem.

2 Preliminaries

Let H be a real Hilbert space and $\tilde{S}$ be a non-empty closed convex subset of H. The symbols $\rightharpoonup$ stands for the weak convergence, the symbols $\rightarrow$ represents the strong convergence.

Lemma 2.1 *Let $\alpha \in (0,1)$, $\forall\ x, y \in H$, then*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2.$$

Definition 2.1 Let $T : H \to H$ is a mapping, then

(a) T is called nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|,\ \forall\ x, y \in H.$$

(b) T is said to be firmly–nonexpansive if

$$\langle Tx - Ty, x - y\rangle \ge \|Tx - Ty\|^2,\ \forall\ x, y \in H.$$

Or equivalently,

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(I-T)x - (I-T)y\|^2,\ \forall\ x, y \in H.$$

Lemma 2.2 *(**Demiclosedness principle**[21]) Assume that $T : \tilde{S} \to \tilde{S}$ is a nonexpansive mapping. Then the following implication holds:*

$$x_n \rightharpoonup x \in \tilde{S} \text{ and } \lim_{n\to\infty}\|x_n - Tx_n\| = 0 \Rightarrow x = Tx.$$

To study the $SVIP$, we denote by $B^{-1}(0) = \{x \in H : 0 \in Bx\}$, $D(T)$ the domain of T and $Fix(T)$ the fixed point set of T, equivalently, $Fix(T) = \{x \in H : x = Tx\}$.

Lemma 2.3 *[22] Let $B : H \to 2^H$ be a set-valued maximal monotone mapping and $\beta > 0$. Then the following statements hold:*

(a) J_β^B is a single-valued and firmly nonexpansive mapping;

(b) $D(J_\beta^B) = H$ and $Fix(J_\beta^B) = \{x \in D(B) : 0 \in Bx\}$;

(c) $\|x - J_\beta^B x\| \le \|x - J_\gamma^B x\|$ for all $0 < \beta < \gamma$ and $x \in H$;

(d) $(I - J_\beta^B)$ is a firmly nonexpansive mapping.

(e) For $B^{-1}(0) \neq \emptyset$, one has

$$\|x - J_\beta^B x\|^2 + \|J_\beta^B x - \tilde{x}\|^2 \le \|x - \tilde{x}\|^2, \forall\ x \in H, \tilde{x} \in B^{-1}(0),$$

and

$$\langle x - J_\beta^B x, J_\beta^B x - \tilde{w}\rangle \ge 0, \forall\ x \in H, \tilde{w} \in B^{-1}(0).$$

Lemma 2.4 *[23] Let H_1 and H_2 be the two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator, A^* be the adjoint of A. Let $\beta > 0$, $\gamma > 0$, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be set-valued maximal monotone mappings. For any $\tilde{x} \in H_1$, we have the following:*

(a) Suppose that $\tilde{x}$ is a solution of $(SVIP)$, then $J_\beta^{B_1}(\tilde{x} - \gamma A^*(I - J_\beta^{B_2})A\tilde{x}) = \tilde{x}$;

(b) If $J_\beta^{B_1}(\tilde{x} - \gamma A^*(I - J_\beta^{B_2})A\tilde{x}) = \tilde{x}$ and the solution set of the $(SVIP)$ is nonempty. Then $\tilde{x}$ is a solution of $(SVIP)$.

Lemma 2.5 *[24] Suppose that the sequences $\{a_n\}$, $\{\sigma_n\}$ and $\{\bar{\beta}_n\} \subset [0, +\infty]$, then there exists $N > 0$ such that for any $n \geq N$, $a_{n+1} \leq a_n + \bar{\beta}_n(a_n - a_{n-1}) + \sigma_n$, $\sum\limits_{n=N}^{+\infty}\{\sigma_n\} < +\infty$, and there exists a real number $\bar{\beta}$ such that for any $n \geq N$, $0 \leq \bar{\beta}_n \leq \bar{\beta} < 1$, then the following results hold that:*

(a) $\sum\limits_{n=N}^{+\infty}[a_n - a_{n-1}]_+ < +\infty$, where $[\gamma]_+ = \max[0, \gamma]$.

(b) There exists $a^* \in [0, +\infty)$ such that $\lim\limits_{n\to\infty} a_n = a^*$.

Lemma 2.6 *[25] Let $\tilde{S} \subset H$ be a non-empty set, and $\{x_n\}$ be a sequence in H that fulfills the conditions below:*

(a) $\lim\limits_{n\to\infty} \|x_n - x\|$ exists for all $x \in \tilde{S}$;

(b) every sequential weak limit sequence of $\{x_n\}$ is in $\tilde{S}$.

Then $\{x_n\}$ converges weakly to a point in $\tilde{S}$.

Lemma 2.7 *[12] Let H_1 and H_2 be real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator, A^* be the adjoint of A and $\beta > 0$. Let $B_2 : H_2 \to 2^{H_2}$ be a set-valued maximal monotone mapping. Define a mapping $T : H_1 \to H_1$ by $Tx = A^*(I - J_\beta^{B_2})Ax$ for all $x \in H_1$. Then, the following statements hold:*

(a) $\|(I - J_\beta^{B_2})Ax - (I - J_\beta^{B_2})Ay\|^2 \leq \langle Tx - Ty, x - y\rangle, \forall\, x, y \in H_1$;

(b) $\|Tx - Ty\|^2 \leq \|A\|^2\langle Tx - Ty, x - y\rangle, \forall\, x, y \in H_1$.

3 Main results

In this section, we introduce an iterative algorithm for solving the problem $(SVIP)$. In the whole paper, let H_1 and H_2 be real Hilbert spaces, $A : H_1 \to H_2$ is a bounded and linear operator, A^* is the adjoint operator of A, $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ be the two set-valued maximal monotone operators, Γ be the solution set of the problem $(SVIP)$, $\Gamma \neq \emptyset$. The designed algorithm is the following:

Algorithm 3.1:

Step0: Given $\lambda_1 > 0$, $\delta \in (0, \mu) \subset (0, 1)$. Let x_0, $x_1 \in H_1$ be arbitrary and $\{\beta_n\}_{n\in\mathbb{N}}$ be a sequence in $(0, \infty)$. Choose a nonnegative real sequence $\{S_n\}$ such that $\sum\limits_{n=1}^{\infty} S_n < \infty$.

Iterative steps: Update x_{n+1} as follows:

Step1: Given the iterates x_{n-1} and x_n $(n \geq 1)$ and compute

$$\begin{aligned} w_n &= x_n + \theta_n(x_n - x_{n-1}), \\ y_n &= J_{\beta_n}^{B_1}(w_n - \lambda_n A^*(I - J_{\beta_n}^{B_2})Aw_n), \end{aligned} \tag{1}$$

if $y_n = w_n$, then stop. Otherwise,

Step2: Update

$$x_{n+1} = w_n - \kappa\alpha_n d(w_n, \lambda_n), \ \kappa \in (0, 2)$$
$$d(w_n, \lambda_n) = w_n - y_n - \lambda_n[A^*(I - J_{\beta_n}^{B_2})Aw_n - A^*(I - J_{\beta_n}^{B_2})Ay_n], \tag{2}$$
$$\alpha_n = \frac{\langle w_n - y_n, d(w_n, \lambda_n)\rangle}{\|d(w_n, \lambda_n)\|^2}.$$

Step3: Update λ_n

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\delta\|w_n - y_n\|^2}{\langle A^*(I - J_{\beta_n}^{B_2})Aw_n - A^*(I - J_{\beta_n}^{B_2})Ay_n, w_n - y_n\rangle}, \lambda_n + S_n\}, & \text{if } \tilde{t}_n \neq 0, \\ \lambda_n + S_n, & \text{otherwise,} \end{cases} \tag{3}$$

where $\tilde{t}_n = \langle A^*(I - J_{\beta_n}^{B_2})Aw_n - A^*(I - J_{\beta_n}^{B_2})Ay_n, w_n - y_n\rangle$.

Step4: Set $n := n + 1$, and go to Step 1.

Remark 3.1 By Lemma 2.4, we observe that if $\Gamma \neq \emptyset$ and $y_n = w_n$, then $w_n \in \Gamma$.

Below, we use the definition of the adjoint operator A^* and the nonexpansiveness of $J_{\beta_n}^{B_2}$ to show that the step size (3) is well defined.

Lemma 3.1 *Let $\{\lambda_n\}$ be the sequence generated by Algorithm 3.1, then we obtain* $\lim\limits_{n\to\infty} \lambda_n = \lambda$ *and* $\lambda \in [\min\{\frac{\delta}{L}, \lambda_1\}, \lambda_1 + S]$. *Where* $S = \sum\limits_{n=1}^{\infty} S_n$ *and* $L = \|A\|^2$.

Proof. Note that since $J_{\beta_n}^{B_2}$ is firmly nonexpansive, then in the case of $\langle A^*(I-J_{\beta_n}^{B_2})Aw_n - A^*(I-J_{\beta_n}^{B_2})Ay_n, w_n - y_n\rangle \neq 0$, we arrive at

$$\begin{aligned}
&\langle A^*(I - J_{\beta_n}^{B_2})Aw_n - A^*(I - J_{\beta_n}^{B_2})Ay_n, w_n - y_n\rangle \\
&\qquad = \langle (I - J_{\beta_n}^{B_2})Aw_n - (I - J_{\beta_n}^{B_2})Ay_n, Aw_n - Ay_n\rangle \\
&\qquad = \|Aw_n - Ay_n\|^2 - \langle J_{\beta_n}^{B_2}Aw_n - J_{\beta_n}^{B_2}Ay_n, Aw_n - Ay_n\rangle \\
&\qquad \leq \|Aw_n - Ay_n\|^2 - \|J_{\beta_n}^{B_2}Aw_n - J_{\beta_n}^{B_2}Ay_n\|^2 \\
&\qquad \leq \|Aw_n - Ay_n\|^2 \\
&\qquad \leq \|A\|^2\|w_n - y_n\|^2,
\end{aligned}$$

which further yields that

$$\frac{\delta\|w_n - y_n\|^2}{\langle A^*(I - J_{\beta_n}^{B_2})Aw_n - A^*(I - J_{\beta_n}^{B_2})Ay_n, w_n - y_n\rangle} \geq \frac{\delta}{\|A\|^2}.$$

From the definition of λ_{n+1} and by induction, it follows that the sequence $\{\lambda_n\}$ has an upper bound $\lambda_1 + S$ and a lower bound $\min\{\frac{\delta}{\|A\|^2}, \lambda_1\}$. Let

$$(\lambda_{n+1} - \lambda_n)^+ = \max\{0, \lambda_{n+1} - \lambda_n\}, \quad (\lambda_{n+1} - \lambda_n)^- = \max\{0, -(\lambda_{n+1} - \lambda_n)\}. \tag{4}$$

Combining (4) together with the given definition of λ_n yields that

$$\sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n)^+ \leq \sum_{n=1}^{\infty} S_n < +\infty. \tag{5}$$

From (5), we deduce that the series $\sum\limits_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n)^+$ is convergence.

In what follows, we establish the convergence of the series $\sum_{n=1}^{\infty}(\lambda_{n+1}-\lambda_n)^-$. We processed by contradiction, suppose that $\sum_{n=1}^{\infty}(\lambda_{n+1}-\lambda_n)^- = +\infty$. From the fact that

$$\lambda_{n+1}-\lambda_n = (\lambda_{n+1}-\lambda_n)^+ - (\lambda_{n+1}-\lambda_n)^-,$$

we arrive at

$$\lambda_{n+1}-\lambda_1 = \sum_{n=1}^{m}(\lambda_{n+1}-\lambda_n) = \sum_{n=1}^{m}(\lambda_{n+1}-\lambda_n)^+ - \sum_{n=1}^{m}(\lambda_{n+1}-\lambda_n)^-. \tag{6}$$

Passing $m \to +\infty$ in (6), we get $\lambda_n \to -\infty (n \to \infty)$, which contradicts the boundness of λ_n. From the convergence of the series $\sum_{n=1}^{m}(\lambda_{n+1}-\lambda_n)^+$ and $\sum_{n=1}^{m}(\lambda_{n+1}-\lambda_n)^-$, taking $m \to +\infty$ again in (6). Consequently, $\lim_{n\to\infty}\lambda_n = \lambda$. On the other hand, the sequence $\{\lambda_n\}$ has a lower bound $\min\{\frac{\delta}{\|A\|^2}, \lambda_1\}$ and an upper bound $\lambda_1 + S$. Finally, we conclude that $\lambda \in [\min\{\frac{\delta}{L}, \lambda_1\}, \lambda_1 + S]$, $L = \|A\|^2$.

Remark 3.2 The step size in Algorithm 3.1 is allowed to increase from iteration to iteration. Thus, Algorithm 3.1 reduces the dependence on the initial step size λ_1.

Theorem 3.1 *Let H_1 and H_2 be the two Hilbert spaces. Let $\{\lambda_n\} \subset [\min\{\frac{\delta}{L}, \lambda_1\}, \lambda_1 + S]$, where $S = \sum_{n=1}^{\infty} S_n$ and $L = \|A\|^2$. $\{\beta_n\}_{n\geq N}$ is a sequence in $[\beta, \infty)$ for some $\beta > 0$, and $\{\theta_n\}$ be a non-decreasing sequence such that $0 \leq \theta_n < \theta < ((2-\kappa)/\kappa)/((2-\kappa)/\kappa+\phi+1))$, $\kappa \in (0,2)$, $\phi = \max\{1, (2-\kappa)/\kappa\}$. Assume that the sequence $\{x_n\}$ generated by Algorithm 3.1 converges weakly to a point $z \in \Gamma$.*

Proof. ***Part 1.*** There exists $N \geq 0$ such that for any $n \geq N$,

$$\|w_n - y_n\|^2 \leq \frac{1+\mu(2+(\lambda_1+S)\|A\|^2)}{(1-\mu)^2}\|x_{n+1}-w_n\|^2.$$

Indeed, by (2) and (3), one has

$$\begin{aligned}
&\langle w_n - y_n, d(w_n, \lambda_n)\rangle \\
&\quad = \langle w_n - y_n, w_n - y_n - \lambda_n[A^*(I - J_{\beta_n}^{B_2})Aw_n - A^*(I - J_{\beta_n}^{B_2})Ay_n]\rangle \\
&\quad = \|w_n - y_n\|^2 - \tfrac{\lambda_n}{\lambda_{n+1}}\lambda_{n+1}\langle w_n - y_n, A^*(I - J_{\beta_n}^{B_2})Aw_n - A^*(I - J_{\beta_n}^{B_2})Ay_n\rangle \\
&\quad \geq \|w_n - y_n\|^2 - \tfrac{\delta\lambda_n}{\lambda_{n+1}}\|w_n - y_n\|^2 \\
&\quad = (1 - \tfrac{\delta\lambda_n}{\lambda_{n+1}})\|w_n - y_n\|^2.
\end{aligned}$$

Since

$$\lim_{n\to\infty}(1 - \frac{\delta\lambda_n}{\lambda_{n+1}}) = 1 - \delta > 1 - \mu,$$

where $\delta \in (0,\mu) \subset (0,1)$. Then, $\exists N \geq 0$ such that $\forall n \geq N$, $1 - \frac{\delta\lambda_n}{\lambda_{n+1}} > 0$. So, for any $n \geq N$,

$$\langle w_n - y_n, d(w_n, \lambda_n)\rangle \geq (1-\mu)\|w_n - y_n\|^2. \tag{7}$$

By Lemma 3.1, we get that

$$\begin{aligned}
\|d(w_n,\lambda_n)\|^2 &= \|w_n - y_n - \lambda_n[A^*(I-J^{B_2}_{\beta_n})Aw_n - A^*(I-J^{B_2}_{\beta_n})Ay_n]\|^2\\
&= \|w_n - y_n\|^2 + \lambda_n^2\|A^*(I-J^{B_2}_{\beta_n})Aw_n - A^*(I-J^{B_2}_{\beta_n})Ay_n\|^2\\
&- \tfrac{2\lambda_n}{\lambda_{n+1}}\lambda_{n+1}\langle w_n - y_n, A^*(I-J^{B_2}_{\beta_n})Aw_n - A^*(I-J^{B_2}_{\beta_n})Ay_n\rangle\\
&\le \|w_n - y_n\|^2 + \tfrac{\|A\|^2\lambda_n^2}{\lambda_{n+1}}\lambda_{n+1}\langle w_n - y_n, A^*(I-J^{B_2}_{\beta_n})Aw_n - A^*(I-J^{B_2}_{\beta_n})Ay_n\rangle\\
&+ \tfrac{2\lambda_n}{\lambda_{n+1}}|\lambda_{n+1}\langle w_n - y_n, A^*(I-J^{B_2}_{\beta_n})Aw_n - A^*(I-J^{B_2}_{\beta_n})Ay_n\rangle|\\
&\le \|w_n - y_n\|^2 + \tfrac{\|A\|^2\delta\lambda_n^2}{\lambda_{n+1}}\|w_n - y_n\|^2 + \tfrac{2\delta\lambda_n}{\lambda_{n+1}}\|w_n - y_n\|^2\\
&\le (1 + \tfrac{\|A\|^2\delta(\lambda_1+S)\lambda_n}{\lambda_{n+1}} + \tfrac{2\delta\lambda_n}{\lambda_{n+1}})\|w_n - y_n\|^2.
\end{aligned}$$

Since

$$\lim_{n\to\infty}(1+\frac{\|A\|^2\delta(\lambda_1+S)\lambda_n}{\lambda_{n+1}}+\frac{2\delta\lambda_n}{\lambda_{n+1}}) = 1+\|A\|^2\delta(\lambda_1+S)+2\delta < 1+\|A\|^2\mu(\lambda_1+S)+2\mu,$$

where $\delta\in(0,\mu)\subset(0,1)$. Then, $\exists N\ge 0$ such that $\forall n\ge N$, $1+\frac{\|A\|^2\delta(\lambda_1+S)\lambda_n}{\lambda_{n+1}}+\frac{2\delta\lambda_n}{\lambda_{n+1}} < 1+\|A\|^2\mu(\lambda_1+S)+2\mu$. Therefore, for all $n\ge N$,

$$\|d(w_n,\lambda_n)\|^2 \le (1+\|A\|^2\mu(\lambda_1+S)+2\mu)\|w_n-y_n\|^2. \tag{8}$$

Combining (7) and (8), it holds that for all $n\ge N$,

$$\alpha_n = \frac{\langle w_n - y_n, d(w_n,\lambda_n)\rangle}{\|d(w_n,\lambda_n)\|^2} \ge \frac{1-\mu}{1+\mu(2+(\lambda_1+S)\|A\|^2)}. \tag{9}$$

Employing (2) and (9) , we obtain that for all $n\ge N$,

$$\begin{aligned}
\langle w_n - y_n, d(w_n,\lambda_n)\rangle &= \alpha_n\|d(w_n,\lambda_n)\|^2\\
&= \tfrac{1}{\alpha_n}\|\alpha_n d(w_n,\lambda_n)\|^2\\
&= \tfrac{1}{\kappa^2\alpha_n}\|x_{n+1}-w_n\|^2\\
&\le \tfrac{1+\mu(2+(\lambda_1+S)\|A\|^2)}{\kappa^2(1-\mu)}\|x_{n+1}-w_n\|^2.
\end{aligned} \tag{10}$$

This, along with (7), verifies that for all $n\ge N$,

$$\|w_n-y_n\|^2 \le \frac{1}{1-\mu}\langle w_n-y_n, d(w_n,\lambda_n)\rangle \le \frac{1+\mu(2+(\lambda_1+S)\|A\|^2)}{\kappa^2(1-\mu)^2}\|x_{n+1}-w_n\|^2.$$

Hence, ***Part 1*** is proved.

Part 2. For any $n\ge N$,

$$\|x_{n+1}-z\|^2 \le (1+\theta_n)\|x_n-z\|^2 - \theta_n\|x_{n-1}-z\|^2 + \theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2 - \|x_{n+1}-w_n\|^2,\ \forall z\in\varGamma.$$

Let $z\in\varGamma$, then $z\in B_1^{-1}(0)$, by Lemma 2.3 (e) and the definition of y_n, we have

$$\langle y_n - z, w_n - y_n - \lambda_n A^*(I-J^{B_2}_{\beta_n})Aw_n\rangle \ge 0. \tag{11}$$

Additionally, one obtain $Az\in B_2^{-1}(0)$ and $Az\in Fix(J^{B_2}_{\beta})$. This verifies that $J^{B_2}_{\beta_n}Az = Az$. By Lemma 2.3 (d), one has

$$\langle y_n - z, \lambda_n (A^*(I - J_{\beta_n}^{B_2})Ay_n - A^*(I - J_{\beta_n}^{B_2})Az\rangle \geq \lambda_n \|(I - J_{\beta_n}^{B_2})Ay_n - (I - J_{\beta_n}^{B_2})Ay\|^2.$$

This means that

$$\langle y_n - z, \lambda_n A^*(I - J_{\beta_n}^{B_2})Ay_n\rangle \geq \lambda_n \|(I - J_{\beta_n}^{B_2})Ay_n\|^2. \tag{12}$$

In view of (11) and (12), we have

$$\begin{aligned}
\langle y_n - z, d(w_n, \lambda_n)\rangle &= \langle y_n - z, w_n - y_n - \lambda_n (A^*(I - J_{\beta_n}^{B_2})Aw_n - A^*(I - J_{\beta_n}^{B_2})Ay_n)\rangle \\
&= \langle y_n - z, w_n - y_n - \lambda_n A^*(I - J_{\beta_n}^{B_2})Aw_n\rangle \\
&+ \lambda_n \langle y_n - z, A^*(I - J_{\beta_n}^{B_2})Ay_n\rangle \\
&\geq \lambda_n \|(I - J_{\beta_n}^{B_2})Ay_n\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
\langle w_n - z, d(w_n, \lambda_n)\rangle &= \langle w_n - y_n + y_n - z, d(w_n, \lambda_n)\rangle \\
&= \langle w_n - y_n, d(w_n, \lambda_n)\rangle + \langle y_n - z, d(w_n, \lambda_n)\rangle \\
&\geq \langle w_n - y_n, d(w_n, \lambda_n)\rangle + \lambda_n \|(I - J_{\beta_n}^{B_2})Ay_n\|^2.
\end{aligned} \tag{13}$$

From (2),(9) and (13), we obtain that $\forall n \geq N$,

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|w_n - \kappa\alpha_n d(w_n, \lambda_n) - z\|^2 \\
&= \|w_n - z\|^2 - 2\kappa\alpha_n \langle w_n - z, d(w_n, \lambda_n)\rangle + \kappa^2\alpha_n^2 \|d(w_n, \lambda_n)\|^2 \\
&\leq \|w_n - z\|^2 - 2\kappa\alpha_n \langle w_n - y_n, d(w_n, \lambda_n)\rangle - 2\kappa\alpha_n\lambda_n \|(I - J_{\beta_n}^{B_2})Ay_n\|^2 \\
&+ \kappa^2\alpha_n^2 \|d(w_n, \lambda_n)\|^2 \\
&= \|w_n - z\|^2 - \kappa\alpha_n^2(2 - \kappa)\|d(w_n, \lambda_n)\|^2 - 2\kappa\alpha_n\lambda_n \|(I - J_{\beta_n}^{B_2})Ay_n\|^2 \\
&\leq \|w_n - z\|^2 - \tfrac{2-\kappa}{\kappa}\|x_{n+1} - w_n\|^2 - 2\kappa\alpha_n\lambda_n \|(I - J_{\beta_n}^{B_2})Ay_n\|^2
\end{aligned} \tag{14}$$

Now, by Lemma 2.1 and (1), we get

$$\|w_n - z\|^2 = (1 + \theta_n)\|x_n - z\|^2 - \theta_n \|x_{n-1} - z\|^2 + \theta_n(1 + \theta_n)\|x_n - x_{n-1}\|^2. \tag{15}$$

Combining (9),(14) and (15) implies that $\forall n \geq N$,

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq (1 + \theta_n)\|x_n - z\|^2 - \theta_n \|x_{n-1} - z\|^2 + \theta_n(1 + \theta_n)\|x_n - x_{n-1}\|^2 \\
&- \tfrac{2-\kappa}{\kappa}\|x_{n+1} - w_n\|^2 - 2\kappa\alpha_n\lambda_n \|(I - J_{\beta_n}^{B_2})Ay_n\|^2 \\
&\leq (1 + \theta_n)\|x_n - z\|^2 - \theta_n \|x_{n-1} - z\|^2 \\
&+ \theta_n(1 + \theta_n)\|x_n - x_{n-1}\|^2 - \tfrac{2-\kappa}{\kappa}\|x_{n+1} - w_n\|^2.
\end{aligned}$$

Hence, we have ***Part 2.***

Part 3. The sequence $\{x_n\}$ is bounded.

Note that

$$\begin{aligned}\|x_{n+1}-w_n\|^2 &= \|x_{n+1}-x_n-\theta_n(x_n-x_{n-1})\|^2\\ &= \|x_{n+1}-x_n\|^2+\theta_n^2\|x_n-x_{n-1}\|^2-2\theta_n\langle x_{n+1}-x_n, x_n-x_{n-1}\rangle\\ &\geq \|x_{n+1}-x_n\|^2+\theta_n^2\|x_n-x_{n-1}\|^2-2\theta_n\|x_{n+1}-x_n\|\|x_n-x_{n-1}\|\\ &\geq \|x_{n+1}-x_n\|^2+\theta_n^2\|x_n-x_{n-1}\|^2-\theta_n\|x_{n+1}-x_n\|^2-\theta_n\|x_n-x_{n-1}\|^2\\ &= (1-\theta_n)\|x_{n+1}-x_n\|^2+(\theta_n^2-\theta_n)\|x_n-x_{n-1}\|^2.\end{aligned} \tag{16}$$

Substituting (16) into ***Part 2***, we obtain that $\forall n \geq N$,

$$\begin{aligned}\|x_{n+1}-z\|^2 \leq\ & (1+\theta_n)\|x_n-z\|^2-\theta_n\|x_{n-1}-z\|^2+\theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2\\ &-(1-\theta_n)\|x_{n+1}-x_n\|^2-(\theta_n^2-\theta_n)\|x_n-x_{n-1}\|^2\\ \leq\ & (1+\theta_n)\|x_n-z\|^2-\theta_n\|x_{n-1}-z\|^2-(1-\theta_n)\|x_{n+1}-x_n\|^2\\ &+\theta_n(1+\theta_n+(1-\theta_n)\tfrac{2-\kappa}{\kappa})\|x_n-x_{n-1}\|^2.\end{aligned} \tag{17}$$

Observe that $1+\theta_n+(1-\theta_n)\frac{2-\kappa}{\kappa} \leq 1+\max\{1, \frac{2-\kappa}{\kappa}\}$, denote $\phi = \max\{1, \frac{2-\kappa}{\kappa}\}$. Therefore, $1+\theta_n+(1-\theta_n)\frac{2-\kappa}{\kappa} \leq 1+\phi$. So, it follows from (17) that $\forall n \geq N$,

$$\begin{aligned}\|x_{n+1}-z\|^2 \leq\ & (1+\theta_n)\|x_n-z\|^2-\theta_n\|x_{n-1}-z\|^2-(1-\theta_n)(\tfrac{2-\kappa}{\kappa})\|x_{n+1}-x_n\|^2\\ &+\theta_n(1+\phi)\|x_n-x_{n-1}\|^2.\end{aligned} \tag{18}$$

Let

$$\varphi_n = \|x_n-z\|^2-\theta_n\|x_{n-1}-z\|^2+\theta_n(1+\phi)\|x_n-x_{n-1}\|^2. \forall n \geq N.$$

Recall that θ_n is non-decreasing, then $\forall n \geq N$,

$$\begin{aligned}&\varphi_{n+1}-\varphi_n\\ &\quad\leq -\tfrac{(1-\theta_n)(2-\kappa)}{\kappa}\|x_{n+1}-x_n\|^2+\theta_n(1+\phi)\|x_n-x_{n-1}\|^2\\ &\quad+\theta_{n+1}(1+\phi)\|x_{n+1}-x_n\|^2-\theta_n(1+\phi)\|x_n-x_{n-1}\|^2\\ &\quad= -[\tfrac{(1-\theta_n)(2-\kappa)}{\kappa}-\theta_{n+1}(1+\phi)]\|x_{n+1}-x_n\|^2\\ &\quad= -[\tfrac{(2-\kappa)}{\kappa}-(\tfrac{(2-\kappa)}{\kappa}\theta_n+\theta_{n+1}(1+\phi))]\|x_{n+1}-x_n\|^2\\ &\quad\leq -[\tfrac{(2-\kappa)}{\kappa}-\theta(\tfrac{(2-\kappa)}{\kappa}+1+\phi)]\|x_{n+1}-x_n\|^2.\end{aligned}$$

Choose $K = \frac{(2-\kappa)}{\kappa}-\theta(\frac{(2-\kappa)}{\kappa}+1+\phi)$, by the condition on θ, we know $K > 0$. Thus,

$$\varphi_{n+1}-\varphi_n \leq -K\|x_{n+1}-x_n\|^2.$$

This shows that $\{\varphi_n\}$ is a non-increasing sequence, then $\varphi_n \geq \|x_n-z\|^2-\theta_n\|x_{n-1}-z\|^2$. Bellow, for any $n \geq N$, one has

$$\begin{aligned}\|x_n - z\|^2 &\le \theta_n \|x_{n-1} - z\|^2 + \varphi_n \\ &\le \theta \|x_{n-1} - z\|^2 + \varphi_N \\ &\le \theta(\theta \|x_{n-2} - z\|^2 + \varphi_{n-1}) + \varphi_N \\ &\le \cdots \le \theta^{n-N} \|x_N - z\|^2 + \varphi_N(\theta^{n-N} + \cdots + \theta + 1) \\ &\le \theta^{n-N} \|x_N - z\|^2 + \tfrac{\varphi_N}{1-\theta}.\end{aligned}$$

Since $\varphi_{n+1} \ge -\theta_{n+1}\|x_n - z\|^2$, for all $n \ge N$, one gets that

$$\begin{aligned}-\varphi_{n+1} \le \theta_{n+1}\|x_n - z\|^2 \le \theta\|x_n - z\|^2 \le \theta^{n-N+1}\|x_N - z\|^2 + \tfrac{\theta\varphi_N}{1-\theta} \\ \le \|x_N - z\|^2 + \tfrac{\theta\varphi_N}{1-\theta}.\end{aligned} \tag{19}$$

it verifies from (19) that

$$K \sum_{j=N}^{n} \|x_{j+1} - x_j\|^2 \le \varphi_N - \varphi_{n+1} \le \|x_N - z\|^2 + \frac{\varphi_N}{1-\theta},$$

which further implies that

$$K \sum_{j=N}^{\infty} \|x_{j+1} - x_j\|^2 \le +\infty.$$

This shows that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{20}$$

Also, since $\theta_n < \theta$, we have

$$\begin{aligned}\|x_{n+1} - w_n\|^2 = \|x_{n+1} - x_n\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 \\ - 2\theta_n \langle x_{n+1} - x_n, x_n - x_{n-1}\rangle \to 0.\end{aligned} \tag{21}$$

For $\forall z \in \Gamma$, using (18),(20) and Lemma 2.6, one arrives at

$$\lim_{n\to\infty} \|x_n - z\|^2 = t.$$

Using(15), one also has

$$\lim_{n\to\infty} \|w_n - z\|^2 = t.$$

This verifies that $\{x_n\}$,$\{w_n\}$ and $\{y_n\}$ are bounded, the proof of ***Part 3*** is completed.

Part 4. We establish the sequential weak limit points of $\{x_n\}$ is in Γ. Now, by (1)and (21), we derive that

$$0 \le \|x_n - w_n\| = \theta_n \|x_{n+1} - x_n\| \le \theta \|x_{n+1} - x_n\| \to 0. \tag{22}$$

Employing ***Part 1*** and (22), we show that

$$\lim_{n\to\infty} \|y_n - w_n\| = 0. \tag{23}$$

By (14), we get

$$\|x_{n+1} - z\|^2 \le \|w_n - z\|^2 - 2\kappa\alpha_n\lambda_n \|(I - J_{\beta_n}^{B_2})Ay_n\|^2,$$

From (9), it verifies that $\forall n \geq N$,

$$\|x_{n+1}-z\|^2 \leq \|w_n-z\|^2 - 2\kappa\frac{1-\mu}{1+\mu(2+(\lambda_1+S)\|A\|^2)}\lambda_n\|(I-J_{\beta_n}^{B_2})Ay_n\|^2,$$

that is, $\forall n \geq N$,

$$2\kappa\frac{1-\mu}{1+\mu(2+\lambda\|A\|^2)}\lambda_n\|(I-J_{\beta_n}^{B_2})Ay_n\|^2 \leq \|w_n-z\|^2 - \|x_{n+1}-z\|^2 \leq \|x_{n+1}-w_n\|(\|w_n-z\|+\|x_{n+1}-z\|).$$

So, by means of (21) and $\lim\limits_{n\to\infty}\lambda_n=\lambda>0$, we conclude that

$$\lim_{n\to\infty}\|(I-J_{\beta_{n_k}}^{B_2})Ay_n\|=0. \tag{24}$$

Moreover, we have

$$\begin{aligned}\|Aw_n-J_{\beta_n}^{B_2}Aw_n\| &\leq \|Aw_n-J_{\beta_n}^{B_2}Aw_n-Ay_n+J_{\beta_n}^{B_2}Ay_n\|+\|(I-J_{\beta_n}^{B_2})Ay_n\|\\ &\leq 2\|A\|\|w_n-y_n\|+\|(I-J_{\beta_n}^{B_2})Ay_n\|.\end{aligned}$$

This implies, by (23) and (24),that

$$\lim_{n\to\infty}\|(I-J_{\beta_n}^{B_2})Aw_n\|=0. \tag{25}$$

From (25) and Lemma 2.3 (c), we deduce that

$$\lim_{n\to\infty}\|(I-J_{\beta}^{B_2})Aw_n\| \leq \lim_{n\to\infty}\|(I-J_{\beta_n}^{B_2})Aw_n\|=0. \tag{26}$$

On the other hand,

$$\begin{aligned}\|y_n-J_{\beta_n}^{B_1}w_n\| &= \|J_{\beta_n}^{B_1}(w_n-\lambda_nA^*(I-J_{\beta_n}^{B_2})Aw_n)-J_{\beta_n}^{B_1}w_n\|\\ &\leq \lambda_n\|A^*\|\|(I-J_{\beta_n}^{B_2})Aw_n\|.\end{aligned} \tag{27}$$

Note that $\lim\limits_{n\to\infty}\lambda_n=\lambda>0$, which together with (25) and (27) implies that

$$\lim_{n\to\infty}\|y_n-J_{\beta_n}^{B_2}w_n\|=0.$$

This along with $\lim\limits_{n\to\infty}\|y_n-w_n\|=0$, further yields that

$$\begin{aligned}\|w_n-J_{\beta_n}^{B_1}w_n\| &= \|w_n-y_n+y_n-J_{\beta_n}^{B_1}w_n\|\\ &\leq \|w_n-y_n\|+\|y_n-J_{\beta_n}^{B_2}w_n\|\\ &\to 0 \quad as \quad n\to\infty.\end{aligned} \tag{28}$$

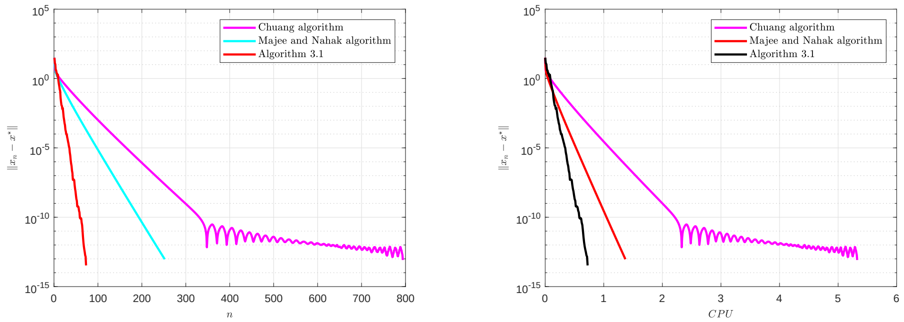
By (28) and Lemma 2.3 (c), we conclude that

$$\lim_{n\to\infty}\|w_n-J_{\beta}^{B_1}w_n\| \leq \lim_{n\to\infty}\|(I-J_{\beta_n}^{B_2})w_n\|=0. \tag{29}$$

Since $\{w_n\}$ is bounded, then there is a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \rightharpoonup z \in H_1$. Note that A is a bounded and linear operator, we get $Aw_{n_k} \rightharpoonup Az$. By (29), Lemma 2.2 and Lemma 2.3(b), we have $z \in Fix(J_{\beta}^{B_1})$. According to (26), Lemma 2.2 and Lemma 2.3(b), we get $Az \in Fix(J_{\beta}^{B_2})$. As a result, we derive that $z \in \mathit{\Gamma}$. Employing Lemma 2.6, we have $w_n \rightharpoonup z \in \mathit{\Gamma}$.

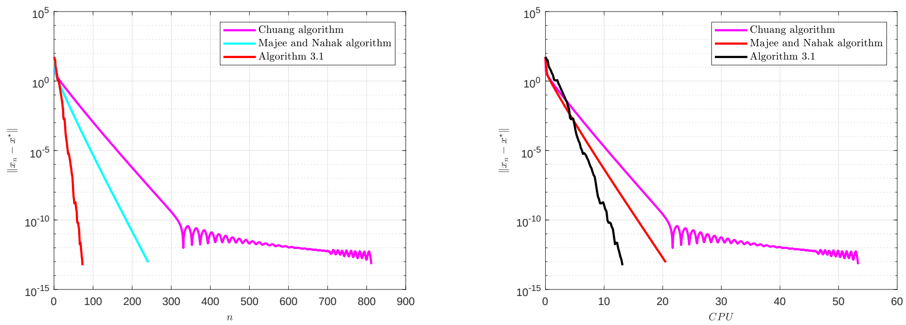
4 Numerical Experiments

In this section, we provide numerical experiments to demonstrate our algorithm and compare them with Chuang's algorithm [12] and the algorithm of Majee and Nahak [16]. We generate random data by the generator *randn* in software MATLAB 2017a. All the numerical experiments are conducted on an Intel(R) Core(TM)i7-6700 CPU@3.40GHZ PC with 8GB of RAM running 64-bit Windows operating system.



(a) Number of Iterations(n) (b) Iterative times(CPU)

Fig. 1 $\|x_n - x^*\|$ is the stopping criterion, n and CPU stand for Number of Iterations and Iterative times measured in seconds, respectively. Comparison with $\epsilon = 10^{-13}$ for the problem 4.1 with $\boldsymbol{m}$**=1000**.



(a) Number of Iterations(n) (b) Iterative times(CPU)

Fig. 2 $\|x_n - x^*\|$ is the stopping criterion, n and CPU stand for Number of Iterations and Iterative times measured in seconds, respectively. Comparison with $\epsilon = 10^{-13}$ for the problem 4.1 with $\boldsymbol{m}$**=3000**.

Example 4.1 [26] Let matrices $A, A_1, A_2 : \mathbb{R}^m \to \mathbb{R}^m$ are generated from a normal distribution with mean zero and unit variance. Let $B_1, B_2 : \mathbb{R}^m \to \mathbb{R}^m$ be defined by $B_1(x) = A_1^* A_1 x$ and $B_2(y) = A_2^* A_2 y$. We mainly find a point $x^* = (x_1, \cdots, x_m)^T \in \mathbb{R}^m$ such that $B_1(x) = (0, \cdots, 0)^T$ and $B_2(Ax) = (0, \cdots, 0)^T$.

In fact, $x_1 = 0, \cdots, x_m = 0$. Let $\epsilon > 0$ and the stopping criterion is given by $\|x_n - x^*\| < \epsilon$. The iterative process is started with initial $x_0 = x_1 = (1, \cdots, 1)^T$, $\theta_n = 0.3$ suggested in [16], and

$$\theta_n = \begin{cases} \frac{1}{n^k \|x_n - x_{n-1}\|^2}, & \text{if } n \geq 2, \\ 0, & \text{if } n = 1. \end{cases}$$

for Chuang's algorithm. Set $\lambda = \lambda_n = \frac{0.3}{\|A\|^2}$, $k = 9$, $\beta_n = 0.19$ for Chuang's algorithm, $\lambda_n = \frac{0.3}{\|A\|^2}, \beta_n = 0.19$, $\kappa = 1.2$, for the algorithm of Majee and Nahak, the initial step size $\lambda_1 = 0.001, \delta = 0.3, S_n = \frac{0.3}{(n+1)^{1.1}}$, $\beta_n = 0.19$, $\kappa = 1.2$ for Algorithm 3.1. The results are shown in **Fig.1** and **Fig.2**.

For each m, as shown in **Fig.1** and **Fig.2**, our method performs better. This shows that the variable step size scheme (3) is highly recommended.

5 Applications

5.1 Split feasibility problem

Let C and Q be non-empty closed convex subsets of infinite dimensional real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a linear and bounded operator. The *split feasibility problem* (SFP) is the following:

$$\text{Find } \tilde{x} \in C \text{ such that } A\tilde{x} \in Q.$$

Let S be the solution set of the problem (SFP). For the study of the problem (SFP), a lot of related results can be found in existing references. Here, the reader can refer to [26–28] for more detail about it. Let $f : H \to (-\infty, \infty)$ be be a proper lower semi-continuous (lsc) convex function. Then the subdifferential ∂f of f is defined by

$$\partial f(x) = \{\xi \in H : f(x) - f(y) \le \langle \xi, x - y\rangle,\ \forall\ y \in H\},\ \forall\ x \in H.$$

Let C be non-empty closed convex subset of an infinite dimensional real Hilbert space H, and δ_C be the indicator function of C, namely,

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{otherwise.} \end{cases}$$

In addition, we define the normal cone $N_C(\tilde{\mu})$ of C at $\tilde{\mu}$ by

$$N_C(\tilde{\mu}) = \{\xi \in H : \langle \xi, \tilde{v} - \tilde{\mu}\rangle \le 0,\ \forall\ \tilde{v} \in C\}.$$

Since δ_C is a proper, lsc and convex function on H. So, the subdifferential $\partial\delta_C$ of δ_C becomes a maximal monotone operator. Hence, we define the resolvent operator $J_\lambda^{\partial\delta_C}$ of $\partial\delta_C$ for arbitrary $\lambda > 0$ by

$$J_\lambda^{\partial\delta_C}(x) = (I + \lambda\partial\delta_C)^{-1}x,\ \forall x \in H.$$

We further see that for all $x \in C$,

$$\begin{aligned} \partial\delta_C(x) &= \{\xi \in H : \delta_C(x) - \delta_C(y) \le \langle \xi, x - y\rangle,\ \forall\ y \in H\} \\ &= \{\xi \in H : \langle \xi, y - x\rangle \le 0,\ \forall\ y \in C\} \\ &= N_C(x). \end{aligned}$$

Thereby, for all $\lambda > 0$, we have

$$\begin{aligned} y = J_\lambda^{\partial\delta_C}(x) &\Leftrightarrow x \in y + \lambda\partial\delta_C(y) \\ &\Leftrightarrow x - y \in \lambda\partial\delta_C(y) \\ &\Leftrightarrow \langle x - y, z - y\rangle \leq 0 \quad \forall z \in C \\ &\Leftrightarrow y = P_C x \end{aligned}$$

The algorithm corresponding to the problem (SFP) is the following :

Algorithm 5.1:

Step0: Given $\lambda_1 > 0$, $\delta \in (0, \mu) \subset (0, 1)$. Let $x_0,\ x_1 \in H_1$ be arbitrary and $\{\beta_n\}_{n\in\mathbb{N}}$ be a sequence in $(0, \infty)$. Choose a nonnegative real sequence $\{S_n\}$ such that $\sum\limits_{n=1}^{\infty} S_n < \infty$.

Iterative steps: Update x_{n+1} as follows:

Step1: Given the iterates x_{n-1} and x_n $(n \geq 1)$, compute

$$\begin{aligned} w_n &= x_n + \theta_n(x_n - x_{n-1}), \\ y_n &= P_C(w_n - \lambda_n A^*(I - P_Q)Aw_n), \end{aligned}$$

if $y_n = w_n$, then stop and y_n is a solution of the problem (SFP). Otherwise,

Step2: Update

$$\begin{aligned} &x_{n+1} = w_n - \kappa\alpha_n d(w_n, \lambda_n),\ \kappa \in (0, 2). \\ &d(w_n, \lambda_n) = w_n - y_n - \lambda_n[A^*(I - P_Q)Aw_n - A^*(I - P_Q)Ay_n], \\ &\alpha_n = \tfrac{\langle w_n - y_n, d(w_n,\lambda_n)\rangle}{\|d(w_n,\lambda_n)\|^2}. \end{aligned}$$

Step3: Update the step size λ_n,

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\delta\|w_n - y_n\|^2}{\langle A^*(I-P_Q)Aw_n - A^*(I-P_Q)Ay_n, w_n - y_n\rangle}, \lambda_n + S_n\}, & \text{if } \langle A^*(I - P_Q)Aw_n - A^*(I - P_Q)Ay_n, w_n - y_n\rangle \neq 0, \\ \lambda_n + S_n, & \text{otherwise.} \end{cases}$$

Step4: Set $n := n + 1$, and go to Step 1.

By theorem 3.1, we can obtain the following weak convergence result of the proposed algorithm for the problem (SFP).

Theorem 5.1 *Let C and Q be non-empty closed convex subsets of infinite dimensional real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a linear and bounded operator. A^* is the adjoint operator of A. Let $0 \leq \theta_n < \theta < ((2 - \kappa)/\kappa)/((2 - \kappa)/\kappa + \phi + 1))$, $\kappa \in (0, 2), \{\beta_n\}_{n\in\mathbb{N}}$ be a sequence in $(\beta, \infty) \subset (0, \infty)$, Γ be the solution set of the problem (SFP) and the sequence $\{\lambda_n\} \subset [\min\{\frac{\delta}{\|A\|^2}, \lambda_1\}, \lambda_1 + S]$. Where $S = \sum\limits_{n=1}^{\infty} S_n$.*

Suppose that $\Gamma \neq \emptyset$, then the sequence $\{x_n\}$ generated by Algorithm 5.1 converges weakly to a point of Γ.

Example 5.1[15] Let $H_1 = H_2 = L^2([0, 1])$ with norm $\|x\|_{L^2} = (\int_0^1 x(t)^2 dt)^{\frac{1}{2}}$ and inner product $\langle x, y\rangle = \int_0^1 x(t)y(t)dt,\ x, y \in L^2([0, 1])$. Let $C = \{x \in L^2([0, 1]) : \|x\|_{L^2} \leq 1\}$ and $Q = \{x \in L^2([0, 1]) : \langle x, t\rangle = 0\}$. Set $Ax(t) = \frac{x(t)}{2}$. Let $\epsilon > 0$ and the stopping criterion is given by $\|x_{n+1} - x_n\|_{L^2} < \epsilon$, $\theta_n = 0.3$ suggested in [16], and

$$\theta_n = \begin{cases} \frac{1}{n^k \|x_n - x_{n-1}\|^2}, & \text{if } n \geq 2, \\ 0, & \text{if } n = 1. \end{cases}$$

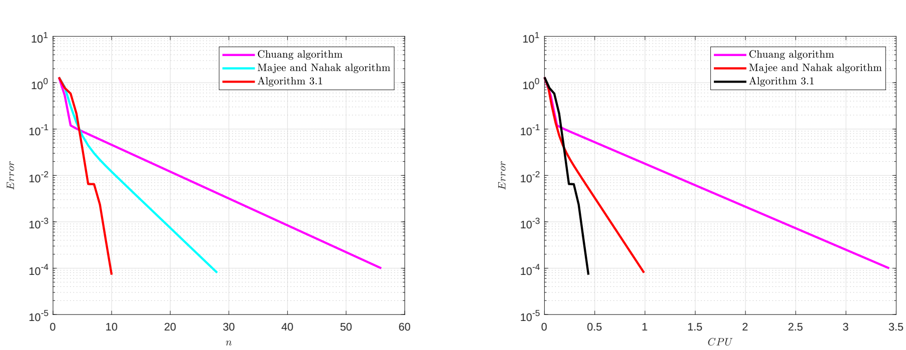
for Chuang'algorithm. Set $\lambda_n = 0.499,\ k = 9$ for Chuang's algorithm, $\lambda_n = 0.499, \kappa = 1.2$ for the algorithm of Majee and Nahak, and the initial step size $\lambda_1 = 0.499,\ \delta = 0.499,\ S_n = \frac{0.3}{(n+1)^{1.1}},\ \kappa = 1.2$ for Algorithm 3.1. we choose two initial points:

(a) $x_0 = t^4,\ x_1 = t + 1$;

(b)$x_0 = 2\sin(t),\ x_1 = \cos(t)$.

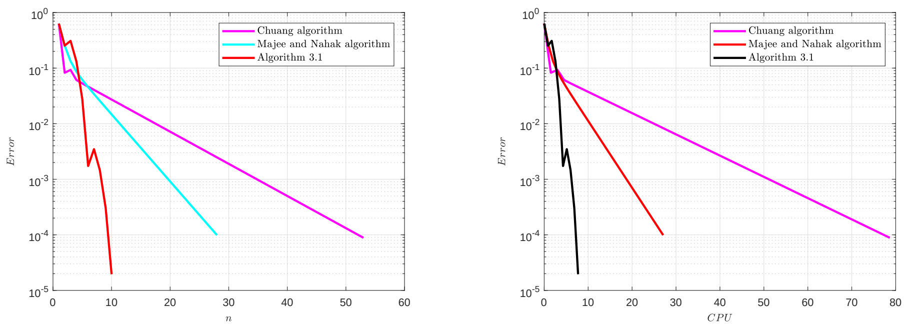
The results are shown in **Fig.3** and **Fig.4**.

From **Fig.3** and **Fig.4**, it is clear that Algorithm 3.1 has a better convergence than other compared algorithms do in number of iterations and iterative times in each given initial points.



(a) Number of Iterations(n) (b) Iterative times(CPU)

Fig. 3 $Error$ is the stopping criterion, n and CPU stand for Number of Iterations and Iterative times measured in seconds, respectively. Comparison with $\epsilon = 10^{-4}$ for the problem 5.1 with the **initial point(a)**.



(a) Number of Iterations(n) (b) Iterative times(CPU)

Fig. 4 $Error$ is the stopping criterion, n and CPU stand for Number of Iterations and Iterative times measured in seconds, respectively. Comparison with $\epsilon = 10^{-4}$ for the problem 5.1 with the **initial point(b)**.

5.2 Split minimization problem

This subsection is devoted to the study of the *split minimization problem* (*SMP*):

$$\text{Find } \tilde{x} \in H_1 \text{ such that } \tilde{x} \in \arg\min_{x \in H_1} \tilde{f}(x) \text{ and } A\tilde{x} \in \arg\min_{y \in H_2} g(y),$$

where $\tilde{f} : H_1 \to \mathbb{R}$ and $g : H_2 \to \mathbb{R}$ are proper,lower semi-continuous convex(lsc)functions.

In a real Hilbert space H, we define the proximal operator of $\tilde{f}$ by

$$\mathrm{prox}_{\beta,\tilde{f}}(x) = \underset{x\in H}{\operatorname{argmin}}\{\tilde{f}(x) + \frac{1}{2\beta}\|x-y\|^2\}, \beta > 0,\ \forall y \in H.$$

Recall that

$$\mathrm{prox}_{\beta,\tilde{f}}(x) = (I + \beta\partial f)^{-1}(x) = J_{\beta}^{\partial\tilde{f}}(x),$$

where $\partial\tilde{f}$ is the subdifferential of $\tilde{f}$ defined as

$$\partial\tilde{f}(x) = \{\tilde{x} \in H : \tilde{f}(x) + \langle y - x, \tilde{x}\rangle \le \tilde{f}(y),\ \forall y \in H\}.$$

In view of [29], we know that $\partial\tilde{f}$ is a maximal monotone operator and $\mathrm{prox}_{\beta,\tilde{f}}$ is firmly nonexpansive.

The algorithm corresponding to the problem (SMP) is the following:

Algorithm 5.2:

Step0: Given $\lambda_1 > 0$, $\delta \in (0,\mu) \subset (0,1)$. Let $x_0,\ x_1 \in H_1$ be arbitrary and $\{\beta_n\}_{n\in\mathbb{N}}$ be a sequence in $(\beta,\infty) \subset (0,\infty)$. Choose a nonnegative real sequence $\{S_n\}$ such that $\sum\limits_{n=1}^{\infty} S_n < \infty$.

Iterative steps: Update x_{n+1} as follows:

Step1: Given the iterates x_{n-1} and x_n $(n \ge 1)$, compute

$$\begin{aligned} w_n &= x_n + \theta_n(x_n - x_{n-1}), \\ y_n &= \mathrm{prox}_{\beta_n,\tilde{f}}(w_n - \lambda_n A^*(I - \mathrm{prox}_{\beta_n,g})Aw_n), \end{aligned}$$

if $y_n = w_n$, then stop and y_n is a solution of the problem (SMP). Otherwise,

Step2: Update

$$\begin{aligned} &x_{n+1} = w_n - \phi\alpha_n d(w_n,\lambda_n),\ \phi \in (0,2). \\ &d(w_n,\lambda_n) = w_n - y_n - \lambda_n[A^*(I - \mathrm{prox}_{\beta_n,g})Aw_n - A^*(I - \mathrm{prox}_{\beta_n,g})Ay_n], \\ &\alpha_n = \tfrac{\langle w_n - y_n, d(w_n,\lambda_n)\rangle}{\|d(w_n,\lambda_n)\|^2}. \end{aligned}$$

Step4: Update the step size λ_n

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\delta\|w_n - y_n\|^2}{\langle A^*(I-\mathrm{prox}_{\beta_n,g})Aw_n - A^*(I-\mathrm{prox}_{\beta_n,g})Ay_n, w_n - y_n\rangle}, \lambda_n + S_n\}, & \text{if } \tilde{w}_n \neq 0, \\ \lambda_n + S_n, & \text{otherwise,} \end{cases}$$

where $\tilde{w}_n = \langle A^*(I - \mathrm{prox}_{\beta_n,g})Aw_n - A^*(I - \mathrm{prox}_{\beta_n,g})Ay_n, w_n - y_n\rangle$.

Step5: Set $n := n + 1$, and go to Step 1.

By theorem 3.1, we get the following weak convergence result of the proposed algorithm for the problem (SMP).

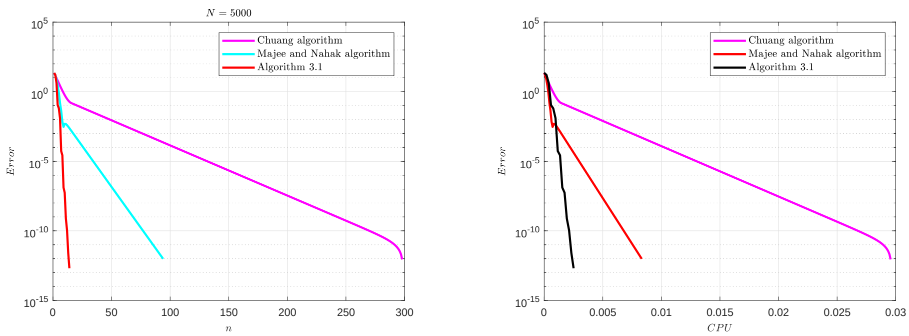
Theorem 5.2 *Let H_1 and H_2 be the two infinite dimensional real Hilbert spaces. Let $\tilde{f}$, g, A be the operators defined as above. Let $\{\beta_n\}_{n\in\mathbb{N}}$ be a sequence in $(\beta,\infty) \subset (0,\infty)$, $0 \le \theta_n < \theta < ((2-\kappa)/\kappa)/((2-\kappa)/\kappa+\phi+1))$, $\kappa \in (0,2)$, and Γ be the solution set of the problem (SMP). The sequence $\{\lambda_n\} \subset [\min\{\frac{\delta}{\|A\|^2}, \lambda_1\}, \lambda_1 + S]$, where $S = \sum\limits_{n=1}^{\infty} S_n$.*

Suppose that $\varGamma \neq \emptyset$, then the sequence $\{x_n\}$ generated by Algorithm 5.2 converges weakly to a point of $\varGamma$.

Example 5.2 [15]Let $H_1 = H_2 = \mathbb{R}^N$ and $\tilde{f}(x) = \frac{1}{2}d_C^2(x)$, where $C \subset \mathbb{R}^N$ is a unit ball and $g(x) = \frac{1}{2}\|x\|^2$. Let $\epsilon > 0$ and the stopping criterion is given by $\|x_n - \mathrm{prox}_{\beta_n,\tilde{f}}(x_n)\| + \|Ax_n - \mathrm{prox}_{\beta_n,g}(Ax_n)\| < \epsilon$. Choose $x_0 = (0,0,0,\cdots,0),\ x_1 = (1,1,1,\cdots,1) \in \mathbb{R}^N, A = I, \theta_n = 0.3$ suggested in [16], and

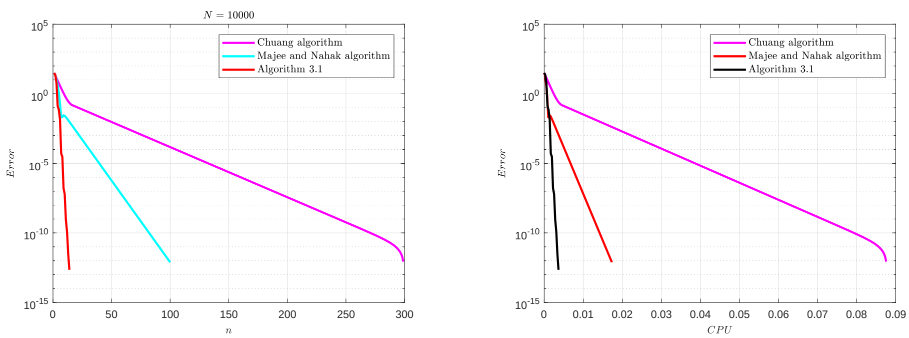
$$\theta_n = \begin{cases} \frac{1}{n^k\|x_n - x_{n-1}\|^2}, & \text{if } n \geq 2, \\ 0, & \text{if } n = 1. \end{cases}$$

for Chuang'algorithm. Set $\lambda_n = 0.499,\ k = 9,\ \beta_n = 0.19$ for Chuang's algorithm, $\lambda_n = 0.499,\ \beta_n = 0.19, \kappa = 1.99$ for the algorithm of Majee and Nahak , and the initial step size $\lambda_1 = 0.499,\ \beta_n = 0.19,\ \delta = 0.499,\ S_n = \frac{2}{(n+1)^{1.1}},\ \phi = 1.99$ for Algorithm 3.1. The results are the following:



(a) Number of Iterations(n) (b) Iterative times(CPU)

Fig. 5 $Error$ is the stopping criterion, n and CPU stand for Number of Iterations and Iterative times measured in seconds, respectively. Comparison with $\epsilon = 10^{-12}$ for the problem 5.2 with ***N*=5000**.



(a) Number of Iterations(n) (b) Iterative times(CPU)

Fig. 6 $Error$ is the stopping criterion, n and CPU stand for Number of Iterations and Iterative times measured in seconds, respectively. Comparison with $\epsilon = 10^{-12}$ for the problem 5.2 with ***N*=10000**.

See **Fig.5** and **Fig.6**, it is shown that great advantages of our iterative algorithms over others in each given N.

Funding Postgraduate Research: Practice Innovation Program of Jiangsu Province (KYCX20-0170).

6 Conclusions

In this paper, we consider the weak convergence results for variational inclusion problems under relaxed conditions. We modify Chuang's algorithm [12] and the algorithm of Majee and Nahak[16] with new and simple step sizes. The weak convergence results are established without any requirement of additional proximities. Also, the application of Algorithm 3.1 is successfully applied for solving the split feasible problem and split minimization problem. Some numerical experiments confirm the efficiency of the algorithm.

References

1. Moudafi, A.: Split monotone variational inclusions. J. Optim. Theory Appl. **150**, 275–283 (2011)
2. Byrne, C.: Iterative oblique projection onto convex subsets and the split feasibility problem. Inverse Probl. **16**, 441–453 (2002)
3. Combettes, P. L.: The convex feasibility problem in image recovery. In: Hawkes, P. (ed.) Advances in Imaging and Electron Physics, pp. 155–270. Academic Press, New York (1996)
4. Tseng, P.: A modified forwardCbackward splitting method for maximal monotone mappings. SIAM J Control Optim. **38**,431–446 (2000)
5. Byrne, C., Censor, Y., Gibali, A.: Weak and strong convergence of algorithms for the split common null point problem. J. Nonlinear Convex Anal. **13**, 759–775 (2011)
6. Chuang, C. S.: Algorithms with new parameter conditions for split variational inclusion problems in Hilbert spaces with application to split feasibility problem. Optimization. **65**, 859–876 (2016)
7. Alvarez, F., Attouch, H.: An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping. Set-Valued Anal. **9**, 3–11 (2001)
8. Alvarez, F.: Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space. SIAM J. Optim. **14**, 773–782 (2004)
9. Moudafi, A., Elisabeth, E.: An approximate inertial proximal method using enlargement of a maximal monotone operator. Int. J. Pure Appl. Math. **5**, 283–299 (2003)
10. Moudafi, A., Oliny, M.: Convergence of a splitting inertial proximal method for monotone operators. J. Comput. Appl. Math. **155**, 447–454 (2003)
11. Thong, D. V., Hieu, D. V.: An inertial method for solving split common fixed point problems. J. fixed point theory appl. **19**, 3029–3051 (2017)
12. Chuang, C. S.: Hybrid inertial proximal algorithm for the split variational inclusion problem in Hilbert spaces with applications. Optimization. **66**, 777–792 (2017)
13. Dang, Y., Sun, J., Xu, H.:Inertial accelerated algorithms for solving a split feasibility problem. J.Ind Manage Optim. **13**,1383–1394 (2017)
14. Dong, Q. L., Yuan, H. B., Cho, Y. J.: Modified inertial Mann algorithm and inertial CQ-algorithm for non expansive mappings. Optim Lett. **12**, 87–102(2018)
15. Kesornprom, S., Cholamjiak, P.: Proximal type algorithms involving linesearch and inertial technique for split variational inclusion problem in hilbert spaces with applications. Optimization. DOI:10.1080/02331934.2019.1638389 (2019)
16. Majee, P., Nahak, C.: On inertial proximal algorithm for split variational inclusion problems. Optimization. **67**, 1701–1716 (2018)
17. Takahashi, W., Xu, H. K., Yao, J. C.: Iterative methods for generalized split feasibility problem in Hilbert spaces. Set Valued Var Anal **23**,205–221 (2015)
18. Tang, Y., Gibali, A.: New self-adaptive step size algorithms for solving split variational inclusion problems and its applications. Num Algor. **83**,305–331 (2020)
19. L*ó*pez, G., Martn-Mrquez, V., Xu, H.K.: Solving the split feasibility problem without prior knowledge of matrix norms. Inverse Prob. **28**,374–389 (2012)

20. Izuchukwu, C., Ogwo, G. N. O., Mewomo,T.: An inertial method for solving generalized split feasibility problems over the solution set of monotone variational inclusions. Optimization. https://doi.org/10.1080/02331934.2020.1808648 (2020)
21. Goebel, K., Kirk, W. A.: Topics in metric fixed point theory. Cambridge: Cambridge University Press; (1990). (Cambridge studies in advanced mathematics; vol. 28).
22. Marino, G., Xu, H. K.: Convergence of generalized proximal point algorithm. Commun. Pure Appl. Anal. **3**, 791–808 (2004)
23. Chuang, C. S.: Strong convergence theorems for the split variational inclusion problem in Hilbert spaces. Fixed Point Theory Appl. **2013**, 350–366 (2013)
24. ALVAREZ, F.: Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert spaces. SIAM J. Optim. **14**,773–782 (2004)
25. Opial, Z.: Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull Am Math Soc. **4**,591–597(1967)
26. Thong, D. V., Dng, V. T., Je, C. Y.: A new strong convergence for solving split variational inclusion problems. Num Algor. DOI:10.1007/s11075-020-00901-0 (2020)
27. Byrne, C.: A unified treatment of some iterative algorithms in signal processing and image reconstruction. Inverse Probl. **20**, 103–120 (2004)
28. Ceng, L. C., Ansari, Q. H., Yao, J. C.: An extragradient method for solving split feasibility and fixed point problems. Comput. Math. Appl. **64**, 633–642 (2012)
29. Butnariu, D., Iusem, A. N.: Totally convex functions for fixed points computation and infinite dimensional optimization. London: Kluwer Academic Publishers. (2000).
30. Saejung, S., Yotkaew, P.: New algorithms for the split variational inclusion problems and application to split feasibility problems. Optimization. DOI:10.1080/02331934.2019.1631821 (2019)