Monotone iterative technique for nonlinear differential equation of fractional order *

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ABSTRACT: In this paper, we mainly study the existence of solution of fractional differential equations. Firstly, the existence of the maxmum solution and minmum solution of the differential equation are proved by using the fixed point theorem and the monotone iteration method. Secondly, the existence of the solution of the original equation is proved by using the newly constructed differential equation. Finally, the application of the monotone iteration method is given through an example.

KEYWORDS: fractional derivatives equations, monotone iterative technique, existence and uniqueness, maximal and minimal solutions

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1. INTRODUCTION

In recent years, fractional differential equations have gained considerable importance duo to their increasing applications in a variety of fields,[1] such as hydromechanics, network traffic, Soliton and chaos etc.[2,3] Many people paid attention to fractional differential equation initial value problems (see,e.g.,Lakshmikantham and Vatsala,[4] Nirto and Rodriguez-Lopez,[5]). In particular, the differential equations involving Riemann-Liouville fractional differential equation of order $0 < \alpha < 1$ obtained many well-know results.[8−11,12−16] The fractional differential equation model can describe phenomena that the general model cannot describe,[17] and researcher use fractional differential models to modeling anomalous behavior and memory effects.[18,19]

In this paper, we use fixed point Theorems to study the existence and uniqueness of solutions of the following initial value problems about Riemann-Liouville fractional differential operator of order $0 < \alpha < 1$. Using the technique of upper and lower solutions and monotone iterative method, we establish an existence result for a Cauchy problem of the Riemann-Liouville type fractional differential equations given by

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), I^{1-\beta}u(t)),\ 0 < t \leq T, \\ t^{1-\alpha}u(t) \mid_{t=0} = u_0, \end{cases} \tag{1}$$

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where $f \in C([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $0 < T < \infty$, D^α is the Riemann-Liouville fractional differential operator of order $0 < \alpha < 1$, $I^{1-\beta}$ is the Riemann-Liouville fractional integral operator of order $0 < 1-\beta < 1$.

The outline of this paper is as follows. In Section 2, we describe the necessary background material related to our problems and prove an auxiliary lemma. Section 3 contains the main results. Section 4 is a example to illustrate our result.

2. PRELIMINARIES

In this section, we recall the definitions and some concepts on fractional integrals and derivatives, and give some lemmas which are useful in next sections.

The Riemann-Liouville fractional integral of order α is defined as

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds,$$

Riemann-Liouville derivative of order α is defined by

$$D^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$. When $0 < \alpha < 1$, the definition of Riemann-Liouville derivative can be written as

$$D^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{y(s)}{(t-s)^\alpha} ds.$$

Obviously, $D^\alpha y(t) = \frac{d}{dt} I^{1-\alpha} y(t)$.

Let $0 < \alpha < 1$, the space $L^\alpha(a,b) = \{u \in L_1(a,b) : D^\alpha u \in L_1(a,b)\}$. Here $L_1(a,b)$ is the space of summable functions on finite interval $[a,b]$.

Lemma 2.1[10] Let $0 < \alpha < 1$. Assume that $f : [0,\infty) \times X \to X$ is continuous. Then the Cauchy problem

$$\begin{cases} D^\alpha_{0+} u(t) = f(t, u(t)), \quad 0 < t < \infty, \\ I^{1-\alpha}_{0+} u(0) = u_0, \end{cases}$$

is equivalent to the integral equation

$$u(t) = u_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x, u(x)) dx, \quad t > 0. \tag{2}$$

provided the right side is point wise defined on $(0, \infty)$.

Lemma 2.2[1] Assume that $0 < \alpha < 1$. Let $c \in \mathbb{R}$ and let $g(x) \in L^1(a,b)$. If $a(x) \in L^\infty(a,b)$ or

$a(x)$ is bounded on $[a,b]$, then the Cauchy type problem

$$\begin{cases} (D^\alpha y)(x) = a(x)y(x) + g(x), \\ \lim\limits_{x\to 0^+} x^{1-\alpha}y(x) = c, \end{cases} \tag{3}$$

has a unique solution $y(x)$ in the space $L^\alpha(a,b)$.

Lemma 2.3[1] Let $0<\alpha<1$ and $y(x)$ be a Lebesgue measurable function on $[0,t]$,

(1)if $\lim\limits_{x\to 0^+} x^{1-\alpha}y(x) = a,\ a\in\mathbb{R}$, then

$$I^{1-\alpha}y(0+) = \lim_{x\to 0^+} I^{1-\alpha}y(x) = a\Gamma(\alpha).$$

(2)if $\lim\limits_{x\to 0+} I^{1-\alpha}y(x) = b,\ b\in\mathbb{R}$ holds, and if the limit $\lim\limits_{x\to 0^+} x^{1-\alpha}y(x)$ exists, then

$$\lim_{x\to 0+} x^{1-\alpha}y(x) = \frac{b}{\Gamma(\alpha)}.$$

3. MAIN RESULTS

Let $0<\alpha<1$, $C^{1-\alpha}(J,\mathbb{R}) = \{u\in C((0,T],\mathbb{R}) : t^{1-\alpha}u \in C([0,T],\mathbb{R})\}$. It is obvious that $C^{1-\alpha}(J,\mathbb{R})$ is a Banach space. For $u\in C^{1-\alpha}(J,\mathbb{R})$, define the norm $\|u\|_1 = \max\limits_{t\in[0,T]} t^{1-\alpha}|u(t)|$. $L_p(J,\mathbb{R})$ is the L_p space with the norm $\|u\|_p = \left(\int_0^T |u(t)|^p dt\right)^{\frac{1}{p}}$, $1\le p<\infty$. In (1), assume $f(x,y,z)=g(z)$, g is a function about $I^{1-\beta}u(t)$. We consider the Cauchy problem

$$\begin{cases} D^\alpha u(t) = g(I^{1-\beta}u(t)),\ 0<t\le T, \\ t^{1-\alpha}u(t)\ |_{t=0} = u_0, \end{cases} \tag{4}$$

Theorem 3.1. Let $0<\alpha,\ \beta<1$. Assume that $g\in C(J,\mathbb{R})\cap L_p(J,\mathbb{R})$, $p\in(1,\frac{1}{\alpha})$ is a constant. Then the Cauchy problem (4) has a unique solution.

proof. Let $0<\alpha<1$, define a set B as follows

$$B = \left\{u\in t^{1-\alpha}C(J,R) : t^{1-\alpha}u(t)\to u_0(t\to 0^+)\right\},$$

It is obvious that B is a Banach space. Define the operator $\mathcal{T}$ on B by

$$(\mathcal{T}u)(t) = u_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g\big(I^{1-\beta}u(s)\big)ds,\quad t\in J, \tag{5}$$

by the boundedness of fractional integration operator I^α, for each $u\in B$, as $t\to 0^+$, we have

$$t^{1-\alpha}(\mathcal{T}u)(t) = u_0 + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g\big(I^{1-\beta}u(s)\big)ds \to u_0,$$

which yields that $\mathcal{T}B\subset B$. Then we consider the equation $u=\mathcal{T}u$. In order to use Schauder's fixed point Theorem, we shall prove $\mathcal{T}$ is a continuous and compact operator. For given $\varepsilon>0$, any $u_n, u_0 \in B, n = 1, 2, \dots$ with $u_n \to u_0 (n \to \infty)$. Let q is a constant such that $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$, we have

$$\begin{aligned}
|(\mathcal{T}u_n)(t) - (\mathcal{T}u_0)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big| g(I^{1-\beta}u_n) - g(I^{1-\beta}u_0) \Big| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \bigg(\int_0^t |(t-s)^{\alpha-1}|^q ds \bigg)^{\frac{1}{q}} \bigg(\int_0^t \Big| g(I^{1-\beta}u_n) - g(I^{1-\beta}u_0) \Big|^p ds \bigg)^{\frac{1}{p}} \\
&\leq \frac{1}{\Gamma(\alpha)} \bigg(\frac{1}{q(\alpha-1)+1} \bigg)^{\frac{1}{q}} \frac{T^{1+\alpha-\beta}}{(1-\beta)\Gamma(1-\beta)} \|g\|_{L_p} \sup_{t \in J} \big| u_n(t) - u_0(t) \big| \\
&\to 0 (n \to \infty).
\end{aligned}$$

Which implies that $\mathcal{T}$ is continuous for $t \in J$.

Next, we prove that $\mathcal{T}$ is equicontinuous on J. Let $t_1, t_2 \in J$, $t_1 < t_2$. Consider the function $f(x) = (t_1 - x)^{\alpha-1} - (t_2 - x)^{\alpha-1}$, $x \in [0, t_1]$, we can obtain that $\|f\|_{L_1} \leq \frac{1}{\Gamma(\alpha+1)} (t_2 - t_1)^{\alpha}$. For given $\varepsilon > 0$, we take

$$\delta = \min \bigg\{ T, \ \bigg(\frac{\varepsilon \Gamma(\alpha+1)}{2(\|u\|_p + |u_0|)} \bigg)^{\frac{1}{\alpha}} \bigg\}.$$

When $|t_1 - t_2| < \delta$, for any $u \in B$. Applying the generalized Minkowski's inequality

$$\begin{aligned}
|(\mathcal{T}u)(t_1) - (\mathcal{T}u)(t_2)| &\leq \frac{1}{\Gamma(\alpha)} \bigg| \int_0^{t_1} (t_1 - s)^{\alpha-1} g(I^{1-\beta}u(s)) ds - \int_0^{t_2} (t_2 - s)^{\alpha-1} g(I^{1-\beta}u(s)) ds \bigg| \\
&\quad + |u_0| |t_1^{\alpha-1} - t_2^{\alpha-1}| \\
&\leq \frac{1}{\Gamma(\alpha)} \bigg| \int_0^{t_1} ((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}) g(I^{1-\beta}u(s)) ds \bigg| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} |g(I^{1-\beta}u(s))| ds + |u_0| |t_1^{\alpha-1} - t_2^{\alpha-1}| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \Big| (t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \Big| ds \bigg(\int_0^T |g(I^{1-\beta}u(s))|^p ds \bigg)^{\frac{1}{p}} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^{t_2 - t_1} \Big| (t_2 - t_1 - x)^{\alpha-1} \Big| dx \bigg(\int_0^T |g(I^{1-\beta}u(s))|^p ds \bigg)^{\frac{1}{p}} \\
&\quad + 2^{\alpha} |u_0| |t_1 - t_2|^{\alpha} \\
&\leq \frac{2\|u\|_p}{\Gamma(\alpha+1)} |t_1 - t_2|^{\alpha} + 2^{\alpha} |u_0| |t_1 - t_2|^{\alpha} < \varepsilon. \qquad (6)
\end{aligned}$$

We see that the operator $\mathcal{T} : B \to B$ is equicontinuous on J, by Arzela-Ascoli's Theorem, $\mathcal{T}$ is a compact operator. And Schauder's fixed point theorem guarantees that $\mathcal{T}$ has a fixed point. Hence Cauchy problem (4) has a unique solution. This proof is completed.

Theorem 3.2. Let $0 < \alpha \leq \frac{1}{2}$, $0 < \beta < 1$ satisfy $2\alpha - \beta > 0$. Assume $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ such that

(H1): exist nonnegative constants K, L satisfy

$$\gamma := \frac{T^{\alpha}\Gamma(\alpha)}{\Gamma(1 + 2\alpha - \beta)} \Big[K + LT^{1-\beta} \Big] < 1,$$

and

$$|f(t,x_1,y_1)-f(t,x_2,y_2)|\le K|x_1-x_2|+L|y_1-y_2|.$$

(H2): $f(t,0,0)\neq 0$ on $t\in(0,T]$, and

$$0<r_0:=\sup_{t\in[0,T]}\left(|u_0|+\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,0,0)|ds\right)<\infty,$$

hold. Then, the initial value problem (1) has a unique solution.

proof. Let $0<\alpha\le\frac{1}{2}, 0<\beta<1$, λ is a constant, $\lambda>\frac{1}{1-\gamma}$. Define a bounded closed subset S of $C^{1-\alpha}(J,R)$ given by

$$S=\left\{u\in C^{1-\alpha}(J,R):t^{1-\alpha}|u(t)|\le\lambda r_0\right\}.$$

Define an operator $\mathcal{A}$ on S as follows

$$\mathcal{A}u(t)=u_0t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,u(s),I^{1-\beta}u(s))ds,\quad t\in J. \tag{7}$$

Consider equation $u=\mathcal{A}u$. In order to use Banach fixed point Theorem, we need to prove $\mathcal{A}$ is a contraction mapping. For any $u\in S$, since (H1), (H2) hold, we have

$$\begin{aligned}
t^{1-\alpha}|\mathcal{A}u(t)| &\le |u_0|+\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big|f(s,u(s),I^{1-\beta}u(s))\Big|ds\\
&\le r_0+\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\Big|f(s,u(s),I^{1-\beta}u(s))-f(s,0,0)\Big|\Big)ds\\
&\le r_0+\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(Ks^{\alpha-1}\lambda r_0+\frac{L\lambda r_0}{\Gamma(1-\beta)}\int_0^s (s-\tau)^{-\beta}\tau^{\alpha-1}d\tau\Big)ds\\
&< r_0+\frac{\lambda r_0\Gamma(\alpha)T^\alpha}{\Gamma(1+2\alpha-\beta)}\Big[K+LT^{1-\beta}\Big]\\
&< (1+\lambda\gamma)r_0<\lambda r_0,\ t\in J,
\end{aligned}$$

which yields that $\mathcal{A}S\subset S$. Next, it will be shown that operator $\mathcal{A}$ is a contraction operator. For any $u,v\in S$, by condition (H2), we obtain

$$\begin{aligned}
\|\mathcal{A}u-\mathcal{A}v\|_1 &\le \frac{1}{\Gamma(\alpha)}\max_{t\in J}t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\Big|f(s,u(s),I^{1-\beta}u(s))-f(s,v(s),I^{1-\beta}v(s))\Big|ds\\
&\le \frac{1}{\Gamma(\alpha)}\max_{t\in J}t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\Big(K\big|u(s)-v(s)\big|\\
&\quad+\frac{L}{\Gamma(1-\beta)}\int_0^s (s-\tau)^{-\beta}\big|u(\tau)-v(\tau)\big|d\tau\Big)ds\\
&\le \frac{1}{\Gamma(\alpha)}\|u-v\|_1\max_{t\in J}t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\Big(Ks^{\alpha-1}+\frac{L\Gamma(\alpha)}{\Gamma(1+\alpha-\beta)}s^{\alpha-\beta}\Big)ds\\
&\le \Big[\frac{KT^\alpha\Gamma(\alpha)}{\Gamma(2\alpha)}+\frac{LT^{1+\alpha-\beta}\Gamma(\alpha)}{\Gamma(1+2\alpha-\beta)}\Big]\|u-v\|_1.
\end{aligned}$$

Since $0 < \alpha \leq \frac{1}{2}$, $0 < \beta < 1$ and $2\alpha - \beta > 0$, we have $\Gamma(\alpha) > \Gamma(2\alpha) > \Gamma(1+2\alpha-\beta)$. Combining the above argument, it is clear that

$$\|\mathcal{A}u - \mathcal{A}v\|_1 \leq \gamma \|u - v\|_1. \tag{8}$$

Therefore, $\mathcal{A}$ is a contraction operator. Applying the Banach fixed point Theorem, it is easy to see that $\mathcal{A}$ has a unique fixed point in S. Thus, we complete this proof.

When $\frac{1}{2} < \alpha < 1$, $0 < \beta < 1$, and the condition (H1), (H2) hold, by using the similar method as in the proof of Theorem 3.1, problem (1) has a unique solution too. Next, the existence result of Cauchy problem (1) needs to use monotonic iterative method. For this reason, we first introduce the concept of upper and lower solutions, and then prove an important inequality.

Definition 3.1. Let $u^*, u_* \in C^{1-\alpha}(J, \mathbb{R})$, if

$$\begin{cases} D^\alpha u^*(s) \geq f(s, u^*(s), I^{1-\beta} u^*(s)), \ 0 < s \leq T, \\ s^{1-\alpha} u^*(s) \mid_{s=0} \geq u_0. \end{cases} \tag{9}$$

we say u^* is a upper solution of Cauchy problem (1), and u_* is a lower solution of Cauchy problem (1), if

$$\begin{cases} D^\alpha u_*(s) \leq f(s, u_*(s), I^{1-\beta} u_*(s)), \ 0 < s \leq T, \\ s^{1-\alpha} u_*(s) \mid_{s=0} \leq u_0. \end{cases} \tag{10}$$

(H3): A function $a(t) \in L^\alpha(J, \mathbb{R})$, and if $a(t) \leq 0$, there exists a nondecreasing function $\overline{a}(t) \in L^{1-\beta}(J, \mathbb{R})$ such that $-a(t) \leq \overline{a}(t)$ and

$$\frac{3(T+1-\alpha)}{T^{3-\beta}\Gamma^2(\alpha)} \max_{t \in (0,T]} \int_0^t (t-s)^{-\beta} \overline{a}(s) ds < 1. \tag{11}$$

Lemma 3.1. Assume that $u^*, w_* \in C^{1-\alpha}(J, \mathbb{R})$ are locally Hölder continuous and satisfy the non-strict inequalities (9) and (10), $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. Suppose further that

$$f(t, x_1, y_1) - f(t, x_2, y_2) \leq a(t)(x_2 - x_1), \tag{12}$$

and $a(t) \in L^\alpha(J, \mathbb{R})$ satisfy condition (H3). Then $u^*(0) \geq w_*(0)$ implies that $u(t) \geq w(t)$.

Lemma 3.2. Let $0 < \alpha, \beta < 1$ and $a(t)$ satisfy (H3), assume $w \in C^{1-\alpha}([0,T])$, and satisfies the problem

$$\begin{cases} D^\alpha w(t) \geq -a(t) w(t), \ 0 < t \leq T, \\ t^{1-\alpha} w(t) \mid_{t=0} \geq 0. \end{cases} \tag{13}$$

Then we have

$$w(t) \geq 0, \quad 0 < t \leq T.$$

Proof. If the assertion is false, we assume that $w(t) < 0, t \in (0,T]$, because $t^{1-\alpha} w(t) \mid_{t=0} \geq 0$

and $w \in C^{1-\alpha}([0,T])$, so exists interval $[t_0, t_1] \subseteq (0,T]$ such that $w(t) < 0, t \in (t_0, t_1]$ and $w(t_0) = 0$; Assume that $t_* \in (t_0, t_1]$ is the first minimal point of $w(t)$ on $(t_0, t_1]$. Therefore, there will be

$$w(t_*) < 0. \tag{14}$$

First, let $a(t) > 0, t \in (0,T]$. From (13), we have $D^\alpha w(t) \geq 0, \quad t \in (t_0, t_1]$, hence

$$D^\alpha w(t_*) \geq 0.$$

From the definition of Riemann-Liouville fractional derivative, we can get

$$\frac{1}{\Gamma(1-\alpha)} w(t_*) t_0^{-\alpha} > D^\alpha w(t_*) \geq 0.$$

Thus $w(t_*) > 0$. It contradicts relation (14), so we obtain the result when $a(t) > 0$.

Next, if $a(t) \leq 0$, nondecreasing function $\overline{a}(t) \in L^1(J, R)$ such that $-a(t) \leq \overline{a}(t)$, by the monotone of Riemann-Liouville fractional integral operator I^α and $I^{1-\beta}$, we apply the fractional integral operator I^α on the both sides of (13),

$$w(t) - [t^{1-\alpha} w(t)|_{t=0}] t^{\alpha-1} \geq -I^\alpha[a(t)w(t)], \ t \in (0,T],$$

because $t^{1-\alpha} w(t) \mid_{t=0} \geq 0$ and by the assertion of t_*, we can obtain that

$$w(t_*) \geq -I^\alpha[a(t_*)w(t_*)].$$

Thus

$$\begin{aligned} w(t_*) + I^\alpha[a(t_*)w(t_*)] &= w(t_*) + \frac{1}{\Gamma(\alpha)} \int_0^{t_*} (t_* - s)^{\alpha-1} a(s) w(s) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{t_*} \Big(\Gamma(\alpha) t_* w(t_*) + (t_* - s)^{\alpha-1} a(s) w(s) \Big) ds \\ &\geq 0, \end{aligned}$$

Note that $w \in C^{1-\alpha}([0,T])$, so there is at least a set $E_1 \subseteq [0, t_*], \ \mu(E_1) > \epsilon > 0$ such that

$$\Gamma(\alpha) t_* w(t_*) + (t_* - s)^{\alpha-1} a(s) w(s) \geq 0, \ s \in E_1,$$

where μ is Lebesgue measure. Thus

$$\Gamma(\alpha) t_* w(t_*) + (t_* - t)^{\alpha-1} a(t) w(t_*) > 0, \ t \in E_1,$$

by assuming that $w(t_*) < 0$, we have

$$\Gamma(\alpha) t_* + (t_* - t)^{\alpha-1} a(t) < 0,$$

using the condition (H3)

$$\overline{a}(t) > \Gamma(\alpha) t_* (t_* - t)^{1-\alpha},$$

because $\overline{a}(s)$ is a nondecreasing function, so $\overline{a}(t) \leq \overline{a}(t_*)$. Taking into account that $t < t_*$ when $t \in E_1$, we have

$$\overline{a}(t_*) > \Gamma(\alpha) t_* (t_* - t)^{1-\alpha}$$

$$> \Gamma(\alpha)t_* \frac{t_* - t}{t_* - t + 1 - \alpha} \geq \Gamma(\alpha)t_* \frac{t_* - t}{\mu(E_1) + 1 - \alpha}.$$

Since $t < t_*, t \in E_1$, we can select a function $\sigma(t), 0 \leq \sigma(t) \leq 1, t \in E_1$ such that $t = \sigma(t)t_*$, therefore, the above inequality can be written as

$$\overline{a}(t_*) > \Gamma(\alpha)\frac{(1-\sigma(t))t_*^2}{\mu(E_1) + 1 - \alpha},\ t \in E_1. \tag{15}$$

We apply fractional integral $I^{1-\beta}$ to the both sides of (15),

$$\begin{aligned} I^{1-\beta}\overline{a}(t_*) &> I^{1-\beta}\Gamma(\alpha)\frac{(1-\sigma(t))t_*^2}{\mu(E_1)+1-\alpha} \\ &= \frac{(1-\sigma(t))\Gamma(\alpha)}{(\mu(E_1)+1-\alpha)\Gamma(1-\beta)}\int_0^{t_*}(t_*-s)^{-\beta}s^2 ds \\ &> \frac{(1-\sigma(t))\Gamma(\alpha)}{3(\mu(E_1)+1-\alpha)\Gamma(1-\beta)}t_*^{3-\beta} \geq \frac{(1-\sigma(t))\Gamma(\alpha)}{3(\mu(E_1)+1-\alpha)\Gamma(1-\beta)}\mu(E_1)^{3-\beta}, \end{aligned}$$

that is

$$\int_0^{t_*}(t_*-s)^{-\beta}\overline{a}(s)ds > \frac{(1-\sigma(t))\Gamma(\alpha)}{3(\mu(E_1)+1-\alpha)}\mu(E_1)^{3-\beta},$$

by inequality (14), we can obtain that

$$\frac{(1-\sigma(t))\Gamma(\alpha)}{\mu(E_1)+1-\alpha}\mu(E_1)^{3-\beta} < \frac{\Gamma^2(\alpha)T^{3-\beta}}{T+1-\alpha},\ t \in E_1. \tag{16}$$

Consider the inequality about x

$$(T+1-\alpha)(1-\sigma(t))x^{3-\beta} - \Gamma(\alpha)T^{3-\beta}x - (1-\alpha)\Gamma(\alpha)T^{3-\beta} < 0, \tag{17}$$

since $2 < 3 - \beta < 3$, it is obvious that inequality (17) has no positive solution, thus when $\mu(E_1) > \epsilon > 0$, inequality (16) is false. It is a contradiction, we complete this proof.

In Lemma 3.3, $a(t) \in L^{\alpha}(J, \mathbb{R})$ satisfies condition (H3), in case $a(t) < 0, t \in J$, there exists a constant M such that $-a(t) \leq M, t \in J$. Then, the inequality (11) becomes $\frac{3(T+1-\alpha)}{(1-\beta)T^2\Gamma(\alpha)^2}M < 1$.

Theorem 3.3. Assume that $u^*, u_* \in C^{1-\alpha}(J, \mathbb{R})$ are upper and lower solutions of Cauchy problem (1), and satisfy $u^* \geq u_*, t^{1-\alpha}u^*(t) \mid_{t=0} \geq t^{1-\alpha}u_*(t) \mid_{t=0}$. In addition, $f \in C(J\times\mathbb{R}\times\mathbb{R}, \mathbb{R})$ satisfies Lemma 3.1 and condition (H3).

Then Cauchy problem (1) has a maximal solution and a minimal solution.

Proof. Let $\Omega = \left\{y \in C^{1-\alpha}(J, R) : u_* \leq y \leq u^*\right\}$. For any $x \in \Omega$, consider the initial value problem

$$\begin{cases} D^{\alpha}u(t) = f(t, x(t), I^{1-\beta}x(t)) - a(t)[u(t) - x(t)],\ t \in (0, T], \\ t^{1-\alpha}u(t) \mid_{t=0} = u_0. \end{cases} \tag{18}$$

By Lemma 2.2 and condition (H3), for every x, boundary value problem (18) exists a unique solution.

Define operator $\mathcal{N}$ by

$$\mathcal{N}u(t) = u_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{F}u(s) ds,$$

where $\mathcal{F}u(s) = f(s, x(s), I^{1-\beta}x(s)) - a(s)[u(s) - x(s)],\ s \in J$, obviously, operator $\mathcal{N}$ is continuous in view of continuity of $\mathcal{F}$. Then the solution initial value problem (18) is a fixed point of operator $\mathcal{N}$.

Define the operator $\mathcal{B} : \overline{\Omega} \to C^{1-\alpha}(J, \mathbb{R})$ by

$$\mathcal{B}x = u,\ x \in \overline{\Omega}.$$

$\mathcal{B}$ is a continuous operator. In fact, operator $\mathcal{B}$ is equicontinuous on J.

We will use operator $\mathcal{B}$ to construct the sequences $\{u_n\}, \{v_n\}$. First, we prove $\mathcal{B}$ is a monotone operator on $\overline{\Omega}$, let $u_1(t), u_2(t) \in C^{1-\alpha}(J, \mathbb{R}), u_1(t) \leq u_2(t)$. Suppose that $u_i = \mathcal{B}x_i, (i = 1, 2)$, set $\mu = u_1 - u_2$. Then

$$\begin{aligned} D^\alpha \mu &= D^\alpha u_1 - D^\alpha u_2 \\ &= f(t, x_1, I^{1-\beta}x_1) - a(t)[u_1 - x_1] - f(t, x_2, I^{1-\beta}x_2) + a(t)[u_2 - x_2] \\ &\leq a(t)[x_2 - x_1] - a(t)[u_1 - x_1] + a(t)[u_2 - x_2] \\ &= -a(t)\mu, \end{aligned}$$

and $t^{1-\alpha}\mu(t)|_{t=0} = 0$.

This conclusion and Lemma 3.2 imply that $\mathcal{B}x_1 \leq \mathcal{B}x_2, t \in J$, so $\mathcal{B}$ is a monotone operator. Next, we prove $u_* \leq \mathcal{B}u_*$ and $u^* \geq \mathcal{B}u^*$, set $\mathcal{B}u_* = u_1$, u_1 is a unique solution of problem (18) with $x = u_*$, put $\nu = u_* - u_1$. Then

$$\begin{aligned} D^\alpha \nu &= D^\alpha u_* - D^\alpha u_1 \\ &\leq f(t, u_*, I^{1-\beta}u_*) - [f(t, u_*, I^{1-\beta}u_*) - a(t)(u_1 - u_*)] \\ &= -a(t)\nu, \end{aligned}$$

and $t^{1-\alpha}(u_*(t) - u_1(t))|_{t=0} \leq 0$.

This shows that $\nu(t) \leq 0, t \in J$, which implies that $u_* \leq \mathcal{B}u_*$. One can show similarly, $u^* \geq \mathcal{B}u^*$. Hence operator $\mathcal{B} : \overline{\Omega} \to \overline{\Omega}$.

Let $u_0 = u_*, v_0 = u^*$. We define the sequences $\{u_n\}, \{v_n\}$ on J by

$$u_n = \mathcal{B}u_{n-1},\ v_n = \mathcal{B}v_{n-1},\quad n = 1, 2, \cdots.$$

Combining the above argument, we can obtain

$$u_* = u_0 \leq u_1 \leq \cdots \leq u_n \leq v_n \leq \cdots \leq v_1 \leq v_0 = u^*. \tag{19}$$

We see that the sequence $\{u_n\}$ is monotone nondecreasing and is bounded on J, the sequence $\{v_n\}$ is monotone nonincreasing and is bounded on J. In fact, the monotone of operator $\mathcal{B}$ implies that $\mathcal{B}^{-1}$ exist, and $\mathcal{B}^{-1} : C^{1-\alpha}(J, \mathbb{R}) \to \overline{\Omega}$ by

$$x = \mathcal{B}^{-1}u.$$

Thus we have

$$\|x\|_\infty = \|\mathcal{B}^{-1}u\|_\infty \leq \|\mathcal{B}^{-1}\|\|u\|_\infty.$$

Consequently, $\mathcal{B}$ is a monotone bounded operator. Furthermore, since $u \in \overline{\Omega}$, $I^{1-\beta}u \in [-c, c]$, where $c = \frac{T^{1-\beta}}{(1-\beta)\Gamma(1-\beta)} \max\{\|u_*\|, \|u^*\|\}$, thus exists a positive constant N such that

$$\max_{s \in J} |\mathcal{F}u(s)| \leq N.$$

Then we prove equicontinuity of operator $\mathcal{B}$ on J. For given $\varepsilon > 0$, take

$$\delta_0 = \min\left\{T,\ \left(\frac{\varepsilon\Gamma(\alpha+1)}{2(N + |u_0|)}\right)^{\frac{1}{\alpha}}\right\}.$$

Let $t_1, t_2 \in J$, $t_1 < t_2$. For each $x \in \overline{\Omega}$, when $|t_1 - t_2| < \delta_0$, we have

$$\begin{aligned}
|\mathcal{B}x(t_1) - \mathcal{B}x(t_2)| &\leq \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}(t_1 - s)^{\alpha-1}\mathcal{F}u(s)ds - \int_0^{t_2}(t_2 - s)^{\alpha-1}\mathcal{F}u(s)ds\right| + |u_0||t_1^{\alpha-1} - t_2^{\alpha-1}| \\
&\leq \frac{N}{\Gamma(\alpha)}\left|\int_0^{t_1}\left((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\right)ds\right| + |u_0||t_1^{\alpha-1} - t_2^{\alpha-1}| \\
&\quad + \frac{N}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}(t_2 - s)^{\alpha-1}ds\right| \\
&\leq \frac{2N}{\Gamma(\alpha+1)}|t_1 - t_2|^\alpha + |u_0|2^\alpha|t_1 - t_2|^\alpha \\
&< \frac{2(N + \Gamma(\alpha+1)|u_0|)}{\Gamma(\alpha+1)}|t_1 - t_2|^\alpha < \varepsilon.
\end{aligned}$$

Hence, operator $\mathcal{B}$ is equicontinuous on J.

Therefore, sequences $\{u_n\}, \{v_n\}$ exist subsequences $\{u_{n_k}\}, \{v_{n_k}\}$, such that $\{u_{n_k}\}, \{v_{n_k}\}$ uniformly converge on J, assume $u_{n_k} \to \lambda (k \to \infty)$, $v_{n_k} \to \kappa (k \to \infty)$, $\lambda \leq \kappa$. It is easy to show that λ, κ are solutions of initial value problem (1), by (18), we have

$$\begin{cases} D^\alpha\lambda(t) = f(t, \lambda(t), I^{1-\beta}\lambda(t)) - a(t)[\lambda(t) - \lambda(t)],\ t \in (0, T], \\ t^{1-\alpha}\lambda(t)\ |_{t=0} = u_0, \end{cases}$$

and

$$\begin{cases} D^\alpha\kappa(t) = f(t, \kappa(t), I^{1-\beta}\kappa(t)) - a(t)[\kappa(t) - \kappa(t)],\ t \in (0, T], \\ t^{1-\alpha}\kappa(t)\ |_{t=0} = u_0, \end{cases}$$

Then we prove λ, κ are minimal and maximal solutions of (1). Let $\theta \in C^{1-\alpha}(J, R)$ is any solution of problem (1) such that $u_* \leq \theta \leq u^*$, $0 < t \leq T$, we shall prove that $\lambda \leq \theta \leq \kappa$, $0 <$

$t \le T$. In fact, by (18), (19), we know that at least exists a k such that $u_k \le \theta \le v_k$, $0 < t \le T$, set $p = u_{k+n} - \theta$. Then

$$\begin{aligned} D^\alpha p(t) &= D^\alpha u_{k+n}(t) - D^\alpha \theta(t) \\ &= [f(t, u_{k+n-1}(t), I^{1-\beta} u_{k+n-1}(t)) - a(t)(u_{k+n}(t) - u_{k+n-1}(t))] - f(t, \theta(t), I^{1-\beta}\theta(t)) \\ &\le a(t)(\theta(t) - u_{k+n-1}(t)) - a(t)(u_{k+n}(t) - u_{k+n-1}(t)) \\ &= -a(t)p(t), \end{aligned}$$

and $t^{1-\alpha}(u_{k+n}(t) - \theta(t))|_{t=0} = 0$.

Which implies that $u_{k+n} \le \theta$, $0 < t \le T$, this prove by induction for all n. Taking limit as $n \to \infty$, we conclude that $\lambda \le \theta$. Similarly, we can prove that $\theta \le \kappa$, $0 < t \le T$. Combining the above argument, we can obtain $\lambda \le \theta \le \kappa,\ 0 < t \le T$. The proof is completed.

Corollary 3.1. Let $u^*, u_* \in C^{1-\alpha}([0,T])$, $a(t) \in C([0,T],[0,\infty))$ such that

$$f(t, x_1, y_1) - f(t, x_2, y_2) \le a(t)|x_2 - x_1|,$$

u^* and u_* are the upper and lower solutions of initial value problem (1), then $\lambda = \theta = \kappa$ is a unique solution of (1).

Proof. Assume that $\theta \le \kappa$, then $\kappa - \theta = s \ge 0$, we consider $D^\alpha s$,

$$D^\alpha s = f(t, \kappa, I^{1-\beta}\kappa) - f(t, \theta, I^{1-\beta}\theta) \le a(t)|\kappa - \theta|,$$

and $t^{1-\alpha}s(t)|_{t=0} = 0$. Lemma 3.2 implies that $s \le 0$ on $[0,T]$. Thus $\lambda = \theta = \kappa$ is the unique solution of (1).

4. EXAMPLE

We now give a example to illustrate our results.

Example 4.1. Consider the following fractional differential equation initial value problem

$$\begin{cases} D^\alpha u(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + (t - u(t))\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}|u(s)|ds,\ 0 < t \le 1, \\ t^{1-\alpha}u(t)\ |_{t=0} = 0, \end{cases} \tag{20}$$

Let $u^*(t) = 1$, $u_*(t) = t - 1$, $t \in [0,1]$. Then, we have

$$\begin{aligned} D^\alpha u^*(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} &\ge \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + (t-1)\frac{t^{1-\beta}}{\Gamma(2-\beta)} \\ &= \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + (t - u^*(t))\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}|u^*(s)|ds, \end{aligned}$$

and

$$D^{\alpha}u_*(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \leq \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{t^{2-\beta}}{\Gamma(3-\beta)} + \frac{t^{1-\beta}}{\Gamma(2-\beta)}$$

$$= \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + (t - u_*(t))\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}|u_*(s)|ds.$$

which implies that u^* is a upper solution and u_* is a lower solution of (20). By the Theorem 3.3, problem (20) has solutions on $[u_*, u^*]$.

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