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RESEARCH ARTICLE

Nonexistence of global solutions of a viscoelastic $p(x)$-Laplacian equation with logarithmic nonlinearity

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Abstract

In this work, we aim to obtain the existence and finite time blow up of weak solutions of a $p(x)$ -Laplacian pseudo-parabolic equation with memory term and logarithmic nonlinearity. Moreover, we extract an upper bound for the blow up time by applying the concavity method and a lower bound using the differential inequality technique.

KEYWORDS:
blow up, weak solution, viscoelastic term, $p(x)$-Laplacian, logarithmic nonlinearity, variable exponents

1 | INTRODUCTION

This work is to study a $p(x)$- Laplacian pseudo parabolic equation. Here, we mainly deal with the existence and finite time blow up of solutions of the problem proposed below,

$$\begin{cases} z_t - \alpha\Delta z_t - \Delta z - \beta\nabla(|\nabla z|^{p(x)-2}\nabla z) + \int\limits_0^t h(t-\tau)\Delta z(x,\tau)d\tau = |z|^{q-2}zlog|z|, & x\in\Omega, t\geq 0 \\ z(x,t)=0, & x\in\partial\Omega, t\geq 0 \\ z(x,0)=z_0(x), & x\in\Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N, (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. $\alpha, \beta \geq 0$ are constants. The variable exponents $p(x) : \overline{\Omega} \longrightarrow [1, \infty)$ is log-Holder continuous and

$$p_- = \text{ess}\inf_{x\in\Omega} p(x),\ \ p_+ = \text{ess}\sup_{x\in\Omega} p(x)$$

Here, we call the nonlinear term $\nabla(|\nabla z|^{p(x)-2}\nabla z)$, the $p(x)-$Laplacian operator. We hypothesize the following,

H_1) $2 < p_- \leq p(x) \leq p_+ < q < p^*$, p^* is given in lemma 2.

H_2) $h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a bounded C^1 function such that

$$h(\tau)\geq 0, \quad h'(\tau)\leq 0, \quad 1-\int\limits_0^t h(\tau)d\tau = l > 0.$$

H_3) $E(0) > 0$, is the energy functional defined by (36).

The integrodifferential equation is a significant area of naturally emerging nonlinear problems. For example, they appear in the fields of heat conduction and viscous flow in materials with memory. Hence the study has a great history too. Existence and regularity results for the following integrodifferential equation

$$z'(t) = Az(t) + \int_0^t K(t-\tau)z(\tau)d\tau + f(t),$$

where A and K are linear operators in a complex Banach space X, was obtained by Prato and Ianelli[20]. They used the Laplace transform approach in order to establish the existence results. The existence and blow-up of solutions of a parabolic equation with nonlinear memory were studied by Bellout[2]. He proved existence using a fixed point method and, after obtaining results on finite time blow up established estimates on the rate of blow up and results concerning the blow up set. The problem

$$\begin{cases} z_t - \Delta z + \int_0^t g(t-\tau)\Delta z(x,\tau)d\tau = f(z), & x \in \Omega, t > 0 \\ z(x,t) = 0, & x \in \partial\Omega, t > 0 \\ z(x,0) = z_0(x), & x \in \Omega \end{cases} \tag{2}$$

was then studied by many authors for different source terms. In 2004, Messaoudi[15] considered the problem with the source given by $|z|^{p-2}z$ and derived blow-up results for positive initial energy solutions under suitable conditions on p and g. He then improved the results and obtained finite time blow up for negative or vanishing initial energy too[16]. For the same problem, in 2017, Tian[21] obtained a new criterion for finite time blow up and established an upper bound for the blow up time. Also obtained a lower bound employing differential inequality technique. For $f(z) \in C(\mathbb{R})$ the nonexistence of global solution with positive initial energy of (2) was established by Fang and Sun[8]. Blow up of solutions for a semilinear heat equation with a viscoelastic term was studied in [[11]] for nonlinear flux on the boundary. The authors obtained blow up results for negative initial energy by employing the concavity method. Li then improved these results, and Han[13] for positive initial energy with the support of potential well method. In 2012, Liu and Chen[14] established the blow-up results of the viscoelastic quasilinear parabolic equation with negative energy and positive initial energy. They proved a general decay of the energy functional for the global solution. Global existence and decay were also studied by Fang and Qiu[9] for a semilinear parabolic equation with mixed boundary conditions and memory term. A Robin boundary value problem associated with the equation

$$z_t - \frac{\partial}{\partial x}\left(\mu(x,t)z_x\right) + \int_0^t g(t-\tau)\frac{\partial}{\partial x}\left[\mu(x,\tau)z_x\right]d\tau = f(z) + f_1(x,t),$$

was analyzed by Ngoc et. al.[18] The authors proved existence, uniqueness, and regularity results for weak solutions using the Faedo-Galerkin method and compactness arguments. They showed the existence of finite time blow up of solutions with initial negative energy. Moreover, established a sufficient condition to ensure the global existence of weak solutions. Antontsev and Shmarev[1] examined an initial boundary value problem associated with

$$z_t - \Delta_p z = \int_0^t g(t-s)\Delta_p z(x,s)ds + \Theta(x,t,z) + f(x,t), \tag{3}$$

Moreover, they established the existence and non-existence of global solutions of the problem without more restrictive assumptions on Θ. The existence of solutions for a pseudo-parabolic equation with memory was deduced by Di and Shang[4] using the Galerkin method and potential well theory. By making a slight change in the source term, Sun et al.[22] studied the problem and came up with existence and finite time blow up results using Galerkin method, concavity argument, and potential well theory. They derived an upper bound for the blow-up time and obtained solutions that blow up in finite time with arbitrary initial energy conditions. In 2019, Messaoudi and Talahmeh[17] studied a semilinear viscoelastic pseudo-parabolic problem with variable exponent and prove that any weak solution initial data at arbitrary energy level blows up in finite time. Furthermore, they obtained an upper bound for the blow up time using the concavity method. Recently Di et. al.[5] analyzed blow-up solutions of a nonlinear pseudo parabolic equation with memory and derived upper and lower bounds for the blow up time. Results on the non-existence of global solutions of the following problem

$$\begin{cases} \partial_t z - \Delta_{q(x)} z - \Delta_x \left(\partial_t z - \sigma(t) \int\limits_0^t g(t-s) z ds \right) = |z|^{p(x)-2} z, & \text{in } \Omega \times (0,T) \\ z = 0, & \text{on } \partial\Omega \times (0,T) \\ z(x,0) = z_0(x), & \text{in } \Omega \end{cases} \tag{4}$$

where $\Omega \subset \mathbb{R}^n (n \geq 2)$ is an open bounded domain and $\sigma \in C^1(\mathbb{R}^+)$, was published by Zennir and Miyasita[23],by obtaining a relation between the relaxation function and $p(x)$. In 2020, Lakshmipriya and Gnananvel[19] studied a $p(x)-$ Laplacian parabolic problem with memory under Dirichlet boundary conditions and established existence and blow up results. Moreover, upper and lower bounds for finite time of blow up were also derived.

All these former works urged us to deal with the problem (1). In this paper, the sections are organized as follows: We call up essential preliminaries in Section 2. We carry on the existence of solutions to the problem(1) in Section 3. In this section, in order to make the calculations more straightforward, we take $\alpha, \beta = 1$. In the absence of the term involving α, actually, the pseudo-parabolic problem turns into a parabolic problem for which the same procedure holds to establish the existence result, with necessary modifications due to the equation. Existence of finite time blow up and an upper bound for blow up time of the solutions for this case, that is when $\alpha = 0$ is established in section 4. Lower bound for blow up time is calculated in section 5. In this paper we use C as a generic constant.

2 | PRELIMINARIES

This section is devoted to call back some essential preliminaries to go forward with the problem (1). However, for more information, one can refer[6,3]. Here, we let $p, q : \Omega \longrightarrow [1, \infty)$ be measurable functions, where Ω is a bounded domain in $\mathbb{R}^N$.

Definition 1. [6] The variable exponent Lebesgue space with exponent $p(x)$ is defined by

$$L^{p(x)}(\Omega) := \{z : \Omega \longrightarrow \mathbb{R} | \rho_{p(x)}(\lambda z) < \infty, \text{ for some } \lambda > 0\},$$

where,

$$\rho_{p(x)}(z) = \int\limits_\Omega |z(x)|^{p(x)} dx.$$

Theorem 1. [6] The space $L^{p(x)}(\Omega)$ set up with the Luxembourg norm

$$\|z\|_{p(x)} = inf\{\lambda > 0 | \rho_{p(x)}\left(\frac{z}{\lambda}\right) \leq 1\},$$

is a Banach space and

$$\min\{\|z\|^{p_-}_{p(x)}, \|z\|^{p_+}_{p(x)}\} \leq \int\limits_\Omega |z|^{p(x)} dx \leq \max\{\|z\|^{p_-}_{p(x)}, \|z\|^{p_+}_{p(x)}\}. \tag{5}$$

Remark 1. [6] $L^{p'(x)}(\Omega)$ stands for the dual space of $L^{p(x)}(\Omega)$, such that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Definition 2. [6] The variable exponent Sobolev space is defined as

$$W^{k,p(x)}(\Omega) = \{z \in L^{p(x)}(\Omega) | D^\alpha z \in L^{p(x)}(\Omega), |\alpha| \leq k\},$$

where $k \geq 1$, $D^\alpha z$ is the α^{th} weak partial derivative with $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ a multi-index, $|\alpha| = \sum\limits_{j=1}^N \alpha_j$.

Theorem 2. [6] The space $W^{k,p(x)}(\Omega)$ set up with the norm
$\|z\|_{k,p(x)} := \sum_{|\alpha| \leq k} \|D^\alpha z\|_{p(x)}$ is a Banach space.

We denote the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$ by $W_0^{k,p(x)}(\Omega)$.

Definition 3. [3] Let X be a Banach space. Then, $L^q(0,T;X)$ is defined as the set of measurable functions $z : [0,T] \longrightarrow X$ such that if $1 \le q < \infty$,

$$\|z\|_{L^q(0,T;X)} = \left(\int_0^T \|z(t)\|_X^q \, dt \right)^{\frac{1}{q}} < \infty,$$

if $q = \infty$,

$$\|z\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0 \le t \le T} \|z(t)\|_X < \infty.$$

Remark 2. For $1 \le q \le \infty$, $L^q(0,T;X)$ endued with the above norms is a Banach space.

Definition 4. [17] The exponent $p(x)$ is said to be log-Holder continuous on Ω if there exists a constant $D > 0$ such that

$$|p(x) - p(y)| \le -\frac{D}{\log|x-y|}, \text{ for } x, y \in \Omega, \text{ with } |x-y| < \delta, 0 < \delta < 1.$$

Lemma 1. [6] If $p(x), q(x)$ are variable exponents satisfying $p(x) \le q(x)$ a.e. in Ω, then there is a continuous embedding from $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$.

Lemma 2. [6] Let the variable exponents $p(x) \in C(\overline{\Omega}), b : \Omega \longrightarrow [1, \infty)$ be a measurable function and satisfy

$$\operatorname*{ess\,inf}_{x \in \overline{\Omega}} (p^*(x) - q(x)) > 0, \text{ where } p^* = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \ge N. \end{cases}$$

Then the Sobolev embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact.

Lemma 3. [10] Let v be a positive number. Then the following inequality

$$z^q log z \le (ev)^{-1} z^{q+v},$$

holds for all $z \in [1, \infty)$.

Here we define a functional, which will be used in the further calculations,

$$J(z) = \frac{1}{2}\left(1 - \int_0^t h(\tau) d\tau\right) \|\nabla z\|_2^2 + \int_\Omega \frac{|\nabla z|^{p(x)}}{p(x)} dx + \frac{1}{q^2}\|z\|_q^q$$
$$-\frac{1}{q}\int_\Omega |z|^q log|z| dx + \frac{1}{2}\int_0^t h(t-\tau)\|\nabla z(x,\tau) - \nabla z(x,t)\|_2^2 d\tau, \tag{6}$$

3 | WEAK SOLUTIONS

In this section, we discuss about the existence of weak solutions for the problem (1), by using the well known Faedo - Galerkin approximation method. Here we assume $\alpha = \beta = 1$.

Definition 5. A function $z(x,t)$ is said to be a weak solution of the problem(1) if
$z(x,t) \in L^2(0,T; W_0^{1,p(x)}(\Omega) \cap L^q(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \cap C(0,T; H_0^1(\Omega))$, $z_t(x,t) \in L^2(0,T; H_0^1(\Omega))$ and $z(x,0) = z_0(x)$ satisfies the following condition

$$\int_0^T\int_\Omega z_t \phi dx dt + \int_0^T\int_\Omega \nabla z_t \nabla\phi dx dt + \int_0^T\int_\Omega \nabla z \nabla\phi dx dt + \int_0^T\int_\Omega |\nabla z|^{p(x)-2}\nabla z \nabla\phi dx dt$$
$$-\int_0^T\int_0^t h(t-\tau)\int_\Omega \nabla z(x,\tau)\nabla\phi dx d\tau dt = \int_0^T\int_\Omega |z|^{q-2} z log|z| \phi dx dt, \tag{7}$$

for all $\phi \in C^\infty(0,T; C_0^\infty(\Omega))$, $[0,T]$ is the maximal interval of existence.

Theorem 3. Assume that $z_0 \in W_0^{1,p(x)}(\Omega) \cap L^q(\Omega) \backslash \{0\}$, then the problem (1) admits a weak solution $z(x,t)$ in the sense of definition(5), $t \in [0,T]$.

Proof. Let $\{\phi_i\}_{i=1}^{\infty}$ be a sequence of eigenfunctions of $-\Delta$ corresponding to the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$, comprising an orthogonal basis of $H_0^1(\Omega)$ and an orthonormal basis of $L^2(\Omega)$. Forthwith, we seek for finite dimensional approximation solutions of (1) as $\{z_n\}$ defined by

$$z_n(x,t) = \sum_{i=1}^{n} a_{ni}(t)\phi_i(x), \tag{8}$$

satisfying

$$\int_\Omega z_n'\phi_i dx + \int_\Omega \nabla z_n'\nabla\phi_i dx + \int_\Omega \nabla z_n\nabla\phi_i dx + \int_\Omega |\nabla z_n|^{p(x)-2}\nabla z_n\nabla\phi_i dx \tag{9}$$

$$-\int_0^t h(t-\tau)\int_\Omega \nabla z_n(x,\tau)\nabla\phi_i dx d\tau = \int_\Omega |z_n|^{q-2}z_n log|z_n|\phi_i dx,$$

and

$$z_n(x,0) = \sum_{i=1}^{n} a_{ni}(0)\phi_i(x) \longrightarrow z_0(x) \text{ in } W_0^{1,p(x)}(\Omega)\cap L^q(\Omega)\backslash\{0\}. \tag{10}$$

Now, we need to get the coefficients $\{a_{ni}\}_{i=1}^{n}$. For $m = 1,2,...n$

$$a_{nm}'(t) = \frac{1}{1+\lambda_m}\Bigg(-\int_\Omega \nabla z_n\nabla\phi_m dx - \int_\Omega |\nabla z_n|^{p(x)-2}\nabla z_n\nabla\phi_m dx \tag{11}$$

$$+\int_0^t h(t-\tau)\int_\Omega \nabla z_n(x,\tau)\nabla\phi_m dx d\tau + \int_\Omega |z_n|^{q-2}z_n log|z_n|\phi_m dx\Bigg).$$

The standard ODE theory gives existence of solution to the problem (11) in a maximal interval $[0,T]$. Multiplying equation (9) by $a_{ni}'(t)$ and summing for $i = 1,2,...n$, to get

$$\int_\Omega |z_n'|^2 dx + \int_\Omega \nabla z_n'\nabla z_n' dx + \int_\Omega \nabla z_n\nabla z_n' dx + \int_\Omega |\nabla z_n|^{p(x)-2}\nabla z_n\nabla z_n' dx$$

$$-\int_0^t h(t-\tau)\int_\Omega \nabla z_n(x,\tau)\nabla z_n' dx d\tau = \int_\Omega |z_n|^{q-2}z_n log|z_n| z_n' dx.$$

This gives

$$\int_\Omega |z_n'|^2 dx + \int_\Omega |\nabla z_n'|^2 dx - \frac{1}{2}\int_0^t h'(t-\tau)\|\nabla z_n(x,\tau)-\nabla z_n(x,t)\|_2^2 d\tau + \frac{1}{2}h(t)\|\nabla z_n\|_2^2 = -\tfrac{d}{dt}J(z_n(t)), \tag{12}$$

Where J is the functional defined by equation (6). From the hypothesis (H_2), we get

$$J'(z_n(t)) = -\left[\int_\Omega |z_n'|^2 dx + \int_\Omega |\nabla z_n'|^2 dx - \frac{1}{2}\int_0^t h'(t-\tau)\|\nabla z_n(x,\tau)-\nabla z_n(x,t)\|_2^2 d\tau + \frac{1}{2}h(t)\|\nabla z_n\|_2^2\right] \le 0. \tag{13}$$

This implies that $J(z_n)$ is non-increasing. So $J(z_n(t)) \le J(z_n(0))$ for all $t \in [0,T)$ and for all $n \in \mathbb{N}$. Now, multiply equation (9) by $a_{ni}(t)$ and sum over i, for $i = 1,2,...n$, to get

$$\int_\Omega z_n' z_n dx \quad \int_\Omega \nabla z_n'\nabla z_n dx + \int_\Omega |\nabla z_n|^2 dx + \int_\Omega |\nabla z_n|^{p(x)} dx - \int_0^t h(t-\tau)\int_\Omega \nabla z_n(x,\tau)\nabla z_n(x,t) dx d\tau = \int_\Omega |z_n|^q log|z_n| dx.$$

Further, we get

$$\frac{d}{dt}\left[\frac{1}{2}\int_{\Omega}|z_n|^2dx\ +\frac{1}{2}\int_{\Omega}|\nabla z_n|^2dx+\int_0^t\int_{\Omega}|\nabla z_n|^2dxd\tau+\int_0^t\int_{\Omega}|\nabla z_n|^{p(x)}dxd\tau\right] \tag{14}$$

$$=\int_0^t h(t-\tau)\int_{\Omega}\nabla z_n(x,\tau)\nabla z_n(x,t)dxd\tau+\int_{\Omega}|z_n|^q log|z_n|dx.$$

We call

$$H_n(t)=\frac{1}{2}\int_{\Omega}|z_n|^2dx+\frac{1}{2}\int_{\Omega}|\nabla z_n|^2dx+\int_0^t\int_{\Omega}|\nabla z_n|^2dxd\tau+\int_0^t\int_{\Omega}|\nabla z_n|^{p(x)}dxd\tau. \tag{15}$$

So,

$$\frac{d}{dt}H_n(t)\ =\int_0^t h(t-\tau)\int_{\Omega}\nabla z_n(x,\tau)\nabla z_n(x,t)dxd\tau+\int_{\Omega}|z_n|^q log|z_n|dx. \tag{16}$$

To carry forward, we apply Young's inequality. Hence we get

$$\begin{aligned}&\int_0^t h(t-\tau)\int_{\Omega}\nabla z_n(x,\tau)\nabla z_n(x,t)dxd\tau\\ &\quad=\int_0^t h(t-\tau)\int_{\Omega}\nabla z_n(x,t)\left(\nabla z_n(x,\tau)-\nabla z_n(x,t)\right)dxd\tau+\int_0^t h(t-\tau)\|\nabla z_n\|_2^2,\\ &\quad\leq\frac{1}{2}\int_0^t h(t-\tau)\|\nabla z_n(x,\tau)-\nabla z_n(x,t)\|_2^2d\tau+\frac{1}{2}\int_0^t h(t-\tau)d\tau\|\nabla z_n\|_2^2+\int_0^t h(t-\tau)d\tau\|\nabla z_n\|_2^2,\end{aligned}$$

this gives

$$\int_0^t h(t-\tau)\int_{\Omega}\nabla z_n(x,\tau)\nabla z_n(x,t)dxd\tau\leq\frac{1}{2}\int_0^t h(t-\tau)\|\nabla z_n(x,\tau)-\nabla z_n(x,t)\|_2^2d\tau+\frac{3}{2}\int_0^t h(t-\tau)d\tau\|\nabla z_n\|_2^2. \tag{17}$$

Hence, by equation (13) and the hypothesis (H_2), we obtain

$$\begin{aligned}\frac{d}{dt}H_n(t)\ &\leq\ J(z_n(t))-\frac{1}{2}\left(1-\int_0^t h(\tau)d\tau\right)\|\nabla z_n\|_2^2-\int_{\Omega}\frac{|\nabla z_n|^{p(x)}}{p(x)}dx-\frac{1}{q^2}\|z_n\|_q^q\\ &\quad+\frac{1}{q}\int_{\Omega}|z_n|^q log|z_n|dx+\frac{3}{2}\int_0^t h(t-\tau)d\tau\|\nabla z_n\|_2^2+\int_{\Omega}|z_n|^q log|z_n|dx,\\ &\leq\ J(z_n(t))-\frac{1}{2}\|\nabla z_n\|_2^2+2\int_0^t h(\tau)d\tau\|\nabla z_n\|_2^2-\int_{\Omega}\frac{|\nabla z_n|^{p(x)}}{p(x)}dx-\frac{1}{q^2}\|z_n\|_q^q+\left(1+\frac{1}{q}\right)\int_{\Omega}|z_n|^q log|z_n|dx,\\ &\leq\ J(z_n(0))+2(1-l)\|\nabla z_n\|_2^2+\left(1+\frac{1}{q}\right)\int_{\Omega}|z_n|^q log|z_n|dx.\end{aligned} \tag{18}$$

Now, the help of lemma (3), we get

$$\int_{\Omega} |z_n|^q log|z_n| dx \leq \int_{\{x\in\Omega:|z_n|\geq 1\}} |z_n|^q log|z_n| dx,$$

$$\leq (ev)^{-1} \int_{\{x\in\Omega:|z_n|\geq 1\}} |z_n|^{q+v} dx,$$

$$\leq (ev)^{-1} \|z_n\|_{q+v}^{q+v}, \tag{19}$$

where $0 < v < q^* - q, q^* = \frac{Nq}{N-q}$. Applying Gagliardo-Nirenberg Interpolation Inequality[3], we get

$$\|z_n\|_{q+v}^{q+v} \leq C\|\nabla z_n\|_{p_-}^{\theta(q+v)} \|z_n\|_2^{(1-\theta)(q+v)}, \tag{20}$$

where $\theta = \frac{Np_-(q+v-2)}{(q+v)(Np_- -2N+2p_-)}$. Now, for $\epsilon > 0$, Young's inequality with ϵ gives

$$\|z_n\|_{q+v}^{q+v} \leq \epsilon\|\nabla z_n\|_{p_-}^{p_-} + C(\epsilon)\|z_n\|_2^{\frac{p_-(1-\theta)(q+v)}{p_- -\theta(q+v)}}, \tag{21}$$

Hence, (18) implies

$$\frac{d}{dt}H_n(t) \leq J(z_n(0)) + 2(1-l)\|\nabla z_n\|_2^2 + \left(1+\frac{1}{q}\right)(ev)^{-1}\left[\epsilon\|\nabla z_n\|_{p_-}^{p_-} + C(\epsilon)\|z_n\|_2^{\frac{p_-(1-\theta)(q+v)}{p_- -\theta(q+v)}}\right]. \tag{22}$$

Set $\rho = \frac{p_-(1-\theta)(q+v)}{2(p_- -\theta(q+v))} > 1$ and assume $\min\{\|\nabla z_n\|_{p(x)}^{p_-}, \|\nabla z_n\|_{p(x)}^{p_+}\} = \|\nabla z_n\|_{p(x)}^{p_-}$, then by (5) we get

$$\frac{d}{dt}H_n(t) \leq J(z_n(0)) + 2(1-l)\|\nabla z_n\|_2^2 + \left(1+\frac{1}{q}\right)(ev)^{-1}\left[\epsilon\int_{\Omega}|\nabla z_n|^{p(x)}dx + C(\epsilon)\|z_n\|_2^{2\rho}\right],$$

integrating this inequality from 0 to t, we get

$$H_n(t) \leq H_n(0) + J(z_n(0))t + 2(1-l)\int_0^t \|\nabla z_n\|_2^2 d\tau$$

$$+\left(1+\frac{1}{q}\right)(ev)^{-1}\left[\epsilon\int_0^t\int_{\Omega}|\nabla z_n|^{p(x)}dxd\tau + 2C(\epsilon)\int_0^t \frac{1}{2}\|z_n\|_2^{2\rho}d\tau\right]. \tag{23}$$

We denote the constant $H_n(0) + J(z_n(0))t = c_1$, which depends on $t, t \in [0, T]$. Now, we Choose $l \geq \frac{3}{4}$ and $\epsilon = \frac{evq}{2(q+1)}$, to obtain

$$H_n(t) \leq c_1(t) + \frac{1}{2}\int_0^t \|\nabla z_n\|_2^2 d\tau + \frac{1}{2}\int_0^t\int_{\Omega}|\nabla z_n|^{p(x)}dxd\tau + c_2\int_0^t \frac{1}{2}\|z_n\|_2^{2\rho}d\tau,$$

$$\leq c_1(t) + \frac{1}{2}H_n(t) + c_2\int_0^t H_n^{\rho}d\tau,$$

where $c_2 = 2C(\epsilon)\left(1+\frac{1}{q}\right)(ev)^{-1}$. Hence we have

$$H_n(t) \leq c_1(t) + c_2\int_0^t H_n^{\rho}(\tau)d\tau. \tag{24}$$

By implementing Gronwall-Bellman-Bihari type inequality, we get

$$H_n(t) = \int_{\Omega}|z_n|^2dx + \int_{\Omega}|\nabla z_n|^2dx + \int_0^t\int_{\Omega}|\nabla z_n|^2dxd\tau + \int_0^t\int_{\Omega}|\nabla z_n|^{p(x)}dxd\tau \leq C_T, \tag{25}$$

C_T depends on T. Our assumption $\min\{\|\nabla z_n\|_{p(x)}^{p_-}, \|\nabla z_n\|_{p(x)}^{p_+}\} = \|\nabla z_n\|_{p(x)}^{p_-}$, together with (H_1) gives $\|\nabla z_n\|_{p(x)}^{2} \leq \|\nabla z_n\|_{p(x)}^{p_-}$. Now, from (5) we get

$$\int_0^T \|\nabla z_n\|_{p(x)}^2 d\tau \leq \int_0^T \|\nabla z_n\|_{p(x)}^{p_-} d\tau \leq \int_0^T \int_\Omega |\nabla z_n|^{p(x)} dx d\tau \leq C_T. \tag{26}$$

Since we have (26) , the Sobolev embedding gives

$$\int_0^T \|z_n\|_q^2 dx \leq C \int_0^T \|\nabla z_n\|_{p(x)}^2 dx \leq C_T. \tag{27}$$

continuity of J gives a constant C and since we have $z_n(x,0) \longrightarrow z_0(x)$ in $W_0^{1,p(x)}(\Omega) \cap L^q(\Omega)$,

$$J(z_n(x,0)) \leq C, \text{ for any } n. \tag{28}$$

Also we have, from equation (12)

$$\int_\Omega |z_{n_t}|^2 dx \leq -\frac{d}{dt} J(z_n),$$

integrating from 0 to t gives

$$\int_0^t \|z_n'(\tau)\|_2^2 d\tau + J(z_n) \leq J(z_n(x,0)) \leq C. \tag{29}$$

The estimates (25),(26), (27) and (29) together with the standard compactness arguments, will give

$$z_n \longrightarrow z \text{ weakly}^* \text{ in } L^\infty(0,T; H_0^1(\Omega)), \tag{30}$$

$$z_n \longrightarrow z \text{ weakly in } L^2\left(0,T; W_0^{1,p(x)}(\Omega) \cap L^q(\Omega)\right) \cap L^2(0,T; H_0^1(\Omega)), \tag{31}$$

$$z_n' \longrightarrow z' \text{ weakly in } L^2(0,T; H_0^1(\Omega)), \tag{32}$$

$$|\nabla z_n|^{p(x)-2}\nabla z_n \longrightarrow \xi \text{ weakly in } L^2(0,T; L^{p'(x)}(\Omega)). \tag{33}$$

Now, by making use of Aubin -Lions lemma, we get

$$z_n \longrightarrow z \text{ in } C(0,T; H_0^1(\Omega)). \tag{34}$$

We get $\xi = |\nabla z|^{p(x)-2}\nabla z$, by the monotonicity of $|s|^{p(x)-2}s$ and using Minty-Browder condition. Hence the proof. □

4 | UPPER BOUND FOR BLOW-UP TIME

Here, the objective is to get an upper bound for the blow-up time of solutions (1). For that we consider the problem (1) for the case, when $\alpha = 0$ and $\beta = 1$. Existence of solutions can be shown following the same steps as above.

Theorem 4. Assume that $(H_1), (H_2)$ and (H_3) hold and weak solution $z(x,t)$ as defined in (5) exists for the problem (1). Then the solutions $z(x,t)$ blows up in finite time T^*. Moreover, there exists an upper bound for the same, given by

$$T^* \leq \frac{(1+\gamma)\left(1+\frac{1}{\eta}\right)\|z_0\|_2^2}{2\psi\gamma E(0)}, \tag{35}$$

where ψ, γ and η are suitable positive constants given later.

Proof. Multiply the equation(1) by z_t and integrating over Ω, will give

$$\int_\Omega z_t^2 dx = -\int_\Omega \nabla z \nabla z_t dx - \int_\Omega |\nabla z|^{p(x)-2}\nabla z \nabla z_t dx - \int_\Omega \int_0^t h(t-\tau)\Delta z(x,\tau) z_t(x,t) d\tau dx + \int_\Omega |z|^{q-2} z \log|z| z_t dx,$$

which implies

$$\int_\Omega z_t^2 dx - \frac{1}{2}\int_0^t h'(t-\tau)\|\nabla z(x,\tau)-\nabla z(x,t)\|_2^2 d\tau + \frac{1}{2}h(t)\|\nabla z\|_2^2$$

$$= \frac{d}{dt}\left[-\frac{1}{2}\|\nabla z\|_2^2 - \int_\Omega \frac{|\nabla z|^{p(x)}}{p(x)}dx - \frac{1}{2}\int_0^t h(t-\tau)\|\nabla z(x,\tau)-\nabla z(x,t)\|_2^2 d\tau \right.$$

$$\left. + \int_0^t h(\tau)d\tau\|\nabla z\|_2^2 - \frac{1}{q^2}\|z\|_q^q + \frac{1}{q}\int_\Omega |z|^q log|z|dx\right].$$

Now, we set a functional

$$E(t) = -\frac{1}{2}\int_0^t h(t-\tau)\|\nabla z(x,\tau)-\nabla z(x,t)\|_2^2 d\tau - \frac{1}{2}\left(1-\int_0^t h(\tau)d\tau\right)\|\nabla z\|_2^2$$

$$- \int_\Omega \frac{|\nabla z|^{p(x)}}{p(x)}dx + \frac{1}{q}\int_\Omega |z|^q log|z|dx - \frac{1}{q}\|z\|_q^q. \tag{36}$$

Hence, by making use of the hypothesis (H_2), we obtain

$$\frac{dE(t)}{dt} = \int_\Omega z_t^2 dx - \frac{1}{2}\int_0^t h'(t-\tau)\|\nabla z(x,\tau)-\nabla z(x,t)\|_2^2 d\tau + \frac{1}{2}h(t)\|\nabla z\|_2^2 \geq 0. \tag{37}$$

Now, to move forward we define an auxiliary functional

$$N(t) = \int_0^t \int_\Omega z^2(x,\tau)dxd\tau + A. \tag{38}$$

Then

$$N'(t) = 2\int_\Omega \int_0^t z(x,\tau)z_t(x,\tau)d\tau dx + \int_\Omega z_0^2(x)dx, \tag{39}$$

and

$$N''(t) = 2\int_\Omega zz_t dx,$$

$$= 2\int_\Omega z(x,t)\left[\Delta z + \nabla\left(|\nabla z|^{p(x)-2}\nabla z\right) - \int_0^t h(t-\tau)\Delta z(x,\tau)d\tau + |z|^{q-2}zlog|z|\right]dx,$$

$$= -2\|\nabla z\|_2^2 - 2\int_\Omega |\nabla z|^{p(x)}dx + 2\int_\Omega |z|^q log|z|dx$$

$$+2\int_0^t h(t-\tau)\int_\Omega \nabla z(x,t)\left(\nabla z(x,\tau)-\nabla z(x,t)\right)dxd\tau + 2\int_0^t h(t-\tau)d\tau\|\nabla z\|_2^2.$$

Now, choose a constantψ such that $p_+ < \psi < q$ and $\psi(\psi-2) > \frac{1-l}{l}$. This gives

$$
\begin{aligned}
N''(t) =\ & 2\psi E(t) + \psi \int_0^t h(t-\tau)\|\nabla z(x,\tau) - \nabla z(x,t)\|_2^2 d\tau \\
& +\psi \left(1 - \int_0^t h(\tau)d\tau\right)\|\nabla z\|_2^2 + 2\psi \int_\Omega \frac{|\nabla z|^{p(x)}}{p(x)} dx - \frac{2\psi}{q}\int_\Omega |z|^q log|z|dx + \frac{2\psi}{q^2}\|z\|_q^q \\
& -2\left(1 - \int_0^t h(t-\tau)d\tau\right)\|\nabla z\|_2^2 - 2\int_\Omega |\nabla z|^{p(x)}dx + 2\int_\Omega |z|^q log|z|dx \\
& +2\int_0^t h(t-\tau)\int_\Omega \nabla z(x,t)\left(\nabla z(x,\tau) - \nabla z(x,t)\right) dxd\tau, \\
\geq\ & 2\psi E(t) + \psi \int_\Omega h(t-\tau)\|\nabla z(x,\tau) - \nabla z(x,t)\| dxd\tau + \frac{2\psi}{q^2}\|z\|_q^q + (\psi-2)\left(1 - \int_0^t h(t-\tau)d\tau\right)\|\nabla z\|_2^2 \\
& + \left(\frac{2\psi}{q} - 2\right)\int_\Omega |z|^q log|z|dx + \left(\frac{2\psi}{p_+} - 2\right)\int_\Omega |\nabla z|^{p(x)}dx + \int_0^t h(t-\tau)\int_\Omega \nabla z(x,t)\left(\nabla z(x,\tau) - \nabla z(x,t)\right) dxd\tau.
\end{aligned}
$$

By using (H_2) and assuming $|z| \geq 1$, we gain

$$
\begin{aligned}
N''(t) \geq\ & 2\psi E(t) + \left[(\psi-2)l + \frac{(l-1)}{\psi}\right]\|\nabla z\|_2^2 + \left(\frac{2\psi}{p_+} - 2\right)\int_\Omega |\nabla z|^{p(x)}dx + \frac{2\psi}{q^2}\|z\|_q^q + 2\left(1 - \frac{\psi}{q}\right)\int_\Omega |z|^q log|z|dx, \\
\geq\ & 2\psi E(t).
\end{aligned}
$$

From (37), we have

$$
E(t) \geq E(0) + \int_0^t \int_\Omega z_t^2(x,\tau)dxd\tau. \tag{40}
$$

Now, for $\eta > 0$ consider

$$
N'(t)^2 \leq 4(1+\eta)\left[\int_\Omega \int_0^t z(x,\tau)z_t(x,\tau)d\tau dx\right]^2 + \left(1 + \frac{1}{\eta}\right)\left[\int_\Omega z_0^2(x)dx\right]^2.
$$

Then, Holder's inequality is used to get

$$
N'^2 \leq 4(1+\eta)\left(\int_\Omega \int_0^t z^2(x,\tau)d\tau dx\right)\left(\int_\Omega \int_0^t z_t^2(x,\tau)d\tau dx\right) + \left(1 + \frac{1}{\eta}\right)\left[\int_\Omega z_0^2(x)dx\right]^2.
$$

Since $\eta > 0$ is arbitrary, we choose $\gamma = \eta = \sqrt{\frac{\psi}{2}} - 1 > 0$, in order to get

$$
N''(t)N(t) - (1+\gamma)N'(t)^2 \geq 2\psi A E(0) - (1+\gamma)\left(1 + \frac{1}{\eta}\right)\left[\int_\Omega z_0^2(x)dx\right]^2. \tag{41}
$$

Since we have (40) and $E(0) > 0$, in equation(41) we choose $A > 0$ large enough such that

$$
N''(t)N(t) - (1+\gamma)N'(t)^2 > 0. \tag{42}
$$

here we conclude that z blows up at a finite time $T^\star$ by Levine's [12] concavity method. By choosing A as,

$$A = \frac{(1+\gamma)\left(1+\frac{1}{\eta}\right)\left[\int_\Omega z_0^2(x)dx\right]^2}{2\psi E(0)},$$

we get an upper bound for the finite time of blow up $T^\star$ given by

$$0 < T^\star \le \frac{(1+\gamma)\left(1+\frac{1}{\eta}\right)\|z_0\|_2^2}{2\psi\gamma E(0)}. \tag{43}$$

□

5 | LOWER BOUND FOR BLOW-UP TIME

Now, our goal is to obtain a lower bound for the blow up time of solutions of (1)when $\alpha = \beta = 1$, when the solutions blow up at a finite time $T^\star$.

Theorem 5. Let z be a weak solution of (1). If z blows up at finite time $T^\star$ then there exists a lower bound for blow up time, given by

$$T^\star \ge \int_{M(0)}^{\infty} \frac{d\sigma}{c_3 + c_4\sigma + c_5(\sigma)^{\frac{q+\nu}{2}}}, \tag{44}$$

where the constants c_3, c_4 and c_5 will be specified later.

Proof. We start by defining

$$M(t) = \int_\Omega |z(x,t)|^2 + |\nabla z(x,t)|^2 dx, \tag{45}$$

which gives

$$\begin{aligned} M'(t) &= 2\int_\Omega zz_t dx + 2\int_\Omega \nabla z \nabla z_t dx, \\ &= -2\int_\Omega |\nabla z|^2 dx - 2\int_\Omega |\nabla z|^{p(x)} dx + 2\int_0^t h(t-\tau)\int_\Omega \nabla z(x,\tau)\nabla z(x,t)dxd\tau + 2\int_\Omega |z|^q log|z|dx. \end{aligned} \tag{46}$$

Now define,

$$\begin{aligned} J_1 &= 2\int_0^t h(t-\tau)\int_\Omega \nabla z(x,\tau)\nabla z(x,t)dxd\tau \\ &= 2\int_0^t h(t-\tau)\int_\Omega \nabla z(x,t)\left(\nabla z(x,\tau) - \nabla z(x,t)\right)dxd\tau + 2\int_0^t h(t-\tau)d\tau\|\nabla z\|_2^2, \\ &\le 2l\int_0^t h(t-\tau)d\tau\|\nabla z\|_2^2 + \frac{1}{2l}\int_0^t h(t-\tau)\|\nabla z(x,\tau) - \nabla z(x,t)\|_2^2 d\tau + 2\int_0^t h(t-\tau)d\tau\|\nabla z\|_2^2, \\ &\le 2l(1-l)\|\nabla z\|_2^2 + 2(1-l)\|\nabla z\|_2^2 + J_2, \end{aligned}$$

where

$$
\begin{aligned}
J_2 &= \frac{1}{2l}\int_0^t h(t-\tau)\|\nabla z(x,\tau)-\nabla z(x,t)\|_2^2 d\tau,\\
&= \frac{1}{l}\left[J(z(t)) - \frac{1}{2}\left(1-\int_0^t h(\tau)d\tau\right)\|\nabla z\|_2^2 - \int_\Omega \frac{|\nabla z|^{p(x)}}{p(x)}dx + \frac{1}{q}\int_\Omega |z|^q log|z|dx - \frac{1}{q^2}\|z\|_q^q\right],\\
&\le \frac{J(z_0)}{l} - \frac{1}{2}\|\nabla z\|_2^2 - \frac{1}{l}\int_\Omega \frac{|\nabla z|^{p(x)}}{p(x)}dx + \frac{1}{lq}\int_\Omega |z|^q log|z|dx - \frac{1}{lq^2}\|z\|_q^q.
\end{aligned}
$$

Substituting the bounds for J_1 and J_2 in (46), we get

$$
\begin{aligned}
M'(t) &\le -2\|\nabla z\|_2^2 - 2\int_\Omega |\nabla z|^{p(x)}dx + 2\int_\Omega |z|^q log|z|dx + 2l(1-l)\|\nabla z\|_2^2 + 2(1-l)\|\nabla z\|_2^2 + \frac{J(z_0)}{l}\\
&\quad -\frac{1}{2}\|\nabla z\|_2^2 - \frac{1}{l}\int_\Omega \frac{|\nabla z|^{p(x)}}{p(x)}dx + \frac{1}{lq}\int_\Omega |z|^q log|z|dx - \frac{1}{lq^2}\|z\|_2^2,\\
&\le \left(2+\frac{1}{lq}\right)\int_\Omega |z|^q log|z|dx + 2(1-l^2)\|\nabla z\|_2^2 + \frac{J(z_0)}{l}.
\end{aligned}
\tag{47}
$$

By the equation(19), we have

$$
\int_\Omega |z|^q log|z|dx \le (ev)^{-1}\|z\|_{q+v}^{q+v}.
$$

Now, Sobolev embedding gives

$$
\int_\Omega |z|^q log|z|dx \le \alpha_1 (ev)^{-1}\|\nabla z\|_2^{q+v},
$$

where α_1 is the embedding constant. Hence we get

$$
\begin{aligned}
M'(t) &\le \left(2+\frac{1}{lq}\right)\alpha_1(ev)^{-1}\|\nabla z\|_2^{q+v} + 2(1-l^2)\|\nabla z\|_2^2 + \frac{J(z_0)}{l},\\
&\le c_3 + c_4 M(t) + c_5 M(t)^{\frac{q+v}{2}},
\end{aligned}
\tag{48}
$$

where $c_3 = \frac{J(z_0)}{l}$, $c_4 = 2(1-l^2)$ and $c_5 = \left(2+\frac{1}{lq}\right)\alpha_1(ev)^{-1}$. Integrating the inequality (48) from 0 to t, we get

$$
\int_{M(0)}^{M(t)} \frac{d\sigma}{c_3 + c_4\sigma + c_5(\sigma)^{\frac{q+v}{2}}} \le t. \tag{49}
$$

Finally, this gives, if the solution blows up at a finite time $T^\star$ then we get a lower bound for $T^\star$ given by

$$
T^\star \ge \int_{M(0)}^{\infty} \frac{d\sigma}{c_3 + c_4\sigma + c_5(\sigma)^{\frac{q+v}{2}}}. \tag{50}
$$

Hence the proof. □

CONCLUSION

We established the existence result for the problem (1) and obtained a lower bound for the blow up time- if the solution blows up at finite time- using differential inequality technique. Here, we felt difficulty in obtaining the estimates properly to show the existence of finite time blow up. However, in the absence of the term involving α, we could show the solutions blow up at finite time for the parabolic problem so obtained. Moreover, an upper bound for this blow up time is also derived.

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