On the degree locating indices of graphs

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Abstract

Graph theory has supported chemist with topological indices which predict different physicochemical properties such as boiling point, entropy, acentric factor etc. of chemical compounds. In this article, we introduce two new topological indices called degree locating indices, based on the degree and location of the vertices, we present exact expressions for some families of standard graphs and we get the exact values of these indices for any graph of diameter two. Finally, we compute these indices for the join of graphs, book graph and firefly graph.

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1 Introduction

Topological indices play a significant role mainly in chemistry, pharmacology, etc.(see [5–9, 12, 17, 18]) Many of the topological indices of current interest in mathematical chemistry are defined in terms of vertex degrees of the molecular graph. Two of the most famous topological indices of graphs are the first and second Zagreb indices which have been introduced by Gutman and Trinajstic in [10], and defined as $M_1(G) = \sum_{u \in V(G)} (d(u))^2$ and $M_2(G) = \sum_{uv \in E(G)} d(u)\, d(v)$, respectively. The Zagreb indices have been studied extensively due to their numerous applications in the place of existing chemical methods which need more time and increase the costs. Many new reformulated and extended versions of the Zagreb indices have been introduced for several similar reasons (cf. [1, 2, 4, 11, 13, 15, 19–21]).

One of the present author Saleh [16] has been recently introduced a new matrix representation for a graph G by defining the locating matrix $\mathbf{Lo}(G)$ over G. We will redefine this representation as in the following.

Definition 1 *Let $G = (V, E)$ be a connected graph with vertex set $V = \{v_1, v_2, ..., v_n\}$. A degree locating function of G denoted by $\mathbf{Ld}(G)$ is a function $\mathbf{Ld}(G) : V(G) \to (\mathbb{Z} \cup \{0\})^d$ where d is the diameter of the graph G such that $\mathbf{Ld}(v_j) = \overrightarrow{v_j} = (\Gamma_0(v_j), \Gamma_1(v_j), ..., \Gamma_d(v_j))$, where $\Gamma_i(v_j)$ is the number of vertices of distance i from the vertex v_j in G. The vector $\overrightarrow{v_j}$ is called the degree locating vector corresponding to the vertex v_j. Then we define first and second degree locating indices of G as*

$$\mathcal{L}_1(G) = \sum_{v_i \in V(G)} (\overrightarrow{v_i})^2 \quad \textit{and} \quad \mathcal{L}_2(G) = \sum_{v_i v_j \in E(G)} \overrightarrow{v_i} \cdot \overrightarrow{v_j} \,,$$

respectively, where $\overrightarrow{v_i} \cdot \overrightarrow{v_j}$ is the dot product of the two vectors $\overrightarrow{v_i}$ and $\overrightarrow{v_j}$.

The above locating function and huge applications of topological indices motivated us to introduce two new topological indices, namely *first* and *second degree locating indices*, based on the degree locating vectors.

All graphs in this paper will be assumed simple, undirected and connected unless stated otherwise.

For graph theoretical terminologies, we refer the reader to [3].

2 Some exact values of locating indices

In this section,we determine the first and second degree locating indices for some standard graphs like K_n, C_n, $K_{n,m}$, W_n, P_n,

Theorem 2 *Let $G \cong K_n$ be the complete graph with a vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$, where $n \geq 2$. Then*

i. $\mathcal{L}_1(G) = n^2 (n^2 - 3n + 3)$.

ii. $\mathcal{L}_2(G) = \frac{n(n-1)\left(n^3-4n^2+7n-2\right)}{2}$.

Proof. Let $G \cong K_n$ be the complete graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$, $n \geq 2$ and Obviously, for any vertex $v_i \in V(G)$ the corresponding degree locating vector $\overrightarrow{v_i}$ has one in the ith component and the other $n-1$ components contain the number $n-1$, so $\overrightarrow{v_i}^2 = 1 + (n-1)^2$. In the same way for any two adjacent vertices v_i, $v_j \in V(G)$, we have $\overrightarrow{v_i}.\overrightarrow{v_j} = 2 + 2(n-1) + (n-1)^2(n-2) = n^3 - 4n^2 + 7n - 2$.

Hence,

$$\mathcal{L}_1(G) = n^2\left(n^2 - 3n + 3\right).$$

Similarly, $\mathcal{L}_2(G) = \frac{n(n-1)}{2}(n^3 - 4n^2 + 7n - 2)$. Therefore,

$$\mathcal{L}_2(G) = \frac{n\,(n-1)\,(n^3 - 4n^2 + 7n - 2)}{2}.$$

■

Proposition 3 *For an even integer $n \geq 2$, let $G \cong C_n$. Then*

$$\mathcal{L}_1(G) = \mathcal{L}_2(G) = 2n(n-1).$$

Proof. By labeling the vertices of the cycle C_n as $\{v_1, v_2, \cdots, v_n\}$ in the anticlockwise direction, we can see that for any vertex v_i the degree locating vector will be in the form

$$\overrightarrow{v_i} = \left(1, \overbrace{2, \cdots, 2}^{n/2-1}, 1\right), \text{for all } i = 1, 2, ..., n.$$

So, $\overrightarrow{v_i}^2 = 2 + 4(n/2 - 1) = 2(n-1)$.
Hence,

$$\mathcal{L}_1(G) = \mathcal{L}_2(G) = 2n(n-1).$$

■

Proposition 4 *For an odd integer $n \geq 3$, let $G \cong C_n$. Then*

$$\mathcal{L}_1(G) = \mathcal{L}_2(G) = n(2n-1).$$

Proof. By labeling the vertices of the cycle C_n as $\{v_1, v_2, \cdots, v_n\}$ in the anticlockwise direction, it is easy to see that for any vertex v_i the degree locating vector will be in the form

$$\overrightarrow{v_i} = \left(1, \overbrace{2, \cdots, 2}^{\frac{n-1}{2}}\right), \text{for all } i = 1, 2, ..., n.$$

So, $\overrightarrow{v_i}^2 = 2n - 1$.
Hence,

$$\mathcal{L}_1(G) = \mathcal{L}_2(G) = n(2n - 1).$$

■

Theorem 5 *Let $G \cong K_{a,b}$, be a complete bipartite graph. Then*

i. $\mathcal{L}_1(G) = a^3 + ab^2 - 2a^2 + 2a + a^2b + b^3 - 2b^2 + 2b.$

ii. $\mathcal{L}_2(G) = ab\,(2ab - a + 2 - b).$

Proof. Let $G \cong K_{a,b}$, with two partite sets A and B such that $|A| = a$ and $|B| = b$,by labeling the vertices as $V(G) = \{v_1, ...v_a, u_1, ...u_b\}$. Obviously for any vertex v_i the corresponding degree degree locating is of the form

$$\overrightarrow{v_i} = (1, b, a - 1)\,, \text{for all } i = 1, 2, ..., a,$$

and

$$\overrightarrow{u_i} = (1, a, b - 1)\,, \text{for all } i = 1, 2, ..., b.$$

Therefore

$$\overrightarrow{v_i}^2 = 1 + b^2(a - 1)^2 = a^2 + b^2 - 2a + 2, \text{for all } i = 1, 2, ..., a,$$

and

$$\overrightarrow{u_i}^2 = 1 + b^2(a - 1)^2 = a^2 + b^2 - 2b + 2, \text{for all } i = 1, 2, ..., b.$$

In the same way we have for any adjacent vertices v_i, u_j,

$$\overrightarrow{v_i}.\overrightarrow{u_i} = 1 + ab + (a - 1)(b - 1), \text{for all } i = 1, 2, ..., b \text{and } j = 1, 2, ..., b$$

Hence,

$$\mathcal{L}_1(G) = a^3 + ab^2 - 2a^2 + 2a + a^2b + b^3 - 2b^2 + 2b,$$

and

$$\mathcal{L}_2(G) = ab\,(2ab - a + 2 - b)\,.$$

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Since the following consequences of Theorem 5 are special cases and clear, we will omit their proofs.

Corollary 6 *Let $G \cong K_{a,a}$, be complete bipartite graph. Then $\mathcal{L}_1(G) = 2a^2(a^2 - a + 1)$, and $\mathcal{L}_2(G) = 4a(a^2 - a + 1)$.*

Corollary 7 *Let $G \cong K_{1,a}$, be a star graph with a+1 vertices. Then $\mathcal{L}_1(G) = a^3 - a^2 + 3a + 1$, and $\mathcal{L}_2(G) = a(a + 1)$.*

Proposition 8 *Let G be wheel graph W_n with $n+1$ vertices, where $(n \geq 4)$. Then we have*

i. $\mathcal{L}_1(G) = n^3 - 5n^2 + 19n + 1$.

ii. $\mathcal{L}_2(G) = n\,(n^2 - 3n + 20)$.

Proof. Let $G \cong W_n$ for $n \geq 4$, where W_n is the wheel graph with $n + 1$ vertices. By labeling the vertices of $V(G)$ in the anticlockwise direction as $v_1, v_2, \cdots, v_n, v_{n+1}$ such that v_{n+1} is the center of the wheel, we obtain

$$\overrightarrow{v_i} = (1, 3, n - 3)\,, \text{for } i = 1, 2, ..., n$$

and

$$\overrightarrow{v_{n+1}} = (1, n, 0)\,.$$

So we get for $i = 1, ..., n$, $\overrightarrow{v_i}^2 = n^2 - 6n + 19$, $\overrightarrow{v_i}.\overrightarrow{v_{i+1}} = n^2 - 6n + 19$, $\overrightarrow{v_i}.\overrightarrow{v_{n+1}} = 3n + 1$ and $\overrightarrow{v_{n+1}}^2 = n^2 + 1$. Hence,

$$\mathcal{L}_1(G) = n^3 - 5n^2 + 19n + 1,$$

and

$$\mathcal{L}_2(G) = n\left(n^2 - 3n + 20\right).$$

■

Theorem 9 *Let $G \cong P_n$ be path of even number of vertices $(n \geq 4)$. Then*

i. $\mathcal{L}_1(G) = \frac{n(3n-2)}{2}$.

ii. $\mathcal{L}_2(G) = \frac{3n(n-2)+4}{2}$.

Proof. Let G is be a path P_n with even number of vertices $(n \geq 4)$. By labeling the vertices from left to right as $v_1, v_2, \cdots, v_n$

$$\overrightarrow{v_1} = \left(\overbrace{1, \cdots, 1}^{n}\right), \qquad \overrightarrow{v_{n-1}} = \left(\overbrace{1, \cdots, 1}^{n}\right),$$

$$\overrightarrow{v_2} = \left(1, 2, \overbrace{1, \cdots, 1}^{n-3}, 0\right), \qquad \overrightarrow{v_{n-1}} = \left(1, 2, \overbrace{1, \cdots, 1}^{n-3}, 0\right),$$

$$\overrightarrow{v_3} = \left(1, 2, 2, \overbrace{1, \cdots, 1}^{n-5}, 0, 0\right), \qquad \overrightarrow{v_{n-2}} = \left(1, 2, 2, \overbrace{1, \cdots, 1}^{n-5}, 0, 0\right),$$

$$\vdots \qquad\qquad \vdots$$

$$\overrightarrow{v_{\frac{n}{2}}} = \left(1, \overbrace{2, \cdots, 2}^{\frac{n}{2}-1}, 1 \overbrace{0, \cdots, 0}^{\frac{n}{2}-1}\right), \quad \overrightarrow{v_{\frac{n}{2}+1}} = \left(1, \overbrace{2, \cdots, 2}^{\frac{n}{2}-1}, 1 \overbrace{0, \cdots, 0}^{\frac{n}{2}-1}\right).$$

By symmetry on components between the vectors easy to see that $\overrightarrow{v_1} = \overrightarrow{v_n}$, $\overrightarrow{v_2} = \overrightarrow{v_{n-1}}$, $\cdots$,$\overrightarrow{v_{\frac{n}{2}}} = \overrightarrow{v_{\frac{n}{2}+1}}$ That means, for $j = 1, 2, ..., \frac{n}{2}$, we have

$$\overrightarrow{v_j} = \left(1, \overbrace{2, \cdots, 2}^{j-1}, \overbrace{1, \cdots, 1}^{n-2j+1}, \overbrace{0, \cdots, 0}^{j-1}\right),$$

and

$$\overrightarrow{v_j}^2 = 1 + 4(j-1) + (n-2j+1) = n + 2j + 1.$$

Therefore,

$$\mathcal{L}_1(G) = 2\sum_{j=1}^{\frac{n}{2}}(n + 2j - 2) = n^2 + 4\frac{n/2(n/2+1)}{2} - 2n.$$

Hence,

$$\mathcal{L}_1(G) = \frac{n(3n-2)}{2}.$$

Now to prove part (ii), we have, for $j = 1, 2, ..., \frac{n}{2} - 1$

$$\overrightarrow{v_i}.\overrightarrow{v_{i+1}} = \left(1, \overbrace{2, \cdots, 2}^{j-1}, \overbrace{1, \cdots, 1}^{n-2j+1}, \overbrace{0, \cdots, 0}^{j-1}\right) . \left(1, \overbrace{2, \cdots, 2}^{j-1}, 2, \overbrace{1, \cdots, 1}^{n-2j-1}, \overbrace{0, \cdots, 0}^{j-1}\right)$$

Thus,

$$\overrightarrow{v_i}.\overrightarrow{v_{i+1}} = 1 + 4(j-1) + +2 + n - 2j - 1 = 2j + n - 2.$$

But, for

$$\overrightarrow{v_{\frac{n}{2}-1}}.\overrightarrow{v_{\frac{n}{2}}} = \left(1, \overbrace{2, \cdots, 2}^{\frac{n}{2}-1}, 1\, \overbrace{0, \cdots, 0}^{\frac{n}{2}-1}\right) . \left(1, \overbrace{2, \cdots, 2}^{\frac{n}{2}-1}, 1\, \overbrace{0, \cdots, 0}^{\frac{n}{2}-1}\right) = 2(n-1).$$

Then we get

$$\mathcal{L}_2(G) = 2(n-1) + 2\sum_{j=1}^{\frac{n-2}{2}}(n + 2j - 2) = \frac{3n\,(n-2)}{2} + 2$$

Hence,

$$\mathcal{L}_2(G) = \frac{3n\,(n-2) + 4}{2}.$$

■

Theorem 10 *Let $G \cong P_n$ be path of odd number of vertices $(n \geq 5)$. Then*

i. $\mathcal{L}_1(G) = \frac{3n^2-2n+1}{2}$.

ii. $\mathcal{L}_2(G) = \frac{3n^2-6n+3}{2}$.

Proof. Let G is be a path P_n with even number of vertices $(n \geq 5)$. By labeling the vertices from left to right as $v_1, v_2, \cdots, v_n$

$$\overrightarrow{v_1} = \left(\overbrace{1, \cdots, 1}^{n}\right), \qquad \overrightarrow{v_{n-1}} = \left(\overbrace{1, \cdots, 1}^{n}\right),$$

$$\overrightarrow{v_2} = \left(1, 2, \overbrace{1, \cdots, 1}^{n-3}, 0\right), \qquad \overrightarrow{v_{n-1}} = \left(1, 2, \overbrace{1, \cdots, 1}^{n-3}, 0\right),$$

$$\overrightarrow{v_3} = \left(1, 2, 2, \overbrace{1, \cdots, 1}^{n-5}, 0, 0\right), \qquad \overrightarrow{v_{n-2}} = \left(1, 2, 2, \overbrace{1, \cdots, 1}^{n-5}, 0, 0\right),$$

$$\vdots \qquad \vdots$$

$$\overrightarrow{v_{\frac{n-1}{2}}} = \left(1, \overbrace{2, \cdots, 2}^{\frac{n-3}{2}}, 1\, \overbrace{0, \cdots, 0}^{\frac{n-3}{2}}\right), \quad \overrightarrow{v_{\frac{n+3}{2}}} = \left(1, \overbrace{2, \cdots, 2}^{\frac{n-3}{2}}, 1\, \overbrace{0, \cdots, 0}^{\frac{n-3}{2}}\right).$$

By symmetry on components between the vectors easy to see that $\overrightarrow{v_1} = \overrightarrow{v_n}$, $\overrightarrow{v_2} = \overrightarrow{v_{n-1}}$, $\cdots$,$\overrightarrow{v_{\frac{n-1}{2}}} = \overrightarrow{v_{\frac{n+3}{2}}}$ That means, for $j = 1, 2, ..., \frac{n-1}{2}$, we have

$$\overrightarrow{v_j} = \left(1, \overbrace{2, \cdots, 2}^{j-1}, \overbrace{1, \cdots, 1}^{n-2j+1}, \overbrace{0, \cdots, 0}^{j-1}\right),$$

and

$$\overrightarrow{v_j}^2 = 1 + 4(j-1) + (n-2j+1) = n + 2j + 1.$$

For the remaining vertex $v_{\frac{n+1}{2}}$, we get

$$\overrightarrow{v_{\frac{n+1}{2}}} = \left(1, \overbrace{2, \cdots, 2}^{\frac{n-1}{2}}, \overbrace{0, \cdots, 0}^{\frac{n-1}{2}}\right),$$

So,

$$\overrightarrow{v_{\frac{n+1}{2}}}^2 = 2n - 1.$$

Therefore,

$$\mathcal{L}_1(G) = 2n - 1 + 2\sum_{j=1}^{\frac{n-1}{2}} (n + 2j - 2) = \frac{3n^2 - 2n + 1}{2}.$$

Now, to prove part (ii), we have, for $j = 1, 2, ..., \frac{n-3}{2}$

$$\overrightarrow{v_i}.\overrightarrow{v_{i+1}} = \left(1, \overbrace{2, \cdots, 2}^{j-1}, \overbrace{1, \cdots, 1}^{n-2j+1}, \overbrace{0, \cdots, 0}^{j-1}\right) . \left(1, \overbrace{2, \cdots, 2}^{j-1}, 2, \overbrace{1, \cdots, 1}^{n-2j-1}, \overbrace{0, \cdots, 0}^{j-1}\right)$$

Thus,

$$\overrightarrow{v_i}.\overrightarrow{v_{i+1}} = 1 + 4(j-1) + +2 + n - 2j - 1 = 2j + n - 2.$$

But, for $i = \frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2}$

$$\overrightarrow{v_{\frac{n-1}{2}}}.\overrightarrow{v_{\frac{n+1}{2}}} = \overrightarrow{v_{\frac{n+1}{2}}}.\overrightarrow{v_{\frac{n+3}{2}}} = \left(1, \overbrace{2, \cdots, 2}^{\frac{n-3}{2}}, 1\, \overbrace{0, \cdots, 0}^{\frac{n-3}{2}}\right) . \left(1, \overbrace{2, \cdots, 2}^{\frac{n-1}{2}}, \overbrace{0, \cdots, 0}^{\frac{n-1}{2}}\right) = 2n-3.$$

Then, we get

$$\mathcal{L}_2(G) = 2(2n-3) + 2\sum_{j=1}^{\frac{n-3}{2}}(n + 2j - 2) = 2n + \frac{3n^2 - 10n + 3}{2}$$

Hence,

$$\mathcal{L}_2(G) = \frac{3n^2 - 6n + 3}{2}.$$

■

Proposition 11 *Let G be k-regular graph of $n \geq 2$ and with diameter two. Then*

i. $\mathcal{L}_1(G) = n\,(n^2 - 2nk - 2n + 2k^2 + 2k + 2)$.

ii. $\mathcal{L}_2(G) = \frac{nk\left(n^2 - 2nk - 2n + 2k^2 + 2k + 2\right)}{2}$.

Proof. Let G be k-regular graph of $n \geq 2$ vertices and with diameter two. Then clearly for any vertex $v \in V(G)$,we have
$\overrightarrow{v} = (1, k, n-k-1)$ and $\overrightarrow{v}^2 = 1 + k^2 + (n-k-1)^2$, Hence,

$$\mathcal{L}_1(G) = n\left(n^2 - 2nk - 2n + 2k^2 + 2k + 2\right)$$

, and

$$\mathcal{L}_2(G) = \frac{nk\,(n^2 - 2nk - 2n + 2k^2 + 2k + 2)}{2}.$$

■

Proposition 12 *Let G be a graph of $n \geq 2$ and with diameter two. Then*

i. $\mathcal{L}_1(G) = n + 2M_1(G) + (n-1)(n-2m-1)$.

ii. $\mathcal{L}_2(G) = 2M_2(G) - (n-1)M_1(G) + mn^2 + 2m - 2mn$.

Proof. Let G be a graph of $n \geq 2$ vertices and with diameter two. Then clearly for any vertex $v \in V(G)$,we have
$\overrightarrow{v} = (1, deg(v), n - deg(v) - 1)$ and $\overrightarrow{v}^2 = 1 + deg(v)^2 + (n - deg(v) - 1)^2$, then

$$\mathcal{L}_1(G) = \sum_{u \in V(G)} \overrightarrow{v}^2 = n + 2M_1(G) + (n-1)(n-2m-1).$$

Similarly,

$$\mathcal{L}_2(G) = \sum_{uv \in E(G)} \overrightarrow{u}.\overrightarrow{v} = \sum_{uv \in E(G)} (1+deg(v)deg(u)+(n-1-deg(v))(n-1-deg(u))).$$

Hence,

$$\mathcal{L}_2(G) = 2M_2(G) - (n-1)M_1(G) + mn^2 + 2m - 2mn.$$

■

Locating indices of the join of two graphs

The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2. In the following theorem we find first and second locating indices for the join graph G.

Theorem 13 *Let $G \cong G_1 + G_2$ such that G_1 and G_2 are both connected graphs, such that G_1 has n_1 vertices and m_1 edges while G_2 has n_2 vertices and m_2 edges. Then*

$$\begin{aligned}\mathcal{L}_1(G) = 2(M_1(G_1) + M_1(G_2)) + (n_2m_1 + n_1m_2) + 4(m_1 + m_2) - 4(n_1m_1 + n_2m_2) \\ +(n_1^3 + n_2^3) + (n_1n_2^2 + n_2n_1^2) - 2(n_1^2 + n_2^2) + 2(n_1 + n_2).\end{aligned}$$

and

$$\mathcal{L}_2(G) = (n_2 - n_1 + 1)(M_1(G_1) + M(G_2)) + 2M_2(G_1) + m_1(n_1^2 + n_2^2 - 2n_1 + 2)$$
$$+ m_2(n_1^2 + n_2^2 - 2n_2 + 2) + n_1 n_2(m_1 + m_2)$$
$$+ 2m_1 m_2 + n_1 n_2(2n_1 n_2 + 2 - n_1 - n_2).$$

Proof. Let $G \cong G_1 + G_2$ such that G_1 and G_2 are both connected graphs, such that G_1 has n_1 vertices and m_1 edges while G_2 has n_2 vertices and m_2 edges. Then.by labeling the vertices of the graph G as

$$V(G) = \{v_1, v_2, \cdots, v_{n_1}, u_1, u_2, \cdots, u_{n_2}\}$$

where $v_1, v_2, \cdots, v_{n_1} \in V(G_1)$ and $u_1, u_2, \cdots, v_{n_2} \in V(G_2)$. we get the degree locating vectors in G as following:

$$\overrightarrow{v_i} = (1, n_2 + deg_{G_1}(v_i), n_1 - deg_{G_1}(v_i) - 1),$$

and

$$\overrightarrow{u_i} = (1, n_1 + deg_{G_2}(u_i), n_2 - deg_{G_2}(u_i) - 1).$$

So,

$$\overrightarrow{v_i}^2 = 2deg_{G_1}(v_i)^2 + 2n_2 deg_{G_1}(v_i) + 2deg_{G_1}(v_i) - 2n_1 deg_{G_1}(v_i) + n_2^2 + n_1^2 + 2 - 2n_1$$

and

$$\sum_{i=1}^{n_1} \overrightarrow{v_i}^2 = 2M_1(G_1) + 4n_2 m_1 + 4m_1 - 4n_1 m_1 + n_1^3 + n_1 n_2^2 - 2n_1^2 + 2n_1.$$

In the same way

$$\overrightarrow{u_i}^2 = 2deg_{G_2}(u_i)^2 + 2n_1 deg_{G_2}(u_i) + 2deg_{G_2}(u_i) - 2n_1 2deg_{G_2}(u_i) + n_2^2 + n_1^2 + 2 - 2n_2$$

S0,

$$\sum_{i=1}^{n_2} \overrightarrow{u_i}^2 = 2M_1(G_2) + 4n_1 m_2 + 4m_2 - 4n_2 m_2 + n_2^3 + n_2 n_1^2 + -2n_2^2 + 2n_2.$$

Clearly,

$$\mathcal{L}_1(G) = \sum_{i=1}^{n_1} \overrightarrow{v_i}^2 + \sum_{i=1}^{n_2} \overrightarrow{u_i}^2,$$

then

$$\mathcal{L}_1(G) = 2(M_1(G_1) + M_1(G_2)) + (n_2m_1 + n_1m_2) + 4(m_1 + m_2) - 4(n_1m_1 + n_2m_2)$$
$$+(n_1^3 + n_2^3) + (n_1n_2^2 + n_2n_1^2) - 2(n_1^2 + n_2^2) + 2(n_1 + n_2)$$

To prove (ii), according to different types of edges in G, we have

$$\mathcal{L}_2(G) = \sum_{v_iv_j \in E(G)} (\overrightarrow{v_i}.\overrightarrow{v_j}) + \sum_{u_iu_j \in E(G)} (\overrightarrow{u_i}.\overrightarrow{u_j}) + \sum_{v_iu_j \in E(G)} (\overrightarrow{v_i}.\overrightarrow{u_j})$$

Since,

$$\overrightarrow{v_i}.\overrightarrow{v_j} = 1 + (n_2 + deg_{G1}(v_i))\,(n_2 + deg_{G_1}(v_j)) + (n_1 - deg_{G1}(v_i) - 1)\,(n_1 - deg_{G1}(v_j) - 1)\,,$$

so,

$$\overrightarrow{v_i}.\overrightarrow{v_j} = n_2deg_{G1}(v_j) + n_2deg_{G1}(v_i) + 2deg_{G1}(v_i)deg_{G1}(v_j)$$
$$+deg_{G1}(v_i) + deg_{G1}(v_j) - n_1deg_{G1}(v_j) - n_1deg_{G1}(v_i)$$
$$+n_2^2 + n_1^2 + 2 - 2n_1.$$

Then

$$\sum_{v_iv_j \in E(G)} (\overrightarrow{v_i}.\overrightarrow{v_j})$$
$$= \sum_{v_iv_j \in E(G)} \Big(n_2(deg_{G_1}(v_i) + deg_{G_1}(v_j)) + 2(deg_{G_1}(v_i)deg_{G_1}(v_j))$$
$$+ (deg_{G_1}(v_i) + deg_{G_1}(v_j)) - n_1(deg_{G_1}(v_i) + deg_{G_1}(v_j)) + n_1^2 + n_2^2 + 2 - 2n_1 \Big)$$
$$= (n_2 - n_1 + 1)M_1(G_1) + 2M_2(G_1) + m_1(n_1^2 + n_2^2 - 2n_1 + 2)$$

Also, since

$$\overrightarrow{u_i}.\overrightarrow{u_j} = 1 + (n_1 + deg_{G_2}(u_i))\,(n_1 + deg_{G_2}(u_j)) + (n_2 - deg_{G_2}(u_i) - 1)\,(n_2 - deg_{G_2}(u_j) - 1)$$

In the same way we get,

$$\sum_{u_iu_j \in E(G)} (\overrightarrow{u_i}.\overrightarrow{u_j}) = (n_1 - n_2 + 1)M_1(G_2) + 2M_2(G_2) + m_2(n_1^2 + n_2^2 - 2n_2 + 2).$$

Similarly, since,

$$\overrightarrow{v_i}.\overrightarrow{u_j} = 1 + \left(n_2 + deg_{G_1}(v_i)\right)\left(n_1 + deg_{G_2}(u_j)\right) + \left(n_1 - deg_{G_1}(v_i) - 1\right)\left(n_2 - deg_{G_2}(v_j) - 1\right)$$

We get

$$\sum_{v_iu_j\in E(G)} (\overrightarrow{v_i}.\overrightarrow{u_j}) = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(\overrightarrow{v_i}.\overrightarrow{u_j}),$$

and

$$\begin{aligned}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(\overrightarrow{v_i}.\overrightarrow{u_j}) &= \sum_{i=1}^{n_1}\Big(n_2m_2 + n_1n_2deg_{G_1}(v_i) + 2m_2deg_{G_1}(v_i) + n_2deg_{G_1}(v_i) \\ &\quad +m_2 - n_2deg_{G_1}(v_i) - n_1m_2 + n_2(2n_1n_2 + 2 - n_2 - n_1)\Big) \\ &= n_1n_2(m_1 + m_2) + 2m_1m_2 + n_1n_2(2n_1n_2 + 2 - n_1 - n_2).\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{L}_2(G) = (n_2 - n_1 + 1)M_1(G_1) + 2M_2(G_1) + m_1(n_1^2 + n_2^2 - 2n_1 + 2) \\ (n_1 - n_2 + 1)M_1(G_2) + 2M_2(G_2) + m_2(n_1^2 + n_2^2 - 2n_2 + 2) + n_1n_2(m_1 + m_2) \\ + 2m_1m_2 + n_1n_2(2n_1n_2 + 2 - n_1 - n_2).\end{aligned}$$

■

Locating indices of the book graph

Theorem 14 *Let G be a book graph $B_t = P_2 \times S_t$ with $(2t + 2)$ vertices. Then,*

i. $\mathcal{L}_1(G) = 4t^3 + 16t + 4.$

ii. $\mathcal{L}_2(G) = 4t^3 + 4t^2 + 14t + 2.$

Proof. Suppose, we have G is the book graph $B_t = P_2 \times S_t$ with $(2t+2)$ by labeling the vertices as $V(G) = \{v, u, v_1, v_2, ...v_t, u_1, u_2, ..., u_t\}$ as in Figure 1, we get,

$$\overrightarrow{v} = (1, t+1, t, 0) \quad , \quad \overrightarrow{u} = (1, t+1, t, 0)$$

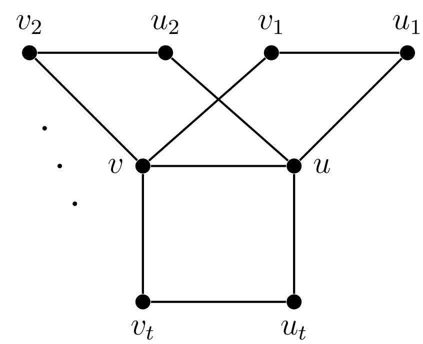

Figure 1: A Book Graph B_t.

And for $i = 1, 2, ..., m$, we have

$$\overrightarrow{v_i} = (1, 2, t, t-1) \quad , \quad \overrightarrow{u} = (1, 2, t, t-1)\,,$$

Then,

$$\overrightarrow{v}^2 = \overrightarrow{u}^2 = 1 + (t+1)^2 + t^2 = 2(t^2 + t + 1),$$

and for $i = 1, 2, 3, ..., t$

$$\overrightarrow{v_i}^2 = \overrightarrow{u_i}^2 = 1 + 4 + t^2 + (t-1)^2 = 2(t^2 - t + 3).$$

Hence,

$$\mathcal{L}_1(G) = 4t^3 + 16t + 4.$$

Similarly, to prove (ii) we have, $\overrightarrow{v}.\overrightarrow{u} = 2(t^2 + t + 1)$, and for any $i = 1, 2, 3, ..., t$,

$$\overrightarrow{v_i}.\overrightarrow{u_i} = 2(t^2 - t + 3)$$

and in the same way,

$$\overrightarrow{v}.\overrightarrow{v_i} = \overrightarrow{u}.\overrightarrow{u_i} = t^2 + 2t + 3.$$

Hence,

$$\mathcal{L}_2(G) = 4t^3 + 4t^2 + 14t + 2.$$

■

Locating indices of firefly graphs

A firefly graph $F_{s,t,n-2s-2t-1}$ ($s \geq 0$, $t \geq 0$ and $n-2s-2t-1 \geq 0$) is a graph of order n that consists of s triangles, t pendant paths of lenght 2 and $n-2s-2t-1$ pendent edges that are sharing a common vertex [14]. Let $\mathcal{F}_n$ be the set of all firefly graphs $F_{s,t,n-2s-2t-1}$.

Theorem 15 *Let $G \cong F_{s,t,l}$ ($s \geq 0$, $t \geq 0$ and $l \geq 0$) be a firefly graph of order n. Then*

i. $\mathcal{L}_1(G) = 20s^2 + 20st + 20ls - 20s + 5l^2 + 10lt + 10t^2 + 25 - 10l - 14t.$

ii. for the second degree locating of G

$$\mathcal{L}_2(G) = 2t^3+3lt^2+8st^2-t^2+l^2t+12s^2t+8st+8lst-3l+8s^3+8ls^2+20s^2+2l^2s+10ls+12s-9.$$

Proof. Let $G \cong F_{s,t,l}$ ($s \geq 0$, $t \geq 0$ and $l \geq 0$) be a firefly graph of $n = 2s+2t+l+1$ vertices, by labeling the vertices of the graph as in Figure 2 with clockwise direction. That means,

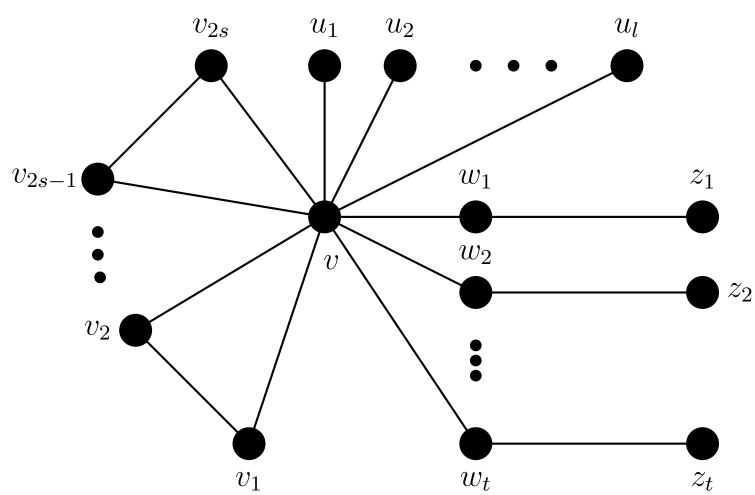

Figure 2: Firefly Graph $F_{s,t,l}$.

$$V(G) = \{v, v_1, v_2, ...v_{2s}, u_1, u_2, ..., u_l, w_1, w_2, ..., w_t, z_1, z_2, ..., z_t\},$$

where v is the center of the firefly graph and $v_1, ...v_{2s}$ the vertices of the triangles, $u_1, ...u_l$ the vertices of the pendent edges, $w_1, ..., w_t$ the first vertices of the pendent paths and $z_1, ..., z_t$ be the the second vertices of the pendent paths. We get

$$\overrightarrow{v} = (1, 2s + l + t, t, 0, 0)\,,$$

and for $i = 1, 2, ..., 2s$,

$$\overrightarrow{v}_i = (1, 2, 2s - 2 + l + t, t, 0)$$

for $i = 1, 2, ..., l$,

$$\overrightarrow{u}_i = (1, 1, 2s + t + l - 1, t, 0)\,,$$

for $i = 1, 2, ..., t$

$$\overrightarrow{w}_i = (1, 1, 2s + l + t - 1, t - 1, 0)\,,$$

and for $i = 1, 2, ..., t$

$$\overrightarrow{z}_i = (1, 1, 1, 2s + l + t - 1, t - 1)\,.$$

By simple calculation we can get

$$\overrightarrow{v}^2 = 1 + (2s + l + t)^2 + t^2,$$

for $i = 1, 2, ..., 2s$

$$\overrightarrow{v_i}^2 = 5 + (2s - 2 + l + t)^2 + t^2,$$

for $i = 1, 2, ..., l$

$$\overrightarrow{u_i}^2 = 2 + (2s + t + l - 1)^2 + t^2,$$

for $i = 1, 2, ..., t$

$$\overrightarrow{w_i}^2 = 5 + (2s + l + t - 1)^2 + (t - 1)^2,$$

and for $i = 1, 2, ..., t$

$$\overrightarrow{z_i}^2 = 3 + (2s + l + t - 1)^2 + (t - 1)^2.$$

Obviously,

$$\mathcal{L}_1(G) = \overrightarrow{v}^2 + \sum_{i=1}^{2s} \overrightarrow{v_i}^2 + \sum_{i=1}^{l} \overrightarrow{u_i}^2 + \sum_{i=1}^{t} \overrightarrow{w_i}^2 + \sum_{i=1}^{t} \overrightarrow{z_i}^2.$$

So

$$\begin{aligned}\mathcal{L}_1(G) = 1 + (2s + l + t)^2 + t^2 + 5 + (2s - 2 + l + t)^2 \\ + t^2 + 2 + (2s + t + l - 1)^2 + t^2 + 5 + (2s + l + t - 1)^2 \\ + (t-1)^2 + 3 + (2s + l + t - 1)^2 + (t-1)^2\end{aligned}$$

Hence,

$$\mathcal{L}_1(G) = 20s^2 + 20st + 20ls - 20s + 5l^2 + 10lt + 10t^2 + 25 - 10l - 14t.$$

To prove (ii), by using the same labeling of the vertices as in Figure 2, we see that

$$\mathcal{L}_2(G) = \sum_{i=1}^{2s-1} \overrightarrow{v_i}.\overrightarrow{v_{i+1}} + \sum_{i=1}^{2s} \overrightarrow{v}.\overrightarrow{v_i} + \sum_{i=1}^{l} \overrightarrow{v}.\overrightarrow{u_i} + \sum_{i=1}^{t} \overrightarrow{v}.\overrightarrow{w_i} + \sum_{i=1}^{t} \overrightarrow{w_i}.\overrightarrow{z_i}.$$

So,

$$\begin{aligned}\mathcal{L}_2(G) = (5 + (2s + l + t + 2)^2)(2s - 1) \\ + (1 + 2(2s + l + t) + t(2s + l + t - 2))(2s) \\ + (1 + 2s + l + t + t(2s + t + l - 1))(l) \\ + (1 + 2(2s + l + t) + t(2s + l + t - 1))(t) \\ + (1 + 2 + 2s + l + t - 1 + (t-1)(2s + t + l - 1))(t).\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{L}_2(G) = 2t^3 + 3lt^2 + 8st^2 - t^2 + l^2t + 12s^2t + 8st + 8lst - 3l + 8s^3 + 8ls^2 \\ + 20s^2 + 2l^2s + 10ls + 12s - 9.\end{aligned}$$

■

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