Dynamical properties in an SVEIR epidemic model with age-dependent vaccination, latent, infected and relapse

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Abstract: In this paper an SVEIR epidemic model with continuous age-dependent vaccination, latent, infected and disease relapse is proposed. The dynamical behavior including the calculation of basic reproduction number $\mathcal{R}_0$, the existence of steady states and their local and global stability, and the uniform persistence are investigated. We derive that if $\mathcal{R}_0 \leq 1$ then the disease-free steady state is globally asymptotically stable, and hence the disease dies out, if $\mathcal{R}_0 > 1$ then the disease in the model is uniform persistence and the endemic steady state also is globally asymptotically stable, and hence the disease becomes endemic. The research shows the global dynamics of the model are sharply determined by its basic reproduction number. Finally, numerical examples support our main analytical results.

Keywords: SVEIR epidemic model; age-dependence; basic reproduction number; local and global stability; uniform persistence.

1 Introduction

Since ancient times, the losses caused by the spread of infectious diseases are far greater than the sum of all wars in history. Infectious diseases not only threaten human health and lives, but also cause severe social decline and even the demise of the country. Therefore, further research in infectious diseases area appears to be very critical and necessary.

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Mathematical modeling has been proven to be important in better understanding the transmission dynamics of epidemic diseases and evaluating the effectiveness of various control and prevention strategies (see [1-5]). For example, measles, tuberculosis, SARS, HBV and HIV etc, will show the influence of age at some time of the disease, so the age-structured disease infection has long been regarded as an important factor affecting the spread of disease (see [6-10]).

By the last 20 years of the mid-20th century, the age-structured of the model has been widely studied. In [11] proposed and analyzed an age-dependent infectious disease model, on this basis, especially in recent years, many authors considered the age-structured model with higher dimensionality, such as [12] gave qualitative analysis of a kind of age-structured SVEIR tuberculosis model; [13] considered a multi-group model with generalized nonlinear incidence and vaccination age; an age-dependent SVEIR model with vaccination, latent, relapse and made corresponding numerical simulation verification theoretical results has been studied in [14-16]; in [17], the authors proposed an SVEIR epidemic model with ages of vaccination and latency. At the same time, many studies considered infection age, Xu [18] studied an epidemiological model with age of infection and disease relapse; [19] showed that a tuberculosis model with fast and slow progression and age-dependent latency and infection etc; an SVEIR model with continuous age-structure in the infectious class are considered in [20,21]; [22] paid attentions to a multi-group SVEIR epidemiological model with the vaccination age and infection age. In comparison, high-dimensional infectious disease model with age of infection is not considered.

To understand the effect of vaccination age, latency age, infection age and relapse age on global dynamics of the model, based on a lot of previous work, in this paper we propose an age-dependent SVEIR model of partial differential equation. The existence, uniqueness, boundedness, asymptotic smoothness and uniform persistence is proved by reformulating it as the so called Volterra integral equations. By calculations, the basic reproduction number of the model was obtained, the local stability of a disease-free steady state and an endemic steady state of the model is established by analyzing corresponding characteristic equations. By constructing suitable Lyapunov functionals and using LaSalle's invariance principle, it is verified that the global dynamics of the model is completely determined by the basic reproduction number.

This work is organized as follows: In Section 2, we propose a novel age-dependent SVEIR model with ages of vaccination, latency, infection and relapse. Some basic properties of the solutions, the existence of steady states and the basic reproduction number for the model is analyzed in Section 3. In Section 4, we prove the local and global stability of the disease-free steady state. In Section 5, we investigate uniform persistence of the model. In Section 6, the stability of the endemic steady state is proved, respectively. Finally, simulation and discussion are made in Section 7-8.

2 Model formulation

In this section, we will construct an SVEIR epidemic model with age-dependent vaccination, latent, infected and relapse. We assume that the total population N is divided into five classes: the susceptible (S), vaccinated (V), latent (E), infected (I) and recovered (R), respectively. Let $S(t)$ be the number of the susceptible at time t, $v(t,a)$ be the density of the vaccinated with age of vaccination a at time t, $e(t,b)$ be the density of the latent with age of latency b at time t, $i(t,c)$ be the density of the infected with age of infection c at time t and $r(t,h)$ be the density of the recovered with age of relapse h at time t.

We assume positive constants Λ, μ, ξ, β to be the birth rate, the per-capita natural death, the vaccination rate of the susceptible individuals and the rate of transmission of the disease. $\delta(c)$ is the disease induced death rate dependent on age c, we here take the nonlinear rate $\beta S f(\bar{I})$. Assume that the newly vaccinated individuals enter the vaccinated class at vaccination age zero, the vaccine-induced immunity wanes rate is dependent on age of vaccination and given by $\omega_1(a)$. Then the total number of vaccinated individuals within the vaccinated subclass at time t is $\int_0^\infty v(t,a)\mathrm{d}a$. Thus the total number of losing immunity which return to the susceptible class alive reads $\int_0^\infty \omega_1(a)v(t,a)\mathrm{d}a$. Similarly, for the density of the latent $e(t,b)$ at time t with latency age b , the density of the infected $i(t,c)$ at time t with infection age c and the density of the recovered $r(t,h)$ at time t with relapse age h, the latency developing the infected class, the infection developing the recovered class and the relapse into the infected class alive write $\int_0^\infty \omega_2(b)e(t,b)\mathrm{d}b$, $\int_0^\infty \omega_3(c)i(t,c)\mathrm{d}c$ and $\int_0^\infty \omega_4(h)r(t,h)\mathrm{d}h$, respectively. Under these assumptions, our model is described by the following diagram in Fig 1.

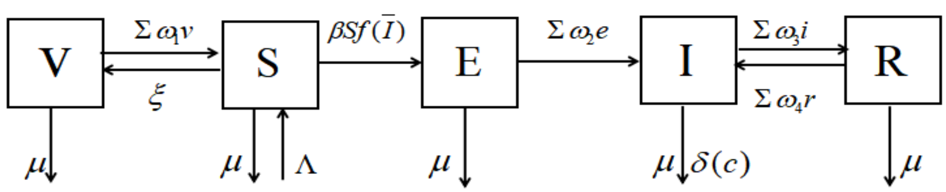

Fig 1. Flow diagram, where $\bar{I}(t) = \int_0^\infty \omega_5(c)i(t,c)\mathrm{d}c$ and $\Sigma\omega_1 v$, $\Sigma\omega_2 e$, $\Sigma\omega_3 i$, $\Sigma\omega_4 r$ represent $\int_0^\infty \omega_1(a)v(t,a)\mathrm{d}a$, $\int_0^\infty \omega_2(b)e(t,b)\mathrm{d}b$, $\int_0^\infty \omega_3(c)i(t,c)\mathrm{d}c$ and $\int_0^\infty \omega_4(h)r(t,h)\mathrm{d}h$.

From Fig 1, the model takes the following form:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - (\mu+\xi)S(t) - \beta S(t) f(\bar{I}(t)) + \displaystyle\int_0^\infty \omega_1(a) v(t,a)\mathrm{d}a, \\ \dfrac{\partial v(t,a)}{\partial t} + \dfrac{\partial v(t,a)}{\partial a} = -(\omega_1(a)+\mu)v(t,a), \\ \dfrac{\partial e(t,b)}{\partial t} + \dfrac{\partial e(t,b)}{\partial b} = -(\omega_2(b)+\mu)e(t,b), \\ \dfrac{\partial i(t,c)}{\partial t} + \dfrac{\partial i(t,c)}{\partial c} = -(\omega_3(c)+\mu+\delta(c))i(t,c), \\ \dfrac{\partial r(t,h)}{\partial t} + \dfrac{\partial r(t,h)}{\partial h} = -(\omega_4(h)+\mu)r(t,h), \end{cases} \tag{1}$$

with the following initial conditions and boundary conditions

$$\begin{aligned} & S(0) = S_0,\ v(0,a) = v_0(a),\ e(0,b) = e_0(b),\ i(0,c) = i_0(c),\ r(0,h) = r_0(h), \\ & v(t,0) = \xi S(t),\ e(t,0) = \beta S(t) f(\bar{I}(t)), \\ & i(t,0) = \int_0^\infty \omega_2(b) e(t,b)\mathrm{d}b + \int_0^\infty \omega_4(h) r(t,h)\mathrm{d}h,\ r(t,0) = \int_0^\infty \omega_3(c) i(t,c)\mathrm{d}c. \end{aligned} \tag{2}$$

where $\bar{I}(t) = \int_0^\infty \omega_5(c) i(t,c)\mathrm{d}c$, $S_0 \in \mathbb{R}_+ = [0,\infty)$ and $v_0(a), e_0(b), i_0(c), r_0(h) \in L_+^1$, where $L_+^1 = L_+^1(0,\infty)$ denotes the space of all Lebesgue integrable functions $\phi : (0,\infty) \to \mathbb{R}_+$.

It is biologically motivated that we always require the following assumptions.

($\mathbf{A}_1$) Functions $\omega_i(l) \in L_+^1$ $(i = 1,2,3,4,5)$ are positive and bounded with the upper bound $\bar{\omega}_i$ and Lipschitz continuous on $\mathbb{R}_+$ with Lipschitz constants M_{ω_i}.

($\mathbf{A}_2$) Function $f(\bar{I})$ is nonnegative and twice differentiable for all $\bar{I} \in [0,\infty)$ with $f(\bar{I}) = 0$ if and only if $\bar{I} = 0$, $f'(\bar{I}) \geq 0$ and $f''(\bar{I}) \leq 0$ for all $\bar{I} \geq 0$.

Remark 1. *It is clear that for the bilinear incidence rate $f(\bar{I}) = \bar{I}$ and the saturated incidence rate $f(\bar{I}) = \frac{\bar{I}}{1+\alpha\bar{I}}$, where $\alpha > 0$ is a constant, assumption ($\boldsymbol{A}_2$) is satisfied.*

Remark 2. *It is easy to see that model (1) generalizes some pre-existing age-dependent SVEIR type epidemic models which are proposed and investigated in [13,14,16,17,22].*

3 Basic properties

In this section we establish some basic properties of solutions for model (1). For the convenience, we give some notations and expressions as follows:

$$\varepsilon_i(l) = \omega_i(l) + \mu\ (i = 1,2,3,4),\ \rho_i(l) = e^{-\int_0^l \varepsilon_i(s)\mathrm{d}s}\ (i = 1,2,4),\ \rho_3(c) = e^{-\int_0^c (\varepsilon_3(s)+\delta(s))\mathrm{d}s},$$

$$\theta_i(\lambda) = \int_0^\infty \omega_i(l) e^{-(\lambda l + \int_0^l \varepsilon_i(s)\mathrm{d}s)}\mathrm{d}l\ (i = 1,2,4),\ \theta_3(\lambda) = \int_0^\infty \omega_3(c) e^{-(\lambda c + \int_0^c (\varepsilon_3(s)+\delta(s))\mathrm{d}s)}\mathrm{d}c,$$

$$\pi_i(l) = \int_l^\infty \omega_i(\tau) e^{-\int_l^\tau \varepsilon_i(s)\mathrm{d}s} \mathrm{d}\tau \ (i = 1, 2, 4), \ \tau_3(\lambda) = \int_0^\infty \omega_5(c) e^{-\int_0^c (\lambda + \varepsilon_3(s) + \delta(s))\mathrm{d}s} \mathrm{d}c,$$

$$\theta_i = \theta_i(0) = \int_0^\infty \omega_i(l)\rho_i(l)\mathrm{d}l \ (i = 1, 2, 3, 4), \ \tau_3 = \tau_3(0) = \int_0^\infty \omega_5(c)\rho_3(c)\mathrm{d}c.$$

It follows from $(\mathbf{A}_1)$ that for every $i = 1, 2, 3, 4$, $0 < \theta_i < 1$ and $0 < \rho_i(s) < 1$ for all $s \geq 0$, $\pi_i(l) > 0$ for all $l \geq 0$, $\pi_i(0) = \theta_i$, $\theta_i(\lambda) \leq \theta_i$, $\frac{\mathrm{d}\theta_i(\lambda)}{\mathrm{d}\lambda} < 0$ and $\tau_3(\lambda) \leq \tau_3$ for all $\lambda \geq 0$. Meanwhile

$$\frac{\mathrm{d}\rho_i(s)}{\mathrm{d}s} = -\varepsilon_i(s)\rho_i(s) \ (i = 1, 2, 4), \ \frac{\mathrm{d}\rho_3(s)}{\mathrm{d}s} = -(\varepsilon_3(s) + \delta(s))\rho_3(s),$$

and

$$\frac{\mathrm{d}\pi_i(l)}{\mathrm{d}l} = \pi_i(l)\varepsilon_i(l) - \omega_i(l), \ i = 1, 2, 4.$$

Furthermore, as the application of Volterra formulation (See [23]), solving $v(t, a)$, $e(t, b)$, $i(t, c)$ and $r(t, h)$ from the second, third, fourth and fifth equations of model (1) along the characteristic line $t - a = const$, we can obtain

$$v(t, a) = \begin{cases} v(t - a, 0)\rho_1(a), \ t > a \geq 0, \\ v_0(a - t)\dfrac{\rho_1(a)}{\rho_1(a - t)}, \ a \geq t \geq 0. \end{cases} \tag{3}$$

$$e(t, b) = \begin{cases} e(t - b, 0)\rho_2(b), \ t > b \geq 0, \\ e_0(b - t)\dfrac{\rho_2(b)}{\rho_2(b - t)}, \ b \geq t \geq 0. \end{cases} \tag{4}$$

$$i(t, c) = \begin{cases} i(t - c, 0)\rho_3(c), \ t > c \geq 0, \\ i_0(c - t)\dfrac{\rho_3(c)}{\rho_3(c - t)}, \ c \geq t \geq 0. \end{cases} \tag{5}$$

$$r(t, h) = \begin{cases} r(t - h, 0)\rho_4(c), \ t > h \geq 0, \\ r_0(h - t)\dfrac{\rho_4(c)}{\rho_4(h - t)}, \ h \geq t \geq 0. \end{cases} \tag{6}$$

The phase space $\mathbb{X}$ of model (1) is defined by $\mathbb{X} = \mathbb{R}_+ \times L^1_+ \times L^1_+ \times L^1_+ \times L^1_+$, equipped with the norm for any $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{X}$ by

$$\|(x_1, x_2, x_3, x_4, x_5)\|_{\mathbb{X}} = x_1 + \int_0^\infty x_2(a)\mathrm{d}a + \int_0^\infty x_3(b)\mathrm{d}b + \int_0^\infty x_4(c)\mathrm{d}c + \int_0^\infty x_5(h)\mathrm{d}h.$$

The initial conditions in model (1) can be rewritten as $x_0 = (S_0, v_0(\cdot), e_0(\cdot), i_0(\cdot), r_0(\cdot)) \in \mathbb{X}$. It is easy to see that $v(0, 0) = v_0(0) = \xi S_0$, $e(0, 0) = e_0(0) = \beta S_0 f(\bar{I}_0)$, $i(0, 0) = i_0(0) = \int_0^\infty \omega_2(b)e_0(b)\mathrm{d}b + \int_0^\infty \omega_4(h)r_0(h)\mathrm{d}h$ and $r(0, 0) = r_0(0) = \int_0^\infty \omega_3(c)i_0(c)\mathrm{d}c$, where $\bar{I}_0 = \int_0^\infty \omega_5(c)i_0(c)\mathrm{d}c$. The standard existence, uniqueness, nonnegativity and continuability of solutions for model (1) are valid (see [24]). Thus, we immediately obtain the following lemma.

Lemma 1. *For any point $x_0 \in \mathbb{X}$, model (1) has a unique nonnegative solution $\Phi(t, x_0) = (S(t), v(t,\cdot), e(t,\cdot), i(t,\cdot), r(t,\cdot)) \in \mathbb{X}$ defined on the maximal existence interval $[0, T_\infty)$ with $T_\infty \leq \infty$ satisfying initial condition $\Phi(0, x_0) = x_0$.*

Clearly, we have

$$\|\Phi(t, x_0)\|_{\mathbb{X}} = S(t) + \int_0^\infty v(t,a)\mathrm{d}a + \int_0^\infty e(t,b)\mathrm{d}b + \int_0^\infty i(t,c)\mathrm{d}c + \int_0^\infty r(t,h)\mathrm{d}h.$$

Define a set Π as follows

$$\begin{aligned}\Pi = \ & \Big\{(S, v(\cdot), e(\cdot), i(\cdot), r(\cdot)) \in \mathbb{X}: \\ & \qquad S + \int_0^\infty v(a)\mathrm{d}a + \int_0^\infty e(b)\mathrm{d}b + \int_0^\infty i(c)\mathrm{d}c + \int_0^\infty r(h)\mathrm{d}h \leq \frac{\Lambda}{\mu}\Big\}.\end{aligned}$$

We have the following result on the global existence, boundedness and invariance of solutions for model (1).

Theorem 1. (i) *For any initial point $x_0 \in \mathbb{X}$, the solution $\Phi(t, x_0)$ is defined for all $t \geq 0$ and is ultimately bounded. That is,*

$$\limsup_{t\to\infty} \|\Phi(t, x_0)\|_{\mathbb{X}} \leq \frac{\Lambda}{\mu}.$$

(ii) *Π is positively invariant for model (1). That is, $\Phi(t, x_0) \in \Pi$ for all $t > 0$ and $x_0 \in \Pi$;*

(iii) *For any constant $M \geq \frac{\Lambda}{\mu}$, if $x_0 \in \mathbb{X}$ satisfies $\|x_0\|_{\mathbb{X}} \leq M$ then for any $t \geq 0$ $\|\Phi(t, x_0)\|_{\mathbb{X}} \leq M$ and*

$$v(t,0) \leq \xi M,\ e(t,0) \leq \beta f'(0)\bar{\omega}_5 M^2,\ i(t,0) \leq (\bar{\omega}_2 + \bar{\omega}_4)M,\ r(t,0) \leq \bar{\omega}_3 M.$$

Proof. For any initial point $x_0 \in \mathbb{X}$, from Lemma 1, model (1) has a unique nonnegative solution $\Phi(t, x_0)$ defined for $t \in [0, T_\infty)$. Calculating the derivative of $\|\Phi(t, x_0)\|_{\mathbb{X}}$, we have

$$\begin{aligned}\frac{\mathrm{d}\|\Phi(t,x_0)\|_{\mathbb{X}}}{\mathrm{d}t} = \ & \Lambda - (\mu+\xi)S(t) - \beta S(t) f(\bar{I}(t)) + \int_0^\infty \omega_1(a) v(t,a)\mathrm{d}a \\ & - \int_0^\infty \varepsilon_1(a) v(t,a)\mathrm{d}a + \xi S - \int_0^\infty \varepsilon_2(b) e(t,b)\mathrm{d}b + \beta S(t) f(\bar{I}(t)) \\ & - \int_0^\infty (\varepsilon_3(c) + \delta(c)) i(t,c)\mathrm{d}c + \int_0^\infty \omega_2(b) e(t,b)\mathrm{d}b \\ & + \int_0^\infty \omega_4(h) r(t,h)\mathrm{d}h - \int_0^\infty \varepsilon_4(h) r(t,h)\mathrm{d}h + \int_0^\infty \omega_3(c) i(t,c)\mathrm{d}c \\ \leq \ & \Lambda - \mu\|\Phi(t,x_0)\|_{\mathbb{X}}.\end{aligned}$$

Then, we further have

$$\|\Phi(t,x_0)\|_{\mathbb{X}} \leq \frac{\Lambda}{\mu} - e^{-\mu t}\Big(\frac{\Lambda}{\mu} - \|x_0\|_{\mathbb{X}}\Big),\ t \in [0, T_\infty). \tag{7}$$

From this, we have that $\Phi(t,x_0)$ is bounded on $[0,T_\infty)$. Hence, $T_\infty = \infty$. That is, the solution $\Phi(t,x_0)$ can be extended to whole $[0,\infty)$.

It follows from (7) that $\limsup_{t\to\infty} \|\Phi(t,x_0)\|_{\mathbb{X}} \leq \frac{\Lambda}{\mu}$ for any $x_0 \in \mathbb{X}$. This gives $\Phi(t,x_0)$ is ultimately bounded for any $x_0 \in \mathbb{X}$.

When $x_0 \in \Pi$, from (7) we directly get $\Phi(t,x_0) \in \Pi$ for all $t \geq 0$. This shows that set Π is a positive invariant set for model (1).

Furthermore, from (7) we also have if $\|x_0\|_{\mathbb{X}} \leq \frac{\Lambda}{\mu}$, then $\|\Phi(t,x_0)\|_{\mathbb{X}} \leq \frac{\Lambda}{\mu} \leq M$, and if $\frac{\Lambda}{\mu} \leq \|x_0\|_{\mathbb{X}} \leq M$, then $\|\Phi(t,x_0)\|_{\mathbb{X}} \leq \frac{\Lambda}{\mu} - e^{-\mu t}(\frac{\Lambda}{\mu} - \|x_0\|_{\mathbb{X}}) \leq \frac{\Lambda}{\mu} - (\frac{\Lambda}{\mu} - \|x_0\|_{\mathbb{X}}) \leq M$.

Besides, we further have $v(t,0) = \xi S(t) \leq \xi M$, $e(t,0) = \beta S(t) f(\bar{I}(t)) \leq \beta S(t) f'(0)\bar{I}(t) \leq \beta f'(0)\bar{\omega}_5 M^2$, $i(t,0) = \int_0^\infty \omega_2(b)e(t,b)\mathrm{d}b + \int_0^\infty \omega_4(h)r(t,h)\mathrm{d}h \leq (\bar{\omega}_2 + \bar{\omega}_4)M$ and $r(t,0) = \int_0^\infty \omega_3(c)i(t,c)\mathrm{d}c \leq \bar{\omega}_3 M$. This completes the proof.

Remark 3. *From Theorem 1 we can obtain that all nonnegative solutions of model (1) generate a solution semiflow* $\Phi(t) : \mathbb{X} \to \mathbb{X}$ *by* $\Phi(t)x_0 = \Phi(t,x_0)$ *for* $t \geq 0$ *and* $x_0 \in \mathbb{X}$.

Furthermore, to discuss the dynamical behavior of solutions for model (1) including stability of steady states and persistence of positive solutions we further need to establish the asymptotically smooth of solutions for model (1). Using the similar argument as in [18] we can state and prove the following result.

Theorem 2. *The semi-flow* $\Phi(t,x_0)$ *generated by model (1) is asymptotically smooth. That is, for any initial point* $x_0 \in \mathbb{X}$ *the solution trajectory* $\Phi(t,x_0)$ *has a compact closure in* $\mathbb{X}$.

As a consequence of Theorems 1 and 2, we have the following corollary.

Corollary 1. *The solution semi-flow* $\Phi(t)$ *of model (1) has a compact and global attractor.*

Proof. From Theorems 1 and 2, we obtain that any solution $\Phi(t,x_0)$ of model (1) is ultimately bounded and has a compact closure in $\mathbb{X}$, which shows the solution semiflow $\Phi(t)$ is point dissipative. By Theorem 6.5 in [25], we can get the $\Phi(t)$ is compact for any $t > 0$. By Theorem 2.6 in [26], we know that $\Phi(t)$ has a compact and global attractor in $\mathbb{X}$. This completes the proof.

Using the next generation operator method which is established in [27], by calculation we can obtain the basic reproduction number of model (1) as follows

$$\mathcal{R}_0 = \beta S^0 f'(0)\theta_2\tau_3 + \theta_3\theta_4.$$

In $\mathcal{R}_0$, we easily see that $S^0 = \frac{\Lambda}{\mu+\xi(1-\theta_1)} = \frac{\Lambda}{\mu}(1 - \frac{\xi(1-\theta_1)}{\mu+\xi(1-\theta_1)})$ indicates the total number of susceptible people remaining after vaccination at the beginning of infectious diseases, β represents the probability for susceptible person contacts infected person and becomes latent person, θ_2 indicates the overall conversion rate of latent with different latency age eventually becoming infected, τ_3 represents the survival period of the infected with different infection age, θ_4 indicates the overall recurrence rate of the recovered with different relapse age. Therefore, $\mathcal{R}_0$ indicates in the initial stage of infectious disease, the sum of the number of infected with different infection age who are susceptible to infection get sick and eventually become infected during their survival period and the number of the recovered with different relapse age. This shows that $\mathcal{R}_0$ happens to be the basic reproduction number of model (1).

Now, we are concerned with the the existence of feasible steady states of model (1) with the boundary conditions.

Clearly, model (1) always has a disease-free steady state $P_0 = (S^0, v^0(a), 0, 0, 0)$, where $v^0(a) = \xi S^0 \rho_1(a)$. If model (1) has an endemic steady state $P^* = (S^*, v^*(a), e^*(b), i^*(c), r^*(h))$, then it must satisfy the following system:

$$\begin{cases} \Lambda = (\mu + \xi)S^* + \beta S^* f(\bar{I}^*) - \displaystyle\int_0^\infty \omega_1(a) v^*(a) \mathrm{d}a, \\ \dfrac{\mathrm{d}v^*(a)}{\mathrm{d}a} = -\varepsilon_1(a) v^*(a), \\ \dfrac{\mathrm{d}e^*(b)}{\mathrm{d}b} = -\varepsilon_2(b) e^*(b), \\ \dfrac{\mathrm{d}i^*(c)}{\mathrm{d}c} = -(\varepsilon_3(c) + \delta(c)) i^*(c), \\ \dfrac{\mathrm{d}r^*(h)}{\mathrm{d}h} = -\varepsilon_4(h) r^*(h), \\ v^*(0) = \xi S^*, e^*(0) = \beta S^* f(\bar{I}^*), \\ i^*(0) = \displaystyle\int_0^\infty \omega_2(b) e^*(b) \mathrm{d}b + \int_0^\infty \omega_4(h) r^*(h) \mathrm{d}h, \\ r^*(0) = \displaystyle\int_0^\infty \omega_3(c) i^*(c) \mathrm{d}c, \end{cases} \tag{8}$$

where $\bar{I}^* = \int_0^\infty \omega_5(c) i^*(c) \mathrm{d}c$. We obtain from system (8) that

$$v^*(a) = v^*(0)\rho_1(a),\ e^*(b) = e^*(0)\rho_2(b),\ i^*(c) = i^*(0)\rho_3(c),\ r^*(h) = r^*(0)\rho_4(h). \tag{9}$$

It follows from the first equation of system (8) that

$$S^* = \frac{\Lambda}{\mu + \xi(1-\theta_1) + \beta f(\bar{I}^*)}. \tag{10}$$

From (8) and (9), we obtain

$$\begin{aligned}
i^*(0) =& e^*(0)\int_0^\infty \omega_2(b)\rho_2(b)\mathrm{d}b + r^*(0)\int_0^\infty \omega_4(h)\rho_4(h)\mathrm{d}h\\
=& i^*(0)\Big(\beta S^*\frac{f(\bar{I}^*)}{\bar{I}^*}\theta_2\tau_3 + \theta_3\theta_4\Big).
\end{aligned}$$

We have $i^*(0) \neq 0$ in the endemic steady state. In fact, if $i^*(0) = 0$, then $i^*(c) \equiv 0$. Moreover, $\bar{I}^* = 0$ and $r^*(0) = 0$. Therefore, $e^*(b) \equiv 0$ and $r^*(h) \equiv 0$. This leads to a contradiction with the definition of endemic steady state. Thus, we further obtain that $\bar{I}^*$ satisfies the equation

$$N(\bar{I}^*) \triangleq \frac{\beta S^* f(\bar{I}^*)\theta_2\tau_3}{\bar{I}^*} + \theta_3\theta_4 - 1 = 0.$$

Obviously, if $\mathcal{R}_0 > 1$ (≤ 1) then $N(0^+) = \lim_{\bar{I}^*\to 0^+} N(\bar{I}^*) = \mathcal{R}_0 - 1 > 0$ (≤ 0) . We also have

$$\begin{aligned}
N(\bar{I}^*) =& \frac{\beta f(\bar{I}^*)}{\bar{I}^*}\frac{\Lambda\theta_2\tau_3}{\mu + \xi(1-\theta_1) + \beta f(\bar{I}^*)} + \theta_3\theta_4 - 1\\
<& \frac{\Lambda\theta_2\tau_3}{\bar{I}^*} + \theta_3\theta_4 - 1 \to \theta_2\tau_3 - 1 \quad \text{as} \quad \bar{I}^* \to +\infty.
\end{aligned}$$

Furthermore, it follows from $(\mathbf{A}_2)$ that $N(\bar{I}^*)$ also is decreasing for $\bar{I}^* > 0$. Therefore $N(\bar{I}^*) = 0$ has a unique positive solution $\bar{I}^*$ if and only if $\mathcal{R}_0 > 1$. Furthermore, from (9) and (10), and choose $i^*(0) = \frac{\bar{I}^*}{\tau_3}$, then model (1) has a unique endemic steady state $P^* = (S^*, v^*(a), e^*(b), i^*(c), r^*(h))$ when $\mathcal{R}_0 > 1$.

4 Stability of disease-free steady state

In this section, we study the stability of disease-free steady state of model (1) with the boundary conditions.

Theorem 3. *The disease-free steady state P_0 is locally asymptotically stable if $\mathcal{R}_0 < 1$, and unstable if $\mathcal{R}_0 > 1$.*

Proof. Linearizing model (1) at the steady state P_0 yields

$$\begin{cases}
\dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -(\mu+\xi)x_1(t) - \beta S^0 f'(0)\displaystyle\int_0^\infty \omega_5(c)x_4(t,c)\mathrm{d}c + \int_0^\infty \omega_1(a)x_2(t,a)\mathrm{d}a,\\
\Big(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\Big)x_2(t,a) = -\varepsilon_1(a)x_2(t,a),\\
\Big(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial b}\Big)x_3(t,b) = -\varepsilon_2(b)x_3(t,b),\\
\Big(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial c}\Big)x_4(t,c) = -(\varepsilon_3(c) + \delta(c))x_4(t,c),\\
\Big(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial h}\Big)x_5(t,h) = -\varepsilon_4(h)x_5(t,h),
\end{cases} \tag{11}$$

with the following boundary conditions

$$\begin{aligned}
x_2(t,0) &= \xi x_1(t),\ x_3(t,0) = \beta S^0 f'(0)\int_0^{\infty} \omega_5(c)x_4(t,c)\mathrm{d}c,\\
x_4(t,0) &= \int_0^{\infty} \omega_2(b)x_3(t,b)\mathrm{d}b + \int_0^{\infty} \omega_4(h)x_5(t,h)\mathrm{d}h,\\
x_5(t,0) &= \int_0^{\infty} \omega_3(c)x_4(t,c)\mathrm{d}c.
\end{aligned}$$

Let $x_1(t) = x_1^0 e^{\lambda t}$, $x_2(t,a) = x_2^0(a)e^{\lambda t}$, $x_3(t,b) = x_3^0(b)e^{\lambda t}$, $x_4(t,c) = x_4^0(c)e^{\lambda t}$ and $x_5(t,h) = x_5^0(h)e^{\lambda t}$ be the solutions of system (11), where x_1^0, $x_2^0(a)$, $x_3^0(b)$, $x_4^0(c)$ and $x_5^0(h)$ are eigenfunctions and not complete zeroes, and λ is eigenvalue. Then

$$\lambda x_1^0 = -(\mu+\xi)x_1^0 - \beta S^0 f'(0)\int_0^{\infty} \omega_5(c)x_4^0(c)\mathrm{d}c + \int_0^{\infty} \omega_1(a)x_2^0(a)\mathrm{d}a, \tag{12}$$

$$\lambda x_2^0(a) + \frac{\mathrm{d}x_2^0(a)}{\mathrm{d}a} = -\varepsilon_1(a)x_2^0(a),\ x_2^0(0) = \xi x_1^0, \tag{13}$$

$$\lambda x_3^0(b) + \frac{\mathrm{d}x_3^0(b)}{\mathrm{d}b} = -\varepsilon_2(b)x_3^0(b),\ x_3^0(0) = \beta S^0 f'(0)\int_0^{\infty} \omega_5(c)x_4^0(c)\mathrm{d}c, \tag{14}$$

$$\begin{cases}
\lambda x_4^0(c) + \dfrac{\mathrm{d}x_4^0(c)}{\mathrm{d}c} = -(\varepsilon_3(c)+\delta(c))x_4^0(c),\\
x_4^0(0) = \displaystyle\int_0^{\infty} \omega_2(b)x_3^0(b)\mathrm{d}b + \int_0^{\infty} \omega_4(h)x_5^0(h)\mathrm{d}h,
\end{cases} \tag{15}$$

$$\lambda x_5^0(h) + \frac{\mathrm{d}x_5^0(h)}{\mathrm{d}h} = -\varepsilon_4(h)x_5^0(h),\ x_5^0(0) = \int_0^{\infty} \omega_3(c)x_4^0(c)\mathrm{d}c. \tag{16}$$

From the first equation of (15) we obtain

$$x_4^0(c) = x_4^0(0)e^{-\int_0^c(\lambda+\delta(s)+\varepsilon_3(s))\mathrm{d}s}. \tag{17}$$

From (14), (16) and (17), we further conclude

$$\begin{aligned}
x_3^0(b) =& e^{-\int_0^b(\lambda+\varepsilon_2(s))\mathrm{d}s}\beta S^0 f'(0)\int_0^{\infty} \omega_5(c)x_4^0(c)\mathrm{d}c\\
=& e^{-\int_0^b(\lambda+\varepsilon_2(s))\mathrm{d}s}\beta S^0 f'(0)x_4^0(0)\int_0^{\infty} \omega_5(c)e^{-\int_0^c(\lambda+\delta(s)+\varepsilon_3(s))\mathrm{d}s}\mathrm{d}c,
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
x_5^0(h) =& e^{-\int_0^h(\lambda+\varepsilon_4(s))\mathrm{d}s}\int_0^{\infty} \omega_3(c)x_4^0(c)\mathrm{d}c\\
=& e^{-\int_0^h(\lambda+\varepsilon_4(s))\mathrm{d}s}x_4^0(0)\int_0^{\infty} \omega_3(c)e^{-\int_0^c(\lambda+\delta(s)+\varepsilon_3(s))\mathrm{d}s}\mathrm{d}c.
\end{aligned} \tag{19}$$

Substituting (18) and (19) into the second equation of (15), we can finally obtain

$$x_4^0(0) = \beta S^0 f'(0)x_4^0(0)\theta_2(\lambda)\tau_3(\lambda) + x_4^0(0)\theta_3(\lambda)\theta_4(\lambda).$$

Therefore, when $x_4^0(c) \neq 0$ we obtain that λ satisfies the equation as follows

$$F_1(\lambda) \triangleq \beta S^0 f'(0)\theta_2(\lambda)\tau_3(\lambda) + \theta_3(\lambda)\theta_4(\lambda) - 1 = 0.$$

When $x_4^0(c) \equiv 0$, then it follows from (15) that $\int_0^\infty \omega_2(b)x_3^0(b)\mathrm{d}b + \int_0^\infty \omega_4(h)x_5^0(h)\mathrm{d}h = 0$, by assumption $(\mathbf{A_1})$, we can get $x_3^0(b) \equiv 0$ and $x_5^0(h) \equiv 0$. Hence, we have $x_1^0 \neq 0$ or $x_2^0(a) \neq 0$. Since $x_2^0(a) = \xi x_1^0 e^{-\int_0^a (\lambda+\varepsilon_1(s))ds}$, it must be $x_1^0 \neq 0$. From (12) we further obtain

$$\lambda x_1^0 = -(\mu+\xi)x_1^0 + \xi x_1^0 \int_0^\infty \omega_1(a)e^{-\int_0^a (\lambda+\varepsilon_1(s))\mathrm{d}s}\mathrm{d}a.$$

Therefore, we obtain that λ satisfies the equation

$$F_2(\lambda) \triangleq \lambda + \mu + \xi - \xi\theta_1(\lambda) = 0.$$

Thus, we finally obtain that the characteristic equation of model (1) at the steady state P_0 is

$$F(\lambda) \triangleq F_1(\lambda)F_2(\lambda) = 0. \tag{20}$$

It is easy to obtain that $F_1(0) = \beta S^0 f'(0)\theta_2\tau_3 + \theta_3\theta_4 - 1 = \mathcal{R}_0 - 1$ and $\lim_{\lambda\to+\infty} F_1(\lambda) = -1$. Clearly, if $\mathcal{R}_0 > 1$, then equation $F_1(\lambda) = 0$ has a positive root. Therefore, the steady state P_0 is unstable.

We now claim that when $\mathcal{R}_0 < 1$, the steady state P_0 is locally asymptotically stable. Otherwise, assume that $\lambda_1 = a_1 + ib_1$ with $a_1 \geq 0$ is a root of $F(\lambda_1) = 0$. Since

$$\begin{aligned}
&|\beta S^0 f'(0)\theta_2(\lambda_1)\tau_3(\lambda_1) + \theta_3(\lambda_1)\theta_4(\lambda_1)| \\
\leq&\left|\beta S^0 f'(0) \int_0^\infty \omega_2(b)e^{-(\lambda_1+\mu)b-\int_0^b \omega_2(s)\mathrm{d}s}\mathrm{d}b \int_0^\infty \omega_5(c)e^{-(\lambda_1+\mu)c-\int_0^c (\delta(s)+\omega_3(s))\mathrm{d}s}\mathrm{d}c\right| \\
&+ \left|\int_0^\infty \omega_3(c)e^{-(\lambda_1+\mu)c-\int_0^c (\delta(s)+\omega_3(s))\mathrm{d}s}\mathrm{d}c\right|\left|\int_0^\infty \omega_4(h)e^{-(\lambda_1+\mu)h-\int_0^h \omega_4(s)\mathrm{d}s}\mathrm{d}h\right| \\
\leq&\beta S^0 f'(0) \int_0^\infty \omega_2(b)e^{-(a_1+\mu)b-\int_0^b \omega_2(s)\mathrm{d}s}\mathrm{d}b \int_0^\infty \omega_5(c)e^{-(a_1+\mu)c-\int_0^c (\delta(s)+\omega_3(s))\mathrm{d}s}\mathrm{d}c \\
&+ \int_0^\infty \omega_3(c)e^{-(a_1+\mu)c-\int_0^c (\omega_3(s)+\delta(S))\mathrm{d}s}\mathrm{d}c \int_0^\infty \omega_4(h)e^{-(a_1+\mu)h-\int_0^h \omega_4(s)\mathrm{d}s}\mathrm{d}h \\
=&\beta S^0 f'(0)\theta_2(a_1)\tau_3(a_1) + \theta_3(a_1)\theta_4(a_1) \\
\leq&\beta S^0 f'(0)\theta_2\tau_3 + \theta_3\theta_4 = \mathcal{R}_0 < 1,
\end{aligned}$$

and

$$\begin{aligned}
&|\lambda_1 + \mu + \xi - \xi\theta_1(\lambda_1)| \geq |\lambda_1 + \mu + \xi| - \left|\xi \int_0^\infty \omega_1(a)e^{-(\lambda_1+\mu)a-\int_0^a \omega_1(s)\mathrm{d}s}\mathrm{d}a\right| \\
&\geq a_1 + \mu + \xi - \xi \int_0^\infty \omega_1(a)e^{-(a_1+\mu)a-\int_0^a \omega_1(s)\mathrm{d}s}\mathrm{d}a \geq \mu + \xi - \xi\theta_1(a_1) > 0.
\end{aligned}$$

This shows that $F_1(\lambda_1) \neq 0$ and $F_2(\lambda_1) \neq 0$, which leads to a contradiction. Therefore, $\mathcal{R}_0 < 1$ implies all roots of equation (20) has negative real parts. Accordingly, disease-free steady state P_0 is locally asymptotically stable. This completes the proof.

Theorem 4. *The disease-free steady state P_0 is globally asymptotically stable if $\mathcal{R}_0 \leq 1$.*

Proof. Firstly, let $G(u) = u - 1 - \ln u$, $U_s(t) = \theta_2 S^0 G(\frac{S}{S^0})$, $U_v(t) = \theta_2 \int_0^\infty v^0(a) G(\frac{v(t,a)}{v^0(a)})\mathrm{d}a$, $U_e(t) = \int_0^\infty \pi_2(b) e(t,b)\mathrm{d}b$, $U_i(t) = \int_0^\infty F(c) i(t,c)\mathrm{d}c$ and $U_r(t) = \int_0^\infty \pi_4(h) r(t,h)\mathrm{d}h$.

Define a Lyapunov function as follows

$$L_0(t) = U_s(t) + U_v(t) + U_e(t) + U_i(t) + U_r(t).$$

By $\mu + \xi = \frac{1}{S^0}(\Lambda + \int_0^\infty \omega_1(a) v^0(a)\mathrm{d}a)$, calculating the derivative of $U_s(t)$ along with any solution of model (1) is given as

$$\begin{aligned}\frac{\mathrm{d}U_s(t)}{\mathrm{d}t} = \quad & -\theta_2 \Lambda G\Big(\frac{S}{S^0}\Big) - \theta_2 \Lambda G\Big(\frac{S^0}{S}\Big) + \theta_2 \beta f(\bar{I}) S^0 - \theta_2 \beta f(\bar{I}) S \\ & + \theta_2 \int_0^\infty \omega_1(a) v^0(a) \Big[\frac{v(t,a)}{v^0(a)} - \frac{S}{S^0} - \frac{S^0 v(t,a)}{S v^0(a)} + 1\Big]\mathrm{d}a.\end{aligned} \tag{21}$$

From $\frac{\mathrm{d}v^0(a)}{\mathrm{d}a} = -\varepsilon_1(a) v^0(a)$, and then

$$\frac{\partial}{\partial a} G(\frac{v(t,a)}{v^0(a)}) = \Big(\frac{v(t,a)}{v^0(a)} - 1\Big)\Big(\frac{1}{v(t,a)}\frac{\partial v(t,a)}{\partial a} + \varepsilon_1(a)\Big),$$

we further obtain

$$\begin{aligned}\frac{\mathrm{d}U_v(t)}{\mathrm{d}t} &= -\theta_2 \int_0^\infty \Big(1 - \frac{v^0(a)}{v(t,a)}\Big)\Big[\frac{\partial v(t,a)}{\partial a} + \varepsilon_1(a) v(t,a)\Big]\mathrm{d}a \\ &= -\theta_2 \int_0^\infty v^0(a) \frac{\partial}{\partial a} G\Big(\frac{v(t,a)}{v^0(a)}\Big)\mathrm{d}a \\ &= -\theta_2 v^0(a) G\Big(\frac{v(t,a)}{v^0(a)}\Big)\Big|^\infty + \theta_2 \xi S^0 G\Big(\frac{S}{S^0}\Big) \\ &\quad - \theta_2 \int_0^\infty v^0(a) \varepsilon_1(a) G\Big(\frac{v(t,a)}{v^0(a)}\Big)\mathrm{d}a.\end{aligned} \tag{22}$$

Since $\frac{\mathrm{d}\pi_2(b)}{\mathrm{d}b} = \pi_2(b)\varepsilon_2(b) - \omega_2(b)$ and $\pi_2(0) = \theta_2$, we have

$$\begin{aligned}\frac{\mathrm{d}U_e(t)}{\mathrm{d}t} &= -\int_0^\infty \pi_2(b) \frac{\partial e(t,b)}{\partial b}\mathrm{d}b - \int_0^\infty \pi_2(b) \varepsilon_2(b) e(t,b)\mathrm{d}b \\ &= -\pi_2(b) e(t,b)\Big|^\infty + \theta_2 \beta f(\bar{I}) S - \int_0^\infty \omega_2(b) e(t,b)\mathrm{d}b.\end{aligned} \tag{23}$$

Since $\frac{\mathrm{d}\pi_4(h)}{\mathrm{d}h} = \pi_4(h)\varepsilon_4(h) - \omega_4(h)$ and $\pi_4(0) = \theta_4$, we have

$$\begin{aligned}\frac{\mathrm{d}U_r(t)}{\mathrm{d}t} =& - \int_0^\infty \pi_4(h)\frac{\partial r(t,h)}{\partial r}\mathrm{d}h - \int_0^\infty \pi_4(h)\varepsilon_4(h)r(t,h)\mathrm{d}h \\ =& - \pi_4(h)r(t,h)\Big|^\infty + \theta_4\int_0^\infty \omega_3(c)i(t,c)\mathrm{d}c - \int_0^\infty \omega_4(h)r(t,h)\mathrm{d}h.\end{aligned} \tag{24}$$

Furthermore, calculating the derivative of $U_i(t)$ is given as

$$\begin{aligned}\frac{\mathrm{d}U_i(t)}{\mathrm{d}t} =& - \int_0^\infty F(c)(\varepsilon_3(c)+\delta(c))i(t,c)\mathrm{d}c - \int_0^\infty F(c)\frac{\partial i(t,c)}{\partial c}\mathrm{d}c \\ =& - \int_0^\infty F(c)(\varepsilon_3(c)+\delta(c))i(t,c)\mathrm{d}c - F(c)i(t,c)\Big|_0^\infty + \int_0^\infty F'(c)i(t,c)\mathrm{d}c \\ =& - F(c)i(t,c)\Big|_0^\infty + \int_0^\infty [F'(c) - (\varepsilon_3(c)+\delta(c))F(c)]i(t,c)\mathrm{d}c.\end{aligned}$$

Choose

$$F(c) = \int_c^\infty \Big[\beta S^0 f'(0)\theta_2\omega_5(u) + \omega_3(u)\theta_4\Big]e^{-\int_c^u(\varepsilon_3(s)+\delta(s))\mathrm{d}s}\mathrm{d}u.$$

A direct calculation shows that

$$F'(c) = -\beta S^0 f'(0)\theta_2\omega_5(c) - \omega_3(c)\theta_4 + (\varepsilon_3(s)+\delta(s))F(c),$$

$$F(0) = \int_0^\infty (\beta S^0 f'(0)\theta_2\omega_5(u) + \omega_3(u)\theta_4)e^{-\int_0^c(\varepsilon_3(s)+\delta(s))\mathrm{d}s}\mathrm{d}u = \beta S^0 f'(0)\theta_2\tau_3 + \theta_3\theta_4 = \mathcal{R}_0,$$

and $\lim_{c\to\infty} F(c) = 0$. Therefore, we further have

$$\begin{aligned}\frac{\mathrm{d}U_i(t)}{\mathrm{d}t} =& - F(c)i(t,c)\Big|_0^\infty + \int_0^\infty \Big[-\beta S^0 f'(0)\theta_2\omega_5(c) - \omega_3(c)\theta_4\Big]i(t,c)\mathrm{d}c \\ =& \mathcal{R}_0\Big[\int_0^\infty \omega_2(b)e(t,b)\mathrm{d}b + \int_0^\infty \omega_4(h)r(t,h)\mathrm{d}h\Big] \\ & - \beta S^0 f'(0)\theta_2\int_0^\infty \omega_5(c)i(t,c)\mathrm{d}c - \theta_4\int_0^\infty \omega_3(c)i(t,c)\mathrm{d}c.\end{aligned} \tag{25}$$

It follows from (21)-(25) that

$$\begin{aligned}\frac{\mathrm{d}L_0(t)}{\mathrm{d}t} =& - \theta_2\Lambda G\Big(\frac{S}{S^0}\Big) - \theta_2\Lambda G\Big(\frac{S^0}{S}\Big) - \theta_2 v^0(a)G\Big(\frac{v(t,a)}{v^0(a)}\Big)\Big|^\infty + \theta_2\xi S^0 G\Big(\frac{S}{S^0}\Big) \\ & - \pi_2(b)e(t,b)\Big|^\infty - \pi_4(h)r(t,h)\Big|^\infty - \theta_2\mu\int_0^\infty v^0(a)G\Big(\frac{v(t,a)}{v^0(a)}\Big)\mathrm{d}a \\ & + (\mathcal{R}_0 - 1)\Big[\int_0^\infty \omega_2(b)e(t,b)\mathrm{d}b + \int_0^\infty \omega_4(h)r(t,h)\mathrm{d}h\Big] \\ & + \theta_2\beta f(\bar{I})S^0 - \theta_2\beta f'(0)\bar{I}S^0 + \theta_2\Sigma_1,\end{aligned} \tag{26}$$

where

$$\Sigma_1 = \int_0^\infty \omega_1(a)v^0(a)\Big[\frac{v(t,a)}{v^0(a)} - \frac{S}{S^0} - \frac{S^0 v(t,a)}{Sv^0(a)} + 1\Big]\mathrm{d}a - \int_0^\infty \omega_1(a)v^0(a)G\Big(\frac{v(t,a)}{v^0(a)}\Big)\mathrm{d}a.$$

Since $f(\bar{I}) \le f'(0)\bar{I}$ for all $\bar{I} \ge 0$, $v^0(a) = \xi S^0 \rho_1(a)$,

$$\begin{aligned}\Sigma_1 &= \int_0^\infty \omega_1(a)v^0(a)\Big[\ln\frac{v(t,a)}{v^0(a)} - \frac{S}{S^0} - \frac{S^0 v(t,a)}{Sv^0(a)} + 2\Big]\mathrm{d}a \\ &= -\int_0^\infty \omega_1(a)v^0(a)G\Big(\frac{S^0 v(t,a)}{Sv^0(a)}\Big)\mathrm{d}a - \theta_1\xi S^0 G\Big(\frac{S}{S^0}\Big),\end{aligned}$$

and

$$\theta_2 G\Big(\frac{S}{S^0}\Big)(\xi S^0 - \Lambda - \theta_1\xi S^0) = -\mu\theta_2 G\Big(\frac{S}{S^0}\Big) \le 0,$$

we can finally get

$$\begin{aligned}\frac{\mathrm{d}L_0(t)}{\mathrm{d}t} \le &- \theta_2\Lambda\Big(\frac{S^0}{S}\Big) - \theta_2 v^0(a)G\Big(\frac{v(t,a)}{v^0(a)}\Big)\Big|^\infty - \pi_2(b)e(t,b)\Big|^\infty - \pi_4(h)r(t,h)\Big|^\infty \\ &- \theta_2\mu\int_0^\infty v^0(a)G\Big(\frac{v(t,a)}{v^0(a)}\Big)\mathrm{d}a + (\mathcal{R}_0 - 1)\Big[\int_0^\infty \omega_2(b)e(t,b)\mathrm{d}b \\ &+ \int_0^\infty \omega_4(h)r(t,h)\mathrm{d}h\Big] - \mu\theta_2 G\Big(\frac{S}{S^0}\Big) - \int_0^\infty \omega_1(a)v^0(a)G\Big(\frac{S^0 v(t,a)}{Sv^0(a)}\Big)\mathrm{d}a.\end{aligned} \tag{27}$$

Therefore, $\frac{\mathrm{d}L_0(t)}{\mathrm{d}t} \le 0$ if $\mathcal{R}_0 \le 1$, and $\frac{\mathrm{d}L_0}{\mathrm{d}t} = 0$ implies that $S = S^0$ and $v(t,a) = v^0(a)$. From model (1), it follows that $e(t,b) \equiv 0$, $i(t,c) \equiv 0$ and $r(t,h) \equiv 0$. By the LaSalle invariance principle (See [28]), P_0 is globally asymptotically stable. This completes the proof.

5 Uniform persistence

In this section, we establish the uniform persistence of the semi-flow $\Phi(t, x_0)$ generated by model (1) when $\mathcal{R}_0 > 1$.

Denote $\hat{\mathbb{X}} = L^1_+ \times L^1_+ \times L^1_+$,

$$\hat{\mathbb{Z}} = \Big\{(e(\cdot), i(\cdot), r(\cdot))^T \in \hat{\mathbb{X}} : \int_0^\infty e(b)\mathrm{d}b > 0, \int_0^\infty i(c)\mathrm{d}c > 0, \int_0^\infty r(h)\mathrm{d}h > 0\Big\},$$

$\partial\hat{\mathbb{Z}} = \hat{\mathbb{X}}\backslash\hat{\mathbb{Z}}$, $\mathbb{Z} = \mathbb{R}_+ \times L^1_+ \times \hat{\mathbb{Z}}$ and $\partial\mathbb{Z} = \mathbb{X}\backslash\mathbb{Z}$.

Lemma 2. *If $\mathcal{R}_0 > 1$, then there is a constant $\varepsilon > 0$ such that for any initial value $x_0 \in \mathbb{X}$ with $e_0(\cdot) \not\equiv 0$, $i_0(\cdot) \not\equiv 0$ and $r_0(\cdot) \not\equiv 0$, the solution $\Phi(t, x_0)$ of model (1) satisfies $\limsup_{t\to\infty} \|\Phi(t, x_0) - P_0\|_{\mathbb{X}} \ge \varepsilon$.*

Proof. Firstly, from $\mathcal{R}_0 > 1$, we can choose an enough small constant $\varepsilon_0 > 0$ such that

$\beta(S^0-\varepsilon_0)(f'(0)-\varepsilon_0)\theta_2\tau_3+\theta_3\theta_4>1$. Moreover, by $\lim_{\bar{I}\to 0}\frac{f(\bar{I})}{\bar{I}}=f'(0)$, there exists a constant $\delta>0$ with $\delta\le\varepsilon_0$ such that $\frac{f(\bar{I})}{\bar{I}}\ge f'(0)-\varepsilon_0$ for all $0\le\bar{I}\le\delta$. That is, $f(\bar{I})\ge(f'(0)-\varepsilon_0)\bar{I}$ for all $0\le\bar{I}\le\delta$.

Assume that the conclusion is not true, there exists a $x_0\in\mathbb{X}$ with $e_0(\cdot)\not\equiv 0$, $i_0(\cdot)\not\equiv 0$ and $r_0(\cdot)\not\equiv 0$ such that $\limsup_{t\to\infty}\|\Phi(t,x_0)-P_0\|_{\mathbb{X}}<\delta$. Then, from the initial and boundary conditions (2) and formulas (4)-(6) we can obtain that $e(t,\cdot)>0$, $i(t,\cdot)>0$ and $r(t,\cdot)>0$ for all $t>0$, and there exists an enough large T such that for any $t>T$,

$$0<S^0-\delta<S(t)<S^0+\delta,\ 0\le\bar{I}(t)=\int_0^\infty\omega_5(c)i(t,c)\mathrm{d}c<\delta.$$

Therefore, by the comparison principle of age-dependent partial differential equations (See [29]), we can obtain

$$e(t,b)\ge\tilde{e}(t,b),\ i(t,c)\ge\tilde{i}(t,c),\ r(t,h)\ge\tilde{r}(t,h), \tag{28}$$

for all $t\ge T$, where $(\tilde{e}(t,b),\tilde{i}(t,c),\tilde{r}(t,h))$ is the solution of the following linear comparison system

$$\begin{cases}\dfrac{\partial\tilde{e}(t,b)}{\partial t}+\dfrac{\partial\tilde{e}(t,b)}{\partial b}=-\varepsilon_2(b)\tilde{e}(t,b),\\ \dfrac{\partial\tilde{i}(t,c)}{\partial t}+\dfrac{\partial\tilde{i}(t,c)}{\partial c}=-(\varepsilon_3(c)+\delta(c))\tilde{i}(t,c),\\ \dfrac{\partial\tilde{r}(t,h)}{\partial t}+\dfrac{\partial\tilde{r}(t,h)}{\partial h}=-\varepsilon_4(h)\tilde{r}(t,h),\\ \tilde{e}(t,0)=\beta(S^0-\varepsilon_0)(f'(0)-\varepsilon_0)\displaystyle\int_0^\infty\omega_5(c)\tilde{i}(t,c)dc,\\ \tilde{i}(t,0)=\displaystyle\int_0^\infty\omega_2(b)\tilde{e}(t,b)\mathrm{d}b+\int_0^\infty\omega_4(h)\tilde{r}(t,h)\mathrm{d}h,\\ \tilde{r}(t,0)=\displaystyle\int_0^\infty\omega_3(c)\tilde{i}(t,c)\mathrm{d}c,\end{cases} \tag{29}$$

with the initial conditions $\tilde{e}(T,b)=e(T,b)$, $\tilde{i}(T,c)=i(T,c)$ and $\tilde{r}(T,h)=r(T,h)$. Assume that system (29) has the solution as follows

$$\tilde{e}(t,b)=\tilde{e}_1(b)e^{\lambda(t-T)},\ \tilde{i}(t,c)=\tilde{i}_1(c)e^{\lambda(t-T)},\ \tilde{r}(t,h)=\tilde{r}_1(h)e^{\lambda(t-T)},$$

where the functions $\tilde{e}_1(b)$, $\tilde{i}_1(c)$ and $\tilde{r}_1(h)$ are eigenfunctions and not complete zeroes, and λ is eigenvalue. Substituting this form of solution into system (29), we obtain the following linear eigenvalue problem:

$$\begin{cases} \dfrac{\mathrm{d}\tilde{e}_1(b)}{\mathrm{d}b} = -(\lambda + \varepsilon_2(b))\tilde{e}_1(b), \\ \dfrac{\mathrm{d}\tilde{i}_1(c)}{\mathrm{d}c} = -(\lambda + \varepsilon_3(c) + \delta(c))\tilde{i}_1(c), \\ \dfrac{\mathrm{d}\tilde{r}_1(h)}{\mathrm{d}h} = -(\lambda + \varepsilon_4(h))\tilde{r}_1(h), \\ \tilde{e}_1(0) = \beta(S^0 - \varepsilon_0)(f'(0) - \varepsilon_0) \displaystyle\int_0^\infty \omega_5(c)\tilde{i}_1(c)\mathrm{d}c, \\ \tilde{i}_1(0) = \displaystyle\int_0^\infty \omega_2(b)\tilde{e}_1(b)\mathrm{d}b + \int_0^\infty \omega_4(h)\tilde{r}_1(h)\mathrm{d}h, \\ \tilde{r}_1(0) = \displaystyle\int_0^\infty \omega_3(c)\tilde{i}_1(c)\mathrm{d}c. \end{cases} \tag{30}$$

It follows from the first, second and third equations of problem (30) that

$$\begin{aligned} &\tilde{e}_1(b) = \tilde{e}_1(0)e^{-\int_0^b(\lambda+\varepsilon_2(s))\mathrm{d}s},\ \tilde{i}_1(c) = \tilde{i}_1(0)e^{-\int_0^c(\lambda+\varepsilon_3(s)+\delta(s))\mathrm{d}s}, \\ &\tilde{r}_1(h) = \tilde{r}_1(0)e^{-\int_0^c(\lambda+\varepsilon_4(s))\mathrm{d}s}. \end{aligned} \tag{31}$$

Then, from (31) and the last three equations of problem (30)

$$\begin{aligned} \tilde{i}_1(0) =& \int_0^\infty \omega_2(b)\tilde{e}_1(0)e^{-\int_0^b(\lambda+\varepsilon_2(s))\mathrm{d}s}\mathrm{d}b + \int_0^\infty \omega_4(h)\tilde{r}_1(0)e^{-\int_0^c(\lambda+\varepsilon_4(s))\mathrm{d}s}\mathrm{d}h \\ =&\theta_2(\lambda)\tilde{e}_1(0) + \theta_4(\lambda)\tilde{r}_1(0) \\ =&\theta_2(\lambda)\beta(S^0 - \varepsilon)(f'(0) - \varepsilon)\int_0^\infty \omega_5(c)\tilde{i}_1(c)\mathrm{d}c + \theta_4(\lambda)\int_0^\infty \omega_3(c)\tilde{i}_1(c)\mathrm{d}c \\ =&\beta(S^0 - \varepsilon)(f'(0) - \varepsilon)\theta_2(\lambda)\tau_3(\lambda)\tilde{i}_1(0) + \theta_3(\lambda)\theta_4(\lambda)\tilde{i}_1(0). \end{aligned}$$

We also can obtain $\tilde{i}_1(0) \neq 0$. Thus, we finally obtain the characteristic equation of system (30) as follows

$$F_3(\lambda) \triangleq \beta(S^0 - \varepsilon_0)(f'(0) - \varepsilon_0)\theta_2(\lambda)\tau_3(\lambda) + \theta_3(\lambda)\theta_4(\lambda) = 1. \tag{32}$$

Clearly, we have $F_3(0) = \beta(S^0 - \varepsilon_0)(f'(0) - \varepsilon_0)\theta_2\tau_3 + \theta_3\theta_4 > 1$ and $\lim_{\lambda\to+\infty} F_3(\lambda) = 0$. Thus, equation (32) has at least one positive root λ_0. This implies that system (29) has the solution as follows

$$\tilde{e}(t, b) = \tilde{e}_1(b)e^{\lambda_0(t-T)},\ \tilde{i}(t, c) = \tilde{i}_1(c)e^{\lambda_0(t-T)},\ \tilde{r}(t, h) = \tilde{r}_1(h)e^{\lambda_0(t-T)}.$$

For this solution, from (31), as long as $(\tilde{e}_1(0), \tilde{i}_1(0), \tilde{r}_1(0)) \neq 0$, then $\int_0^\infty \tilde{e}(t, b)\mathrm{d}b + \int_0^\infty \tilde{i}(t, c)\mathrm{d}c + \int_0^\infty \tilde{r}(t, h)\mathrm{d}h$ is unbounded on $t \in [T, \infty)$. From (28), we further obtain that $\int_0^\infty e(t, b)\mathrm{d}b + \int_0^\infty i(t, c)\mathrm{d}c + \int_0^\infty r(t, h)\mathrm{d}h$ is also unbounded for on $t \in [T, \infty)$. This leads to a contradiction with the boundedness of $\Phi(t, x_0)$. This completes the proof.

Theorem 5. *If $\mathcal{R}_0 > 1$, then there exists a constant ε_1 such that for any initial value $x_0 \in \mathbb{X}$ with $e_0(\cdot) \not\equiv 0$, $i_0(\cdot) \not\equiv 0$ and $r_0(\cdot) \not\equiv 0$, the solution $\Phi(t, x_0)$ of model (1) satisfies*

$$\liminf_{t\to\infty} S(t) \geq \varepsilon_1, \quad \liminf_{t\to\infty} \|v(t,\cdot)\|_{L^1} \geq \varepsilon_1, \quad \liminf_{t\to\infty} \|e(t,\cdot)\|_{L^1} \geq \varepsilon_1,$$

$$\liminf_{t\to\infty} \|i(t,\cdot)\|_{L^1} \geq \varepsilon_1, \quad \liminf_{t\to\infty} \|r(t,\cdot)\|_{L^1} \geq \varepsilon_1.$$

Proof. From Theorem 1, there exists a constant $M > 0$ such that for any solution $(S(t), v(t,\cdot), e(t,\cdot), i(t,\cdot), r(t,\cdot))$, there is a $t_0 > 0$ such that $\int_0^\infty i(t,c)\mathrm{d}c \leq M$ for all $c \geq 0$ and $t \geq t_0$. Thus, from the first equation of model (1) we have

$$\frac{\mathrm{d}S}{\mathrm{d}t} \geq \Lambda - (\mu+\xi)S - \beta S f'(0)\bar{\omega}_5 M, \ t \geq t_0.$$

Consider following comparison system

$$\frac{\mathrm{d}\nu}{\mathrm{d}t} = \Lambda - (\mu+\xi)\nu - \beta \nu f'(0)\bar{\omega}_5 M, \ t \geq t_0.$$

It has the position solution $\nu^* = \frac{\Lambda}{\mu+\xi+\beta f'(0)\bar{\omega}_5 M}$ which is globally asymptotically stable. By the comparison principle we can obtain $\liminf_{t\to\infty} S(t) \geq \nu^*$. This shows that $S(t)$ in model (1) is uniformly persistent.

For any initial value $(S_0, v_0(\cdot), e_0(\cdot), i_0(\cdot), i_0(\cdot)) \in \mathbb{Z}$ with $e_0(\cdot) \not\equiv 0$, $i_0(\cdot) \not\equiv 0$ and $r_0(\cdot) \not\equiv 0$. From the formulas (4)-(6), we have $\int_0^\infty e(t,b)\mathrm{d}b > 0$, $\int_0^\infty i(t,c)\mathrm{d}c > 0$ and $\int_0^\infty r(t,h)\mathrm{d}h > 0$ for all $t > 0$. Therefore, set $\mathbb{Z}$ is the positive invariant set of semi-flow $\Phi(t)$ of model (1). Define the set

$$M_\partial = \Big\{x_0 = (S_0, v_0(\cdot), e_0(\cdot), i_0(\cdot), r_0(\cdot)) \in \mathbb{X} : \Phi(t, x_0) \in \partial\mathbb{Z} \ \text{ for all } \ t \geq 0\Big\}.$$

Let $\omega(x_0)$ be the omega limit set of $\Phi(t, x_0)$ and the set $M_1 = \{P_0\}$. Since $\Phi(t, P_0) = P_0$ for all $t \geq 0$, we have $M_1 \subset \bigcup_{x_0 \in M_\partial} \omega(x_0)$.

Next, we prove $\bigcup_{x_0\in M_\partial} \omega(x_0) \subset M_1$. For any $x_0 \in M_\partial$, since $\Phi(t, x_0) \in \partial\mathbb{Z}$ for all $t \geq 0$, we have $\int_0^\infty e(t,b)\mathrm{d}b \equiv 0$ or $\int_0^\infty i(t,c)\mathrm{d}c \equiv 0$ or $\int_0^\infty r(t,h)\mathrm{d}h \equiv 0$ for all $t \geq 0$. If $\int_0^\infty e(t,b)\mathrm{d}b \equiv 0$ for all $t \geq 0$, then we have $e(t,b) \equiv 0$ for all $b \geq 0$ and $t \geq 0$. Combining with $e(t,0) = \beta S(t) f(\bar{I}(t))$ and the uniform persistence of $S(t)$, we derive that $f(\bar{I}(t)) \equiv 0$ for all $t \geq 0$ which implies that $\bar{I}(t) = \int_0^\infty \omega_5(c) i(t,c)\mathrm{d}c \equiv 0$ for all $t \geq 0$. Then, we further have $i(t,c) \equiv 0$ for all $c \geq 0$ and $t \geq 0$. From assumption $(\mathbf{A_1})$ and $i(t,0) = \int_0^\infty \omega_2(b) e(t,b)\mathrm{d}b + \int_0^\infty \omega_4(h) r(t,h)\mathrm{d}h$, we further obtain $r(t,h) \equiv 0$ for all $t \geq 0$

and $h \geq 0$. Furthermore, model (1) degrades into the following subsystem

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - (\mu + \xi)S(t) + \displaystyle\int_0^\infty \omega_1(a)v(t,a)\mathrm{d}a, \\ \dfrac{\partial v(t,a)}{\partial t} + \dfrac{\partial v(t,a)}{\partial a} = -(\omega_1(a) + \mu)v(t,a). \end{cases} \tag{33}$$

It is clear from (33) that $\lim_{t\to\infty} S(t) = S^0$ and $\lim_{t\to\infty} v(t,a) = v^0(a)$. This shows $\omega(x_0) = \{P_0\}$. Similarly, if $\int_0^\infty i(t,c)\mathrm{d}c \equiv 0$ or $\int_0^\infty r(t,h)\mathrm{d}h \equiv 0$, we also can obtain that $e(t,b) \equiv 0$, $i(t,c) \equiv 0$ and $r(t,h) \equiv 0$ for all $t \geq 0$, $b \geq 0$, $c \geq 0$ and $h \geq 0$, respectively, and model (1) also degrades into subsystem (33). Consequently, $\lim_{t\to\infty} S(t) = S^0$ and $\lim_{t\to\infty} v(t,a) = v^0(a)$. This shows $\omega(x_0) = P_0$, and hence $\bigcup_{x_0\in M_\partial} \omega(x_0) \subset M_1$. Thus, we finally obtain $\bigcup_{x_0\in M_\partial} \omega(x_0) = M_1$.

From $\bigcup_{x_0\in M_\partial} \omega(x_0) = M_1$, we know that all solutions on boundary $\partial\mathbb{Z}$ of model (1) tend to P_0 when $t \to \infty$. From Lemma 2, we also know that P_0 is an isolated invariant set in $\mathbb{X}$, and $W^s(P_0) \cap \mathbb{Z} = \emptyset$, where $W^s(P_0)$ is the stable set of P_0.

Furthermore, from the above arguments, we can easily observe that no subset of M_1 forms a cycle in $\partial\mathbb{Z}$. By Corollary 1 and the theory of persistence for dynamical systems in [30,31], it follows that the semi-flow $\Phi(t)$ of model (1) is uniformly persistent. This completes the proof.

6 Stability of endemic steady state

Based on the above discussions, we now investigate the local stability and the global stability of endemic steady state for model (1). Firstly, we give a result on the local stability of endemic steady state P^*.

Theorem 6. *The endemic steady state P^* is locally asymptotically stable if $\mathcal{R}_0 > 1$.*

Proof. Linearizing model (1) at the steady state P^* yields

$$\begin{cases} \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -(\mu + \xi + \beta f(\bar{I}^*))x_1(t) - \beta S^* f'(\bar{I}^*) \displaystyle\int_0^\infty \omega_5(c)x_4(t,c)\mathrm{d}c \\ \qquad\qquad + \displaystyle\int_0^\infty \omega_1(a)x_2(t,a)\mathrm{d}a, \\ \Big(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\Big)x_2(t,a) = -\varepsilon_1(a)x_2(t,a), \\ \Big(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial b}\Big)x_3(t,b) = -\varepsilon_2(b)x_3(t,b), \\ \Big(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial c}\Big)x_4(t,c) = -(\varepsilon_3(c) + \delta(c))x_4(t,c), \\ \Big(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial h}\Big)x_5(t,h) = -\varepsilon_4(h)x_5(t,h), \end{cases} \tag{34}$$

with the following boundary conditions

$$x_2(t,0)=\xi x_1(t),\ x_3(t,0)=\beta f(\bar{I}^*)x_1(t)+\beta S^* f'(\bar{I}^*)\int_0^\infty \omega_5(c)x_4(t,c)\mathrm{d}c,$$
$$x_4(t,0)=\int_0^\infty \omega_2(b)x_3(t,b)\mathrm{d}b+\int_0^\infty \omega_4(h)x_5(t,h)\mathrm{d}h,\ x_5(t,0)=\int_0^\infty \omega_3(b)x_4(t,c)\mathrm{d}c.$$

Let $x_1(t)=x_1^0e^{\lambda t}$, $x_2(t,a)=x_2^0(a)e^{\lambda t}$, $x_3(t,b)=x_3^0(b)e^{\lambda t}$, $x_4(t,c)=x_4^0(c)e^{\lambda t}$ and $x_5(t,h)=x_5^0(h)e^{\lambda t}$ be the solutions of system (34), where x_1^0, $x_2^0(a)$, $x_3^0(b)$, $x_4^0(c)$ and $x_5^0(h)$ are eigenfunctions and not complete zeroes, and λ is eigenvalue. We obtain the following linear eigenvalue problem:

$$\lambda x_1^0=-(\mu+\xi+\beta f(\bar{I}^*))x_1^0-\beta S^* f'(\bar{I}^*)\int_0^\infty \omega_5(c)x_4^0(c)\mathrm{d}c+\int_0^\infty \omega_1(a)x_2^0(a)\mathrm{d}a, \tag{35}$$

$$\lambda x_2^0(a)+\frac{\mathrm{d}x_2^0(a)}{\mathrm{d}a}=-\varepsilon_1(a)x_2^0(a),\ x_2^0(0)=\xi x_1^0, \tag{36}$$

$$\lambda x_3^0(b)+\frac{\mathrm{d}x_3^0(b)}{\mathrm{d}b}=-\varepsilon_2(b)x_3^0(b),\ x_3^0(0)=\beta f(\bar{I}^*)x_1^0+\beta S^* f'(\bar{I}^*)\int_0^\infty \omega_5(c)x_4^0(c)\mathrm{d}c, \tag{37}$$

$$\begin{cases}\lambda x_4^0(c)+\dfrac{\mathrm{d}x_4^0(c)}{\mathrm{d}b}=-(\varepsilon_3(c)+\delta(c))x_4^0(c),\\ x_4^0(0)=\displaystyle\int_0^\infty \omega_2(b)x_3^0(b)\mathrm{d}b+\int_0^\infty \omega_4(h)x_5^0(h)\mathrm{d}h,\end{cases} \tag{38}$$

$$\lambda x_5^0(h)+\frac{\mathrm{d}x_5^0(h)}{\mathrm{d}h}=-\varepsilon_4(h)x_5^0(h),\ x_5^0(0)=\int_0^\infty \omega_3(c)x_4^0(c)\mathrm{d}c. \tag{39}$$

It follows from (36)-(39) that

$$x_2^0(a)=x_2^0(0)e^{\int_0^a(\lambda+\varepsilon_1(s))\mathrm{d}s},\ x_3^0(b)=x_3^0(0)e^{-\int_0^b(\lambda+\varepsilon_2(s))\mathrm{d}s}, \tag{40}$$

$$x_4^0(c)=x_4^0(0)e^{-\int_0^c(\lambda+\varepsilon_3(s)+\delta(s))\mathrm{d}s},\ x_5^0(h)=x_5^0(0)e^{-\int_0^h(\lambda+\varepsilon_4(s))\mathrm{d}s}. \tag{41}$$

Then from (35) we deduce that

$$x_1^0=\frac{-\beta S^* f'(\bar{I}^*)\int_0^\infty \omega_5(c)x_4^0(c)\mathrm{d}c}{\lambda+\mu+\xi+\beta f(\bar{I}^*)-\xi\theta_1(\lambda)}. \tag{42}$$

Furthermore, combining (40)-(42) and (36)-(39) we can obtain $x_4^0(0)\neq 0$ and

$$\begin{aligned}x_4^0(0)=&\int_0^\infty \omega_2(b)x_3^0(b)\mathrm{d}b+\int_0^\infty \omega_4(h)x_5^0(h)\mathrm{d}h\\ =&\int_0^\infty \omega_2(b)x_3^0(0)e^{-\int_0^b(\lambda+\varepsilon_2(s))\mathrm{d}s}\mathrm{d}b+\int_0^\infty \omega_4(h)x_5^0(0)e^{-\int_0^h(\lambda+\varepsilon_4(s))\mathrm{d}s}\mathrm{d}h\\ =&\theta_2(\lambda)\Big[\beta f(\bar{I}^*)x_1^0+\beta S^* f'(\bar{I}^*)\int_0^\infty \omega_5(c)x_4^0(c)\mathrm{d}c\Big]+\theta_4(\lambda)\int_0^\infty \omega_3(c)x_4^0(c)\mathrm{d}c\end{aligned}$$

$$
\begin{aligned}
=&\theta_2(\lambda)\Big[\frac{-\beta f(\bar{I}^*)\beta S^* f'(\bar{I}^*)\int_0^\infty \omega_5(c)x_4^0(c)\mathrm{d}c}{\lambda+\mu+\xi+\beta f(\bar{I}^*)-\xi\theta_1(\lambda)}+\beta S^* f'(\bar{I}^*)\int_0^\infty \omega_5(c)x_4^0(c)\mathrm{d}c\Big]\\
&+\theta_4(\lambda)\int_0^\infty \omega_3(c)x_4^0(0)e^{-\int_0^c(\lambda+\varepsilon_3(s)+\delta(s))\mathrm{d}s}\mathrm{d}c\\
=&\frac{\beta S^* f'(\bar{I}^*)(\lambda+\mu+\xi-\xi\theta_1(\lambda))}{\lambda+\mu+\xi+\beta f(\bar{I}^*)-\xi\theta_1(\lambda)}\theta_2(\lambda)\tau_3(\lambda)x_4^0(0)+\theta_3(\lambda)\theta_4(\lambda)x_4^0(0).
\end{aligned}
$$

Thus, the characteristic equation of model (1) at the steady state P^* is calculated as follows

$$
F_4(\lambda)\triangleq\frac{\lambda+\mu+\xi-\xi\theta_1(\lambda)}{\lambda+\mu+\xi+\beta f(\bar{I}^*)-\xi\theta_1(\lambda)}\beta S^* f'(\bar{I}^*)\theta_2(\lambda)\tau_3(\lambda)+\theta_3(\lambda)\theta_4(\lambda)=1. \tag{43}
$$

Now, we claim that all roots of equation (43) have negative real parts. Otherwise, equation (43) has a root $\lambda_2=a_2+ib_2$ with $a_2\geq 0$. We have

$$
\begin{aligned}
|F_4(\lambda_2)|\leq&\frac{\beta S^* f'(\bar{I}^*)|a_2+ib_2+\mu+\xi-\xi\theta_1(\lambda_2)|}{|a_2+ib_2+\mu+\xi+\beta f(\bar{I}^*)-\xi\theta_1(\lambda_2)|}\\
&\times\Big|\int_0^\infty \omega_5(c)e^{-\int_0^c(\varepsilon_3(s)+a_2+ib_2+\delta(s))\mathrm{d}s}\mathrm{d}c\int_0^\infty \omega_2(b)e^{-\int_0^b(\varepsilon_2(s)+a_2+ib_2)\mathrm{d}s}\mathrm{d}b\Big|\\
&+\Big|\int_0^\infty \omega_3(c)e^{-\int_0^c(\varepsilon_3(s)+a_2+ib_2+\delta(s))\mathrm{d}s}\mathrm{d}c\int_0^\infty \omega_4(h)e^{-\int_0^h(\varepsilon_4(s)+a_2+ib_2)\mathrm{d}s}\mathrm{d}h\Big|\\
<&\beta S^* f'(\bar{I}^*)\int_0^\infty \omega_5(c)e^{-\int_0^c(\varepsilon_3(s)+\delta(s))\mathrm{d}s}\mathrm{d}c\int_0^\infty \omega_2(b)e^{-\int_0^b\varepsilon_2(s)\mathrm{d}s}\mathrm{d}b\\
&+\int_0^\infty \omega_3(c)e^{-\int_0^c(\varepsilon_3(s)+\delta(s))\mathrm{d}s}\mathrm{d}c\int_0^\infty \omega_4(h)e^{-\int_0^h\varepsilon_4(s)\mathrm{d}s}\mathrm{d}h\\
=&\beta S^* f'(\bar{I}^*)\theta_2\tau_3+\theta_3\theta_4\leq\beta S^*\frac{f(\bar{I}^*)}{\bar{I}^*}\theta_2\tau_3+\theta_3\theta_4=1.
\end{aligned}
$$

which leads to a contradiction. Therefore, if $\mathcal{R}_0>1$, the endemic steady state P^* is locally asymptotically stable. This completes the proof.

In order to guarantee the Lyapunov functional in proving the global stability of P^* to be well-defined, we introduce the following assumption.

($\mathbf{A}_3$) $\int_0^\infty e^{-\int_0^a\varepsilon_1(s)\mathrm{d}s}\ln v_0(a)\mathrm{d}a<+\infty$, $\int_0^\infty e^{-\int_0^b\varepsilon_2(s)\mathrm{d}s}\ln e_0(b)\mathrm{d}b<+\infty$,
$\int_0^\infty e^{-\int_0^c(\varepsilon_3(s)+\delta(s))\mathrm{d}s}\ln i_0(c)\mathrm{d}c<+\infty$, $\int_0^\infty e^{-\int_0^h\varepsilon_4(s)\mathrm{d}s}\ln r_0(h)\mathrm{d}h<+\infty$.

Lemma 3. *If assumption* ($\boldsymbol{A}_3$) *holds, then*

$$
\int_0^\infty v^*(a)\ln\frac{v(t,a)}{v^*(a)}\mathrm{d}a<+\infty,\ \int_0^\infty e^*(b)\ln\frac{e(t,b)}{e^*(b)}\mathrm{d}b<+\infty,
$$

$$
\int_0^\infty i^*(c)\ln\frac{i(t,c)}{i^*(c)}\mathrm{d}c<+\infty,\ \int_0^\infty r^*(h)\ln\frac{r(t,h)}{r^*(h)}\mathrm{d}h<+\infty.
$$

Proof. Let $(S(t),v(t,a),e(t,b),i(t,c),r(t,h))$ be any positive solution of model (1). It follows from (3) that

$$
\begin{aligned}
&\int_0^{\infty} v^*(a)\ln\frac{v(t,a)}{v^*(a)}\mathrm{d}a\\
=&\int_0^{t} v^*(a)\ln\frac{\xi S(t-a)\rho_1(a)}{v^*(a)}\mathrm{d}a+\int_t^{\infty} v^*(a)\ln\frac{v_0(a-t)\frac{\rho_1(a)}{\rho_1(a-t)}}{v^*(a)}\mathrm{d}a\\
=&v^*(0)\int_0^{t} e^{-\int_0^a \varepsilon_1(s)\mathrm{d}s}[\xi\ln S(t-a)-\ln v^*(0)]\mathrm{d}a\\
&+v^*(0)\int_t^{\infty} e^{-\int_0^a \varepsilon_1(s)\mathrm{d}s}\Big[\ln v_0(a-t)+\int_0^{a-t}\varepsilon_1(s)\mathrm{d}s-\ln v^*(0)\Big]\mathrm{d}a.
\end{aligned}
\tag{44}
$$

Letting $a-t=u$, then from $(\mathbf{A}_3)$ one has

$$
\int_t^{\infty} e^{-\int_0^a \varepsilon_1(s)\mathrm{d}s}\ln v_0(a-t)\mathrm{d}a=\int_0^{\infty} e^{-\int_0^{t+u} \varepsilon_1(s)\mathrm{d}s}\ln v_0(u)\mathrm{d}u<+\infty.
$$

Hence, it follows from (44) that $\int_0^\infty v^*(a)\ln\frac{v(t,a)}{v^*(a)}\mathrm{d}a<+\infty$. In a similar way one can show that $\int_0^\infty e^*(b)\ln\frac{e(t,b)}{e^*(b)}\mathrm{d}b<+\infty$, $\int_0^\infty i^*(c)\ln\frac{i(t,c)}{i^*(c)}\mathrm{d}c<+\infty$ and $\int_0^\infty r^*(h)\ln\frac{r(t,h)}{r^*(h)}\mathrm{d}h<+\infty$. This completes the proof.

Theorem 7. *Assume that $(\boldsymbol{A}_3)$ holds. Then the endemic steady state P^* is globally asymptotically stable if $\mathcal{R}_0>1$.*

Proof. Define a Lyapunov function

$$
L^*(t)=U_s(t)+U_v(t)+U_e(t)+U_i(t)+U_r(t),
$$

where

$$
U_s(t)=\theta_2 S^* G\Big(\frac{S}{S^*}\Big),\ U_v(t)=\theta_2\int_0^{\infty} v^*(a)G\Big(\frac{v(t,a)}{v^*(a)}\Big)\mathrm{d}a,
$$

$$
U_e(t)=\int_0^{\infty}\pi_2(b)e^*(b)G\Big(\frac{e(t,b)}{e^*(b)}\Big)\mathrm{d}b,\ U_i(t)=\int_0^{\infty}F_1(c)i^*(c)G\Big(\frac{i(t,c)}{i^*(c)}\Big)\mathrm{d}c,
$$

and

$$
U_r(t)=\int_0^{\infty}\pi_4(h)r^*(h)G\Big(\frac{r(t,h)}{r^*(h)}\Big)\mathrm{d}h.
$$

From $(\mathbf{A}_3)$ and Lemma 3, all integrals involved in $L^*(t)$ are finite. Therefore, $L^*(t)$ is defined for any positive solution $(S(t),v(t,a),e(t,b),i(t,c),r(t,h))$ of model (1).

By $\mu+\xi=\frac{1}{S^*}(\Lambda-\beta S^*f(\bar{I}^*)+\int_0^\infty\omega_1(a)v^*(a)\mathrm{d}a)$, the derivative of $U_s(t)$ along with the solutions of model (1) is given as

$$
\begin{aligned}
\frac{\mathrm{d}U_s}{\mathrm{d}t}=&-\theta_2\Lambda\Big[G\Big(\frac{S^*}{S}\Big)+G\Big(\frac{S}{S^*}\Big)\Big]+\theta_2\beta Sf(\bar{I}^*)-\theta_2\beta S^*f(\bar{I}^*)-\theta_2\beta Sf(\bar{I})\\
&+\theta_2\beta S^*f(\bar{I})+\theta_2\int_0^{\infty}\omega_1(a)v^*(a)\Big[\frac{v(t,a)}{v^*(a)}-\frac{S}{S^*}-\frac{S^*v(t,a)}{Sv^*(a)}+1\Big]\mathrm{d}a.
\end{aligned}
\tag{45}
$$

Calculating the derivative of $U_v(t)$ along the solution of model (1), we have

$$\frac{\mathrm{d}U_v}{\mathrm{d}t} = -\theta_2 \int_0^{\infty} v^*(a)\Big(\frac{v(t,a)}{v^*(a)} - 1\Big)\Big(\frac{v_a(t,a)}{v(t,a)} + \varepsilon_1(a)\Big)\mathrm{d}a,$$

where $v_a(t,a) = \frac{\partial}{\partial a}v(t,a)$. Noticing that $\frac{\partial}{\partial a}G\Big(\frac{v(t,a)}{v^*(a)}\Big) = \Big(\frac{v(t,a)}{v^*(a)} - 1\Big)\Big(\frac{v_a(t,a)}{v(t,a)} + \varepsilon_1(a)\Big)$ and $\frac{\mathrm{d}v^*(a)}{\mathrm{d}a} = -\varepsilon_1(a)v^*(a)$, using the integral by parts and the fact $G(\frac{v(t,0)}{v^*(0)}) = G(\frac{S}{S^*})$, we can obtain

$$\begin{aligned}\frac{\mathrm{d}U_v}{\mathrm{d}t} =& -\theta_2 \int_0^{\infty} v^*(a)\frac{\partial}{\partial a}G\Big(\frac{v(t,a)}{v^*(a)}\Big)\mathrm{d}a \\ =& -\theta_2 v^*(a)G\Big(\frac{v(t,a)}{v^*(a)}\Big)\Big|^{\infty} + \theta_2\xi S^* G\Big(\frac{S}{S^*}\Big) - \theta_2\int_0^{\infty}\varepsilon_1(a)v^*(a)G\Big(\frac{v(t,a)}{v^*(a)}\Big)\mathrm{d}a.\end{aligned} \tag{46}$$

Noticing that $\frac{\mathrm{d}e^*(b)}{\mathrm{d}b} = -\varepsilon_2(b)e^*(b)$ and $\frac{\mathrm{d}\pi_2(b)}{\mathrm{d}b} = \pi_2(b)\varepsilon_2(b) - \omega_2(b)$, calculating the derivative of $U_e(t)$, we obtain

$$\begin{aligned}\frac{\mathrm{d}U_e}{\mathrm{d}t} =& -\pi_2(b)e^*(b)G\Big(\frac{e(t,b)}{e^*(b)}\Big)\Big|^{\infty} + \theta_2\beta S^* f(\bar{I}^*)G\Big(\frac{Sf(\bar{I})}{S^*f(\bar{I}^*)}\Big) \\ & -\int_0^{\infty}\omega_2(b)e^*(b)G\Big(\frac{e(t,b)}{e^*(b)}\Big)\mathrm{d}b.\end{aligned} \tag{47}$$

Noticing that $\frac{\mathrm{d}r^*(h)}{\mathrm{d}h} = -\varepsilon_4(h)r^*(h)$ and $\frac{\mathrm{d}\pi_4(h)}{\mathrm{d}h} = \pi_4(h)\varepsilon_4(h) - \omega_4(h)$, calculating the derivative of $U_r(t)$, we obtain

$$\begin{aligned}\frac{\mathrm{d}U_r}{\mathrm{d}t} =& -\pi_4(h)r^*(h)G\Big(\frac{r(t,h)}{r^*(h)}\Big)\Big|^{\infty} + \theta_4\int_0^{\infty}\omega_3(c)i^*(c)\mathrm{d}cG\Big(\frac{r(t,0)}{r^*(0)}\Big) \\ & -\int_0^{\infty}\omega_4(h)r^*(h)G\Big(\frac{r(t,h)}{r^*(h)}\Big)\mathrm{d}h.\end{aligned} \tag{48}$$

Since $\frac{\partial}{\partial c}G\Big(\frac{i(t,c)}{i^*(c)}\Big) = \Big(\frac{i(t,c)}{i^*(c)} - 1\Big)\Big(\frac{i_c(t,c)}{i(t,c)} + (\varepsilon_3(c) + \delta(c))\Big)$, where $i_c(t,c) = \frac{\partial}{\partial c}i(t,c)$, calculating the derivative of $U_i(t)$, we obtain

$$\begin{aligned}\frac{\mathrm{d}U_i(t)}{\mathrm{d}t} =& -\int_0^{\infty} F_1(c)\Big(1 - \frac{i^*(c)}{i(t,c)}\Big)\Big[(\varepsilon_3(c) + \delta(c))i(t,c) + \frac{\partial i(t,c)}{\partial c}\Big]\mathrm{d}c \\ =& -\int_0^{\infty}F_1(c)i^*(c)\frac{\partial}{\partial c}G\Big(\frac{i(t,c)}{i^*(c)}\Big)\mathrm{d}c \\ =& -F_1(c)i^*(c)G\Big(\frac{i(t,c)}{i^*(c)}\Big)\Big|_0^{\infty} \\ & + \int_0^{\infty}G\Big(\frac{i(t,c)}{i^*(c)}\Big)[F_1'(c) - (\varepsilon_3(c) + \delta(c))F_1(c)]i^*(c)\mathrm{d}c.\end{aligned}$$

Choosing

$$F_1(c) = \int_c^\infty \Big[\beta S^* \frac{f(\bar{I}^*)}{\bar{I}^*}\theta_2\omega_5(u) + \omega_3(u)\theta_4\Big] e^{-\int_c^u(\varepsilon_3(s)+\delta(s))\mathrm{d}s}\mathrm{d}u,$$

a direct calculation shows that $\lim_{c\to\infty} F_1(c) = 0$,

$$F_1'(c) = -\beta S^* \frac{f(\bar{I}^*)}{\bar{I}^*}\theta_2\omega_5(c) - \omega_3(c)\theta_4 + (\varepsilon_3(c)+\delta(c))F_1(c),$$

and

$$F_1(0) = \int_0^\infty \Big[\beta S^* \frac{f(\bar{I}^*)}{\bar{I}^*}\theta_2\omega_5(u) + \omega_3(u)\theta_4\Big] e^{-\int_0^u(\varepsilon_3(s)+\delta(s))\mathrm{d}s}\mathrm{d}u = \beta S^* \frac{f(\bar{I}^*)}{\bar{I}^*}\theta_2\tau_3 + \theta_3\theta_4 = 1.$$

Hence, we further have

$$\begin{aligned}
\frac{\mathrm{d}U_i(t)}{\mathrm{d}t} =& i^*(0)G\Big(\frac{i(t,0)}{i^*(0)}\Big) - \int_0^\infty \Big[\beta S^* \frac{f(\bar{I}^*)}{\bar{I}^*}\theta_2\omega_5(c) + \omega_3(c)\theta_4\Big] i^*(c)G\Big(\frac{i(t,c)}{i^*(c)}\Big)\mathrm{d}c \\
=& \int_0^\infty \omega_2(b)e(t,b)\mathrm{d}b + \int_0^\infty \omega_4(h)r(t,h)\mathrm{d}h - \int_0^\infty \omega_2(b)e^*(b)\mathrm{d}b \\
&- \int_0^\infty \omega_4(h)r^*(h)\mathrm{d}h - \Big[\int_0^\infty \omega_2(b)e^*(b)\mathrm{d}b \\
&+ \int_0^\infty \omega_4(h)r^*(h)\mathrm{d}h\Big] \ln\frac{i(t,0)}{i^*(0)} - \theta_2\beta S^* f(\bar{I}^*) \\
&+ \theta_2\beta S^* \frac{f(\bar{I}^*)}{\bar{I}^*}\bar{I} - \theta_4\int_0^\infty \omega_3(c)i(t,c)\mathrm{d}c + \theta_4\int_0^\infty \omega_3(c)i^*(c)\mathrm{d}c \\
&+ \int_0^\infty \Big[\beta S^* \frac{f(\bar{I}^*)}{\bar{I}^*}\theta_2\omega_5(c) + \omega_3(c)\theta_4\Big] i^*(c)\ln\frac{i(t,c)}{i^*(c)}\mathrm{d}c.
\end{aligned} \tag{49}$$

From (45)-(49), we finally obtain that

$$\begin{aligned}
\frac{\mathrm{d}L^*(t)}{\mathrm{d}t} =& -\theta_2\Lambda G\Big(\frac{S^*}{S}\Big) - \theta_2 v^*(a)G\Big(\frac{v(t,a)}{v^0(a)}\Big)\Big|^\infty - \pi_2(b)e^*(b)G\Big(\frac{e(t,b)}{e^*(b)}\Big)\Big|^\infty \\
&- \pi_4(h)r^*(h)G\Big(\frac{r(t,h)}{r^*(h)}\Big)\Big|^\infty + \sum_{l=1}^5 B_l,
\end{aligned}$$

where

$$\begin{aligned}
B_1 =& \theta_2\int_0^\infty \omega_1(a)v^*(a)\Big[\frac{v(t,a)}{v^*(a)} - \frac{S}{S^*} - \frac{S^*v(t,a)}{Sv^*(a)} + 1\Big]\mathrm{d}a \\
&- \theta_2\int_0^\infty v^*(a)\varepsilon_1(a)G\Big(\frac{v(t,a)}{v^*(a)}\Big)\mathrm{d}a, \\
B_2 =& \int_0^\infty \omega_2(b)e^*(b)\ln\frac{e(t,b)}{e^*(b)}\mathrm{d}b - \Big[\int_0^\infty \omega_2(b)e^*(b)\mathrm{d}b \\
&+ \int_0^\infty \omega_4(h)r^*(h)\mathrm{d}h\Big]\ln\frac{i(t,0)}{i^*(0)} + \int_0^\infty \omega_4(h)r^*(h)\ln\frac{r(t,h)}{r^*(h)}\mathrm{d}h,
\end{aligned}$$

$$
\begin{aligned}
B_3 =& -\theta_2\Lambda G\Big(\frac{S}{S^*}\Big) + \theta_2\beta S f(\bar{I}^*) - \theta_2\beta S f(\bar{I}) - \theta_2\beta S^* f(\bar{I}^*) \\
&+ \theta_2\beta S^* f(\bar{I}) + \theta_2\beta S^* f(\bar{I}^*) G\Big(\frac{S f(\bar{I})}{S^* f(\bar{I}^*)}\Big) + \theta_2\xi S^* G\Big(\frac{S}{S^*}\Big), \\
B_4 =& \theta_2\beta S^* \frac{f(\bar{I}^*)}{\bar{I}^*}\int_0^\infty \omega_5(c) i^*(c) \ln\frac{i(t,c)}{i^*(c)}\mathrm{d}c + \theta_2\beta S^*\frac{f(\bar{I}^*)}{\bar{I}^*}\bar{I} - \theta_2\beta S^* f(\bar{I}^*), \\
B_5 =& \theta_4\int_0^\infty \omega_3(c) i^*(c)\ln\frac{i(t,c)}{i^*(c)}\mathrm{d}c - \theta_4\int_0^\infty \omega_3(c) i(t,c)\mathrm{d}c \\
&+ \theta_4\int_0^\infty \omega_3(c) i^*(c)\mathrm{d}c + \theta_4\int_0^\infty \omega_3(c) i^*(c)\mathrm{d}c G\Big(\frac{r(t,0)}{i^*(0)}\Big).
\end{aligned}
$$

By calculating we can further obtain that

$$
\begin{aligned}
B_1 =& -\theta_2\mu\int_0^\infty v^*(a) G\Big(\frac{v(t,a)}{v^*(a)}\Big)\mathrm{d}a - \theta_1\theta_2\xi S^* G\Big(\frac{S}{S^*}\Big) \\
&- \theta_2\int_0^\infty \omega_1(a) v^*(a) G\Big(\frac{S^* v(t,a)}{S v^*(a)}\Big)\mathrm{d}a, \\
B_2 =& -\int_0^\infty \omega_2(b) e^*(b) G\Big(\frac{e(t,b) i^*(0)}{e^*(b) i(t,0)}\Big)\mathrm{d}b - \int_0^\infty \omega_4(h) r^*(h) G\Big(\frac{r(t,h) i^*(0)}{r^*(h) i(t,0)}\Big)\mathrm{d}h \\
&- \int_0^\infty \omega_2(b) e^*(b)\Big[1 - \frac{e(t,b) i^*(0)}{e^*(b) i(t,0)}\Big]\mathrm{d}b - \int_0^\infty \omega_4(h) r^*(h)\Big[1 - \frac{r(t,h) i^*(0)}{r^*(h) i(t,0)}\Big]\mathrm{d}h \\
=& -\int_0^\infty \omega_2(b) e^*(b) G\Big(\frac{e(t,b) i^*(0)}{e^*(b) i(t,0)}\Big)\mathrm{d}b - \int_0^\infty \omega_4(h) r^*(h) G\Big(\frac{r(t,h) i^*(0)}{r^*(h) i(t,0)}\Big)\mathrm{d}h, \\
B_3 =& \theta_2\beta S^* f(\bar{I}^*)\Big[\frac{S}{S^*} - \frac{S f(\bar{I})}{S^* f(\bar{I}^*)} - 1 + \frac{f(\bar{I})}{f(\bar{I}^*)} + G\Big(\frac{S f(\bar{I})}{S^* f(\bar{I}^*)}\Big)\Big] + \theta_2 G\Big(\frac{S}{S^*}\Big)(\xi S^* - \Lambda) \\
=& \theta_2\beta S^* f(\bar{I}^*) G\Big(\frac{f(\bar{I})}{f(\bar{I}^*)}\Big) + \theta_2 G\Big(\frac{S}{S^*}\Big)(\beta f(\bar{I}^*) S^* + \xi S^* - \Lambda), \\
B_4 =& -\theta_2\beta S^*\frac{f(\bar{I}^*)}{\bar{I}^*}\int_0^\infty \omega_5(c) i^*(c) G\Big(\frac{i(t,c)}{i^*(c)}\Big)\mathrm{d}c, \\
B_5 =& \theta_4\int_0^\infty \omega_3(c) i^*(c)\Big[\frac{r(t,0)}{r^*(0)} - \frac{i(t,c)}{i^*(c)} + \ln\frac{i(t,c) r^*(0)}{i^*(c) r(t,0)}\Big]\mathrm{d}c \\
=& -\theta_4\int_0^\infty \omega_3(c) i^*(c) G\Big(\frac{i(t,c) r^*(0)}{i^*(c) r(t,0)}\Big)\mathrm{d}c + \theta_4\int_0^\infty \omega_3(c) i^*(c)\Big[\frac{i(t,c) r^*(0)}{i^*(c) r(t,0)} - 1\Big]\mathrm{d}c \\
&+ \theta_4\int_0^\infty \omega_3(c) i^*(c)\Big[\frac{r(t,0)}{r^*(0)} - \frac{i(t,c)}{i^*(c)}\Big]\mathrm{d}c \\
=& -\theta_4\int_0^\infty \omega_3(c) i^*(c) G\Big(\frac{i(t,c) r^*(0)}{i^*(c) r(t,0)}\Big)\mathrm{d}c.
\end{aligned}
$$

By calculating, we also can obtain

$$
\begin{aligned}
&\theta_2\beta S^* f(\bar{I}^*) G\Big(\frac{f(\bar{I})}{f(\bar{I}^*)}\Big) + \theta_2 G\Big(\frac{S}{S^*}\Big)(\beta f(\bar{I}^*)S^* + \xi S^* - \Lambda) - \theta_1\theta_2\xi S^* G\Big(\frac{S}{S^*}\Big)\\
&= \theta_2\beta S^* f(\bar{I}^*) G\Big(\frac{f(\bar{I})}{f(\bar{I}^*)}\Big) - \theta_2\mu G\Big(\frac{S}{S^*}\Big).
\end{aligned}
$$

From Appendix in [32] it follows that

$$
\begin{aligned}
&\theta_2\beta S^* f(I^*) G\Big(\frac{f(\bar{I})}{f(\bar{I}^*)}\Big) - \theta_2\beta S^* \frac{f(\bar{I}^*)}{\bar{I}^*}\int_0^\infty \omega_5(c) i^*(c) G\Big(\frac{i(t,c)}{i^*(c)}\Big)\mathrm{d}c\\
&= \theta_2\beta S^* \frac{f(\bar{I}^*)}{\bar{I}^*}\int_0^\infty \omega_5(c) i^*(c)\Big[G\Big(\frac{f(\bar{I})}{f(\bar{I}^*)}\Big) - G\Big(\frac{i(t,c)}{i^*(c)}\Big)\Big]\mathrm{d}c \le 0.
\end{aligned}
$$

Therefore, we finally derive

$$
\begin{aligned}
\frac{\mathrm{d}L^*(t)}{\mathrm{d}t} \le &- \theta_2\Lambda G\Big(\frac{S^*}{S}\Big) - \theta_2 v^*(a) G\Big(\frac{v(t,a)}{v^*(a)}\Big)\Big|^\infty - \pi_2(b) e^*(b) G\Big(\frac{e(t,b)}{e^*(b)}\Big)\Big|^\infty\\
&- \pi_4(h) r^*(h) G\Big(\frac{r(t,h)}{r^*(h)}\Big)\Big|^\infty - \mu\theta_2 G\Big(\frac{S}{S^*}\Big) - \mu\theta_2\int_0^\infty v^*(a) G\Big(\frac{v(t,a)}{v^*(a)}\Big)\mathrm{d}a\\
&- \theta_2\int_0^\infty \omega_1(a) v^*(a) G\Big(\frac{S^* v(t,a)}{S v^*(a)}\Big)\mathrm{d}a - \int_0^\infty \omega_2(b) e^*(b) G\Big(\frac{e(t,b) i^*(0)}{e^*(b) i(t,0)}\Big)\mathrm{d}b \qquad (50)\\
&- \int_0^\infty \omega_4(h) r^*(h) G\Big(\frac{r(t,h) i^*(0)}{r^*(h) i(t,0)}\Big)\mathrm{d}h\\
&- \theta_4\int_0^\infty \omega_3(c) i^*(c) G\Big(\frac{i(t,c) r^*(0)}{i^*(c) r(t,0)}\Big)\mathrm{d}c.
\end{aligned}
$$

Obviously, we have that $\frac{\mathrm{d}L^*(t)}{\mathrm{d}t} \le 0$, and $\frac{\mathrm{d}L^*}{\mathrm{d}t} = 0$ implies that $S = S^*$, $v(t,a) = v^*(a)$, $e(t,b) = e^*(b)$, $i(t,c) = i^*(c)$ and $r(t,h) = r^*(h)$. By the LaSalle invariance principle, the steady state P^* is globally asymptotically stable. This completes the proof.

Remark 4. *It can be easily found that the main research techniques and methods proposed and used in [12,14-16,19,21] are improved and developed in this paper.*

Remark 5. *Comparing the main results obtained in [13,14,16,17,22], we see that the same sharp threshold criteria for the local and global stability of disease-free and endemic steady state and the uniform persistence are established in this paper.*

7 Numerical examples

In this section, we carry out two numerical examples to testify Theorems 4 and 7. Firstly, following [14,33-34], we can derive the vaccine-induced wanes rate, the age-dependent removal rate, the age-dependent infection rate and the age-dependent relapse rate are

$\omega_1(a) = 1\times 10^{-7} + 4.902\times 10^{-9}a + 1.18\times 10^{-10}a^2 + 2\times 10^{-12}a^3$, $\omega_2(b) = 1\times 10^{-4} + 4.167\times 10^{-7}b + 6.9\times 10^{-9}b^2 + 1.6\times 10^{-10}b^3$, $\omega_3(c) = 0.2 + 9.804\times 10^{-3}c + 2.35\times 10^{-4}c^2 + 4\times 10^{-6}c^3$, $\omega_4(h) = 7\times 10^{-6} + 2.917\times 10^{-8}h + 4.9\times 10^{-10}h^2 + 1.1\times 10^{-11}h^3$ and $\omega_5(c) \equiv 1$. The disease induced death rate is $\delta(c) = 6.8627 + 0.3364c + 8.1\times 10^{-3}c^2 + 1\times 10^{-4}c^3$. By calculating, we obtain $\theta_1 \approx 0.3004$, $\theta_2 \approx 0.6729$, $\theta_3 \approx 0.029$, $\theta_4 \approx 0.452$ and $\tau_3 \approx 0.1441$. The initial values are given by $S_0 = 350$, $v_0(a) = 0.02\exp\{-0.4a\} + 0.7(\sin(0.01a))^2$, $e_0(b) = 0.02\exp\{-0.1b\} + 0.7(\sin(0.05b))^2$, $i_0(c) = 0.02\exp\{-0.1c\} + 0.7(\sin(0.03c))^2$ and $r_0(h) = 0.02\exp\{-0.5h\} + 0.7(\sin(0.055h))^2$. Furthermore, we choose the parameters in model (1) as follows: $\Lambda = 1$, $\beta = 0.06$, $\mu = 0.0012$ and $f(\bar{I}) = \frac{\bar{I}}{1+50\bar{I}}$. Let $(S(t), v(t,a), e(t,b), i(t,c), r(t,h))$ be the solution of model (1) with the initial value $(S_0, v_0(a), e_0(b), i_0(c), r_0(h))$.

Example 1. Take $\xi = 0.08$, we have $\mathcal{R}_0 = 0.1149 < 1$, by Theorem 4 the disease-free steady state $P_0 = (S^0, v^0(a), 0, 0, 0)$ is globally asymptotically stable, where $S^0 \approx 14$, $v^0(a) \approx 1.12\rho_1(a)$. Fig 2 shows that over time, the number of people in each converge to the disease-free steady state.

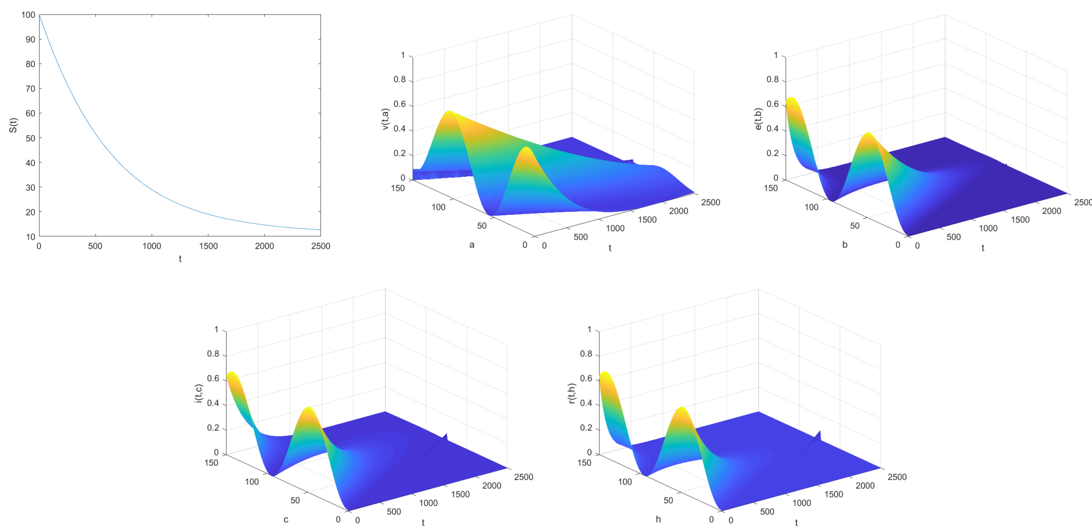

Fig 2. The solution $(S(t), v(t,a), e(t,b), i(t,c), r(t,h))$ converge to the disease-free steady state P_0 as $t \to \infty$.

Example 2. Take $\xi = 0.004$, we have $\mathcal{R}_0 = 1.4682 > 1$, by Theorem 7 the endemic steady state $P^* = (S^*, v^*(a), e^*(b), i^*(c), r^*(h))$ is globally asymptotically stable, where $S^* \approx 232$, $v^*(a) \approx 0.928\rho_1(a)$, $e^*(b) \approx 0.072\rho_2(b)$, $i^*(c) \approx 0.049\rho_3(c)$, $r^*(h) \approx 0.001\rho_4(h)$. Fig 3 shows that over time, the number of people in each converge to the endemic steady state.

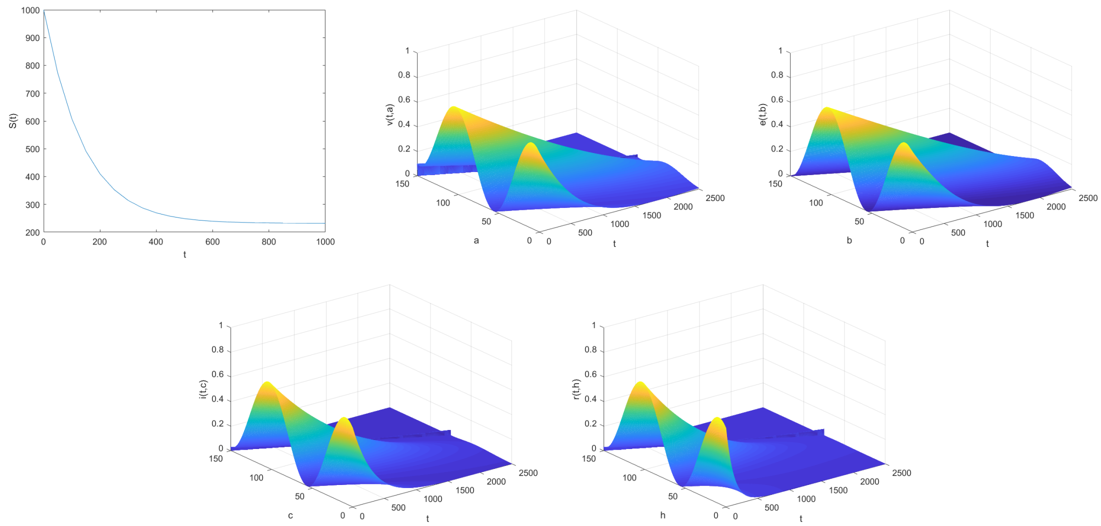

Fig 3. The solution $(S(t), v(t,a), e(t,b), i(t,c), r(t,h))$ converge to the endemic steady state P^* as $t \to \infty$.

8 Conclusions

In this paper, we investigated an SVEIR type epidemic model with continuous age-dependent vaccination, latent, infected, recovered and disease relapse. Through the research, we find that the global dynamics of the model is fully determined by the basic reproduction number. That is, when the basic reproduction number is less than unity, the disease-free steady state is globally asymptotically stable, i.e., the disease died out; and when the basic reproduction number is greater than unity, the disease in the model in accordance with age-distribution is uniform persistence and the endemic steady state is globally asymptotically stable, i.e., the disease becomes endemic. Furthermore, we offered the numerical examples to illustrate the theoretical results established in this paper.

From the expression of the basic reproduction number $\mathcal{R}_0$ and the main results in this article, we see that age-dependent vaccination, latent, infected, recovered and recurrence have an impact on the global dynamics of infectious diseases. It is not difficult to see that these effects are reflected among the parameters $\xi(1-\theta_1)$, θ_i $(i = 1, 2, 3, 4)$ and τ_3, respectively.

Consider multiple class ages is a prominent feature of our model, and the part of investigate uniform persistence is the novel result of our paper. Of course, other factors, such as a general nonlinear incidence, saturation therapy of disease, diseases infection during latency period (such as AIDS, tuberculosis, hepatitis B, see [35]), diseases infection during vaccination period (because the protection rate of vaccines is usually less than 100% and individual differences in vaccinators, see [36]), multi-group SVEIR age-dependent model (decompose the heterogeneous population into several subgroups on the basis of modes of transmission, contact patterns, or geographic distribution, so that effects of both the intra-group and inter-group infection are considered, see [22]) and age-structured epidemic model with environmental virus infectious (such as SARS, AIDS, Ebola and COVID-19, see [37]) should be integrated into the model to make it more realistic. Consider the above factors and establish more accurate model, we leave this for future research.

In general, we can take a general nonlinear incidence, such as $f(S,\bar{I})$ (for example, standard incidence $\frac{S\bar{I}}{S+\bar{I}}$ and Beddington-DeAngelis incidence $\frac{S\bar{I}}{1+\omega_1 S+\omega_2\bar{I}}$, where ω_1 and ω_2 are positive constants, see [38]). For the variable $\bar{I}(t)$, in model (1) we have taken $\bar{I}(t)=\int_0^\infty \omega_5(c)i(t,c)\mathrm{d}c$ which is linear for $i(t,c)$. However, in general case, we can consider the nonlinear expression $\bar{I}(t)=\int_0^\infty g(c,i(t,c))\mathrm{d}c$, where $g(c,i)$ is assumed to be nonnegative and continuously differentiable. Particularly, we can choose $g(c,i(t,c)) = \frac{\omega_5(c)i(t,c)}{b+i(t,c)}$. Meanwhile, we can using a continuously differentiable saturation treatment function describes the effect of delaying treatment when medical conditions are limited and the number of infected people increases (see [39]). That extend the model (1) to the following form

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \Lambda-(\mu+\xi)S-\beta f(S,\bar{I})+\displaystyle\int_0^\infty \omega_1(a)v(t,a)\mathrm{d}a, \\ \dfrac{\partial v(t,a)}{\partial t}+\dfrac{\partial v(t,a)}{\partial a} = -(\omega_1(a)+\mu)v(t,a), \\ \dfrac{\partial e(t,b)}{\partial t}+\dfrac{\partial e(t,b)}{\partial b} = -(\omega_2(b)+\mu)e(t,b), \\ \dfrac{\partial i(t,c)}{\partial t}+\dfrac{\partial i(t,c)}{\partial c} = -(\mu+\delta(c))i(t,c)-\omega_3(c)\dfrac{i(t,c)}{1+\alpha i(t,c)}, \\ \dfrac{\partial r(t,h)}{\partial t}+\dfrac{\partial r(t,h)}{\partial h} = -(\omega_4(h)+\mu)r(t,h), \\ \bar{I} = \displaystyle\int_0^\infty g(c,i(t,c))\mathrm{d}c. \end{cases}$$

For this model whether we also can establish the similar conclusions as in this paper for model (1) still is an interesting open problem. We will leave these subjects for future research.

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