THERMAL STRESSES ANALYSIS IN AN ISOTROPIC MATERIAL DISC SUBJECTED TO THERMAL LOAD AND DENSITY WITH RIGID INCLUSION

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Abstract:
The objective of this paper is to present the study thermal stresses in an isotropic material disc with rigid inclusion and subjected to mechanical load and density parameter by using transition theory. The transition theory includes classical macroscopic solving problems in plasticity, creep relaxation, and semi-empirical yield conditions. It has been observed that the radial stress has a maximum value at the internal surface of the disc made of compressible material (*i.e.* copper) as compared to the disc made of incompressible material (*i.e.* rubber). With the introduction of thermal condition, density parameter and load, the values of radial and circumferential stress increase of the internal surface of compressible/ incompressible materials. The displacement component increased on the outer surface of the disc made of compressible /incompressible materials and fully plastic stage. Results have been discussed numerical and depicted graphically.

Key words: Stresses, displacement, disc, load, shaft, temperature, density.

Introduction

Rotating disc plays an important role in machine design. The isotropic materials disc varying properties are mainly constructed to work in high temperature environments that find their application in turbine motor, rotors, flywheel, gears, compressors, turbojet engines, sink fits, computer's disc and many other applications in mechanical, aerospace industries and chemical processing. Such disc work under complex thermo-mechanical loading conditions. In the past two decades, there have been numerous works identifying the investigations on isotropic material disc. Elasto–plastic stress distribution in a rotating disc having variable thickness and other condition has been investigated extensively by Guven [8, 9]. On the other hand, thermally induced elastic–plastic deformations of stationary heat generating disks have been studied analytically by Guven and Altay [10] employing different boundary conditions. Eraslan et al. [11] investigated elastic–plastic Deformation of a Rotating Disk Subjected to a Radial Temperature Gradient by using of Tresca and von Mises criteria. Parmaksizoglu *et al.* [7] also investigated plastic stress analysis in a rotating disc with inclusion under a temperature gradient based on Tresca's yield condition. Nayak *et al.* [21] obtained the stress distribution in elasto-plastic functionally graded disc subjected to thermo-mechanically loaded by using an iterative variational method. Sethi *et al.* [17] have investigated Elastoplastic deformation in an isotropic material disk with shaft subjected to mechanical load and density by using transition theory. Eldeeb *et al.* [22] evaluated the problem of thermo-elastoplastic behaviors of the rotating sandwich non-uniform thickness annular disc made of functionally graded materials by using finite difference method.

Weighted Integral Measures representations: Cauchy's measure of uniaxial is given by $\int_{s_0}^{s} \frac{ds}{s_0} = \frac{s-s_0}{s_0}$; where s and s_0 are deformed and un-deformed lengths respectively. The first weighted measure of Hencky's measure can be written as $\int_{s_0}^{s} \left(\frac{s_0}{s}\right) \frac{ds}{s_0} = \ln \frac{s}{s_0}$ and is widely used in plasticity problems. But in the creep problems, it is found useful only in the secondary or stationary creep not in the transient or fracture stages. The second weighted measure of Swainger

[2] by is $\int_{s_0}^{s}\left(\frac{s_0}{s}\right)^2\frac{ds}{s_0}=\frac{s-s_0}{s_0}$. In finite elasticity, Almansi and Green measures, the deformed and un-deformed states are taken as reference frameworks respectively and are extensively used. The third weighted measures are given as $\int_{s_0}^{s}\left(\frac{s_0}{s}\right)^3\frac{ds}{s_0}=\frac{1}{2}\left[1-\left(\frac{s_0}{s}\right)^2\right]$, $\int_{s_0}^{s}\left(\frac{s}{s_0}\right)^3\frac{ds_0}{s}=\frac{1}{2}\left[\left(\frac{s}{s_0}\right)^2-1\right]$ and in this case, the weighted functions are $\left(s_0/s\right)^3$ and $\left(s/s_0\right)^3$ respectively. Obviously, generalization of these measures are $\int_{s_0}^{s}\left(\frac{s_0}{s}\right)^{n+1}\frac{ds}{s_0}=\frac{1}{n}\left[1-\left(\frac{s_0}{s}\right)^n\right]$ in which the weighted function is $\left(s_0/s\right)^{n+1}$. Putting n = -2, -1, 0, 1, 2, it gives Green, Cauchy, Hencky, Swainger and Almansi measures respectively. Thus, in the general case, if the principal Almansi and Green measures are denoted by $\overset{A}{\varepsilon}_{ii}$ and $\overset{G}{\varepsilon}_{ii}$ then the generalized measures in Cartesian co-ordinates maybe written in the form:

$$\varepsilon_{ii}^{M}=\int_0^{\overset{A}{\varepsilon}_{ii}}\left[1-2\varepsilon_{ii}^{A}\right]^{\frac{n}{2}-1}d\overset{A}{\varepsilon}_{ii}=\frac{1}{n}\left[1-\left(1-2\varepsilon_{ii}^{A}\right)^{\frac{n}{2}}\right],(i=1,2,3) \tag{1}$$

and
$$\varepsilon_{ii}^{M}=\int_0^{\overset{G}{\varepsilon}_{ii}}\left[1+2\varepsilon_{ii}^{G}\right]^{\frac{n}{2}-1}d\varepsilon_{ii}^{G}=\frac{1}{n}\left[\left(1+2\varepsilon_{ii}^{G}\right)^{\frac{n}{2}}-1\right]$$

The main objective of the present paper is to develop a consistent analytical model capable to resolve a class of control problems rotating disc under thermal effect. The novelty of the current research is to be including three control factors such as thermal condition, rotating speed, density and load in the consideration of the optimal performance of the disc. Here, we assumed that the density of disc varies along the radius in the form:

$$\rho=\rho_0\left(r/r_0\right)^{-m} \tag{2}$$

where ρ_0 is the constant density at $r=r_0$ and m is the density variation parameter. Result obtained have been numerically and depicted graphically.

Mathematical model

Consider a thin disc of isotropic and homogeneous material having variable density with central bore of inner radius r_i and outer radius r_0. The annular disc is mounted on a rigid shaft. The disc is rotating with angular speed ω of gradually increasing magnitude about an axis perpendicular to its plane and passed through the center as shown in Fig. 1. The thickness of disc is assumed to be constant and sufficiently small so that it is effectively in a state of plane stress. We assume that steady state temperature Θ_0 is applied on the inner surface of the disc.

Formulation of the Problem

Displacement components in cylindrical polar coordinates (r,θ,z), as:

$$u = r(1-\eta) \quad , \quad v = 0 \quad , \quad w = dz \tag{3}$$

where η is function of $r=\sqrt{x^2+y^2}$ only and d is a constant. The finite strain components are given by Seth [5]:

$$\overset{A}{\varepsilon}_{rr} \equiv \frac{\partial u}{\partial r} - \frac{1}{2}\left(\frac{\partial u}{\partial r}\right)^2 = \frac{1}{2}\left[1-(r\eta' + \eta)^2\right] \quad , \quad \overset{A}{\varepsilon}_{\theta\theta} \equiv \frac{u}{r} - \frac{u^2}{2r^2} = \frac{1}{2}\left[1-\eta^2\right]$$

$$\overset{A}{\varepsilon}_{zz} \equiv \frac{\partial w}{\partial z} - \frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2 = \frac{1}{2}\left[1-(1-d)^2\right] \quad , \quad \overset{A}{\varepsilon}_{r\theta} = \overset{A}{\varepsilon}_{\theta z} = \overset{A}{\varepsilon}_{zr} = 0 \tag{4}$$

where $\eta' = d\eta/dr$ and meaning of superscripts 'A' is Almansi. Substituting Eq. (3) in Eq. (1) ,the generalized components of strain which are given by [5]:

$$\varepsilon_{rr} = \frac{1}{n}\left[1-(r\eta' + \eta)^n\right] \quad , \quad \varepsilon_{\theta\theta} = \frac{1}{n}\left[1-\eta^n\right] \quad , \quad \varepsilon_{zz} = \frac{1}{n}\left[1-(1-d)^n\right] \quad , \quad \varepsilon_{r\theta} = \varepsilon_{\theta z} = \varepsilon_{zr} = 0 \tag{5}$$

where $\eta' = d\eta/dr$. The stress –strain relations for thermo elastic isotropicmaterial are given by [1]:

$$\tau_{ij} = \lambda\delta_{ij} I_1 + 2\mu\varepsilon_{ij} - \xi\Theta\delta_{ij} \quad , \quad (i, j = 1,2,3) \tag{6}$$

where τ_{ij} is the stress components, λ and μ are Lame's constants and $I_1 = e_{kk}$ is the first strain invariant, δ_{ij} is the Kronecker's delta and $\xi = \alpha(3\lambda + 2\mu)$, α being the coefficient of thermal expansion and Θ is the rise of temperature. Further, Θ has to satisfy [6]:

$$\nabla^2\Theta = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\Theta}{dr}\right) = 0 \tag{7}$$

Eq. (6) for this problem become:

$$\tau_{rr} = \frac{2\lambda\mu}{\lambda + 2\mu}\left[\varepsilon_{rr} + \varepsilon_{\theta\theta}\right] + 2\mu\varepsilon_{rr} - \frac{2\mu\xi\,\Theta}{(\lambda+2\mu)} \quad , \quad \tau_{\theta\theta} = \frac{2\lambda\mu}{\lambda + 2\mu}\left[\varepsilon_{rr} + \varepsilon_{\theta\theta}\right] + 2\varepsilon_{\theta\theta} - \frac{2\mu\xi\,\Theta}{(\lambda+2\mu)} \quad ,$$

$$\tau_{r\theta} = \tau_{\theta z} = \tau_{zr} = \tau_{zz} = 0 \tag{8}$$

Substituting Eq. (4) in Eq. (6), the strain components in terms of stresses are obtained as [6]:

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} - \frac{1}{2}\left(\frac{\partial u}{\partial r}\right)^2 = \frac{1}{2}\left[1-(r\eta'+\eta)^2\right] = \frac{1}{E}\left[\tau_{rr} - \left(\frac{1-c}{2-c}\right)\tau_{\theta\theta}\right] + \alpha\,\Theta,$$

$$\varepsilon_{\theta\theta} = \frac{u}{r} - \frac{u^2}{2r^2} = \frac{1}{2}\left[1-\eta^2\right] = \frac{1}{E}\left[\tau_{\theta\theta} - \left(\frac{1-c}{2-c}\right)\tau_{\theta\theta}\right] + \alpha\,\Theta,$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} - \frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2 = \frac{1}{2}\left[1-(1-d)^2\right] = -\frac{(1-c)}{E(2-c)}\left[\tau_{rr} - \tau_{\theta\theta}\right] + \alpha\,\Theta,$$

$$\varepsilon_{r\theta} = \varepsilon_{\theta z} = \varepsilon_{zr} = 0 \tag{9}$$

where E is the Young's modulus and c is compressibility factor of the material in term of Lame's constant, there are given by $E=\mu(3\lambda+2\mu)/(\lambda+\mu)$ and $c=2\mu/(\lambda+2\mu)$. Substituting Eq.(5) in Eq. (8), we get

$$\tau_{rr}=\frac{2\mu}{n}\left[3-2c-\eta^n\left\{1-c+(2-c)(T+1)^n+\frac{nc\xi\Theta}{2\mu\eta^n}\right\}\right],$$

$$\tau_{\theta\theta}=\frac{2\mu}{n}\left[3-2c-\eta^n\left\{2-c+(1-c)(T+1)^n+\frac{nc\xi\Theta}{2\mu\eta^n}\right\}\right],$$

$$\tau_{r\theta}=\tau_{\theta z}=\tau_{zr}=\tau_{zz}=0 \tag{10}$$

where $r\eta'=\eta T$. The equations of equilibrium are all satisfied except:

$$\frac{d}{dr}(r\tau_{rr})-\tau_{\theta\theta}+\rho\omega^2 r^2=0 \tag{11}$$

where ρ is variable density of the material of the rotating disc. The temperature field satisfying Laplace Eq. (7) with boundary condition

$\Theta=\Theta_0$ at $r=r_i$, $\Theta=0$ at $r=r_0$ where Θ_0 is constant,

is given by: $$\Theta=\Theta_0\frac{\ln(r/r_0)}{\ln(r_i/r_0)} \tag{12}$$

Using Eqs. (10), (11) and (12), we get a non- linear differential equation in η as:

$$(2-c)n\eta^{n+1}T(T+1)^{n-1}\frac{dT}{d\eta}=¿¿ \tag{13}$$

where $\overline{\Theta_0}=\Theta_0/\ln(r_i/r_0)$ and $r\eta'=\eta T$ (T is function of η and η is function of r) and $\eta'=d\eta/dr$ (T is function of η and η is function of r only).

Boundary conditions

The disc is considered in the present study having variable density and subjected to thermal load and inner surface of the disc is assumed to be fixed to a shaft. The outer surface of the disc is free from mechanical load. Thus, the boundary conditions of the problem are given by:

$$r=r_i, u=0 \quad , \quad r=r_0, \tau_{rr}=l_0 \tag{14}$$

where u and τ_{rr} denote displacement component and stress along the radial direction and l_0 load applied at the outer surface of the disc.

Solution through the Problem

For finding the plastic stress, the transition function is taken through the principal stresses [3, 4, 12 - 20) at the transition point $T\to\pm\infty$. The transition function ψ is defined as:

$$\psi=\frac{n}{2\mu}\left[\tau_{\theta\theta}-c\xi\Theta\right]=\left[(3-2c)-\eta^{n}\left\{2-c+(1-c)(T+1)^{n}\right\}-\frac{nc\,\xi\Theta}{\mu}\right]$$
(15)

Taking the logarithmic differentiation of Eq. (15) with respect to *r*, we get:

$$\frac{d(\log\psi)}{dr}=-\left(\frac{n\eta^{n}T}{r}\right)\left[\frac{2-c+(1-c)(T+1)^{n-1}\left\{(T+1)+\eta\frac{dT}{d\eta}\right\}}{(3-2c)-\eta^{n}\left\{2-c+(1-c)(T+1)^{n}\right\}-\frac{nc\,\xi\Theta}{\mu}}\right]$$
(16)

Substituting the value $dT/d\eta$ from Eq. (13) in Eq. (16) and taking the asymptotic value $T\to\pm\infty$ and integrating, one gets:

$$\psi=Ar^{v-1} \qquad (17)$$

where $v=1-c/2-c$ and A is a constant of integration can be determined by boundary conditions. Eqs. (15) and (17) give:

$$\tau_{\theta\theta}=\left(\frac{2\mu}{n}\right)Ar^{v-1}+\frac{c\xi\Theta_0\ln(r/r_0)}{\ln(r_i/r_0)} \qquad (18)$$

Substituting Eq. (18) in Eq. (11) then using Eq. (2) and integrating, we get:

$$\tau_{rr}=\left(\frac{2\mu}{nv}\right)Ar^{v-1}+\frac{c\xi\Theta_0\ln(r/r_0)}{\ln(r_i/r_0)}-\frac{c\xi\Theta_0}{\ln(r_i/r_0)}-\frac{\rho_0\omega^2 r_{0^m}r^{2-m}}{(3-m)}+\frac{B}{r}$$
(19)

where *B* is a constant of integration can be determined by boundary conditions. Substituting Eqs. (18) and (19) in second equation of Eq. (9), we get:

$$\eta=\sqrt{1-\frac{2v}{E}\left[\frac{\rho_0\omega^2 r_{0^m}r^{2-m}}{(3-m)}-\frac{B}{r}+\frac{\alpha E(2-c)\Theta_0}{\ln(r_i/r_0)}+\frac{2\alpha E(2-c)\Theta_0\ln(r/r_0)}{(1-c)\ln(r_i/r_0)}\right]} \qquad (20)$$

where $c\xi=\alpha E(2-c)$ .Substituting Eq. (20) in Eq. (3), we get:

$$u=r-r\sqrt{1-\frac{2v}{E}\left[\frac{\rho_0\omega^2 r_{0^m}r^{2-m}}{(3-m)}-\frac{B}{r}+\frac{\alpha E(2-c)\Theta_0}{\ln(r_i/r_0)}+\frac{2\alpha E(2-c)\Theta_0\ln(r/r_0)}{(1-c)\ln(r_i/r_0)}\right]} \qquad (21)$$

where $E=2\mu(3-2c)/(2-c)$ is the Young's modulus in term of compressibility factor. Using boundary condition (14) and (12) in Eqs. (19) and (21), we get:

$$A=\frac{nv}{2\mu r_{0^v}}\left[l_0 r_0+\frac{\rho_0\omega^2 r_{0^m}\left(r_{0^{3-m}}-r_{i^{3-m}}\right)}{(3-m)}+\frac{\alpha E(2-c)\Theta_0}{\ln(r_i/r_0)}(r_0-r_i)-\frac{2r_i\alpha E(2-c)\Theta_0}{(1-c)}\right] \qquad (22)$$

$$B=\frac{\rho_0 \omega^2 r_0{}^m r_i{}^{3-m}}{(3-m)}+\frac{\alpha Ea(2-c)\Theta_0}{\ln(r_i/r_0)}+\frac{2r_i\alpha E\Theta_0(2-c)}{(1-c)}$$

(23)

Substituting Eqs. (22) and (23) in Eqs. (18), (19), and (21) respectively, we get the transitional stresses and displacement as:

$$\tau_{\theta\theta}=valignl\left[l_0\left(\frac{r}{r_0}\right)^{\nu-1}+\left(\frac{r}{r_0}\right)^{\nu}\frac{\rho_0\omega^2 r_0{}^m\left(r_0{}^{3-m}-r_i{}^{3-m}\right)}{r(3-m)}+\text{¿}\right]\text{¿¿¿}$$

$$\text{¿}$$

(24)

$$\tau_{rr}=\text{¿}\left[l_0\left(\frac{r}{r_0}\right)^{\nu-1}+\frac{\rho_0\omega^2 r_0{}^m}{r(3-m)}\left[\left(r_0{}^{3-m}-r_i{}^{3-m}\right)\left(\frac{r}{r_0}\right)^{\nu}-r^{3-m}+r_i{}^{3-m}\right]+\text{¿}\right]\text{¿¿¿}$$

$$\text{¿}$$

(25)

$$u=r-r\sqrt{1-\frac{2\nu}{E}\text{¿}}\left[\frac{\rho_0\omega^2 r_0{}^m}{r(3-m)}\left[r^{3-m}-r_i{}^{3-m}\right]+\frac{\alpha E(2-c)\Theta_0(r-r_i)}{r\ln(r_i/r_0)}+\text{¿}\right]\text{¿¿¿}$$

$$\text{¿}$$

(26)

and

$$\tau_{rr}-\tau_{\theta\theta}=\left[\begin{array}{l} l_0\left(\frac{r}{r_0}\right)^{\nu-1}(1-\nu)+\frac{\rho_0\omega^2 r_0{}^m}{r(3-m)}\left[\left(r_0{}^{3-m}-r_i{}^{3-m}\right)\left(\frac{r}{r_0}\right)^{\nu}(1-\nu)-r^{3-m}+r_i{}^{3-m}\right]+ \\ \alpha E(2-c)\Theta_0\left[\begin{array}{l}\left(\frac{r_0-r_i}{r\ln(r_i/r_0)}\right)\left(\frac{r}{r_0}\right)^{\nu}(1-\nu)+\frac{2r_i}{r(1-c)}+\frac{1}{\ln(r_i/r_0)}\left(\frac{r_i}{r}-1\right) \\ +\frac{2r_i}{r(1-c)}\left(\frac{r}{r_0}\right)^{\nu}(\nu-1)\end{array}\right]\end{array}\right]$$

(27)

Initial Yielding Stage: The maximum value $|\tau_{rr}-\tau_{\theta\theta}|$ occurs at the radius $r=r_i$ (say), which depends upon the value of *m* and *c*. For yielding at $r=r_i$ .Eq. (27) becomes:

$$
\left|\tau_{rr}-\tau_{\theta\theta}\right|_{r=r_i}=\left[\begin{array}{l} l_0\left(\dfrac{r_i}{r_0}\right)^{\nu-1}(1-\nu)+\dfrac{\rho_0\omega^2 r_0^{\,m}}{r_i(3-m)}\left[\left(r_0^{\,3-m}-r_i^{\,3-m}\right)\left(\dfrac{r_i}{r_0}\right)^{\nu}(1-\nu)\right]+ \\ \alpha E(2-c)\Theta_0\left[\begin{array}{l}\left(\dfrac{r_0-r_i}{r_i\ln(r_i/r_0)}\right)\left(\dfrac{r_i}{r_0}\right)^{\nu}(1-\nu)+\dfrac{2}{(1-c)} \\ +\dfrac{2}{(1-c)}\left(\dfrac{r_i}{r_0}\right)^{\nu}(\nu-1)\end{array}\right]\end{array}\right]\equiv Y
$$

where Y is the yielding stress. The angular speed necessary for initial yielding is given by:

$$
\Omega_i^2=\frac{\rho_0\omega_i^2 r_0^{\,2}}{Y}=\frac{(3-m)}{\left(1-R_0^{\,3-m}\right)\left(\dfrac{r_i}{r_0}\right)^{\nu-1}(1-\nu)}\text{¿}\left[1-\left(\frac{L_0}{Y}\right)\left(\frac{r_i}{r_0}\right)^{\nu-1}(1-\nu)-\text{¿}\right]\text{¿}\,\text{¿}\,\text{¿}
$$

(28)

We introduce the following non-dimensional components:

$R_0=r_i/r_0$, $\sigma_r=\tau_{rr}/Y$, $\sigma_\theta=\tau_{\theta\theta}/Y$, $U=u/r_0$, $\Theta_1=\alpha E\Theta_0/Y$, $\Omega^2=\rho_0\omega^2 r_0^{\,2}/Y$, $H=Y/E$ and $L_0=l_0/Y$. Elastic-plastic transitional stresses, angular speed and displacement from Eqs. (24), (25), (28) and (26) in non-dimensional form become:

$$
\sigma_\theta=\nu\left[L_0R^{\nu-1}+\frac{\Omega_i^2R^{\nu-1}\left(1-R_0^{3-m}\right)}{(3-m)}\right]+\Theta_1(2-c)\left[\frac{\ln R}{\ln R_0}-\frac{2R_0}{(2-c)}R^{\nu-1}+\frac{(1-R_0)\nu}{\ln R_0}R^{\nu-1}\right] \quad (29)
$$

$$
\sigma_r=\text{¿}\left[L_0R^{\nu-1}+\frac{\Omega_i^2}{R(3-m)}\left[\left(1-R_0^{3-m}\right)R^{\nu}-R^{3-m}+R_0^{3-m}\right]\text{¿}\right]\left[+\frac{\Theta_1(2-c)}{\ln R_0}\left[(1-R_0)R^{\nu-1}+\ln R-1+\frac{R_0}{R}\right]\text{¿}\right]\text{¿}\,\text{¿}\,\text{¿}\;\text{¿}
$$

(30)

$$\Omega_i^2 = \frac{\rho_o \omega_i^2 r_{0^2}}{Y} = \frac{(3-m)}{(1-R_{0^{3-m}}) R_{0^{\nu-1}} (1-\nu)} ¿ \left[1 - L_0 R_{0^{\nu-1}} (1-\nu) - ¿\right] ¿ ¿ ¿$$

(31)

$$U_i = R - R\sqrt{1-2\nu Halignl \left[\frac{\Omega_i^2}{R(3-m)}\left[R^{3-m} - R_{0^{3-m}}\right] + \frac{(2-c)\Theta_1 (R-R_0)}{R \ln R_0} ¿ \right] ¿¿¿¿} ¿$$

and (32)

Fully Plastic Stage: For fully-plastic stage $c \to 0$ *i.e.* $\nu = 1/2$ at the outer surface and Eq. (27) obtained:

$$|T_{rr} - T_{\theta\theta}|_{R=R_0} = \left[\frac{L_0}{2} + \frac{\rho_0 \omega_f^2 r_{0^2}}{(3-m)}\left[\frac{(1-R_{0^{3-m}})}{2} + R_0^{3-m} - 1\right] + \alpha E \Theta_0 \left[\frac{(1-R_0)}{\ln R_0} + 2R_0 + \frac{2(1-R_0)}{\ln R_0}\right]\right] ¿ Y^{¿}$$

and the angular speed required for fully plastic state is given by:

$$\Omega_{f^{¿}}^2 = \frac{\rho_o \omega_f^2 r_{0^2}}{Y^{¿}} = \left|\frac{2(3-m)}{(1-R_{0^{3-m}})}\right| \left[\left|1 - \frac{L_0}{2}\right| + \Theta_1 \left\{\frac{(1-R_0)}{\ln R_0} + 2R_0 + \frac{2(1-R_0)}{\ln R_0}\right\}\right]$$

(33)

where $\omega_{f^{¿}} = \frac{\Omega_{f^{¿}}}{r_0}\sqrt{\frac{Y}{\rho_0}}$ and $\frac{\alpha E \Theta_0}{Y^{¿}} = \Theta_1$.

Using Eq. (33) into Eqs. (29), (30), (32) by taking $c \to 0$ *i.e.* $\nu = 1/2$, we get the stresses and displacement for the inner surface as:

$$\sigma_{\theta^{¿}} = \frac{1}{2}\left[\frac{L_0}{\sqrt{R}} + \frac{\Omega_{f^{¿}}^2 (1-R_0^{3-m})}{\sqrt{R}(3-m)}\right] + 2\Theta_1 \left[\frac{\ln R}{\ln R_0} - \frac{R_0}{\sqrt{R}} + \frac{(1-R_0)}{2\sqrt{R}\ln R_0}\right]$$

(34)

$$\sigma_{r^{¿}} = ¿ \left[\frac{L_0}{\sqrt{R}} + \frac{\Omega_{f^{¿}}^2}{R(3-m)}\left[\sqrt{R}(1-R_0^{3-m}) - R^{3-m} + R_0^{3-m}\right] + ¿\right] ¿¿¿ ¿$$

(35)

and

$$U_f = R - R\sqrt{1 - H\left[\frac{\Omega_{f^{¿}}^2}{R(3-m)}\left[R^{3-m} - R_{0^{3-m}}\right] + \frac{2\Theta_1 (R-R_0)}{R \ln R_0} + 4\Theta_1 \left[\frac{\ln R}{\ln R_0} - \frac{R_0}{R}\right]\right]}$$

(36)

Eqs. (33)-(36) are same as given by Sethi *et al.* [17] ,when we neglecting thermal condition.

Results and Discussion

For calculating the stresses, angular speed for initial, fully plastic stage and displacement of the disc made of incompressible material (*i.e.* c = 0 or v= 0.5; rubber) and compressible material (*i.e.* c = 0.5 or v = 0.33 ; copper) based on the above analysis, the following values have been taken as: m = -2,0,2 (density parameter); H = 1, ½; $L_0=50,75$ (load), $\alpha=5.0\times10^{-5}\deg F^{-1}$ for Methyl Methacrylate [5], Θ_0 = 0 ^0F , 10000^0F , 15000^0F ; $\Theta_1=\alpha\Theta_0=0.0.5,0.75$ respectively.

From Table 1, shows that the percentage increase in angular speed required for initial yielding to fully-plastic stage for the rotating disc made of incompressible material (*i.e.* v = 0.5) and compressible material (*i.e.* v = 0.333) has been discussed. It can also be seen from the Table 1, that for compressible material (*i.e.* v = 0.33, copper) required higher percentage values (*say P* = 59.36%, 59.37%, 59.38% at $\Theta_1=0$) of the angular speed to become fully plastic in comparison to the disc made of incompressible material (*i.e.* v = 0.5, rubber) and the percentage values are (*say P* = 42.64 %, 42.65 %, 42. 66 % at Θ_1 = 0) respectively. With increased the values of mechanical load and temperature, the ratio of angular speed for fully-plastic with respect to initial plastic are decreased.

Table 1. Values of angular speed for initial yielding Ω_i^2 and fully plastic stage Ω_f^2 : **(a)** v = 0.5; **(b)** v = 0.33

Density	Angular speed	v = 0.5 ; rubber (incompressible material)					
		Θ_1 = 0		Θ_1 = 0.5		Θ_1 = 0.75	
		Yielding starts at the bore ($r = r_i$)					
		L_o= 50	L_o = 75	L_o = 50	L_o = 75	L_o = 50	L_o = 75
m= -2	Ω_i^2	121.733	186.249	132.608	197.124	138.046	202.562
	Ω_f^2	247.7419	376.7742	252.1841	381.2164	254.4052	383.4375
m= 0	Ω_i^2	80.8656	123.723	88.0898	130.947	91.702	134.559
	Ω_f^2	164.5714	250.2857	167.5223	253.2366	168.9978	254.712
m= 2	Ω_i^2	47.1716	72.1716	51.3857	76.3857	53.4928	78.4928
	Ω_f^2	96	146	97.72135	147.7213	98.58202	148.582
m = -2	P %	42.64	42.21	37.88	39.04	35.75	37.58
m = 0	P %	42.65	42.22	37.89	39.05	35.76	37.59
m= 2	P %	42.66	42.23	37.90	39.06	35.77	37.60

(a)

Density	Angular speed	v = 0.333;copper (compressible material)					
		Θ_1 =0		Θ_1 = 0.5		Θ_1 = 0.75	
		Yielding starts at the bore ($r = r_i$)					
		L_o= 50	L_o = 75	L_o = 50	L_o = 75	L_o = 50	L_o = 75
m= -2	Ω_i^2	97.5359	148.742	109.375	160.581	115.294	166.501
	Ω_f^2	247.7419	376.7742	252.1841	381.2164	254.4052	383.4375

$m = 0$	Ω_i^2	64.7917	98.8074	72.6562	106.672	76.5884	110.604
	Ω_f^2	164.5714	250.2857	167.5223	253.2366	168.9978	254.712
$m = 2$	Ω_i^2	37.7951	57.6377	42.3828	62.2253	44.6766	64.5191
	Ω_f^2	96	146	97.72135	147.7213	98.58202	148.582
$m = -2$	P %	59.36	59.15	51.84	54.07	48.55	51.74
$m = 0$	P %	59.37	59.16	51.85	54.08	48.56	51.75
$m = 2$	P %	59.38	59.17	51.86	54.09	48.57	51.76

(b)

where $P = \left(\sqrt{\Omega_f^2/\Omega_i^2} - 1\right) * 100$ is the percentage increase in angular speed from initial yielding to become fully plastic state having density m, Load L_0 , Θ_1 = 0, 0.5, 0.75 and $v = 0.5, 0.33$.

In Fig. 2- Fig. 3, curves have been drawn between angular speed required for initial and fully-plastic stage and radii ratio $R = r_i/r_0$ for the disc made of compressible and incompressible materials and having Possion's ratio $v = 0.5, 0.333$, density (m = -2, 0, 2) and mechanical load (*i.e.* L_0 = 50, 75). It has been seen from Fig. 2, that the rotating disc made of incompressible material (*i.e.* v = 0.5: rubber) requires higher angular speed to yield at the internal surface as compared to the disc made of compressible material.

With the introduction of thermal condition, mechanical load and density parameter, the value of angular speed are increased at the internal surface of the disc made of incompressible material and also in compressible material. It has been observed from Fig. 3, that the value of angular speed for fully-plastic state increases at the internal surface with increased the value of mechanical load, thermal condition and density parameter. Curves are drawn between the stress distribution along the radii ratios $R = r/r_0$ (*see* Fig. 4) at the transition stage. From Fig. 4, it has been seen that the radial stress has a maximum value at the internal surface of the disc made of compressible material as compare to the disc made of incompressible material. With the introduction of thermal condition, density parameter and load, the values of radial and circumferential stress increase in the internal surface for compressible and incompressible materials. Curves are drawn between displacement component and radii ratio $R = r/r_0$ at the transition and fully-plastic stage (*see* Fig. 5). It has been seen that the value of the displacement component increased on the outer surface of the disc made of compressible /incompressible materials and fully plastic stage. With the increased value of the mechanical load and thermal conditions, the displacement component is increased on the outer surface but reverse results are obtained in case of density parameter.

Conclusion

It has been observed that the rotating disc made of incompressible material (*i.e*. rubber) requires higher angular speed to yield at the internal surface as compared to the disc made of compressible material (*i.e*. copper). With the introduction of thermal condition, mechanical load and density parameter, the value of angular speed are increased at the internal surface of the disc made of incompressible material and also in compressible material. The value of radial stress has a maximum at the internal surface of the disc made of compressible material as compare to the disc made of incompressible material.

Funding: This research received no specific grant from any funding agency in the public, commercial, or not for profit sectors.

Compliance with ethical standards

Conflict of interest: The authors certify that they have no affiliations with or involvement in any organization or entity with any financial or nonfinancial interest in the subject matter or materials discussed in this manuscript.

Nomenclature

r_i, r_0 -Inner and outer radii of the disc [m],

ω - Angular velocity of rotation,[s^{-1}]

u, v, w -Displacement components, [m]

ρ -Density of material,[kgm^{-3}]

ρ_0 - Constant density

c -Compressibility, [-]

λ, μ -Lame's constants

m - Density parameter

$\tau_{ij}, \varepsilon_{ij}$ -Stress and Strain components

Y -Yield stress,[$kgm^{-1}s^{-2}$]

Ω_i^2 -Angular speed for initial stage

Ω_f^2 -Angular speed for fully plastic stage

Greek letters

$R = r/r_0, R_0 = r_i/r_0$ Radii ratio,[-]

σ_r - Radial stress component (τ_{rr}/Y),[-]

σ_θ - Circumferential stress component ($\tau_{\theta\theta}/Y$),[-]

Θ - Temperature,[$^0 F$]

A, B, d -Constants of integration,[-]

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