G-CONTINUOUS MAPPINGS AND G-QUOTIENT MAPPINGS

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Abstract. As a generalization of the usual convergence in topological spaces, a method G on a set X is a function $G : c_G(X) \to X$ defined on a subset $c_G(X)$ which is constituted by some sequences in X. In this paper, we mainly study the G-continuous mappings and the G-quotient mappings determined by G-methods and their connections with continuous mappings and quotient mappings in topological spaces. At the same time, we also discuss some properties of G-open mappings and G-closed mappings, and unify some results of several important convergence of sequences involving continuous mappings and quotient mappings.

1. Introduction

Convergence of sequences in topological spaces is a basic concept in mathematics. For a long time, the deepening of the concept of convergence has been an interesting subject in topology and analysis. For example, as a generalization of the usual convergence, Fast [13] and Steinhaus [30] introduced the concepts of statistical convergence in real and complex spaces, independently; thereafter, Di Maio and Kočinac [11] developed the concept into statistical convergence in topological spaces. Recently, Kostyrko, Šalát and Wilczynski [15] proposed the ideal convergence in metric spaces; Lahiri and Das [16] further discussed the ideal convergence in topological spaces. In addition to the above convergence, there are various kinds of convergence in theoretical research and mathematical applications. We mention, for example, A-convergence of a matrix method in summability theory [9], Cesàro convergence and and statistical convergence in real analysis [8], almost convergence in functional analysis [3] and so on.

Based on several kinds of convergent properties in real analysis and considering the importance of sequence convergence in continuous discussion and the concept of continuity and any concept related to continuity play a very important role not only in pure mathematics but also in other branches of science involving mathematics especially in computer science, information theory, biological science, and dynamical systems. Connor and Grosse-Erdmann [9] introduced G-methods defined on a linear subspace of the vector space of all real sequences and G-convergence on real spaces and G-continuity for real functions, studied the relationship among G-continuous functions, linear functions and continuous functions, established the dichotomy theorem of G-continuity and extended several known results in the literature. Although the notion of G-continuity was introduced for the first time, its thought has appeared many times before. Many authors (e.g., Posner [24], Iwiński [14], Srinivasan [29], Antoni and Šalát [1, 2], Spigel

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and Krupnik [28], Çakallı and Khan [7]) have studied the G-continuity defined by regular summability matrix A, in this case the G-continuity is usually called A-continuity. Some authors (e.g., Öztürk [23], Savaş and Das [25, 26], Borsík and Šalát [5]) have studied the G-continuity for the method of almost convergence or for related methods. Schoenberg [27] and Demirci [10] considered the G-continuity for methods of statistical convergence. Based on the above results, people begin to pay attention to the G-methods and the G-continuity on Hausdorff topological groups satisfying the first axiom of countability [6, 22]. Recently, Lin and Liu [17] introduced the concepts of G-methods and G-convergence in arbitrary sets, discussed some relations between mappings which preserves G-convergence and continuous mappings. It lays a foundation for the further study of all kinds of convergence and topological properties of special spaces.

As the concepts of G-methods and G-convergence proposed in the analysis background, how can they be better placed in the scope of the general topology? General topology mainly studies the structure of topological spaces and continuous mappings on these spaces. As special G-methods and related mappings, Tang et al [31] and Liu et al [21] discussed mappings which preserves statistical convergence and statistically sequentially quotient mappings in topological spaces, respectively. For those reasons, this paper is interested in G-continuous mappings and G-quotient mappings determined by G-methods, and their connections with continuous mappings and quotient mappings in topological spaces. In view of the close relationship among open mappings, closed mappings and quotient mappings [12], we also discuss some properties of G-open mappings and G-closed mappings. The quotient spaces are important tools to construct new topological spaces through the known topological spaces, and quotient mappings are also a kind of important mappings widely used in topology and analysis [18]. Therefore, the discussion on G-quotient mappings are necessary for the development of the theory of G-methods, and it provides general methods for the study of quotient mappings of statistical convergence, ideal convergence and other concrete convergence.

2. Preliminaries

We denote by $\mathbb{N}$ the set of all natural numbers. Let X be a set, $s(X)$ denote the set of all X-valued sequences, i.e., $\boldsymbol{x} \in s(X)$ if and only if $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}}$ is a sequence with each $x_n \in X$. If $f : X \to Y$ is a mapping, then $f(\boldsymbol{x}) = \{f(x_n)\}_{n\in\mathbb{N}}$ for each $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in s(X)$. A *method* on X, we mean a function $G : c_G(X) \to X$ defined on a subset $c_G(X)$ of $s(X)$ into X. A sequence $\boldsymbol{x}$ on X is said to be *G-convergent* to a point $l \in X$ if $\boldsymbol{x} \in c_G(X)$ and $G(\boldsymbol{x}) = l$ [17]. A method $G : c_G(X) \to X$ is called a *pointwise method* [20], if for each $x \in X$, the constant sequence $\boldsymbol{x} = \{x, x, x, \cdots\}$ is G-convergent to x. If X is a topological space, $c(X)$ denotes the set of all X-valued convergent sequences. A method $G : c_G(X) \to X$ is called *regular* if $c(X) \subset c_G(X)$ and $G(\boldsymbol{x}) = \lim \boldsymbol{x}$ for each $\boldsymbol{x} \in c(X)$. A method $G : c_G(X) \to X$ is called *subsequential* if whenever $\boldsymbol{x} \in c_G(X)$ is G-convergent to $l \in X$, then there exists a subsequence $\boldsymbol{x}' \in c(X)$ of $\boldsymbol{x}$ with $\lim \boldsymbol{x}' = l$. Obviously, a regular method on a topological space X is a pointwise method. The G-methods with the name "convergence" is only a function relation. It is not related with the topology on a space X. Based on the regular or subsequential methods, we can establish close ties between G-convergent and convergent sequences on X [17].

Next, we recall some concepts and lemmas related to the G-methods.

Definition 2.1. [17] Let G be a method on a set X. For each $A \subset X$,

(1) the set A is called a *G-closed set* of X if, whenever $\boldsymbol{x} \in s(A) \cap c_G(X)$, then $G(\boldsymbol{x}) \in A$;
(2) the *G-closure* of A is defined as the intersection of all G-closed sets containing A, and the G-closure of A is denoted by $\mathrm{cl}_G(A)$ or $\overline{A}^G$;
(3) the *G-hull* of A is defined as the set $\{G(\boldsymbol{x}) : \boldsymbol{x} \in s(A) \cap c_G(X)\}$, and the G-hull of A is denoted by $\mathrm{hu}_G(A)$ or $[A]_G$.

Lemma 2.2. [17] *Let G be a method on a set X. If $A \subset X$, then A is a G-closed set if and only if $[A]_G \subset A$, if and only if $\overline{A}^G \subset A$, i.e., $\overline{A}^G = A$.*

Definition 2.3. [17] Let G be a method on a set X. For each $A \subset X$,
(1) the set A is called a *G-open set* of X if $X \setminus A$ is G-closed in X;
(2) the *G-interior* of A is defined as the union of all G-open sets contained in A, and the G-interior of A is denoted by $\mathrm{int}_G(A)$ or $A^{\circ G}$;
(3) the *G-kernel* of A is defined as the set

$$\{l \in X : \text{ there is no } \boldsymbol{x} \in s(X \setminus A) \cap c_G(X) \text{ with } l = G(\boldsymbol{x})\},$$

and the G-kernel of A is denoted by $\ker_G(A)$ or $(A)_G$.

Lemma 2.4. [17] *Let G be a method on a set X. If $A \subset X$, then A is a G-open set if and only if $A \subset (A)_G$, if and only if $A \subset A^{\circ G}$, i.e., $A^{\circ G} = A$.*

Lemma 2.5. [17] *Let G be a method on a set X and $A \subset X$. Then*
(1) $(A)_G = X \setminus [X \setminus A]_G$.
(2) $A^{\circ G} = X \setminus \overline{X \setminus A}^G$.

Definition 2.6. Let G be a method on a set X. For each $A \subset X$ and $x \in X$,
(1) the set A is called a *G-kernel-open set* of X [20] if $A = (A)_G$;
(2) the set A is called a *G-hull-closed set* of X [20] if $A = [A]_G$;
(3) the set A is called a *G-neighborhood* of the point $x \in X$ [17] if there exists a G-open set U with $x \in U \subset A$;
(4) the set A is called a *G-kernel-neighborhood* of the point $x \in X$ [20] if there exists a G-kernel-open set U with $x \in U \subset A$.

By Lemma 2.5, a set A is a G-kernel-open set of X if and only if $X \setminus A$ is a G-hull-closed set of X [20].

Lemma 2.7. [17] *Let G be a method on a set X. A subset U of X is a G-open set if and only if U is a G-neighborhood of each point in U.*

Lemma 2.8. [17] *Let G be a method on a set X. For each $A \subset X$ and $x \in X$,*
(1) *$x \in [A]_G$ if and only if the set A intersects any subset U of X with $x \in (U)_G$;*
(2) *$x \in \overline{A}^G$ if and only if the set A intersects any subset U of X with $x \in U^{\circ G}$.*

Definition 2.9. [17] Let G be a method on a topological space X. X is said to be a *G-sequential space* if any subset A of X with $[A]_G \subset A$ is closed in X, i.e., every G-closed set in X is closed.

Obviously, X is a G-sequential space if and only if every G-open set in X is open [17]. A subset A of a topological space X is called a *sequentially closed set* of X if, whenever $\boldsymbol{x} \in s(A) \cap c(X)$, then $\lim \boldsymbol{x} \in A$. A subset A of X is called a *sequentially open set* of X if $X \setminus A$ is sequentially closed. If G is a subsequential method on a topological space X, then every sequentially closed set is G-closed [17].

Readers may refer to [12, 17] for some terminology unstated here.

3. G-continuous mappings

Let X and Y be topological spaces. A mapping $f : X \to Y$ is called *sequentially continuous* [4] if, whenever a sequence $\boldsymbol{x}$ of X converges to a point $x \in X$, then the sequence $f(\boldsymbol{x})$ of Y converges to $f(x) \in Y$. It is easy to verify that a mapping $f : X \to Y$ is sequentially continuous if and only if $f^{-1}(U)$ is a sequentially open set of X for each sequentially open set U of Y [4]. Therefore, sequentially continuous mappings can be defined in two ways. Corresponding to statistical convergence, Z.B. Tang and F.C. Lin [31] used the concept of "preserves limits of statistical sequences" in the discussion of continuity, which was called a mapping which preserves statistical convergence in [21]. As an extension of statistical convergence, B.K. Lahiri and P. Das [16] used the concept of "preserves $\mathcal{I}$-convergence" when discussing continuity with ideal convergence. To maintain consistency with the first paper on the methods [9], we defined the G-continuous mapping in [17] as follows. Let G_1, G_2 be methods on sets X and Y, respectively. A mapping $f : X \to Y$ is called (G_1, G_2)-*continuous* if $f(\boldsymbol{x}) \in c_{G_2}(Y)$ and $G_2(f(\boldsymbol{x})) = f(G_1(\boldsymbol{x}))$ for each $\boldsymbol{x} \in c_{G_1}(X)$. Combined with the above factors, G-continuous mappings and preserving G-convergence mappings are redefined as follows.

Definition 3.1. Let G_1, G_2 be methods on sets X and Y, respectively. For a mapping $f : X \to Y$,

(1) $f : X \to Y$ is called (G_1, G_2)-*continuous* at a point $x \in X$ if $f^{-1}(U)$ is a G_1-neighborhood of x for each G_2-neighborhood U of the point $f(x) \in Y$; for short, $f : X \to Y$ is called G-*continuous* at $x \in X$;
(2) $f : X \to Y$ is called (G_1, G_2)-*continuous*, if $f^{-1}(U)$ is a G_1-open set of X for each G_2-open set U of Y; for short, $f : X \to Y$ is called G-*continuous*;
(3) $f : X \to Y$ is called *preserving* (G_1, G_2)-*convergence*, if $f(\boldsymbol{x}) \in c_{G_2}(Y)$ and $G_2(f(\boldsymbol{x})) = f(G_1(\boldsymbol{x}))$ for each $\boldsymbol{x} \in c_{G_1}(X)$; for short, $f : X \to Y$ is called *preserving* G-*convergence*.

The following are well known.

Lemma 3.2. [17] *Let G_1, G_2 be methods on sets X and Y, respectively. If $f : X \to Y$ is a mapping, then* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *in the following conditions, and the contrary don't hold.*

(1) *f is a preserving (G_1, G_2)-convergence mapping.*
(2) *$f([A]_{G_1}) \subset [f(A)]_{G_2}$ for each $A \subset X$.*
(3) *$f^{-1}(F)$ is a G_1-closed set of X for each G_2-closed set F of Y.*

Next, we discuss some properties of conditions in Lemma 3.2.

Theorem 3.3. *Let G_1, G_2 be methods on sets X and Y, respectively. $f : X \to Y$ is G-continuous at $x \in X$ if and only if for every G_2-neighborhood U of $f(x)$, there exists a G_1-neighborhood V of x with $f(V) \subset U$.*

Proof. Suppose that the mapping $f : X \to Y$ satisfies the sufficient condition, and $x \in X$. For every G_2-neighborhood U of $f(x)$, there exists a G_1-neighborhood V of x with $f(V) \subset U$ by the sufficient condition. Thus $V \subset f^{-1}(f(V)) \subset f^{-1}(U)$, that is $f^{-1}(U)$ is a G_1-neighborhood of x. On the other hand, suppose that $f : X \to Y$ is G-continuous at $x \in X$. For every G_2-neighborhood U of $f(x)$, $f^{-1}(U)$ is a G_1-neighborhood of x, and $f(f^{-1}(U)) \subset U$. □

Theorem 3.4. *Let G_1, G_2 be methods on sets X and Y, respectively. The following are equivalent for a mapping $f : X \to Y$.*

(1) *f is a G-continuous mapping.*
(2) *$f^{-1}(F)$ is a G_1-closed set of X for every G_2-closed set F of Y.*
(3) *f is G-continuous at x for every $x \in X$.*
(4) *$f^{-1}(\overline{B}^{G_2}) \supset \overline{f^{-1}(B)}^{G_1}$ for every $B \subset Y$.*
(5) *$f(\overline{A}^{G_1}) \subset \overline{f(A)}^{G_2}$ for every $A \subset X$.*
(6) *For every $x \in X$ if U is a G_2-neighborhood of $f(x)$, then there exists a G_1-neighborhood V of x with $f(V) \subset U$.*
(7) *$f^{-1}(B^{\circ G_2}) \subset (f^{-1}(B))^{\circ G_1}$ for every $B \subset Y$.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) were proved in [17]. It is easy to know (3) $\Leftrightarrow$ (6) by Theorem 3.3. To complete the proof, we prove (5) $\Rightarrow$ (4) $\Rightarrow$ (7) $\Rightarrow$ (1) next.

(5) $\Rightarrow$ (4). For every $B \subset Y$, by condition (5), $f(\overline{f^{-1}(B)}^{G_1}) \subset \overline{f(f^{-1}(B))}^{G_2} \subset \overline{B}^{G_2}$, thus $\overline{f^{-1}(B)}^{G_1} \subset f^{-1}(\overline{B}^{G_2})$.

(4) $\Rightarrow$ (7). For every $B \subset Y$, by condition (4) and lemma 2.8, $f^{-1}(\overline{(Y \setminus B)}^{G_2}) = f^{-1}(Y \setminus B^{\circ G_2}) = X \setminus f^{-1}(B^{\circ G_2}) \supset \overline{f^{-1}(Y \setminus B)}^{G_1} = \overline{X \setminus f^{-1}(B)}^{G_1} = X \setminus (f^{-1}(B))^{\circ G_1}$, hence $f^{-1}(B^{\circ G_2}) \subset (f^{-1}(B))^{\circ G_1}$.

(7) $\Rightarrow$ (1). For every G_2-open set B of Y, then $B = B^{\circ G_2}$. By condition (7), $f^{-1}(B) = f^{-1}(B^{\circ G_2}) \subset (f^{-1}(B))^{\circ G_1}$, hence $f^{-1}(B)$ is a G_1-open set of X. Namely, f is a G-continuous mapping. □

Theorem 3.5. *Let G_1, G_2 be methods on sets X and Y, respectively. The following are equivalent for a mapping $f : X \to Y$.*

(1) *$f([A]_{G_1}) \subset [f(A)]_{G_2}$ for every $A \subset X$.*
(2) *$f^{-1}([B]_{G_2}) \supset [f^{-1}(B)]_{G_1}$ for every $B \subset Y$.*
(3) *$f^{-1}((B)_{G_2}) \subset (f^{-1}(B))_{G_1}$ for every $B \subset Y$.*
(4) *For every $x \in X$ and $V \subset Y$, if $f(x) \in (V)_{G_2}$, then there exists $U \subset X$ with $x \in (U)_{G_1}$ and $f(U) \subset V$.*

Proof. (1) $\Rightarrow$ (2). For every $B \subset Y$, by condition (1), $f([f^{-1}(B)]_{G_1}) \subset [f(f^{-1}(B))]_{G_2} \subset [B]_{G_2}$, hence $f^{-1}([B]_{G_2}) \supset [f^{-1}(B)]_{G_1}$.

(2) $\Rightarrow$ (3). For every $B \subset Y$, we have $f^{-1}([Y \setminus B]_{G_2}) \supset [f^{-1}(Y \setminus B)]_{G_1}$ from condition (2). By Lemma 2.5, $X \setminus f^{-1}((B)_{G_2}) \supset [X \setminus f^{-1}(B)]_{G_1} = X \setminus (f^{-1}(B))_{G_1}$, hence $f^{-1}((B)_{G_2}) \subset (f^{-1}(B))_{G_1}$.

(3) $\Rightarrow$ (4). For every $x \in X$ and $V \subset Y$, if $f(x) \in (V)_{G_2}$, then $x \in f^{-1}((V)_{G_2})$. By condition (3), $f^{-1}((V)_{G_2}) \subset (f^{-1}(V))_{G_1}$. Thus the point $x \in (f^{-1}(V))_{G_1}$ and $f(f^{-1}(V)) \subset V$.

(4) $\Rightarrow$ (1). Let $A \subset X$ and $x \in [A]_{G_1}$. For any subset V of Y with $f(x) \in (V)_{G_2}$, there exists $U \subset X$ with $x \in (U)_{G_1}$ and $f(U) \subset V$ from condition (4). By Lemma 2.8, $U \bigcap A \neq \emptyset$, thus $V \bigcap f(A) \supset f(U) \bigcap f(A) \neq \emptyset$. By Lemma 2.8, $f(x) \in [f(A)]_{G_2}$; $f([A]_{G_1}) \subset [f(A)]_{G_2}$. □

Does the G-continuity be characterized by "G-open sets" or "G-kernel-open sets"? In order to discuss this problem, we cite the following lemma first.

Lemma 3.6. [20] *Let G be a pointwise method on a set X. For every $A \subset X$, we have*

(1) *$(A)_G \subset A \subset [A]_G$;*
(2) *A is a G-open (resp., G-closed) set if and only if A is a G-kernel-open (resp., G-hull-closed) set.*

Thus, the subset A of X is a G-kernel-neighborhood of a point $x \in X$ if and only if A is a G-neighborhood of x [20].

Corollary 3.7. *Let G_1, G_2 be pointwise methods on sets X and Y, respectively. The following are equivalent for a mapping $f : X \to Y$.*

(1) *f is a G-continuous mapping.*
(2) *$f^{-1}(F)$ is a G_1-closed set of X for every G_2-closed set F of Y.*
(3) *f is G-continuous at x for every $x \in X$.*
(4) *$f^{-1}(\overline{B}^{G_2}) \supset \overline{f^{-1}(B)}^{G_1}$ for every $B \subset Y$.*
(5) *$f(\overline{A}^{G_1}) \subset \overline{f(A)}^{G_2}$ for every $A \subset X$.*
(6) *For every $x \in X$, if U is a G_2-neighborhood of $f(x)$, then there exists a G_1-neighborhood V of x with $f(V) \subset U$.*
(7) *$f^{-1}(B^{\circ G_2}) \subset (f^{-1}(B))^{\circ G_1}$ for every $B \subset Y$.*
(8) *For every $x \in X$, if U is a G_2-kernel-neighborhood of $f(x)$, then there exists a G_1-kernel-neighborhood V of x with $f(V) \subset U$.*
(9) *$f^{-1}(F)$ is a G_1-hull-closed set of X for every G_2-hull-closed set F of Y.*
(10) *$f^{-1}(U)$ is a G_1-kernel-open set of X for every G_2-kernel-open set U of Y.*

Proof. The equivalence of $(1) - (7)$ come from the Theorem 3.4. By Definition 2.6, $(9) \Leftrightarrow (10)$. And from Lemma 3.6, we have $(6) \Leftrightarrow (8)$ and $(2) \Leftrightarrow (9)$. $\square$

The following examples illustrate that the pointwise method in Corollary 3.7 is essential.

Example 3.8. [17, Example 7.4] *There exists a mapping $f : X \to Y$ which satisfies Corollary 3.7(7), but does not satisfy Theorem 3.5(1), where G_2 is not a pointwise method on Y.*

Let $X = \mathbb{N}$. Put $c_{G_1}(X) = \{\{x_n\}_{n\in\mathbb{N}} \in s(X)$: there exists an $m \in \mathbb{N}$ such that $\{x_n - x_{n-1}\}_{n>m}$ is a constant sequence$\}$. Define $G_1 : c_{G_1}(X) \to X$ by $G_1(\boldsymbol{x}) = \lim\limits_{n\to\infty}(x_{n+1} - x_n)$ for each $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in c_{G_1}(X)$.

Let $Y = \{0, 1\}$. Put $c_{G_2}(Y) = \{\boldsymbol{x} \in s(Y) \cap c_{G_1}(X) : G_1(\boldsymbol{x}) \in Y\}$, and define a function $G_2 : c_{G_2}(Y) \to Y$ by $G_2(\boldsymbol{x}) = G_1(\mathbf{x})$, $\boldsymbol{x} \in c_{G_2}(Y)$. Then G_2 is a method on the subset Y of X. Since the constant sequence $\{1, 1, 1, \cdots\}$ of Y is G_2-convergent to 0, G_2 is not a pointwise method and the G_2-kernel-open sets of Y are only $\{1\}$ and Y.

Define a mapping $f : X \to Y$ as follows: $f(x) = 0$ if and only if $x = 2k,\ k \in \mathbb{N}$. If $f(x) = 0$, since the G_2-kernel-neighborhood V of 0 is Y, there exists a G_1-kernel-neighborhood U of x with $f(U) \subset V$. If $f(x) = 1$, then $x \in \{2k + 1 : k \in \mathbb{N}\}$. Suppose that V is a G_2-kernel-neighborhood of 1 in Y. Then V is $\{1\}$ or Y. We can assume that $V = \{1\}$. Put $U = \{2k + 1 :\ k \in \mathbb{N}\}$, then $(U)_{G_1} = U$, and U is a G_1-kernel-neighborhood of x and $f(U) = \{1\} \subset V$. To sum up, the mapping f satisfies Corollary 3.7(7). But, f does not satisfy Theorem 3.5(1). In fact, $f([X]_{G_1}) = f(X) = Y \not\subset \{0\} = [Y]_{G_2} = [f(X)]_{G_2}$.

Example 3.9. *There exists a mapping $f : X \to Y$ which satisfies Theorem 3.5(2), but does not satisfy Corollary 3.7(8), where G_1 is not a pointwise method on X.*

Let $X = \{0, 1\}$. Put $c_{G_1}(X) = \{\{x_n\}_{n\in\mathbb{N}} \in s(X)$: there exists an $m \in \mathbb{N}$ such that $\{x_n - x_{n-1}\}_{n>m}$ is a constant sequence$\}$. Define $G_1 : c_{G_1}(X) \to X$ by $G_1(\boldsymbol{x}) = \lim\limits_{n\to\infty}(x_{n+1} - x_n)$ for each $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in c_{G_1}(X)$. Obviously, G_1 is not pointwise from Example 3.8.

Let $Y = \{0\}$. The method G_2 on Y be defined by $G_2(\{0\}) = 0$, then G_2 is a pointwise method on Y.

Define a mapping $f : X \to Y$ by $f(x) = 0$. It is easy to see that $[f^{-1}(\emptyset)]_{G_1} = [\emptyset]_{G_1} = \emptyset \subset f^{-1}([\emptyset]_{G_2})$, and $[f^{-1}(Y)]_{G_1} = [X]_{G_1} = \{0\} \subset X = f^{-1}(Y) = f^{-1}([Y]_{G_2})$. Thus $[f^{-1}(B)]_{G_1} \subset f^{-1}([B]_{G_2})$ for every $B \subset Y$, i.e., f satisfies Theorem 3.5(2).

Obviously, Y is a G_2-hull-closed set, but $[f^{-1}(Y)]_{G_1} = [X]_{G_1} = \{0\} \neq f^{-1}(Y)$, hence the inverse of the G_2-hull-closed set Y is not a G_1-hull-closed set. Thus f does not satisfy Corollary 3.7(8).

Question 3.10. *Let G_1, G_2 be pointwise methods on sets X and Y, respectively. Does Theorem 3.5 be equivalent to Corollary 3.7 for a mapping $f : X \to Y$?*

Theorem 3.11. *Let G_1, G_2 and G_3 be methods on sets X, Y and Z, respectively.*

(1) *If $f : X \to Y$ and $g : Y \to Z$ are preserving G-convergence mappings, then $g \circ f : X \to Z$ is a preserving G-convergence mapping.*
(2) *If $f : X \to Y$ and $g : Y \to Z$ are G-continuous mappings, then $g \circ f : X \to Z$ is a G-continuous mapping.*

Proof. (1) Let $\boldsymbol{x} \in c_{G_1}(X)$. Since $f : X \to Y$ is a preserving G-convergence mapping, $f(\boldsymbol{x}) \in c_{G_2}(Y)$ and $G_2(f(\boldsymbol{x})) = f(G_1(\boldsymbol{x}))$. And because $g : Y \to Z$ is a preserving G-convergence mapping, $g(f(\boldsymbol{x})) \in c_{G_3}(Z)$ and $G_3(g(f(\boldsymbol{x}))) = g(f(G_1(\boldsymbol{x})))$. Namely, $g \circ f : X \to Z$ is a preserving G-convergence mapping.

(2) Let U be a G_3-open set of Z. Since g is a G-continuous mapping, $g^{-1}(U)$ is a G_2-open set of Y. From the G-continuity of f, we know that $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is a G_1-open set of X. Hence $g \circ f : X \to Z$ is a G-continuous mapping. □

Example 3.12. *Let G_1, G_2 and G_3 be methods on sets X, Y and Z, respectively. There exist preserving G-convergence mappings $f : X \to Y$ and $g \circ f : X \to Z$ such that $g : Y \to Z$ is not a G-continuous mapping.*

Let $X = (0, +\infty)$, $c_{G_1}(X) = s(X)$, and $G_1(\boldsymbol{x}) = 1$ for every $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in c_{G_1}(X)$.
Let $Y = \mathbb{R}$, $c_{G_2}(Y) = s(Y)$, and for every $\boldsymbol{y} = \{y_n\}_{n\in\mathbb{N}} \in c_{G_2}(Y)$,

$$G_2(\boldsymbol{y}) = \begin{cases} 1, & \text{all } y_n > 0, \\ 0, & \text{others.} \end{cases}$$

Let $Z = \mathbb{R}$, $c_{G_3}(Z) = s(Z)$, and for every $\boldsymbol{z} = \{z_n\}_{n\in\mathbb{N}} \in c_{G_3}(Z)$,

$$G_3(\boldsymbol{z}) = \begin{cases} 1, & \text{all } z_n > 0, \\ -1, & \text{others.} \end{cases}$$

$f : X \to Y$ and $g : Y \to Z$ all are defined as injective. For every $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in s(X) \bigcap c_{G_1}(X)$, it is easy to see that $G_2(f(\boldsymbol{x})) = 1 = f(1) = f(G_1(\boldsymbol{x}))$, and $G_3((g \circ f)(\boldsymbol{x})) = 1 = (g \circ f)(1) = (g \circ f)(G_1(\boldsymbol{x}))$. Then $f : X \to Y$ and $g \circ f : X \to Z$ all are preserving G-convergence mappings. But the mapping $g : Y \to Z$ is not a G-continuous mapping. In fact, let $A = \{-1\} \subset Z$. Then $[A]_{G_3} = \{-1\} \subset A$, thus A is a G_3-closed set of Z. But $[g^{-1}(A)]_{G_2} = \{0\} \not\subset g^{-1}(A)$, so $g^{-1}(A)$ is not a G_2-closed set of Y. Hence $g : Y \to Z$ is not a G-continuous mapping, and g is not a preserving G-convergence mapping.

Theorem 3.13. *Let G_1, G_2 be methods on topological spaces X and Y, respectively. $f : X \to Y$ is a mapping.*

(1) *If X is a G_1-sequential space, G_2 is a subsequential method and f is a G-continuous mapping, then f is a continuous mapping.*
(2) *If Y is a G_2-sequential space, G_1 is a subsequential method and f is a continuous mapping, then f is a G-continuous mapping.*

Proof. (1) Let U be an open set of Y. Then U is a sequentially open set. Since G_2 is a subsequential method, U is a G_2-open set. Because f is a G-continuous mapping, $f^{-1}(U)$ is a G_1-open set of X, and U is an open set by X is a G_1-sequential space. Namely, f is a continuous mapping.

(2) Let U be a G_2-open set of Y. Since Y is a G_2-sequential space, U is an open set. Because f is a continuous mapping, $f^{-1}(U)$ is an open set of X, and hence $f^{-1}(U)$ is a sequentially open set. Since G_1 is a subsequential method, $f^{-1}(U)$ is a G_1-open set of X. Namely, f is a G-continuous mapping. □

4. G-open mappings and G-closed mappings

Open mappings and closed mappings are basic and important mappings in topology, and they can be characterized by open sets, closed sets, interiors or closures in topological spaces [12]. As an extension of general convergence, we introduces the concepts of G-open mappings and G-closed mappings on this section first, and then try to discuss the basic properties of the above mappings from G-open sets, G-closed sets, G-interiors and G-closures.

Definition 4.1. Let G_1, G_2 be methods on sets X and Y, respectively. Suppose that $f : X \to Y$ is a mapping.

(1) f is called a (G_1, G_2)-*open mapping*, if for every G_1-open set U of X, $f(U)$ is G_2-open set of Y; for short, f is called a G-*open mapping*;

(2) f is called a (G_1, G_2)-*closed mapping*, if for every G_1-closed set F of X, $f(F)$ is G_2-closed set of Y; for short, f is called a G-*closed mapping.*

Theorem 4.2. *Let G_1, G_2 be methods on sets X and Y, respectively. The following are equivalent for a mapping $f : X \to Y$.*

(1) *f is a G-open mapping.*

(2) *$f(A^{\circ G_1}) \subset (f(A))^{\circ G_2}$ for every $A \subset X$.*

(3) *For every $x \in X$, if $U \subset X$ is a G_1-neighborhood of x, then $f(U)$ is a G_2-neighborhood of $f(x)$.*

(4) *$f^{-1}(\overline{B}^{G_2}) \subset \overline{f^{-1}(B)}^{G_1}$ for every $B \subset Y$.*

(5) *$f^{-1}(B^{\circ G_2}) \supset (f^{-1}(B))^{\circ G_1}$ for every $B \subset Y$.*

Proof. (1) ⇒ (2). Let $A \subset X$. Since $A^{\circ G_1} \subset A$, $f(A^{\circ G_1}) \subset f(A)$. By condition (1), $f(A^{\circ G_1})$ is a G_2-open set which is contained in $f(A)$, thus $f(A^{\circ G_1}) \subset (f(A))^{\circ G_2}$.

(2) ⇒ (3). Let $x \in X$ and $U \subset X$ be a G_1-neighborhood of x. Then $x \in U^{\circ G_1}$. By condition (2), $f(x) \in f(U^{\circ G_1}) \subset (f(U))^{\circ G_2} \subset f(U)$, hence $f(U)$ is a G_2-neighborhood of $f(x)$.

(3) ⇒ (4). Let $B \subset Y$. If $x \in f^{-1}(\overline{B}^{G_2})$, then $f(x) \in \overline{B}^{G_2}$. For any G_1-neighborhood U of x, $f(U)$ is a G_2-neighborhood of $f(x)$ by condition (3), thus $f(U) \bigcap B \neq \emptyset$ by Lemma 2.8. So, $U \bigcap f^{-1}(B) \neq \emptyset$, that is $x \in \overline{f^{-1}(B)}^{G_1}$.

(4) ⇒ (5). Let $B \subset Y$. By condition (4) and lemma 2.8, $f^{-1}(\overline{(Y \setminus B)}^{G_2}) = f^{-1}(Y \setminus B^{\circ G_2}) = X \setminus f^{-1}(B^{\circ G_2}) \subset \overline{f^{-1}(Y \setminus B)}^{G_1} = \overline{X \setminus f^{-1}(B)}^{G_1} = X \setminus (f^{-1}(B))^{\circ G_1}$, hence $f^{-1}(B^{\circ G_2}) \supset (f^{-1}(B))^{\circ G_1}$.

(5) ⇒ (1). For every G_1-open set A of X, then $A = A^{\circ G_1}$ and $f(A) \subset Y$. By condition (5), $f^{-1}((f(A))^{\circ G_2}) \supset (f^{-1}(f(A)))^{\circ G_1} \supset A^{\circ G_1}$, then$f(A) = f(A^{\circ G_1}) \subset (f(A))^{\circ G_2}$, hence $f(A)$ is a G_2-open set of Y. Namely, f is a G-open mapping.

□

Theorem 4.3. *Let G_1, G_2 be methods on sets X and Y, respectively. The following are equivalent for a mapping $f : X \to Y$.*

(1) *f is a G-closed mapping.*
(2) *$f(\overline{A}^{G_1}) \supset \overline{f(A)}^{G_2}$ for every $A \subset X$.*

Proof. (1) $\Rightarrow$ (2). For every $A \subset X$, since $A \subset \overline{A}^{G_1}$, $f(A) \subset f(\overline{A}^{G_1})$. By condition (1) and Definition 2.1(2), $\overline{f(A)}^{G_2} \subset f(\overline{A}^{G_1})$.

(2) $\Rightarrow$ (1). Let A be a G_1-closed set of X. Then $A = \overline{A}^{G_1}$. By condition (2), $f(A) = f(\overline{A}^{G_1}) \supset \overline{f(A)}^{G_2}$. It follows from Lemma 2.2 that $f(A)$ is a G_2-closed set of Y. Hence f is a G-closed mapping. □

Theorem 4.4. *Let G_1, G_2 be methods on sets X and Y, respectively. $f : X \to Y$ is a G-closed mapping if and only if for every $y \in Y$ and G_1-open set $U \supset f^{-1}(y)$ of X, there exists a G_2-open set W of Y with $y \in W$ and $f^{-1}(W) \subset U$.*

Proof. Suppose that f is a G-closed mapping. For every $y \in Y$ and G_1-open set $U \supset f^{-1}(y)$ of X, let $W = Y \setminus f(X \setminus U)$, then $y \in W$ and $f^{-1}(W) = f^{-1}(Y \setminus f(X \setminus U)) = X \setminus f^{-1}(f(X \setminus U)) \subset U$. Since f is a G-closed mapping, W is a G_2-open set of Y, and W is required.

On the other hand, suppose that f satisfies the sufficient condition. Let F be a G_1-closed set of X and $y \in Y \setminus f(F)$. Then $f^{-1}(y) \subset X \setminus F$. Hence there exists a G_2-open set W of Y with $y \in W$ and $f^{-1}(W) \subset X \setminus F$ by the sufficient condition, so W is a G_2-neighborhood of y and $W \subset Y \setminus f(F)$, then $W \bigcap f(F) = \emptyset$, thus $y \notin \overline{f(F)}^{G_2}$ by Lemma 2.8. In summary, $\overline{f(F)}^{G_2} \subset f(F)$, i.e., $f(F)$ is a G_2-closed set of Y. So f is a G-closed mapping. □

Corollary 4.5. *Let G_1, G_2 be methods on sets X and Y, respectively. If $f : X \to Y$ is a surjective G-continuous mapping, then the following are equivalent:*

(1) *f is a G-closed mapping.*
(2) *For every $E \subset Y$ and G_1-open set $U \supset f^{-1}(E)$ of X, there exists a G_1-open set V of X with $f^{-1}(E) \subset V \subset U$, $V = f^{-1}(f(V))$ and $f(V)$ is a G_2-open set of Y.*
(3) *For every $y \in Y$ and G_1-open set $U \supset f^{-1}(y)$ of X, there exists a G_1-open set V of X with $f^{-1}(y) \subset V \subset U$, $V = f^{-1}(f(V))$ and $f(V)$ is a G_2-open set of Y.*

Proof. (1) $\Rightarrow$ (2). Suppose that f is a surjective G-continuous and G-closed mapping. For every $E \subset Y$ and G_1-open set $U \supset f^{-1}(E)$ of X, arbitrarily pick $y \in E$, then $f^{-1}(y) \subset U$. By Theorem 4.4, there exists a G_2-open set W_y of Y with $y \in W_y$ and $f^{-1}(W_y) \subset U$. Let $V = \bigcup_{y \in E} f^{-1}(W_y)$. Since f is G-continuous, the set V is a G_1-open set of X. And since $f : X \to Y$ is surjective, $f(V) = \bigcup_{y \in E} W_y$, hence $f(V)$ is a G_2-open set of Y, $f^{-1}(E) \subset V \subset U$ and $f^{-1}(f(V)) = f^{-1}(\bigcup_{y \in E} W_y) = \bigcup_{y \in E} f^{-1}(W_y) = V$.

(2) $\Rightarrow$ (3). Obviously.

(3) $\Rightarrow$ (1). For every $y \in Y$ and G_1-open set $U \supset f^{-1}(y)$ of X, there exists a G_1-open set V of X with $f^{-1}(y) \subset V \subset U$, $V = f^{-1}(f(V))$ and $f(V)$ is a G_2-open set of Y by condition (3). Let $W = f(V)$. Then W is a G_2-open set of Y, $y \in W$ and $f^{-1}(W) = f^{-1}(f(V)) = V \subset U$. By Theorem 4.4, f is a G-closed mapping. □

Comparison Theorems 3.5 and 4.2, we have the following question.

Question 4.6. *Let G_1, G_2 be methods on sets X and Y, respectively. If $f : X \to Y$ is a mapping, what kind of mapping properties can be characterized by condition "$f((A)_{G_1}) \subset (f(A))_{G_2}$ for every $A \subset X$"?*

5. G-quotient mappings

As we all know, quotient spaces are important tools constructing new topological spaces through the known topological spaces in topology, and quotient mappings are also an essential part of the research on quotient spaces [12]. Therefore, we will introduce the concepts of G-quotient mappings and G-quotient spaces, discuss some basic properties and characterizations of G-quotient mappings, and studies its relationship with quotient mappings in this section.

Definition 5.1. Let G_1, G_2 be methods on sets X and Y, respectively. A surjective mapping $f : X \to Y$ is called a (G_1, G_2)-*quotient mapping*, if $f^{-1}(U)$ is a G_1-open set of X, then U is a G_2-open set of Y; for short, f is called a G-*quotient mapping*, and Y is called a G-*quotient space*.

We know that quotient mappings in topological spaces require continuity. The continuity under the G-methods has at least three cases described in Lemma 3.2. So, as analogy with the usual quotient mappings, there are at least three forms of G-quotient mappings. For this reason, we divides continuous mappings from G-quotient mappings in this paper, and G-quotient mappings do not require a certain continuity. In order to make it easier to compare, quotient mappings in topological spaces take the following form: Let X, Y be topological spaces, a surjective mapping $f : X \to Y$ is called a *quotient mapping* [21], if $f^{-1}(U)$ is an open set of X, then U is an open set of Y. That is, the usual quotient mappings are equivalent to the continuous quotient mappings in this paper.

Statistically sequentially quotient mappings and quotient mappings are not implied each other [21]. At this time, if we define the G-methods as the statistical convergence methods in [21, Examples 4.7 and 4.8], it is easy to know that G-quotient mappings and quotient mappings are also not implied each other.

Theorem 5.2. *Let G_1, G_2 be methods on sets X and Y, respectively. If $f : X \to Y$ is a surjective mapping, then*

(1) *f is a G-quotient mapping if and only if for every subset F of Y, if $f^{-1}(F)$ is a G_1-closed set of X, then F is a G_2-closed set of Y;*
(2) *If f is a G-closed (G-open) mapping, then f is a G-quotient mapping.*

Proof. (1) Let f be a G-quotient mapping. For every subset F of Y, if $f^{-1}(F)$ is a G_1-closed set of X, then $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is a G_1-open set of X. Since f is a G-quotient mapping, $Y \setminus F$ is a G_2-open set of Y, thus F is a G_2-closed set of Y. On the other hand, suppose that f satisfies the sufficient condition. For every subset U of Y, if $f^{-1}(U)$ is a G_1-open set of X, then $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is a G_1-closed set of X. By the sufficient condition, $Y \setminus U$ is a G_2-closed set of Y, thus U is a G_2-open set of Y. By Definition 5.1, f is a G-quotient mapping.

(2) Suppose that f is a G-closed (G-open) mapping. For every subset F of Y, if $f^{-1}(F)$ is a G_1-closed (G_1-open) set of X, since $f : X \to Y$ is a surjective G-closed (G-open) mapping, then $F = f(f^{-1}(F))$ is a G_2-closed (G_2-open) set of Y. By Definition 5.1, f is a G-quotient mapping. □

Example 5.3. [17, Example 7.4] *There exists a G-quotient mapping $f : X \to Y$ which satisfies Lemma 3.2(3), but does not satisfy Lemma 3.2(2).*

The methods and mapping on sets X, Y are the same as Example 3.8. Let $F \subset Y$. Then $F = \emptyset$, $\{0\}$, $\{1\}$ or Y. It is clear to see that $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{0\}) = \{2k : k \in \mathbb{N}\}$, $f^{-1}(\{1\}) = \{2k+1 : k \in \mathbb{N}\}$, $f^{-1}(Y) = X$. Since $0 \in [\{2k+1 : k \in \mathbb{N}\}]_{G_1}$ and $0 \notin \{2k+1 : k \in \mathbb{N}\}$, $f^{-1}(\{1\})$ is not a G_1-closed set of X. It is easy to see that $[\{2k : k \in \mathbb{N}\}]_{G_1} = \{2k : k \in \mathbb{N}\}$. Therefore, $f^{-1}(F)$ is a G_1-closed set of X only when F is equal to $\emptyset$, $\{0\}$ or Y, then F is a G_2-closed set of Y. By Theorem 5.2(1), $f : X \to Y$ is a G-quotient mapping.

From the above explanation, we know that $f : X \to Y$ is G-continuous. But f is not satisfy Lemma 3.2(2). In fact, let $A = \mathbb{N}$, then $f([A]_{G_1}) = f(X) = Y \not\subset \{0\} = [Y]_{G_2} = [f(A)]_{G_2}$.

Theorem 5.4. *Let G_1, G_2 and G_3 be methods on sets X, Y and Z, respectively. If $f : X \to Y$ is a G-continuous and G-quotient mapping, then*

(1) *$g : Y \to Z$ is a G-quotient mapping if and only if $g \circ f : X \to Z$ is a G-quotient mapping;*

(2) *$g : Y \to Z$ is a G-continuous mapping if and only if $g \circ f : X \to Z$ is a G-continuous mapping.*

Proof. (1) On the one hand, let $U \subset Z$ such that $(g \circ f)^{-1}(U)$ is a G_1-open set of X. Since $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ and $f : X \to Y$ is a G-quotient mapping, $g^{-1}(U)$ is a G_2-open set of Y. And because $g : Y \to Z$ is a G-quotient mapping, U is a G_3-open set of Z. By Definition 5.1, $g \circ f : X \to Z$ is a G-quotient mapping. On the other hand, let $g^{-1}(U)$ be a G_2-open set of Y for some $U \subset Z$. Since $f : X \to Y$ is a G-continuous mapping, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is a G_1-open set of X. And because $g \circ f : X \to Z$ is a G-quotient mapping, U is a G_3-open set of Z. By Definition 5.1, $g : Y \to Z$ is a G-quotient mapping.

(2) The necessity comes from Theorem 3.11(2). Next, we prove the sufficiency. Suppose that $g \circ f : X \to Z$ is a G-continuous mapping. For every G_3-open set U of Z, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is a G_1-open set of X. Since $f : X \to Y$ is a G-quotient mapping, $g^{-1}(U)$ is a G_2-open set of Y, thus g is a G-continuous mapping. □

Question 5.5. *Let G_1, G_2 and G_3 be methods on sets X, Y and Z, respectively. Suppose that $f : X \to Y$ is a G-quotient and preserving G-convergence mapping. If $g \circ f : X \to Z$ is a preserving G-convergence mapping, does $g : Y \to Z$ be a preserving G-convergence mapping?*

Theorem 5.6. *Let G_1, G_2 be methods on topological spaces X and Y, respectively. If X is a G_1-sequential space and $f : X \to Y$ is a G-continuous quotient mapping, then Y is a G_2-sequential space.*

Proof. Let U be a G_2-open set of Y. Since $f : X \to Y$ is a G-continuous mapping, $f^{-1}(U)$ is a G_1-open set of X. And because X is a G_1-sequential space, it is clear that $f^{-1}(U)$ is an open set. It follows that U is an open set of Y because f is a quotient mapping. By Definition 2.9, Y is a G_2-sequential space. □

Finally, we discuss the conversion relationship between quotient mappings and G-quotient mappings.

Theorem 5.7. *Let G_1, G_2 be methods on topological spaces X and Y, respectively. $f : X \to Y$ is a surjective mapping.*

(1) *If X is a G_1-sequential space, G_2 is a subsequence method and f is a quotient mapping, then f is a G-quotient mapping.*

(2) *If Y is a G_2-sequential space, G_1 is a subsequence method and f is a G-quotient mapping, then f is a quotient mapping.*

Proof. (1) Suppose that X is a G_1-sequential space, G_2 is a subsequence method and f is a quotient mapping. Let $f^{-1}(U)$ be a G_1-open set of X for some $U \subset Y$. Since X is a G_1-sequential space, $f^{-1}(U)$ is an open set. And because f is a quotient mapping, U is an open set of Y, thus U is a sequentially open set. Since G_2 is a subsequence method, U is a G_2-open set [17, Lemma 2.11(2)]. By Definition 5.1, f is a G-quotient mapping.

(2) Suppose that Y is a G_2-sequential space, G_1 is a subsequence method and f is a G-quotient mapping. Let $f^{-1}(U)$ be an open set of X for some $U \subset Y$, then $f^{-1}(U)$ is a sequentially open set. Since G_1 is a subsequence method, $f^{-1}(U)$ is a G_1-open set [17, Lemma 2.11(2)]. And because f is a G-quotient mapping, U is a G_2-open set of Y. Since Y is a G_2-sequential space, U is an open set. Therefore, f is a quotient mapping. □

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