Existence solutions of integral equation and initial value questions for differential equation of fractional order *

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Abstract: In this paper, using the method of upper and lower solutions and fixed point Theorem, consider the existence result of the initial value problem for fractional differential equation of Riemann-Liouville fractional derivative, obtained the existence results for maximal and minimal solutions.

Keywords: Fractional derivatives equations; Monotone iterative technique; Existence and uniqueness; Maximal and minimal solutions

1. Introduction

In recent years, the tools of fractional calculus have been available and applicable to various fields of study, much attention has been paid to study fractional differential equation initial value questions, such as hydromechanics, network traffic, soliton and chaos etc. Especially, the differential equations involving Riemann-Liouville fractional differential equation of order $0 < \alpha < 1$ obtained many well-know result, see [2-6].

Many authors obtained their results under the assumption that $f(t, y)$ satisfy Lipschitz condition about y. Monotone iterative method is a powerful tool for differential equation, such as [2-3], and [6].

In this paper, we will use fixed point Theorems to study the existence and uniqueness of solution of following initial value problem about Riemann-Liouville fractional differential operator of order $0 < \alpha < 1$. Using the technique of upper and lower solutions and monotone iterative method establish an existence result

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), I^{1-\beta}u(t)),\ 0 < t \leq T, \\ t^{1-\alpha}u(t) \mid_{t=0} = u_0, \end{cases} \tag{1.1}$$

there $f \in C([0, T] \times R \times R, R)$, $0 < \alpha, \beta < 1$ is real number, $0 < T < \infty$, D^{α} is criterion Riemann-Liouville fractional differential operator.

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2. Preliminaries

In this section, we recall the definition and some concepts on fractional integrals and derivatives, and give some lemma which are useful in next section.

The Riemann-Liouville fractional integral of order α is defined as

$$I^{\alpha}y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds,$$

Riemann-Liouville derivative of order α is defined by

$$D^{\alpha}y(t)=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_0^t\frac{y(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n=[\alpha]+1$.

When $0<\alpha<1$, the definition of Riemann-Liouville derivative turn into

$$D^{\alpha}y(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{y(s)}{(t-s)^{\alpha}}ds.$$

Obviously, $D^{\alpha}y(t)=\frac{d}{dt}I^{1-\alpha}y(t)$.

Let $0<\alpha<1$, the space $L^{\alpha}(a,b)=\{u\in L^1(a,b):D^{\alpha}u\in L^1(a,b)\}$. Here $L^1(a,b)$ is the space of summable functions in a finite interval $[a,b]$.

Theorem 2.1 ([1]) Let $0<\alpha<1$. Assume $f(x)\in C(R^+)\cap L^1_{loc}(R^+)$. Then for all $(a,b)\in R^+\times R$, $f(x)\in C(R^+)\cap L^1_{loc}(R^+)$, Cauchy question

$$\begin{cases} D^{\alpha}y(x)=f(x),\\ y(a)=b,\end{cases}$$

has a unique solution in $C(R^+)\cap L^1_{loc}(R^+)$ given by

$$y(x)=(b-\frac{1}{\Gamma(\alpha)}\int_0^a(a-t)^{\alpha-1}f(t)dt)\frac{x^{\alpha-1}}{a^{\alpha-1}}+\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)dt. \tag{2.1}$$

Theorem 2.2 ([1]) Assume $0<\alpha<1$. Let $c\in R$ and let $g(x)\in L^1(a,b)$. If $a(x)\in L^{\infty}(a,b)$ or $a(x)$ is bounded on $[a,b]$, they the Cauchy type question

$$\begin{cases} (D^{\alpha}y)(x)=a(x)y(x)+g(x),\\ \lim\limits_{x\to0^+}x^{1-\alpha}y(x)=c,\end{cases} \tag{2.2}$$

has a unique solution $y(x)$ in the space $L^{\alpha}(a,b)$.

Lemma 2.1 ([1]) Let $0<\alpha<1$ and let $y(x)$ be a Lebesgue measurable function on $[0,T]$,

(1)if

$$\lim_{x\to0^+}x^{1-\alpha}y(x)=a,\ \ a\in R$$

then

$$I^{1-\alpha}y(0+) = \lim_{x\to 0^+} I^{1-\alpha}y(x) = a\Gamma(\alpha).$$

(2)if

$$\lim_{x\to 0+} I^{1-\alpha}y(x) = b,\ \ b \in R$$

hold, and if there exists the limit $\lim\limits_{x\to 0^+} x^{1-\alpha}y(x)$, then

$$\lim_{x\to 0+} x^{1-\alpha}y(x) = \frac{b}{\Gamma(\alpha)}.$$

Lemma 2.2 ([1]) If $f(x) \in L_1(a,b)$ and $0 < \alpha < 1$, then

$$I^\alpha D^\alpha f(x) = f(x) - \frac{f_{1-\alpha}(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1}$$

holds almost everywhere on (a,b), where $f_{1-\alpha}(x) = I^{1-\alpha}(x)$.

Lemma 2.3 ([1]) If $\alpha > 0$ and $\beta > 0$, then the equation

$$I^\alpha I^\beta f(x) = I^{\alpha+\beta}f(x)$$

is satisfied at almost every point $x \in [a,b]$ for $f(x) \in L^p[a,b](p \geq 1)$. If $\alpha + \beta > 1$, then the relation hold at any point of $[a,b]$.

3. Main results

Let $0 < \alpha < 1$, $C^{1-\alpha}(J,R) = \{u \in C((0,T],R) : t^{1-\alpha}u \in C([0,T],R)\}$. It is obviously that $C^{1-\alpha}(J,R)$ is a Banach space. For $u \in C^{1-\alpha}(J,R)$, we define the norm

$$\|u\|_{C^{1-\alpha}(J,R)} = \max_{t\in[0,T]} t^{1-\alpha}|u(t)|.$$

Theorem 3.1. Let $0 < \alpha \leq \frac{1}{2}$, $0 < \beta < 1$ satisfy $2\alpha - \beta > 0$. Assume $f \in C(J \times R \times R, R)$ such that
(H1): exist nonnegative constants K, L satisfy

$$\gamma := \frac{T^\alpha\Gamma(\alpha)}{\Gamma(1+2\alpha-\beta)}[K + LT^{1-\beta}] < 1,$$

and

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq K|x_1 - x_2| + L|y_1 - y_2|.$$

(H2): $f(t,0,0) \neq 0$ on $t \in (0,T]$, and

$$0 < r_0 := \sup_{t\in[0,T]} (|u_0| + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,0,0)|ds) < \infty.$$

holds, then initial value question (1.1) has a unique solution.

proof. Let $0 < \alpha \le \frac{1}{2}, 0 < \beta < 1$, λ is a constant, $\lambda > \frac{1}{1-\gamma}$. Define a bounded closed subset S of $C^{1-\alpha}(J,R)$ given by

$$S = \{u \in C^{1-\alpha}(J,R) : t^{1-\alpha}|u(t)| \le \lambda r_0\}.$$

Define an operator $\mathcal{A}$ on S as follows

$$\mathcal{A}u(t) = u_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s),I^{1-\beta}u(s))ds, \quad t \in J. \tag{3.1}$$

Consider question $u = \mathcal{A}u$, we have to show that operator $\mathcal{A}$ has a fixed point, In order to use Banach fixed point Theorem, need to prove $\mathcal{A}$ is a contraction mapping. For any $u \in S$, since (H1), (H2) holds, we have

$$\begin{aligned}
t^{1-\alpha}|\mathcal{A}u(t)| &\le |u_0| + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,u(s),I^{1-\beta}u(s))|ds \\
&\le r_0 + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[|f(s,u(s),I^{1-\beta}u(s)) - f(s,0,0)|]ds \\
&\le r_0 + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[K|u(s)| + \frac{L}{\Gamma(1-\beta)}\int_0^s (s-\tau)^{-\beta}|u(\tau)|d\tau]ds \\
&\le r_0 + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[Ks^{\alpha-1}\lambda r_0 + \frac{L\lambda r_0}{\Gamma(1-\beta)}\int_0^s (s-\tau)^{-\beta}\tau^{\alpha-1}d\tau]ds \\
&< r_0 + \frac{\lambda r_0 \Gamma(\alpha) T^{\alpha}}{\Gamma(1+2\alpha-\beta)}[K + LT^{1-\beta}] \\
&< (1+\lambda\gamma)r_0 < \lambda r_0, \ t \in J,
\end{aligned}$$

which yields that $\mathcal{A}S \subset S$.

Nextly, we show that operator $\mathcal{A}$ is contraction operator.

For any $u, v \in S$, using condition (H2), we get

$$\begin{aligned}
\|\mathcal{A}u - \mathcal{A}v\|_{C^{1-\alpha}(J,R)} &\le \frac{1}{\Gamma(\alpha)}\max_{t\in J} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}|f(s,u(s),I^{1-\beta}u(s)) - f(s,v(s),I^{1-\beta}v(s))|ds \\
&\le \frac{1}{\Gamma(\alpha)}\max_{t\in J} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}[K|u(s)-v(s)| \\
&\quad + \frac{L}{\Gamma(1-\beta)}\int_0^s (s-\tau)^{-\beta}|u(\tau)-v(\tau)|d\tau]ds \\
&\le \frac{1}{\Gamma(\alpha)}\|u-v\|_{C^{1-\alpha}}\max_{t\in J} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}[Ks^{\alpha-1} \\
&\quad + \frac{L}{\Gamma(1-\beta)}\int_0^s (s-\tau)^{-\beta}\tau^{\alpha-1}d\tau]ds \\
&= \frac{1}{\Gamma(\alpha)}\|u-v\|_{C^{1-\alpha}}\max_{t\in J} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}[Ks^{\alpha-1} + \frac{L\Gamma(\alpha)}{\Gamma(1+\alpha-\beta)}s^{\alpha-\beta}]ds
\end{aligned}$$

$$\leq [\frac{KT^{\alpha}\Gamma(\alpha)}{\Gamma(2\alpha)} + \frac{LT^{1+\alpha-\beta}\Gamma(\alpha)}{\Gamma(1+2\alpha-\beta)}]\|u-v\|_{C^{1-\alpha}}.$$

Since $0 < \alpha \leq \frac{1}{2}$, $0 < \beta < 1$ and $2\alpha - \beta > 0$, we have $\Gamma(\alpha) > \Gamma(2\alpha) > \Gamma(1+2\alpha-\beta)$. Combining the above argument, it is clearly that

$$\|\mathcal{A}u - \mathcal{A}v\|_{C^{1-\alpha}} \leq \gamma\|u-v\|_{C^{1-\alpha}}. \tag{3.2}$$

Hence, $\mathcal{A}$ is a contraction operator. Apply the Banach fixed point Theorem, it is easy to see that $\mathcal{A}$ has a unique fixed point in S. Thus, we complete this proof. $\square$

Remark 1. When $\frac{1}{2} < \alpha < 1$, $0 < \beta < 1$, and the condition (H1), (H2) holds, by using the similar method as in the proof of Theorem 3.1, question (1,1) have a unique solution too.

For $u \in L^p$, we define the norm $\|u\|_p = (\int_0^T |u(t)|^p dt)^{\frac{1}{p}}$, $1 \leq p < \infty$. In (1.1), assume $f(x,y,z) = g(z)$, g is a function about $I^{1-\beta}u(t)$. Then we consider Cauchy question

$$\begin{cases} D^{\alpha}u(t) = g(I^{1-\beta}u(t)),\ 0 < t \leq T, \\ t^{1-\alpha}u(t)\mid_{t=0} = u_0, \end{cases} \tag{3.3}$$

Theorem 3.2. Let $0 < \alpha,\ \beta < 1$. Assume $g \in C(R) \cap L^p(R)$, there $p \in (1, \frac{1}{\alpha})$ is a constant. Then the Cauchy question (3.3) has a unique solution.

proof. Let $0 < \alpha < 1$, define a set B as follows

$$B = \{u \in t^{1-\alpha}C(J,R) : t^{1-\alpha}u(t) \to u_0(t \to 0^+)\},$$

It is obvious that B is a Banach space.

Define the operator $\mathcal{T}$ on B by

$$(\mathcal{T}u)(t) = u_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(I^{1-\beta}u(s))ds, \quad t \in J, \tag{3.4}$$

by the boundedness of fractional integration operator I^{α}, for each $u \in B$, as $t \to 0^+$, we have

$$t^{1-\alpha}(\mathcal{T}u)(t) = u_0 + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(I^{1-\beta}u(s))ds \to u_0,$$

which yields that $\mathcal{T}B \subset B$.

Then we consider question $u = \mathcal{T}u$. In order to use Schauder's fixed point Theorem, we must prove $\mathcal{T}$ is a continuous and compact operator.

For given $\varepsilon > 0$, any $u_n, u_0 \in B, n = 1, 2, \ldots$ with $u_n \to u_0 (n \to \infty)$. Let q is a constant such that $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$, we have

$$|(\mathcal{T}u_n)(t) - (\mathcal{T}u_0)(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g(I^{1-\beta}u_n) - g(I^{1-\beta}u_0)|ds$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(\alpha)}(\int_0^t |(t-s)^{\alpha-1}|^q ds)^{\frac{1}{q}}(\int_0^t |g(I^{1-\beta}u_n)-g(I^{1-\beta}u_0)|^p ds)^{\frac{1}{p}}\\
&\le \frac{1}{\Gamma(\alpha)}(\frac{1}{q(\alpha-1)+1})^{\frac{1}{q}}\frac{T^{1+\alpha-\beta}}{(1-\beta)\Gamma(1-\beta)}\|g\|_{L^p}\sup_{t\in J}|u_n(t)-u_0(t)|\\
&\to 0,
\end{aligned}
$$

as $n\to\infty$. Which implies that $\mathcal{T}$ is continuous for $t\in J$.

Secondly, we prove that operator $\mathcal{T}$ is equicontinuous on J.

Let $t_1, t_2 \in J$, $t_1 < t_2$. Consider the function $f(x) = (t_1-x)^{\alpha-1} - (t_2-x)^{\alpha-1}$, $x\in[0,t_1]$, for any $1\le p<\infty$, exist constant C_1 such that $\|f\|_{L^p}\le C_1\|f\|_{L^1}$. $C_2 = (\frac{1}{q(\alpha-1)+1})^{\frac{1}{q}}$, $M = (\int_0^T |g(I^{1-\beta}u(s))|^p ds)^{\frac{1}{p}}$ are constants. For given $\varepsilon>0$, we take

$$
\delta = \min\{T,\ [\frac{\varepsilon\Gamma(\alpha+1)}{M(C_1+C_2)+2^\alpha|u_0|}]^{\frac{1}{\alpha}}\}.
$$

Then, when $|t_1-t_2|<\delta$, for each $u\in B$, we have

$$
\begin{aligned}
|(\mathcal{T}u)(t_1)-(\mathcal{T}u)(t_2)| &\le \frac{1}{\Gamma(\alpha)}|\int_0^{t_1}(t_1-s)^{\alpha-1}g(I^{1-\beta}u(s))ds - \int_0^{t_2}(t_2-s)^{\alpha-1}g(I^{1-\beta}u(s))ds|\\
&\quad + |u_0||t_1^{\alpha-1}-t_2^{\alpha-1}|\\
&\le \frac{1}{\Gamma(\alpha)}|\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]g(I^{1-\beta}u(s))ds|\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|g(I^{1-\beta}u(s))|ds + |u_0||t_1^{\alpha-1}-t_2^{\alpha-1}|\\
&\le \frac{1}{\Gamma(\alpha)}(\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]^q ds)^{\frac{1}{q}}(\int_0^{t_1}|g(I^{1-\beta}u(s))|^p ds)^{\frac{1}{p}}\\
&\quad + \frac{1}{\Gamma(\alpha)}(\int_{t_1}^{t_2}[(t_2-s)^{\alpha-1}]^q ds)^{\frac{1}{q}}(\int_{t_1}^{t_2}|g(I^{1-\beta}u(s))|^p ds)^{\frac{1}{p}}\\
&\quad + 2^\alpha|u_0||t_1-t_2|^\alpha\\
&\le \frac{C_1}{\alpha\Gamma(\alpha)}[(t_2-t_1)^\alpha - (t_2^\alpha - t_1^\alpha)](\int_0^T |g(I^{1-\beta}u(s))|^p ds)^{\frac{1}{p}}\\
&\quad + \frac{C_2}{\Gamma(\alpha)}(t_2-t_1)^{\frac{q(\alpha-1)+1}{q}}(\int_0^T |g(I^{1-\beta}u(s))|^p ds)^{\frac{1}{p}} + 2^\alpha|u_0||t_1-t_2|^\alpha\\
&< \frac{M}{\Gamma(\alpha)}[\frac{C_1}{\alpha}+C_2]|t_2-t_1|^\alpha + 2^\alpha|u_0||t_1-t_2|^\alpha,
\end{aligned}
$$

because $q>1$, $\delta>0$, then $\delta^\alpha \ge \delta^{\frac{q(\alpha-1)+1}{q}}$. Consequently, we obtain that

$$
|(\mathcal{T}u)(t_1)-(\mathcal{T}u)(t_2)| < \varepsilon. \tag{3.5}
$$

We see that the operator $\mathcal{T}: B\to B$ is equicontinuous on J, by Arzela-Ascoli's Theorem, $\mathcal{T}$ is a compact operator. Schauder's fixed point theorem guarantees operator $\mathcal{T}$ has a fixed point. Hence Cauchy question (3.3) has a unique solution. This proof is completed. □

The following is an existence of solution of the Cauchy question for fractional differential equation use the monotone iterative method, we must introduce the upper and lower solution.

Definition 3.1. A function $u^* \in C^{1-\alpha}(J,R)$ is called a upper solution of Cauchy problem (1.1), if it satisfies

$$\begin{cases} D^\alpha u^*(t) \geq f(t, u^*(t), I^{1-\beta} u^*(t)), \ 0 < t \leq T, \\ t^{1-\alpha} u^*(t) \mid_{t=0} \geq u_0. \end{cases} \tag{3.6}$$

And a function $u_* \in C^{1-\alpha}(J,R)$ is called a lower solution of Cauchy problem (1.1), if it satisfies

$$\begin{cases} D^\alpha u_*(t) \leq f(t, u_*(t), I^{1-\beta} u_*(t)), \ 0 < t \leq T, \\ t^{1-\alpha} u_*(t) \mid_{t=0} \leq u_0. \end{cases} \tag{3.7}$$

(H3): A function $a(t) \in L^\alpha(J,R)$, and if $a(t) \leq 0$, there exist a nondecreasing function $\overline{a}(t) \in L^{1-\beta}(J,R)$ such that $-a(t) \leq \overline{a}(t)$ and

$$\frac{3(T+1-\alpha)}{T^{3-\beta}\Gamma^2(\alpha)} \max_{t\in(0,T]} \int_0^t (t-s)^{-\beta} \overline{a}(s) ds < 1. \tag{3.8}$$

Lemma 3.1. Assume that $u^*, w_* \in C^{1-\alpha}(J,R)$ are locally Hölder continuous and satisfy the non-strict inequalities (3.6) and (3.7), $f \in C(J \times R \times R, R)$. Suppose further that

$$f(t, x_1, y_1) - f(t, x_2, y_2) \leq a(t)(x_2 - x_1), \tag{3.9}$$

and $a(t) \in L^\alpha(J,R)$ satisfy condition (H3). Then $u^*(0) \geq w_*(0)$ implies that $u(t) \geq w(t)$.

Lemma 3.2. Let $0 < \alpha, \beta < 1$ and $a(t)$ satisfy (H3), if $w \in C^{1-\alpha}([0,T])$ satisfies the problem

$$\begin{cases} D^\alpha w(t) \geq -a(t) w(t), \ 0 < t \leq T, \\ t^{1-\alpha} w(t) \mid_{t=0} \geq 0. \end{cases} \tag{3.10}$$

Then $w(t) \geq 0$ for $t \in (0,T]$.

Proof. If the assertion is false, similar to the proof of Lemma 2.1 in [2], we assume $w(t) < 0, t \in (0,T]$, because $t^{1-\alpha} w(t) \mid_{t=0} \geq 0$ and $w \in C^{1-\alpha}([0,T])$), so exist interval $[t_0, t_1] \subseteq (0,T]$ such that $w(t) < 0, t \in (t_0, t_1]$ and $w(t_0) = 0$; assume that $t_* \in (t_0, t_1]$ is the first minimal point of $w(t)$ on $(t_0, t_1]$. Therefore, there will be

$$w(t_*) < 0. \tag{3.11}$$

Firstly, let $a(t) > 0, t \in (0,T]$. From (3.10), we have

$$D^\alpha w(t) \geq 0, \quad t \in (t_0, t_1]$$

hence

$$D^\alpha w(t_*) \geq 0,$$

Apply the definition of Riemann-Liouville fractional derivative, we can obtain that

$$\frac{1}{\Gamma(1-\alpha)}w(t_*)t_0^{-\alpha} > D^\alpha w(t_*) \geq 0.$$

Thus

$$w(t_*) > 0.$$

It contradicts relation (3.11), so we obtain the result when $a(t) > 0$.

Secondly, if $a(t) \leq 0$, nondecreasing function $\overline{a}(t) \in L^1(J,R)$ such that $-a(t) \leq \overline{a}(t)$, by the monotone of Riemann-Liouville fractional integral operator I^α and $I^{1-\beta}$, use the fractional integral operator I^α to the both sides of (3.10), we have

$$w(t) - [t^{1-\alpha}w(t)|_{t=0}]t^{\alpha-1} \geq -I^\alpha[a(t)w(t)],\ t \in (0,T],$$

because $t^{1-\alpha}w(t)\ |_{t=0} \geq 0$ and by the assertion of t_*, we can obtain that

$$w(t_*) \geq -I^\alpha[a(t_*)w(t_*)].$$

Thus

$$\begin{aligned}
w(t_*) + I^\alpha[a(t_*)w(t_*)] &= w(t_*) + \frac{1}{\Gamma(\alpha)}\int_0^{t_*}(t_*-s)^{\alpha-1}a(s)w(s)ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^{t_*}[\Gamma(\alpha)t_*w(t_*) + (t_*-s)^{\alpha-1}a(s)w(s)]ds \\
&\geq 0,
\end{aligned}$$

Note that $w \in C^{1-\alpha}([0,T])$, so at least exist a set $E_1 \subseteq [0,t_*],\ \mu(E_1) > \epsilon > 0$ such that

$$\Gamma(\alpha)t_*w(t_*) + (t_*-s)^{\alpha-1}a(s)w(s) \geq 0,\ s \in E_1,$$

there μ is Lebesgue measure. And so

$$\Gamma(\alpha)t_*w(t_*) + (t_*-t)^{\alpha-1}a(t)w(t_*) > 0,\ t \in E_1,$$

by assuming that $w(t_*) < 0$, we have

$$\Gamma(\alpha)t_* + (t_*-t)^{\alpha-1}a(t) < 0,$$

using the condition (H3)

$$\overline{a}(t) > \Gamma(\alpha)t_*(t_*-t)^{1-\alpha},$$

because $\overline{a}(s)$ is nondecreasing function, so $\overline{a}(t) \leq \overline{a}(t_*)$.

Taking into account that $t < t_*$ when $t \in E_1$, we have

$$\begin{aligned}
\overline{a}(t_*) &> \Gamma(\alpha)t_*(t_*-t)^{1-\alpha} \\
&> \Gamma(\alpha)t_*\frac{t_*-t}{t_*-t+1-\alpha}
\end{aligned}$$

$$\geq \Gamma(\alpha)t_*\frac{t_*-t}{\mu(E_1)+1-\alpha},\ t\in E_1.$$

Since $t<t_*, t\in E_1$, we can select a function $\sigma(t), 0\leq\sigma(t)\leq 1, t\in E_1$ such that $t=\sigma(t)t_*$, therefore, the above inequality can be written as

$$\overline{a}(t_*)>\Gamma(\alpha)\frac{(1-\sigma(t))t_*^2}{\mu(E_1)+1-\alpha},\ t\in E_1. \tag{3.12}$$

Apply fractional integral $I^{1-\beta}$ to the both sides of (3.12), we obtain

$$\begin{aligned}
I^{1-\beta}\overline{a}(t_*) &> I^{1-\beta}\Gamma(\alpha)\frac{(1-\sigma(t))t_*^2}{\mu(E_1)+1-\alpha}\\
&= \frac{(1-\sigma(t))\Gamma(\alpha)}{(\mu(E_1)+1-\alpha)\Gamma(1-\beta)}\int_0^{t_*}(t_*-s)^{-\beta}s^2ds\\
&= \frac{(1-\sigma(t))\Gamma(\alpha)\Gamma(3)}{(\mu(E_1)+1-\alpha)\Gamma(4-\beta)}t_*^{3-\beta}\\
&> \frac{(1-\sigma(t))\Gamma(\alpha)}{3(\mu(E_1)+1-\alpha)\Gamma(1-\beta)}t_*^{3-\beta}\\
&\geq \frac{(1-\sigma(t))\Gamma(\alpha)}{3(\mu(E_1)+1-\alpha)\Gamma(1-\beta)}\mu(E_1)^{3-\beta},
\end{aligned}$$

that is

$$\int_0^{t_*}(t_*-s)^{-\beta}\overline{a}(s)ds>\frac{(1-\sigma(t))\Gamma(\alpha)}{3(\mu(E_1)+1-\alpha)}\mu(E_1)^{3-\beta}.$$

by inequality (3.8), we can obtain that

$$\frac{(1-\sigma(t))\Gamma(\alpha)}{\mu(E_1)+1-\alpha}\mu(E_1)^{3-\beta}<\frac{\Gamma^2(\alpha)T^{3-\beta}}{T+1-\alpha},\ t\in E_1. \tag{3.13}$$

Consider the inequality about x

$$(T+1-\alpha)(1-\sigma(t))x^{3-\beta}-\Gamma(\alpha)T^{3-\beta}x-(1-\alpha)\Gamma(\alpha)T^{3-\beta}<0, \tag{3.14}$$

since $2<3-\beta<3$, it is obviously that inequality (3.14) has no positive solution, thus when $\mu(E_1)>\epsilon>0$, inequality (3.13) is false. It is a contradiction, we complete this proof. $\square$

Remark 2. In Lemma 3.3, $a(t)\in L^{\alpha}(J,R)$ and satisfy condition (H3), in case $a(t)<0, t\in J$, assume there exists a constant M such that $-a(t)\leq M, t\in J$. Then, the inequality (3.8) become $M<\frac{(1-\beta)T^2\Gamma^2(\alpha)}{3(T+1-\alpha)}$.

Theorem 3.3. Assume that $u^*, u_*\in C^{1-\alpha}(J,R)$ are upper and lower solutions of Cauchy question (1,1), and satisfy $u^*\geq u_*$, $t^{1-\alpha}u^*(t)\mid_{t=0}\geq t^{1-\alpha}u_*(t)\mid_{t=0}$. In addition, $f\in C(J\times R\times R,R)$ satisfy Lemma 3.1 and condition (H3).

Then Cauchy question (1,1) has a maximal solution and a minimal solution.

Proof. Let $\Omega = \{x \in C^{1-\alpha}(J,R) : u_* \le x \le u^*\}$. For any $x \in \Omega$, consider the boundary value problems

$$\begin{cases} D^\alpha u(t) = f(t,x(t),I^{1-\beta}x(t)) - a(t)[u(t)-x(t)],\ t \in (0,T], \\ t^{1-\alpha}u(t)\ |_{t=0} = u_0, \end{cases} \tag{3.15}$$

By Theorem 2.2 and condition (H3), for every x, boundary value problem (3,15) exists a unique solution.

Define operator $\mathcal{N}$ by

$$\mathcal{N}u(t) = u_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathcal{F}u(s)ds,$$

where $\mathcal{F}$ is defined by $\mathcal{F}u(s) = f(s,x(s),I^{1-\beta}x(s)) - a(s)[u(s)-x(s)],\ s \in J$, obviously operator $\mathcal{N}$ is continuous in view of continuity of $\mathcal{F}$. Then the solution initial value problem (3.15) is a fixed point of operator $\mathcal{N}$.

Define the operator $\mathcal{B} : \overline{\Omega} \to C^{1-\alpha}(J,R)$ by

$$\mathcal{B}x = u,\ x \in \overline{\Omega}.$$

$\mathcal{B}$ is a continuous operator. In fact, operator $\mathcal{B}$ is equicontinuous on J.

We will use operator $\mathcal{B}$ to construct the sequences $\{u_n\},\{v_n\}$. Firstly, we prove $\mathcal{B}$ is a monotone operator in $\overline{\Omega}$, let $u_1(t), u_2(t) \in C^{1-\alpha}(J,R), u_1(t) \le u_2(t)$. Suppose that $u_i = \mathcal{B}x_i, (i = 1,2)$, set $\mu = u_1 - u_2$. Then

$$\begin{aligned} D^\alpha \mu &= D^\alpha u_1 - D^\alpha u_2 \\ &= f(t,x_1,I^{1-\beta}x_1) - a(t)[u_1-x_1] - f(t,x_2,I^{1-\beta}x_2) + a(t)[u_2-x_2] \\ &\le a(t)[x_2-x_1] - a(t)[u_1-x_1] + a(t)[u_2-x_2] \\ &= -a(t)\mu, \end{aligned}$$

and $t^{1-\alpha}\mu(t)|_{t=0} = 0$.

This conclusion and Lemma 3.2 implies that $\mathcal{B}x_1 \le \mathcal{B}x_2, t \in J$, so $\mathcal{B}$ is monotone operator. Secondly, we prove $u_* \le \mathcal{B}u_*$ and $u^* \ge \mathcal{B}u^*$, set $\mathcal{B}u_* = u_1$, u_1 is s unique solution of question (3.15) with $x = u_*$, put $\nu = u_* - u_1$. Then

$$\begin{aligned} D^\alpha \nu &= D^\alpha u_* - D^\alpha u_1 \\ &\le f(t,u_*,I^{1-\beta}u_*) - [f(t,u_*,I^{1-\beta}u_*) - a(t)(u_1-u_*)] \\ &= -a(t)\nu, \end{aligned}$$

and $t^{1-\alpha}(u_*(t) - u_1(t))|_{t=0} \le 0$.

This shows that $\nu(t) \le 0, t \in J$, which implies that $u_* \le \mathcal{B}u_*$. One can show similarly, $u^* \ge \mathcal{B}u^*$. Hence operator $\mathcal{B} : \overline{\Omega} \to \overline{\Omega}$.

Let $u_0 = u_*, v_0 = u^*$. We define the sequences $\{u_n\}, \{v_n\}$ on J by

$$u_n = \mathcal{B}u_{n-1},\ v_n = \mathcal{B}v_{n-1},\quad n = 1, 2, \cdots.$$

Combining the above argument, we can obtain

$$u_* = u_0 \le u_1 \le \cdots \le u_n \le v_n \le \cdots \le v_1 \le v_0 = u^*. \tag{3.16}$$

We see that the sequence $\{u_n\}$ is monotone nondecreasing and is bounded sequences on J, the sequence $\{v_n\}$ is monotone nonincreasing and is bounded sequence on J. In fact, the monotone of operator $\mathcal{B}$ implies that $\mathcal{B}^{-1}$ exist, and $\mathcal{B}^{-1} : C^{1-\alpha}(J, R) \to \overline{\Omega}$ by

$$x = \mathcal{B}^{-1}u.$$

As a result

$$\|x\|_\infty = \|\mathcal{B}^{-1}u\|_\infty \le \|\mathcal{B}^{-1}\|\|u\|_\infty.$$

Consequently, $\mathcal{B}$ is a monotone bounded operator. Furthermore, since $u \in \overline{\Omega}$, $I^{1-\beta}u \in [-c, c]$, there $c = \frac{T^{1-\beta}}{(1-\beta)\Gamma(1-\beta)} \max\{\|u_*\|, \|u^*\|\}$, thus exists a positive constant N such that

$$\max_{s\in J} |\mathcal{F}u(s)| \le N.$$

Then we prove equicontinuity of operator $\mathcal{B}$ on J. For given $\varepsilon > 0$, take

$$\delta_0 = \min\{T,\ [\frac{\varepsilon\Gamma(\alpha+1)}{2(N + |u_0|)}]^{\frac{1}{\alpha}}\}.$$

Let $t_1, t_2 \in J$, $t_1 < t_2$. For each $x \in \overline{\Omega}$, when $|t_1 - t_2| < \delta_0$, we see that

$$\begin{aligned}
|\mathcal{B}x(t_1) - \mathcal{B}x(t_2)| &\le \frac{1}{\Gamma(\alpha)}|\int_0^{t_1} (t_1 - s)^{\alpha-1}\mathcal{F}u(s)ds - \int_0^{t_2} (t_2 - s)^{\alpha-1}\mathcal{F}u(s)ds| + |u_0||t_1^{\alpha-1} - t_2^{\alpha-1}| \\
&\le \frac{N}{\Gamma(\alpha)}|\int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}]ds| + |u_0||t_1^{\alpha-1} - t_2^{\alpha-1}| \\
&\quad + \frac{N}{\Gamma(\alpha)}|\int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds| \\
&\le \frac{2N}{\Gamma(\alpha+1)}|t_1 - t_2|^\alpha + |u_0|2^\alpha|t_1 - t_2|^\alpha \\
&< \frac{2(N + \Gamma(\alpha+1)|u_0|)}{\Gamma(\alpha+1)}|t_1 - t_2|^\alpha < \varepsilon.
\end{aligned}$$

Hence, operator $\mathcal{B}$ is equicontinuous on J.

Therefore, sequences $\{u_n\}, \{v_n\}$ exists subsequences $\{u_{n_k}\}, \{v_{n_k}\}$ uniformly converge on J, assume $u_{n_k} \to \lambda(k \to \infty)$, $v_{n_k} \to \kappa(k \to \infty)$, $\lambda \le \kappa$. It is easy to show that λ, κ are solutions of initial value questions (1.1), by (3.15), we have

$$\begin{cases} D^\alpha\lambda(t) = f(t, \lambda(t), I^{1-\beta}\lambda(t)) - a(t)[\lambda(t) - \lambda(t)],\ t \in (0, T], \\ t^{1-\alpha}\lambda(t)\mid_{t=0} = u_0, \end{cases}$$

and

$$\begin{cases} D^\alpha \kappa(t) = f(t, \kappa(t), I^{1-\beta}\kappa(t)) - a(t)[\kappa(t) - \kappa(t)],\ t \in (0, T], \\ t^{1-\alpha}\kappa(t) \mid_{t=0} = u_0, \end{cases}$$

Then we prove $\lambda,\ \kappa$ is minimal solution and maximal solution of (1.1). Let $\theta \in C^{1-\alpha}(J, R)$ is any solution of question (1.1) such that $u_* \leq \theta \leq u^*$, $0 < t \leq T$, we have to prove that $\lambda \leq \theta \leq \kappa,\ 0 < t \leq T$. In fact, by (3.15), (3.16), we know that at least exist a k such that $u_k \leq \theta \leq v_k$, $0 < t \leq T$, set $p = u_{k+n} - \theta$. Then

$$\begin{aligned} D^\alpha p(t) &= D^\alpha u_{k+n}(t) - D^\alpha \theta(t) \\ &= [f(t, u_{k+n-1}(t), I^{1-\beta}u_{k+n-1}(t)) - a(t)(u_{k+n}(t) - u_{k+n-1}(t))] - f(t, \theta(t), I^{1-\beta}\theta(t)) \\ &\leq a(t)(\theta(t) - u_{k+n-1}(t)) - a(t)(u_{k+n}(t) - u_{k+n-1}(t)) \\ &= -a(t)p(t), \end{aligned}$$

and $t^{1-\alpha}(u_{k+n}(t) - \theta(t))|_{t=0} = 0$.

Which implies that $u_{k+n} \leq \theta$, $0 < t \leq T$, this prove by induction for all n. Taking limit as $n \to \infty$, we conclude that $\lambda \leq \theta$. Similarly, we can prove that $\theta \leq \kappa$, $0 < t \leq T$. Combining the above argument, we can obtain $\lambda \leq \theta \leq \kappa,\ 0 < t \leq T$. The proof is completed. □

Corollary 3.1. In Theorem 3.3, if $a(t) \in C([0, T], [0, \infty))$ such that

$$f(t, x_1, y_1) - f(t, x_2, y_2) \leq a(t)|x_2 - x_1|,$$

$u^*, u_* \in C^{1-\alpha}([0, T])$ are upper and lower solutions of initial value question (1.1), then $\lambda = \theta = \kappa$ is a unique solution of (1.1).

Proof. Assume $\theta \leq \kappa$, then $\kappa - \theta = s \geq 0$, we consider $D^\alpha s$,

$$D^\alpha s = f(t, \kappa, I^{1-\beta}\kappa) - f(t, \theta, I^{1-\beta}\theta) \leq a(t)|\kappa - \theta|,$$

and $t^{1-\alpha}s(t)|_{t=0} = 0$. This implies by Lemma 3.2 that $s \leq 0$ on $[0, T]$. Thus $\lambda = \theta = \kappa$ is the unique solution of (1.1).

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