$\mathcal{I}$-CONVERGENT FUNCTIONS DEFINED ON AMENABLE SEMIGROUPS

ERDINÇ DÜNDAR*, FATIH NURAY*, UĞUR ULUSU**

Abstract. In this paper we firstly introduce and study the concepts of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence, $\mathcal{I}$-Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence of functions defined on discrete countable amenable groups, where $\mathcal{I}$ is an ideal of subsets of the amenable semigroup G. Secondly, we introduce and examine $\mathcal{I}$-limit points and $\mathcal{I}$-Cluster points of functions defined on discrete countable amenable groups. Finally, we introduce and investigate $\mathcal{I}$-limit superior and $\mathcal{I}$-limit inferior of functions defined on discrete countable amenable groups.

1. Introduction and Background

The concept of statistical convergence was introduced by Fast [5] and this concept has been studied by many others (see, [6, 7]). The idea of $\mathcal{I}$-convergence which is a generalization of statistical convergence was introduced by Kostyrko et al. [8] which is based on the structure of the ideal $\mathcal{I}$ of subset of the set natural numbers $\mathbb{N}$. After than, Demirci [2] introduced the concepts of $\mathcal{I}$-limit superior and $\mathcal{I}$-limit inferior of real sequences and investigated the relationships between this concepts. Then, Nabiev et al. [12] introduced the concepts of $\mathcal{I}$-Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence of real sequences and investigated some properties this concepts.

In [1], Day studied on the concept of amenable semigroups. Then, the concepts of summability in amenable semigroups were studied in [4, 10, 11]. Douglas [3] extended the notion of arithmetic mean to amenable semigroups and obtained a characterization for almost convergence in amenable semigroups. In [14], Nuray and Rhoades introduced the notions of convergence and statistical convergence in amenable semigroups. Also, the notion of almost statistical convergence in amenable semigroups studied by Nuray and Rhoades [15].

The aim of this study is to introduce the concepts of $\mathcal{I}$-convergence, $\mathcal{I}$-Cauchy sequence and $\mathcal{I}$-limit points for functions defined on discrete countable amenable semigroups and to examine some properties of these concepts. For the particular case; when the amenable semigroup is the additive positive integers, our definitions and theorems yield the results of [2, 8, 9].

Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold, and $w(G)$ and $m(G)$ denote the spaces of all real valued functions and all bounded real functions on G, respectively. $m(G)$ is a

2010 *Mathematics Subject Classification.* 40A05, 40A35, 43A07.

Key words and phrases. $\mathcal{I}$-convergence, $\mathcal{I}$-Cauchy sequence, $\mathcal{I}$-limit points, Folner sequence, Amenable semigroups.

Banach space with the supremum norm

$$\|f\|_\infty = \sup\big\{|f(g)| : g \in G\big\}.$$

Namioka [13] showed that, if G is a countable amenable group, there exists a sequence $\{S_n\}$ of finite subsets of G such that

i) $G = \bigcup_{n=1}^{\infty} S_n$,
ii) $S_n \subset S_{n+1} \quad (n = 1, 2, ...)$,
iii) $\lim\limits_{n\to\infty} \frac{|S_n g \cap S_n|}{|S_n|} = 1, \quad \lim\limits_{n\to\infty} \frac{|g S_n \cap S_n|}{|S_n|} = 1$,

for all $g \in G$, where $|A|$ denotes the number of elements the finite set A.

Any sequence of finite subsets of G satisfying $(i), (ii)$ and (iii) is called a Folner sequence for G.

The sequence $S_n = \{0, 1, 2, ..., n-1\}$ is a familiar Folner sequence giving rise to the classical Cesàro method of summability.

Now, we recall the basic definitions and concepts (See, [6, 8, 14]).

Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. A function $f \in w(G)$ is said to be convergent to s for any Folner sequence $\{S_n\}$ for G if, for every $\varepsilon > 0$ there exists a $k_0 \in \mathbb{N}$ such that $|f(g) - s| < \varepsilon$, for all $m > k_0$ and $g \in G \setminus S_m$.

Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. A function $f \in w(G)$ is said to be a Cauchy sequence for any Folner sequence $\{S_n\}$ for G if, for every $\varepsilon > 0$ there exists a $k_0 \in \mathbb{N}$ such that $|f(g) - f(h)| < \varepsilon$, for all $m > k_0$ and $g, h \in G \setminus S_m$.

The upper and lower Folner densities of a set $S \subset G$ are defined by

$$\overline{\delta}(S) = \limsup_{n\to\infty} \frac{1}{|S_n|} \big|\{g \in S_n : g \in S\}\big|$$

and

$$\underline{\delta}(S) = \liminf_{n\to\infty} \frac{1}{|S_n|} \big|\{g \in S_n : g \in S\}\big|,$$

respectively. If $\overline{\delta}(S) = \underline{\delta}(S)$, then

$$\delta(S) = \lim_{n\to\infty} \frac{1}{|S_n|} \big|\{g \in S_n : g \in S\}\big|,$$

is called Folner density of S. Take $G = \mathbb{N}$, $S_n = \{0, 1, 2, ..., n-1\}$ and S be the set of positive integers with leading digit 1 in the decimal expansion. The set S has no Folner density with respect to the Folner sequence $\{S_n\}$, since $\underline{\delta}(S) = \frac{1}{9}$ and $\overline{\delta}(S) = \frac{5}{9}$.

Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. A function $f \in w(G)$ is said to be statistically convergent to s for any Folner sequence $\{S_n\}$ for G if, for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{|S_n|} \big|\big\{g \in S_n : |f(g) - s| \geq \varepsilon\big\}\big| = 0.$$

Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. A function $f \in w(G)$ is said to be a statistically

Cauchy sequence for any Folner sequence $\{S_n\}$ for G if, for every $\varepsilon > 0$ and $m \geq 0$, there exists an $h \in G \setminus S_m$ such that

$$\lim_{n\to\infty} \frac{1}{|S_n|}\big|\big\{g \in S_n : |f(g) - f(h)| \geq \varepsilon\big\}\big| = 0.$$

Let X is a non-empty set. A family of sets $\mathcal{I} \subseteq 2^X$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I} \subseteq 2^X$ is called admissible if and only if $\mathcal{I} \supset \big\{\{x\} : x \in X\big\}$. All ideals in this paper are assumed to be admissible.

A family of sets $\mathcal{F} \subseteq 2^X$ is called a filter on X if and only if

(i) $\emptyset \notin \mathcal{F}$, (ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, (iii) For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

$\mathcal{I}$ is a non-trivial ideal in X, then the set

$$\mathcal{F}(\mathcal{I}) = \big\{M \subset X : (\exists A \in \mathcal{I})(M = X\backslash A)\big\}$$

is a filter in X, called the filter associated with $\mathcal{I}$.

Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. A sequence $x = (x_n)$ is said to be $\mathcal{I}$-convergent to L if for every $\varepsilon > 0$

$$A(\varepsilon) = \big\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\big\} \in \mathcal{I}.$$

An admissible ideal $\mathcal{I} \subset 2^X$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_i \Delta B_i$ is a finite set for $j \in X$ and $B = \bigcup_{i=1}^{\infty} B_i \in \mathcal{I}$.

Lemma 1.1. *[12] Let $\{P_i\}_{i\in\mathbb{N}}$ be a countable collection of subsets of $\mathbb{N}$ such that $\{P_i\} \in \mathcal{F}(\mathcal{I})$ for each i, where $\mathcal{F}(\mathcal{I})$ is a filter associate with an admissible ideal $\mathcal{I}$ with property (AP). Then there exists a set $P \subset \mathbb{N}$ such that $\{P_i\} \in \mathcal{F}(\mathcal{I})$ and the set $P \setminus P_i$ is finite for all i.*

2. $\mathcal{I}$-Convergent Functions and $\mathcal{I}$-Cauchy Sequence

In this section, we let $\mathcal{I} \subseteq 2^G$ be an admissible ideal for amenable semigroup G.

Definition 2.1. Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. A function $f \in w(G)$ is said to be $\mathcal{I}$-convergent to s for any Folner sequence $\{S_n\}$ for G, if for every $\varepsilon > 0$

$$\big\{g \in G : |f(g) - s| \geq \varepsilon\big\} \in \mathcal{I}.$$

In this case, we write $\mathcal{I} - \lim f(g) = s$.

Example 2.2. .

(a) If we let $\mathcal{I} = \mathcal{I}_f$ be an ideal of all finite subsets of G, then we get usual convergence with respect to Folner sequence.

(b) Let $\mathcal{I}_d = \big\{H \subset G : \delta(H) = 0\big\}$. Then, $\mathcal{I}_d$ is an admissible ideal and $\mathcal{I}_d$-convergence coincides with statistical convergence with respect to the Folner sequence.

Theorem 2.3. *The $\mathcal{I}$-convergence of $f \in w(G)$ depends on the particular choice of the Folner sequence.*

By assuming $\mathcal{I} = \mathcal{I}_d$, let us show this by an example.

Example 2.4. Let $G = \mathbb{Z}^2$ and take two Folner sequences as follows:
$\{S_n^1\} = \big\{(i,j) \in \mathbb{Z}^2 : |i| \leq n, |j| \leq n\big\}$ and $\{S_n^2\} = \big\{(i,j) \in \mathbb{Z}^2 : |i| \leq n, |j| \leq n^2\big\}$
and define $f(g) \in w(G)$ by

$$f = \begin{cases} 1 & , \text{ if } (i,j) \in A, \\ 0 & , \text{ if } (i,j) \notin A. \end{cases}$$

where

$$A = \big\{(i,j) \in \mathbb{Z}^2 : i \leq j \leq n,\ i = 0,1,2,...,n;\ n = 1,2,...\big\}.$$

Since for the Folner sequence $\{S_n^2\}$

$$\lim_{n\to\infty} \frac{1}{|S_n^2|}\big|\big\{g \in S_n^2 : |f(g) - 0| \geq \varepsilon\big\}\big| = \lim_{n\to\infty} \frac{\frac{(n+1)(n+2)}{2}}{(2n+1)(2n^2+1)} = 0,$$

then $f(g)$ is $\mathcal{I}_d$-convergent to 0. But, since for the Folner sequence $\{S_n^1\}$

$$\lim_{n\to\infty} \frac{1}{|S_n^1|}\big|\big\{g \in S_n^1 : |f(g) - 0| \geq \varepsilon\big\}\big| = \lim_{n\to\infty} \frac{\frac{(n+1)(n+2)}{2}}{(2n+1)^2} \neq 0,$$

then $f(g)$ is not $\mathcal{I}_d$-convergent to 0.

Theorem 2.5. *Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. A function $f \in w(G)$ is $\mathcal{I}$-convergent if and only if for every $\varepsilon > 0$ there exists $g_\varepsilon \in G$ such that*

$$\big\{g \in G : |f(g) - f(g_\varepsilon)| < \varepsilon\big\} \in \mathcal{F}(\mathcal{I}). \tag{2.1}$$

Proof. Let $f \in w(G)$ is $\mathcal{I}$-convergent to s. Then,

$$A_\varepsilon = \Big\{g \in G : |f(g) - s| < \frac{\varepsilon}{2}\Big\} \in \mathcal{F}(\mathcal{I})$$

for all $\varepsilon > 0$. Fix a $g_\varepsilon \in A_\epsilon$. Then, for all $g \in A_\epsilon$ we have

$$|f(g_\varepsilon) - f(g)| \leq |f(g_\varepsilon) - s| + |s - f(g)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, (2.1) holds.

Now, we assume that (2.1) holds for all $\varepsilon > 0$. Then, the set

$$B_\varepsilon = \big\{g \in G : f(g) \in [f(g_\varepsilon) - \varepsilon, f(g_\varepsilon) + \varepsilon]\big\} \in \mathcal{F}(\mathcal{I})$$

for all $\varepsilon > 0$. Denote $J_\varepsilon = [f(g_\varepsilon) - \varepsilon, f(g_\varepsilon) + \varepsilon]$. Fix an $\varepsilon > 0$. Then, $B_\varepsilon \in \mathcal{F}(\mathcal{I})$ and $B_{\varepsilon/2} \in \mathcal{F}(\mathcal{I})$. Hence, $B_\varepsilon \cap B_{\varepsilon/2} \in \mathcal{F}(\mathcal{I})$. This implies that

$$J = J_\varepsilon \cap J_{\varepsilon/2} \neq \emptyset,\ \big\{g \in G : f(g) \in J\big\} \in \mathcal{F}(\mathcal{I}) \text{ and } diam(J) \leq \frac{1}{2}diam(J_\varepsilon).$$

This way, by induction, we can construct the sequence of closed intervals

$$J_\varepsilon = J_0 \supseteq J_1 \supseteq J_2 \supseteq \ldots \supseteq J_n \supseteq \ldots$$

with property

$$diam(J_n) \leq \frac{1}{2} diam(J_{n-1}) \text{ and } \{g \in G : f(g) \in J_n\} \in \mathcal{F}(\mathcal{I}), \ (n = 1, 2, 3, ...).$$

Then, there exist a $s \in \bigcap_{n\in\mathbb{N}} J_n$ and it is a routine verification work to verify that f is $\mathcal{I}$-convergent to s. □

Let M be a subset of G such that $|M| = \infty$.

Definition 2.6. Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. A function $f \in w(G)$ is said to be $\mathcal{I}^*$-convergent to s, for any Folner sequence $\{S_n\}$ for G if there exists $M \subset G$, $M \in \mathcal{F}(\mathcal{I})$ (i.e., $G \setminus M \in \mathcal{I}$) and a $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that for every $\varepsilon > 0$

$$|f(g) - s| < \varepsilon,$$

for all $n > k_0$ and all $g \in M \setminus S_n$. In this case, we write $\mathcal{I}^* - \lim f(g) = s$.

Theorem 2.7. *If $f \in w(G)$ is $\mathcal{I}^*$-convergent to s, then f is $\mathcal{I}$-convergent to s.*

Proof. Suppose that $f \in w(G)$ is $\mathcal{I}^*$-convergent to s. Then, there exists $M \subset G$, $M \in \mathcal{F}(\mathcal{I})$ (i.e., $H = G \setminus M \in \mathcal{I}$) and a $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that for every $\varepsilon > 0$

$$|f(g) - s| < \varepsilon,$$

for all $n > k_0$ and all $g \in M \setminus S_n$. Therefore obviously,

$$A(\varepsilon) = \{g \in G : |f(g) - s| \geq \varepsilon\} \subset H \cup S_{k_0}.$$

Since $\mathcal{I}$ is admissible, $H \cup S_{k_0} \in \mathcal{I}$, so $A(\varepsilon) \in \mathcal{I}$. Hence, $f \in w(G)$ is $\mathcal{I}$-convergent to s. □

Theorem 2.8. *Let $\mathcal{I} \subset 2^G$ be an admissible ideal with the property (AP). If $f(g) \in w(G)$ is $\mathcal{I}$-convergent to s, then f is $\mathcal{I}^*$-convergent to s.*

Proof. Suppose that $\mathcal{I}$ satisfies condition (AP) and $f(g) \in w(G)$ is $\mathcal{I}$-convergent to s. Then, $A(\varepsilon) = \{g \in G : |f(g) - s| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$. Put

$$A_1 = \{g \in G : |f(g) - s| \geq 1\} \text{ and } A_n = \left\{g \in G : \frac{1}{n} \leq |f(g) - s| < \frac{1}{n+1}\right\}$$

for $n \geq 2, n \in \mathbb{N}$. Obviously $A_i \cap A_j = \emptyset$ for $i \neq j$. By condition (AP), there exists a sequence of sets $(B_n)_{n\in\mathbb{N}}$ such that $A_j \bigtriangleup B_j$ are infinite sets for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$. It is sufficient to prove that there exist $M \subset G$, $M \in \mathcal{F}(\mathcal{I})$ (i.e., $M = G \setminus B$) and a $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for every $\varepsilon > 0$

$$|f(g) - s| < \varepsilon,$$

for all $n > n_0$ and all $g \in M \setminus S_n$.

Let $\xi > 0$. Choose $k \in \mathbb{N}$ such that $\frac{1}{k+1} < \xi$. Then,

$$\{g \in G : |f(g) - s| \geq \xi\} \subset \bigcup_{j=1}^{k+1} A_j.$$

Since $A_j \triangle B_j, j = 1, 2, ..., k+1$ are finite sets, there exists n_0 such that

$$\bigcup_{j=1}^{k+1} B_j \cap (M \setminus S_{n_0}) = \bigcup_{j=1}^{k+1} A_j \cap (M \setminus S_{n_0}). \tag{2.2}$$

If $g \in M \setminus S_{n_0}$ and $g \notin \bigcup_{j=1}^{k+1} B_j$, then $g \notin \bigcup_{j=1}^{k+1} A_j$ by (2.2). But then

$$|f(g) - s| < \frac{1}{n+1} < \xi.$$

Hence, f is $\mathcal{I}^*$-convergent to s. □

Definition 2.9. Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. A function $f \in w(G)$ is said to be $\mathcal{I}$-Cauchy sequence, for any Folner sequence $\{S_n\}$ for G if for every $\varepsilon > 0$, there exists an $h = h(\varepsilon) \in G$ such that

$$\big\{g \in G : |f(g) - f(h)| \geq \varepsilon\big\} \in \mathcal{I}.$$

Theorem 2.10. *If $f \in w(G)$ is $\mathcal{I}$-convergent for Folner sequence $\{S_n\}$ for G, then it is $\mathcal{I}$-Cauchy for same sequence.*

Proof. Let $f \in w(G)$ is $\mathcal{I}$-convergent to s for Folner sequence $\{S_n\}$ for G. Then for every $\varepsilon > 0$, we have

$$A_\varepsilon = \big\{g \in G : |f(g) - s| \geq \varepsilon\big\} \in \mathcal{I}.$$

Since $\mathcal{I}$ is an admissible ideal there exists an $h \in G$ such that $h \notin A_\varepsilon$. Let

$$B_\varepsilon = \big\{g \in G : |f(g) - f(h)| \geq 2\varepsilon\big\}.$$

Taking into account the inequality

$$|f(g) - f(h)| \leq |f(g) - s| + |f(h) - s|,$$

we observe that if $g \in B_\varepsilon$, then

$$|f(g) - s| + |f(h) - s| \geq 2\varepsilon.$$

On the other hand, since $h \notin A_\varepsilon$ we have $|f(h) - s| < \varepsilon$ and so $|f(g) - s| > \varepsilon$. Hence $g \in A_\varepsilon$ and so we have

$$B_\varepsilon \subset A_\varepsilon \in \mathcal{I}.$$

Thus $B_\varepsilon \in \mathcal{I}$, i.e., f is $\mathcal{I}$-Cauchy sequence. □

Definition 2.11. Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. A function $f \in w(G)$ is said to be $\mathcal{I}^*$-Cauchy sequence, for any Folner sequence $\{S_n\}$ for G if there exists $M \subset G$, $M \in \mathcal{F}(\mathcal{I})$ (i.e., $G \setminus M \in \mathcal{I}$) and a $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that for every $\varepsilon > 0$

$$|f(g) - f(h)| < \varepsilon,$$

for all $n > k_0$ and $g, h \in M \setminus S_n$.

Theorem 2.12. *If $f \in w(G)$ is $\mathcal{I}^*$-Cauchy for Folner sequence $\{S_n\}$ for G, then it is $\mathcal{I}$-Cauchy for same sequence.*

Proof. Let $f \in w(G)$ be an $\mathcal{I}^*$-Cauchy for Folner sequence $\{S_n\}$ for G. Then by definition, there exists $M \subset G$, $M \in \mathcal{F}(\mathcal{I})$ (i.e., $G \setminus M \in \mathcal{I}$) and a $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that for every $\varepsilon > 0$

$$|f(g) - f(h)| < \varepsilon,$$

for all $n > k_0$ and $g, h \in M \setminus S_n$. Let $H = G \setminus M$. It is clearly $H \in \mathcal{I}$ and

$$A_\varepsilon = \{g \in G : |f(g) - f(h)| \geq \varepsilon\} \subset H \cup S_{k_0}. \tag{2.3}$$

Since $\mathcal{I}$ is admissibe ideal, the set on the right side of (2.3) belongs to $\mathcal{I}$ and so $A_\varepsilon \in \mathcal{I}$. Consequently, f is $\mathcal{I}$-Cauchy for same sequence. □

Following Lemma can be proved similar to the Lemma 1.1.

Lemma 2.13. *Let $\{P_i\}_{i\in\mathbb{N}}$ be a countable collection of subsets of G such that $\{P_i\} \in \mathcal{F}(\mathcal{I})$ for each i, where $\mathcal{F}(\mathcal{I})$ is a filter associate with an admissible ideal $\mathcal{I}$ with property (AP). Then there exists a set $P \subset G$ such that $\{P_i\} \in \mathcal{F}(\mathcal{I})$ and the set $P \setminus P_i$ is finite for all i.*

Theorem 2.14. *If $\mathcal{I} \subseteq 2^G$ is an admissible ideal with property (AP) then, the concepts $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy coincide for any Folner sequence $\{S_n\}$ for G.*

Proof. If $f \in w(G)$ be an $\mathcal{I}^*$-Cauchy for Folner sequence $\{S_n\}$ for G, then it is $\mathcal{I}$-Cauchy by Theorem (2.12), where $\mathcal{I}$ need not have the (AP) property. Now it is sufficient to prove that f is $\mathcal{I}^*$-Cauchy under assumption that f is an $\mathcal{I}$-Cauchy sequence.

Let f be an $\mathcal{I}$-Cauchy sequence. Then, by definition, for every $\varepsilon > 0$ there exists an $h = h(\varepsilon) \in G$ such that

$$\{g \in G : |f(g) - f(h)| \geq \varepsilon\} \in \mathcal{I}.$$

Let

$$P_i = \left\{g \in G : |f(g) - f(m_i)| < \frac{1}{i}\right\}, \quad i = 1, 2, ...$$

where $m_i = h(\frac{1}{i})$. It is clear that $P_i \in \mathcal{F}(\mathcal{I})$ for $i = 1, 2, \dots$. Since $\mathcal{I}$ has the property (AP), then by Lemma 2.13 there exists a set $P \subset G$ such that $P \in \mathcal{F}(\mathcal{I})$ and $P \setminus P_i$ is finite for all i. Now we show that for every $\varepsilon > 0$ there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that for $g, m \in P \setminus S_n$

$$|f(g) - f(m)| < \varepsilon,$$

for all $n > k_0$. To prove this, let $\varepsilon > 0$ and $j \in \mathbb{N}$ such that $j > \frac{2}{\varepsilon}$. If $g, m \in P \setminus S_n$, then $P \setminus P_i$ is a finite set, so there exists $k_0 = k_0(j)$ such that $g, m \in P_j \setminus S_n$ for all $n > k_0$. Therefore, for $g, m, m_j \in P_j \setminus S_n$ we have

$$|f(g) - f(m_j)| < \frac{1}{j} \text{ and } |f(m) - f(m_j)| < \frac{1}{j},$$

for all $n > k_0$. Hence, for $g, m, m_j \in P_j \setminus S_n$ it follows that

$$|f(g) - f(m)| < |f(g) - f(m_j)| + |f(m) - f(m_j)| < \varepsilon,$$

for all $n > k_0$. Thus, for any $\varepsilon > 0$ there exists $k_0 = k_0(\varepsilon)$ such that for $g, m \in P \setminus S_n \in \mathcal{F}(\mathcal{I})$

$$|f(g) - f(m)| < \varepsilon,$$

for all $n > k_0$. Hence, f is $\mathcal{I}^*$-Cauchy. □

3. $\mathcal{I}$-Limit Points and $\mathcal{I}$-Cluster Points

In this section, we introduce the notions of $\mathcal{I}$-limit points and $\mathcal{I}$-cluster points of the functions defined on discrete countable amenable semigroups.

Definition 3.1. The $s \in \mathbb{R}$ is is said to $\mathcal{I}$-limit point of the $f \in w(G)$, for any Folner sequence $\{S_n\}$ for G, provided that there is a set $M \subset G$, $M \notin \mathcal{I}$ such that $\lim f(g) = s$ $(g \in M \setminus S_n)$ for any Folner sequence (S_n) for G.

Definition 3.2. The number $c \in \mathbb{R}$ is said to be an $\mathcal{I}$-cluster point of the $f \in w(G)$ provided that for each $\varepsilon > 0$ we have $\{g \in G : |f(g) - c| < \varepsilon\} \notin \mathcal{I}$ for any Folner sequence $\{S_n\}$ for G.

For an $f \in w(G)$, let $\mathcal{I}^f_\Lambda(G)$ and $\mathcal{I}^f_\Gamma(G)$ denote the set of all $\mathcal{I}$-limit points and $\mathcal{I}$-cluster points of f, respectively.

Theorem 3.3. *Let $\mathcal{I}$ be an admissible ideal. Then, for each $f \in w(G)$ we have $\mathcal{I}^f_\Lambda(G) \subseteq \mathcal{I}^f_\Gamma(G)$.*

Proof. Let $s \in \mathcal{I}^f_\Lambda(G)$. Then, there exists a set $M \notin \mathcal{I}$ such that $\lim f(g) = s$ $(g \in M \setminus S_n)$. Hence, for every $\delta > 0$ there exists a $k_0 = k_0(\delta) \in \mathbb{N}$ such that for $g \in M \setminus S_n$ we have $|f(g) - s| < \delta$, for all $n > k_0$. Hence,

$$\big\{g \in G : |f(g) - s| < \delta\big\} \supset M \setminus S_n$$

and so

$$\big\{g \in G : |f(g) - s| < \delta\big\} \notin \mathcal{I},$$

which means that $s \in \mathcal{I}^f_\Gamma(G)$. □

Theorem 3.4. *Let $\mathcal{I}$ be an admissible ideal. The set $\mathcal{I}^f_\Lambda(G)$ is closed set in $\mathbb{R}$ for each function $f(g) \in w(G)$.*

Proof. Let $r \in \overline{\mathcal{I}^f_\Lambda(G)}$ and $\varepsilon > 0$. There exists $r_0 \in \mathcal{I}^f_\Lambda(G) \cap B(r, \varepsilon)$. Choose $\delta \geq 0$ such that $B(r_0, \delta) \subset B(r, \varepsilon)$. We obviously have

$$\big\{g \in G : |r - f(g)| < \varepsilon\big\} \supset \big\{g \in G : |r_0 - f(g)| < \delta\big\}.$$

Hence, $\big\{g \in G : |r - f(g)| < \varepsilon\big\} \notin \mathcal{I}$ and $r \in \mathcal{I}^f_\Lambda(G)$. □

4. $\mathcal{I}$-Limit Superior and $\mathcal{I}$-Limit Inferior

For $f \in w(G)$, we define the following sets:

$$B_f = \big\{b \in \mathbb{R} : \{g \in G : f(g) > b\} \notin \mathcal{I}\big\},$$

similarly

$$A_f = \big\{a \in \mathbb{R} : \{g \in G : f(g) < a\} \notin \mathcal{I}\big\}.$$

Definition 4.1. For an $f \in w(G)$, the $\mathcal{I}$-limit superior for any Folner sequence (S_n) for G is given by

$$\mathcal{I} - \limsup f = \begin{cases} \sup B_f, & B_f \neq \emptyset \\ -\infty, & B_f = \emptyset. \end{cases}$$

Also $\mathcal{I}$-limit inferior for any Folner sequence (S_n) for G is given by

$$\mathcal{I}-\liminf f = \begin{cases} \inf A_f, & A_f \neq \emptyset \\ \infty, & A_f = \emptyset. \end{cases}$$

It is easy to see that, for an $f \in w(G)$ and for any Folner sequence $\{S_n\}$ for G, $\mathcal{I}-\liminf f \leq \mathcal{I}-\limsup f$.

Definition 4.2. The function $f \in w(G)$ is said to be $\mathcal{I}$-bounded for any Folner sequence $\{S_n\}$ for G if there is a number K such that $\big\{g \in G : |f(g)| > K\big\} \in \mathcal{I}$.

Note that $\mathcal{I}$-boundedness implies that $\mathcal{I}-\limsup f$ and $\mathcal{I}-l\liminf f$ finite. The following theorem can be proved by a straightforward least upper bound argument.

Theorem 4.3. *For any Folner sequence (S_n) for G if $\mu = \mathcal{I}-\limsup f$ is finite, then for every $\varepsilon > 0$*

$$\big\{g \in G : f(g) > \mu - \varepsilon\big\} \notin \mathcal{I} \ \ \textit{and} \ \ \big\{g \in G : f(g) > \mu + \varepsilon\big\} \in \mathcal{I} \tag{4.1}$$

Conversely, if (4.1) *holds for every $\varepsilon > 0$, then $\mu = \mathcal{I}-\limsup f$.*

The dual statement for $\mathcal{I}-\liminf f$ is as follows.

Theorem 4.4. *For any Folner sequence $\{S_n\}$ for G, if $\lambda = \mathcal{I}-\liminf f$ is finite, then for every $\varepsilon > 0$*

$$\big\{g \in G : f(g) < \lambda + \varepsilon\big\} \notin \mathcal{I} \ \ \textit{and} \ \ \big\{g \in G : f(g) < \lambda - \varepsilon\big\} \in \mathcal{I} \tag{4.2}$$

Conversely, if (4.2) *holds for every $\varepsilon > 0$, then $\lambda = \mathcal{I}-\liminf f$.*

Theorem 4.5. *For any Folner sequence $\{S_n\}$ for G the $\mathcal{I}$-bounded function is $\mathcal{I}$-convergent if and only if $\mathcal{I}-\limsup f = \mathcal{I}-\liminf f$.*

Proof. For any Folner sequence $\{S_n\}$ for G, let $\lambda = \mathcal{I}-\liminf f$ and $\mu = \mathcal{I}-\limsup f$. First assume that $\mathcal{I}-\lim f(g) = s$ and $\varepsilon > 0$, then

$$\big\{g \in G : |f(g) - s| \geq \varepsilon\big\} \in \mathcal{I}$$

and so

$$\big\{g \in G : f(g) > s + \varepsilon\big\} \in \mathcal{I},$$

which implies that $\mu \leq s$. Also, we have

$$\big\{g : f(g) < s - \varepsilon\big\} \in \mathcal{I},$$

which implies that $s \leq \lambda$. Therefore $\mu \leq \lambda$, which we combine with the fact that $\mathcal{I}-\liminf f \leq \mathcal{I}-\limsup f$, to conclude that $\mu = \lambda$.

Now, we assume that for any Folner sequence (S_n) for G, $\mathcal{I}-\liminf f = \mathcal{I}-\limsup f$. If $\varepsilon > 0$, then (4.1) and (4.2) imply

$$\Big\{g : f(g) > s + \frac{\varepsilon}{2}\Big\} \in \mathcal{I} \ \text{ and } \ \Big\{g : f(g) < s - \frac{\varepsilon}{2}\Big\} \in \mathcal{I}.$$

Hence, for any Folner sequence (S_n) for G, we have $\mathcal{I}-\lim f = s$. $\square$

References

[1] M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544.

[2] K. Demirci, *$\mathcal{I}$-limit superior and limit inferior*, Mathematical Communications, **6** (2001), 165–172.

[3] S. A. Douglass, *On a concept of Summability in Amenable Semigroups*, Math. Scand. **28** (1968), 96–102.

[4] S. A. Douglass, *Summing Sequences for Amenable Semigroups*, Michigan Math. J. **20** (1973), 169–179.

[5] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

[6] J. A. Fridy, *On statistical convergence*, Analysis, **5** (1985), 301–313.

[7] J. A. Fridy, C. Orhan *Statistical limit superior and limit inferior*, Proc. Amer. Math. Soc. **125** (1997), 2625–3631.

[8] P. Kostyrko, T. Šalát, W. Wilczyński, *$\mathcal{I}$-Convergence*, Real Anal. Exchange, **26**(2) (2000), 669–686.

[9] P. Kostyrko, M. Mačaj, T. Šalát, M. Sleziak, *$\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points*, Mathematica Slovaca, **55** (2005), 443–464.

[10] P. F. Mah, *Summability in amenable semigroups*, Trans. Amer. Math. Soc. **156** (1971), 391–403.

[11] P. F. Mah, *Matrix summability in amenable semigroups*, Proc. Amer. Math. Soc. **36** (1972), 414–420.

[12] A. Nabiev, S. Pehlivan, M. Gürdal, *On $\mathcal{I}$-Cauchy Sequences*, Taiwanese J. Math. **11**(2) (2007), 569–576.

[13] I. Namioka, *Følner's conditions for amenable semigroups*, Math. Scand. **15** (1964), 18–28.

[14] F. Nuray, B. E. Rhoades, *Some kinds of convergence defined by Folner sequences*, Analysis, **31**(4) (2011), 381–390.

[15] F. Nuray, B. E. Rhoades, *Almost statistical convergence in amenable semigroups*, Math. Scand. **111** (2012), 127–134.

*DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND LITERATURE, AFYON KOCATEPE UNIVERSITY, 03200, AFYONKARAHISAR, TURKEY

**SIVAS CUMHURIYET UNIVERSITY, SIVAS, TURKEY

E-mail address: `edundar@aku.edu.tr`

E-mail address: `fnuray@aku.edu.tr`

E-mail address: `ugurulusu@cumhuriyet.edu.tr`