Modified subgradient extragradient algorithms for pseudomonotone equilibrium problems and fixed point problem

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Abstract: In this paper, a new algorithm is considered to find a common element of the solution set of a pseudomonotone equilibrium problem and fixed point set for a quasi-nonexpansive mapping in a real Hilbert space. The algorithm is based on subgradient extragradient method, inertial method and viscosity method. The adaptive step size ensures that the algorithm does not need to know apriori the Lipschitz constants of the associated bifunction. Under standard assumptions, the strong convergence of the proposed algorithm was studied . Moreover, numerical experiments on several specific problems and comparison with other algorithms show the superiority of the algorithm.

Keywords： Subgradient extragradient method; Inertial method; Viscosity method; Pseudomonotone equilibrium problem; Fixed point problem

1 Introduction

Let H be a real Hilbert space endowed with inner product $\langle \cdot,\cdot \rangle$ and the induced norm $\|\cdot\|$. Let C be a nonempty, closed and convex subset of H and assume $f : H \times H \to \mathbb{R}$ is a bifunction with $f(x,x) = 0$, $\forall x \in C$. The equilibrium problem (EP) for f on C is stated as follows:

$$\text{Find } x^* \in C\text{, such that } f(x^*, y) \quad 0 \text{ for all } y \in C. \tag{1.1}$$

We denoted the solution set of equilibrium problem (1.1) by $EP(f)$. Equilibrium problem is also regarded as the Ky Fan inequality [1] due to his contribution to the field. Numerous known models such as optimization problems, variational inequalities, fixed point problems, Nash equilibrium problems and others can be turned into finding a solution of the equilibrium problem (1.1). [2-6] In recent decades, many methods have been constructed for approximating solution of equilibrium problems. [7-16] One of the most popular methods is the proximal point method, [7,8,9] but the method cannot be applied to peudomonotone equilibrium problems. [10]

Another method is the proximal-like method (the extragradient method). [11-14] It is necessary to calculate two strongly convex programming problems of the algorithm in each iteration. However, the evaluation of the subprograms involved in the algorithm may be expensive, in the case where the bifunction and/or the feasible set have complicated structures. In order to improve this method, many authors have studied the subgradient extragradient method. [15-18] In this new method, the second strongly convex programming problems is not performed over onto the closed convex set but on a half space and allows a clear computation. On the other hand, iterative methods for finding a common element of the set of solutions of variational inequality and the set of fixed points of nonlinear mappings have been extensively studied by many authors, where variational inequalities are a special case of equilibrium problems. Therefore, it is very attractive to find a common element of the set of solutions of equilibrium problems and the set of fixed points of operators in a real Hilbert space. [18, 20-22]

Motivated and inspired by the works, [17-19, 23, 24] this paper proposes a new algorithm for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of pseudomonotone equilibrium problems. The algorithm is constructed based on subgradient extragradient method, inertial method and viscosity method. The main advantages of our method are: the self adaptive step-size which avoids the need to know apriori the Lipschitz constant of the associated monotone operator, the strong convergence and the inertial technique employed which speeds up the rate of convergence of the algorithm. Finally, the results of several numerical experiments show that the new algorithm has better convergence than the existing algorithm.

The paper is organized as follows. In Section 2, we give the basic concepts and preliminary results for our analysis. Sections 3 study the strong convergence of the proposed algorithm. Finally, in Section 4, several numerical experiments are reported to show the behavior of the new algorithm.

2 Preliminaries

In this section, we provide some basic concepts, definitions, and lemmas which will be used in later proofs. The strong converge and weak converge of the sequence $\{x_n\}_{n=1}^{\infty}$ to x are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. For each $x, y, z \in H$ and $\mu \in \mathbb{R}$, we have

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle, \tag{2.1}$$

$$\|\mu x + (1-\mu) y\|^2 = \mu \|x\|^2 + (1-\mu)\|y\|^2 - \mu(1-\mu)\|x-y\|^2, \tag{2.2}$$

$$2\langle x-y, x-z \rangle = \|x-y\|^2 + \|x-z\|^2 - \|y-z\|^2. \tag{2.3}$$

For a closed and convex set $C \subseteq H$, the (metric) projection $P_C : H \to C$ is defined, for all $x \in H$ such that $P_C(x) = \arg\min\left\{ \|y-x\| : y \in C \right\}$. It is known that P_C has the following property.

Lemma 2.1 [25] Let C be a nonempty closed convex subset of a real Hilbert space H. For any $x \in H$, we have

$$z = P_C x \Leftrightarrow \langle x-z, y-z \rangle \leq 0, \forall y \in C.$$

Definition 2.1 A bifunction $f : C \times C \to \mathbb{R}$ is said to be as follows:

• pseudomonotone on C, if $f(x,y) \quad 0 \Rightarrow f(y,x) \leq 0$, $\forall x, y \in C$;

• Lipschitz-type condition on C, if there exist two positive constants c_1 and c_2 such that

$$f(x,y) + f(y,z) \quad f(x,z) - c_1\|x-y\| - c_2\|y-z\|, \forall x,y,z \in C.$$

Further, we recall that the subdifferential of a convex function $g : C \to \mathbb{R}$ at $x \in C$ is defined by

$$\partial g(x) = \left\{ v \in H : g(y) - g(x) \quad \langle v, y-x \rangle, \forall y \in C \right\},$$

and the normal cone N_C to C at a point $x \in C$ is defined by

$$N_C(x) = \left\{ \varsigma \in H : \langle \varsigma, y-x \rangle \leq 0, y \in C \right\}.$$

Lemma 2.2 [26, Chapter 7] Let C be a nonempty convex subset of a real Hilbert space H and $g : C \to \mathbb{R}$ be a convex subdifferentiable and lower semicontinuous function on C. Then, x^* is a solution to the convex problem $\min\left\{ g(x) : x \in C \right\}$ if and only if $0 \in \partial g(x^*) + N_C(x^*)$, where $\partial g(\cdot)$ denotes the subdifferential of g and $N_C(x^*)$ is the normal cone of C at x^*.

Definition 2.2 Let $U : H \to H$ is a mapping with $Fix(T) \neq \varnothing$. Then

(i) U is called quasi-nonexpansive, if $\|U(x) - y\| \leq \|x-y\|, \forall x \in H, y \in Fix(U)$, where $Fix(U)$ is denoted the fixed point set of U.

(ii) $I - U$ is called demiclosed at zero, if $\{x_n\} \subset H$, $x_n \rightharpoonup x$ and $\|U(x_n) - x\| \to 0$, it follows that $x \in Fix(U)$.

We also recall the definition of the proximal operator which is the basic tool for the proposed algorithm. For a proper, convex and lower semicontinuous function $g : C \to (-\infty, +\infty]$ and $\lambda > 0$, the proximal mapping of g associated with λ is defined by

$$prox_{\lambda g}(x) = \arg\min\left\{ \lambda g(y) + \tfrac{1}{2}\|x-y\|^2 : y \in C \right\}, x \in H.$$

The following lemma gives an important property of the proximal mapping.

Lemma 2.3 [27] For all $x \in H$, $y \in C$ and $\lambda > 0$, the following inequality holds:

$$\lambda\left\{ g(y) - g\left(prox_{\lambda g}(x)\right) \right\} \quad \left\langle x - prox_{\lambda g}(x), y - prox_{\lambda g}(x) \right\rangle. \tag{2.4}$$

Remark 2.1 From Lemma 2.3, we note that if $x = prox_{\lambda g}(x)$, then

$$x = \arg\min\left\{ g(y) : y \in C \right\} := \left\{ x \in C : g(x) = \min_{y \in C} g(y) \right\}.$$

Lemma 2.4 (Peter-Paul inequality) For any $a, b \in \mathbb{R}$, and $\varepsilon > 0$. Then,

$$2ab \le \frac{a^2}{\varepsilon} + \varepsilon b^2. \tag{2.5}$$

Lemma 2.5 (Opial) Let $\{x_n\}$ be a sequence in H such that $x_n \rightharpoonup x$. Then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \ \forall y \ne x. \tag{2.6}$$

Lemma 2.6 [28] Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence in (0, 1) such that $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $\{b_n\}$ be a sequence of real numbers. Suppose that $a_{n+1} \le (1-\alpha_n) a_n + \alpha_n b_n, \forall n \quad 1.$ If $\limsup_{k\to\infty} b_{n_k} \le 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\limsup_{k\to\infty}\left(a_{n_k+1} - a_{n_k}\right) \quad 0$ then $\lim_{n\to\infty} a_n = 0$.

3 The convergence of the Algorithm

In this section, we present our algorithm and discuss its convergence analysis. For ease of notation we denote $\Omega = EP(f) \cap Fix(U)$, which we will replace it in the following proof. We establish the convergence of the algorithm under the following conditions:

Condition (A):

(A1) f is pseudomonotone on C and $f(x,x) = 0$ for all $x \in C$;

(A2) $f(x,\cdot)$ is convex and subdifferentiable on C for every fixed $x \in H$;

(A3) $\limsup_{n\to\infty} f(x_n, y) \le f(x, y)$ for every sequence $\{x_n\} \subset C$ which converges weakly to x and for each $y \in C$;

(A4) f satisfies the Lipschitz-type condition on H with constants c_1 and c_2.

Condition (B):

(B1) $U : H \to H$ is a quasi-nonexpansive mapping;

(B2) $f : H \to H$ is a contraction mapping with contraction parameter $\kappa \in [0,1)$.

Condition (C):

(C1) $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\} \subset (0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$;

(C2) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(C4) $\varepsilon_n = o(\alpha_n)$, i.e. $\lim_{n\to\infty} (\varepsilon_n / \alpha_n) = 0$.

Now, we introduce the following algorithm:

Algorithm 3.1

(Step 0) Choose $x_0, x_1 \in C$, $\mu \in (0,1)$, $\delta \in (0,1)$, $\lambda_1 > 0$. Choose a non-negative real sequence $\{p_n\}$, such that $\sum_{n=0}^{\infty} p_n < +\infty$.

(Step 1) Given the current iterate x_{n-1}, x_n, choose θ_n such that $0 \le \theta_n \le \overline{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{ \dfrac{e_n}{\|x_n - x_{n-1}\|}, \dfrac{n-1}{n+\alpha-1} \right\}, & \text{if } \ x_n \neq x_{n-1}, \\ \dfrac{n-1}{n+\alpha-1}, & \text{otherwise.} \end{cases}$$

(Step 2) Set $w_n = x_n + \theta_n (x_n - x_{n-1})$ and compute

$$y_n = \arg\min \left\{ \lambda_n f(w_n, y) + \frac{1}{2} \|w_n - y\|^2 , y \in C \right\} = prox_{\lambda_n f(w_n, \cdot)} (w_n),$$

If $y_n = w_n$ then stop (y_n is a solution to the problem (EP)). Otherwise, go to Step 3.

(Step 3) Choose $v_n \in \partial_2 f (w_n, y_n)$ such that $w_n - \lambda_n v_n - y_n \in N_C (y_n)$, compute

$$z_n = \arg\min \left\{ \delta\lambda_n f (y_n, z) + \frac{1}{2} \|w_n - z\|^2 , z \in T_n \right\} = prox_{\delta\lambda_n f(y_n, \cdot)} (w_n),$$

where $T_n = \left\{ x \in H \,\middle|\, \langle w_n - \lambda_n v_n - y_n, x - y_n \rangle \leq 0 \right\}$.

(Step 4) Compute $x_{n+1} = \alpha_n f (x_n) + \beta_n z_n + \gamma_n U z_n$ and

$$\lambda_{n+1} = \begin{cases} \min\left\{ \dfrac{\mu\left(\|w_n - y_n\|^2 + \|y_n - z_n\|^2 \right)}{2\left[f (w_n, z_n) - f (w_n, y_n) - f (y_n, z_n) \right]}, \lambda_n + p_n \right\}, & \text{if } \ f (w_n, z_n) - f (w_n, y_n) - f (y_n, z_n) > 0 \\ \lambda_n + p_n, & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and return to step 1.

Remark 3.1 It is easy to see that, by the definition of θ_n we obtain $\lim\limits_{n\to\infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0.$ Indeed, we have $\theta_n \|x_n - x_{n-1}\| \leq e_n$ for all n, which together with $\lim\limits_{n\to\infty} \frac{e_n}{\alpha_n} = 0$ implies that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{e_n}{\alpha_n} = 0.$$

Lemma 3.1 Algorithm 3.1 generates sequence $\{ \lambda_n \}$, which is monotonically decreasing and has a lower bound $\min\left\{ \frac{\mu}{2\max\{ c_1, c_2 \}}, \lambda_0 \right\}$.

Proof It is easy to see that $\{ \lambda_n \}$ is a monotonically decreasing sequence. Since f satisfies the Lipschitz-type condition with constants c_1 and c_2, in the case of $f (w_n, z_n) - f (w_n, y_n) - f (y_n, z_n) > 0$, we have

$$\frac{\mu\left(\|w_n - y_n\|^2 + \|y_n - z_n\|^2 \right)}{2\left(f (w_n, z_n) - f (w_n, y_n) - f (y_n, z_n) \right)} \quad \frac{\mu\left(\|w_n - y_n\|^2 + \|y_n - z_n\|^2 \right)}{2\left(c_1 \|w_n - y_n\|^2 + c_2 \|y_n - z_n\|^2 \right)} \quad \frac{\mu}{2\max\{ c_1, c_2 \}}.$$

Hence, the sequence $\{ \lambda_n \}$ has a lower bound $\min\left\{ \frac{\mu}{2\max\{ c_1, c_2 \}}, \lambda_0 \right\}$.

Remark 3.2 It is deduced that the limit of $\{ \lambda_n \}$ exists and we denote $\lambda = \lim\limits_{n\to\infty} \lambda_n$. Clearly $\lambda > 0$. If $\lambda_0 < \frac{\mu}{2\max\{ c_1, c_2 \}}$, Then $\{ \lambda_n \}$ is a constant sequence.

The following lemma plays a crucial role in the proof the convergence result.

Lemma 3.2 Let $\{ w_n \}$, $\{ y_n \}$, $\{ z_n \}$ be the sequences generated by Algorithm3.1. Then

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - (1-\delta)\|w_n - z_n\|^2$$
$$- \delta\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right)\left(\|w_n - y_n\|^2 + \|z_n - y_n\|^2\right), \forall p \in \Omega.$$

Proof From $z_n = \arg\min\left\{\delta\lambda_n f(y_n, z) + \frac{1}{2}\|w_n - z\|^2, z \in T_n\right\}$ and using Lemma 2.3, we derive

$$\delta\lambda_n\left(f(y_n, z) - f(y_n, z_n)\right) \quad \langle w_n - z_n, z - z_n\rangle,\ \forall z \in T_n. \tag{3.3}$$

Note that $\Omega \subset EP(f) \subset C \subset T_n$. Let $p \in \Omega$, substituting $z = p$ into (3.3), we deduce

$$\delta\lambda_n\left(f(y_n, p) - f(y_n, z_n)\right) \quad \langle w_n - z_n, p - z_n\rangle. \tag{3.4}$$

It follows from $p \in EP(f)$ and the pseudomonotonicity of f that $f(y_n, p) \le 0$. Then the inequality (3.4) implies

$$-\delta\lambda_n f(y_n, z_n) \quad \langle w_n - z_n, p - z_n\rangle. \tag{3.5}$$

By using the definition of the subdifferential and $v_n \in \partial_2 f(w_n, y_n)$, we obtain

$$f(w_n, y) - f(w_n, y_n) \quad \langle v_n, y - y_n\rangle, \forall y \in H.$$

In particular, taking $y = z_n$ in the last inequality, we have

$$f(w_n, z_n) - f(w_n, y_n) \quad \langle v_n, z_n - y_n\rangle.$$

From the definition of T_n, we have $\langle w_n - \lambda_n v_n - y_n, z_n - y_n\rangle \le 0$, which implies that $\lambda_n\langle v_n, z_n - y_n\rangle \quad \langle w_n - y_n, z_n - y_n\rangle$. Hence, we achieve the following

$$\lambda_n\left(f(w_n, z_n) - f(w_n, y_n)\right) \quad \langle w_n - y_n, z_n - y_n\rangle, \tag{3.6}$$

which together with (3.5) and (2.3) give us

$$\begin{aligned}
&2\lambda_n\delta\left(f(w_n, z_n) - f(w_n, y_n) - f(y_n, z_n)\right)\\
&2\delta\langle w_n - y_n, z_n - y_n\rangle + 2\langle w_n - z_n, p - z_n\rangle\\
&= \delta\left(\|w_n - y_n\|^2 + \|z_n - y_n\|^2 - \|w_n - z_n\|^2\right)\\
&\quad + \|w_n - z_n\|^2 + \|z_n - p\|^2 - \|w_n - p\|^2.
\end{aligned} \tag{3.7}$$

Owing to the definition of λ_{n+1} and (3.7), we obtain

$$\begin{aligned}
&\|z_n - p\|^2\\
&\le \|w_n - p\|^2 - \delta\left(\|w_n - y_n\|^2 + \|z_n - y_n\|^2\right) - (1-\delta)\|w_n - z_n\|^2\\
&\quad + 2\lambda_n\delta\left(f(w_n, z_n) - f(w_n, y_n) - f(y_n, z_n)\right)\\
&= \|w_n - p\|^2 - \delta\left(\|w_n - y_n\|^2 + \|z_n - y_n\|^2\right) - (1-\delta)\|w_n - z_n\|^2\\
&\quad + 2\frac{\lambda_n}{\lambda_{n+1}}\delta\lambda_{n+1}\left(f(w_n, z_n) - f(w_n, y_n) - f(y_n, z_n)\right)\\
&\le \|w_n - p\|^2 - \delta\left(\|w_n - y_n\|^2 + \|z_n - y_n\|^2\right) - (1-\delta)\|w_n - z_n\|^2\\
&\quad + \frac{\lambda_n}{\lambda_{n+1}}\delta\mu\left(\|w_n - y_n\|^2 + \|z_n - y_n\|^2\right)\\
&= \|w_n - p\|^2 - (1-\delta)\|w_n - z_n\|^2\\
&\quad - \delta\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right)\left(\|w_n - y_n\|^2 + \|z_n - y_n\|^2\right).
\end{aligned} \tag{3.8}$$

Lemma 3.3 Let sequences $\{x_n\}$, $\{w_n\}$ and $\{y_n\}$ be generated by Algorithm 3.1. Assume that $\lim_{k\to\infty}\|x_{n_k} - w_{n_k}\| = 0$, $\lim_{k\to\infty}\|w_{n_k} - y_{n_k}\| = 0$,

$\lim_{k\to\infty}\left\|w_{n_k}-z_{n_k}\right\|=0$, $\lim_{k\to\infty}\left\|y_{n_k}-z_{n_k}\right\|=0$ and $\lim_{n\to\infty}\left\|Uz_{n_k}-z_{n_k}\right\|=0$. If $\{x_{n_k}\}$ converges weakly to some $z\in H$, then $z\in\Omega$.

Proof It is clear that $w_{n_k}\rightharpoonup z$, $y_{n_k}\rightharpoonup z$ and $z\in C$. From the relation of (3.3), we deduce

$$\delta\lambda_{n_k}\left(f\left(y_{n_k},x\right)-f\left(y_{n_k},z_{n_k}\right)\right)\quad\left\langle w_{n_k}-z_{n_k},x-z_{n_k}\right\rangle,\ \forall x\in T_n. \tag{3.9}$$

On the other hand, since f satisfies the Lipschitz-type condition on C, we arrive at

$$\begin{aligned}\lambda_{n_k}f\left(y_{n_k},z_{n_k}\right)\quad&\lambda_{n_k}\left(f\left(w_{n_k},z_{n_k}\right)-f\left(w_{n_k},y_{n_k}\right)\right)\\&-\lambda_{n_k}c_1\left\|y_{n_k}-w_{n_k}\right\|^2-\lambda_{n_k}c_2\left\|y_{n_k}-z_{n_k}\right\|^2.\end{aligned} \tag{3.10}$$

According to the relations (3.6) and (3.10), we obtain

$$\begin{aligned}\lambda_{n_k}f\left(y_{n_k},z_{n_k}\right)\quad&\left\langle w_{n_k}-y_{n_k},z_{n_k}-y_{n_k}\right\rangle\\&-\lambda_{n_k}c_1\left\|y_{n_k}-w_{n_k}\right\|^2-\lambda_{n_k}c_2\left\|y_{n_k}-z_{n_k}\right\|^2.\end{aligned} \tag{3.11}$$

Combining (3.9) an9d (3.11) and $C\subseteq T_n$, we get that, for all $x\in C$,

$$\begin{aligned}f\left(y_{n_k},x\right)\quad&\frac{1}{\delta\lambda_{n_k}}\left\langle w_{n_k}-z_{n_k},x-z_{n_k}\right\rangle+\frac{1}{\lambda_{n_k}}\left\langle w_{n_k}-y_{n_k},z_{n_k}-y_{n_k}\right\rangle\\&-c_1\left\|y_{n_k}-w_{n_k}\right\|^2-c_2\left\|y_{n_k}-z_{n_k}\right\|^2.\end{aligned}$$

Let $k\to\infty$, using the facts that $\lim_{k\to\infty}\left\|w_{n_k}-y_{n_k}\right\|=\lim_{k\to\infty}\left\|w_{n_k}-z_{n_k}\right\|=\lim_{k\to\infty}\left\|y_{n_k}-z_{n_k}\right\|=0$, $\{x_n\}$ is bounded, $\lim_{n\to\infty}\lambda_n=\lambda>0$, $\delta\in(0,1)$ and the assumption (A3), we obtain $f(z,x)\quad 0$, $\forall x\in C$. That is $z\in EP(f)$. Moreover, since $z_{n_k}\rightharpoonup z$ and $I-U$ is demi-closed at zero, we get that $z\in Fix(U)$. Then, $z\in\Omega$. This completes the proof.

Theorem 3.1 Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Suppose conditions (A), (B), (C) are satisfied and $\Omega\neq\varnothing$. Then, $\{x_n\}$ converges strongly to a point $p\in\Omega$, where $p\in P_\Omega\circ f(p)$.

Proof Now we show the sufficiency of the theorem.

Step 1. We show that $\{x_n\}$ is bounded. From the definition of $\{\lambda_n\}$, we deduced $1-\lambda_n\frac{\mu}{\lambda_{n+1}}>0$, $\forall n$. Using (3.8), we obtain

$$\left\|z_n-p\right\|\le\left\|w_n-p\right\|,\ \forall n. \tag{3.12}$$

From the definition of w_n, we get

$$\begin{aligned}\left\|w_n-p\right\|&=\left\|x_n+\theta_n\left(x_n-x_{n-1}\right)-p\right\|\\&\le\left\|x_n-p\right\|+\theta_n\left\|x_n-x_{n-1}\right\|\\&=\left\|x_n-p\right\|+\alpha_n\frac{\theta_n}{\alpha_n}\left\|x_n-x_{n-1}\right\|.\end{aligned} \tag{3.13}$$

According to Remark 3.1, we have $\frac{\theta_n}{\alpha_n}\left\|x_n-x_{n-1}\right\|\to 0$, it follows that there exists a constant $M_1>0$ such that

$$\frac{\theta_n}{\alpha_n}\left\|x_n-x_{n-1}\right\|\le M_1,\ \forall n\quad 1. \tag{3.14}$$

Combining (3.12), (3.13) and (3.14), we obtain

$$\|z_n - p\| \le \|w_n - p\| \le \|x_n - p\| + \alpha_n M_1, \forall n \quad N. \tag{3.15}$$

By means of the definition of x_{n+1} and (3.15), we get

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n f(x_n) + \beta_n z_n + \gamma_n U z_n - p\| \\
&\le \alpha_n \left(\|f(x_n) - f(p)\| + \|f(p) - p\| \right) + \beta_n \|z_n - p\| + \gamma_n \|z_n - p\| \\
&\le \alpha_n \left(\kappa \|x_n - p\| + \|f(p) - p\| \right) + (1 - \alpha_n) \|z_n - p\| \\
&\le \alpha_n \left(\kappa \|x_n - p\| + \|f(p) - p\| \right) + (1 - \alpha_n) \left(\|x_n - p\| + \alpha_n M_1 \right) \\
&\le \left(1 - (1 - \kappa) \alpha_n\right) \|x_n - p\| + \alpha_n \left(M_1 + \|f(p) - p\| \right) \\
&= \left(1 - (1 - \kappa) \alpha_n\right) \|x_n - p\| + (1 - \kappa) \alpha_n \frac{M_1 + \|f(p) - p\|}{1 - \kappa} \\
&\le \max\left\{ \|x_n - p\|, \frac{M_1 + \|f(p) - p\|}{1 - \kappa} \right\} \\
&\le \cdots \\
&\le \max\left\{ \|x_N - p\|, \frac{M_1 + \|f(p) - p\|}{1 - \kappa} \right\}.
\end{aligned}$$

This implies $\{x_n\}$ is bounded. We also get $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, $\{Uz_n\}$, $\{f(x_n)\}$ are bounded.

Step 2 We prove that

$$\begin{aligned}
&\beta_n \gamma_n \|U z_n - z_n\|^2 + (1 - \alpha_n)(1 - \delta) \|w_n - z_n\|^2 \\
&+ (1 - \alpha_n) \left[\delta \left(1 - \frac{\lambda_n}{\lambda_{n+1}} \mu \right) \left(\|w_n - y_n\|^2 + \|z_n - y_n\|^2 \right) \right] \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4.
\end{aligned}$$

for some $M_4 > 0$. Indeed, we get

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\alpha_n (f(x_n) - p) + \beta_n (z_n - p) + \gamma_n (U z_n - p)\|^2 \\
&\le \alpha_n \|f(x_n) - p\|^2 + \beta_n \|z_n - p\|^2 + \gamma_n \|U z_n - p\|^2 - \beta_n \gamma_n \|U z_n - z_n\|^2 \\
&\le \alpha_n \left(\|f(x_n) - f(p)\| + \|f(p) - p\| \right)^2 + (1 - \alpha_n) \|z_n - p\|^2 - \beta_n \gamma_n \|U z_n - z_n\|^2 \\
&\le \alpha_n \left(\kappa \|x_n - p\| + \|f(p) - p\| \right)^2 + (1 - \alpha_n) \|z_n - p\|^2 - \beta_n \gamma_n \|U z_n - z_n\|^2 \\
&\le \alpha_n \left(\|x_n - p\| + \|f(p) - p\| \right)^2 + (1 - \alpha_n) \|z_n - p\|^2 - \beta_n \gamma_n \|U z_n - z_n\|^2 \\
&= \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|z_n - p\|^2 - \beta_n \gamma_n \|U z_n - z_n\|^2 \\
&+ \alpha_n \left(2 \|x_n - p\| . \|f(p) - p\| + \|f(p) - p\|^2 \right) \\
&\le \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|z_n - p\|^2 + \alpha_n M_2 - \beta_n \gamma_n \|U z_n - z_n\|^2.
\end{aligned} \tag{3.16}$$

for some $M_2 > 0$. By injecting (3.8) into (3.16), we find

$$\begin{aligned}
\|x_{n+1} - p\|^2 \le \alpha_n \|x_n - p\|^2 &+ (1 - \alpha_n) \|w_n - p\|^2 + \alpha_n M_2 \\
&- (1 - \alpha_n) \delta \left(1 - \frac{\lambda_n}{\lambda_{n+1}} \mu \right) \left(\|w_n - y_n\|^2 + \|z_n - y_n\|^2 \right) \\
&- (1 - \alpha_n)(1 - \delta) \|w_n - z_n\|^2 - \beta_n \gamma_n \|U z_n - z_n\|^2.
\end{aligned} \tag{3.17}$$

Also, from (3.15) we have

$$\begin{aligned}
&\|w_n - p\|^2 \le \left(\|x_n - p\| + \alpha_n M_1 \right)^2 \\
&= \|x_n - p\|^2 + \alpha_n \left(2 M_1 \cdot \|x_n - p\| + \alpha_n M_1^2 \right) \\
&\le \|x_n - p\|^2 + \alpha_n M_3,
\end{aligned} \tag{3.18}$$

for some $M_3 > 0$. Combining (3.17) and (3.18), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 \le\ & \|x_n-p\|^2+\alpha_n M_3+\alpha_n M_2\\ & -(1-\alpha_n)\,\delta\left(1-\frac{\lambda_n}{\lambda_{n+1}}\mu\right)\left(\|w_n-y_n\|^2+\|z_n-y_n\|^2\right)\\ & -(1-\alpha_n)(1-\delta)\,\|w_n-z_n\|^2-\beta_n\gamma_n\|Uz_n-z_n\|^2 .\end{aligned}$$

This implies that

$$\begin{aligned}&\beta_n\gamma_n\|Uz_n-z_n\|^2+(1-\alpha_n)(1-\delta)\|w_n-z_n\|^2+(1-\alpha_n)\left[\delta\left(1-\frac{\lambda_n}{\lambda_{n+1}}\mu\right)\left(\|w_n-y_n\|^2+\|z_n-y_n\|^2\right)\right]\\ &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n M_4.\end{aligned}$$

where $M_4 = M_2 + M_3$.

Step 3 We prove that

$$\begin{aligned}\|x_{n+1}-p\|^2 \le\ & \left(1-(1-\kappa)\alpha_n\right)\|x_n-p\|^2\\ & +(1-\kappa)\alpha_n\left[\frac{2}{1-\kappa}\left\langle f(p)-p,x_{n+1}-p\right\rangle+\frac{3M}{1-\kappa}\cdot\frac{\alpha_n}{\beta_n}\cdot\|x_n-x_{n-1}\|\right].\end{aligned}$$

for some $M > 0$. Indeed, we have

$$\begin{aligned}\|w_n-p\|^2 &\le \left[\|x_n-p\|+\theta_n\|x_n-x_{n-1}\|\right]^2\\ &\le \|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|\left(2\|x_n-p\|+\theta_n\|x_n-x_{n-1}\|\right).\end{aligned} \tag{3.19}$$

Combining (2.1) and (3.19), we have

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &=\|\alpha_n f(x_n)+\beta_n z_n+\gamma_n Uz_n-p\|^2\\ &=\|\alpha_n(f(x_n)-f(p))+\beta_n(z_n-p)+\gamma_n(Uz_n-p)+\alpha_n(f(p)-p)\|^2\\ &\le\|\alpha_n(f(x_n)-f(p))+\beta_n(z_n-p)+\gamma_n(Uz_n-p)\|^2+2\alpha_n\left\langle f(p)-p,x_{n+1}-p\right\rangle\\ &\le\alpha_n\|f(x_n)-f(p)\|^2+(1-\alpha_n)\|z_n-p\|^2+2\alpha_n\left\langle f(p)-p,x_{n+1}-p\right\rangle\\ &\le\alpha_n\kappa\|x_n-p\|^2+(1-\alpha_n)\|w_n-p\|^2+2\alpha_n\left\langle f(p)-p,x_{n+1}-p\right\rangle\\ &\le\alpha_n\kappa\|x_n-p\|^2+(1-\alpha_n)\|x_n-p\|^2+2\alpha_n\left\langle f(p)-p,x_{n+1}-p\right\rangle\\ &\quad+\theta_n\|x_n-x_{n-1}\|\left(2\|x_n-p\|+\theta_n\|x_n-x_{n-1}\|^2\right)\\ &=\left(1-(1-\kappa)\alpha_n\right)\|x_n-p\|^2+2\alpha_n\left\langle f(p)-p,x_{n+1}-p\right\rangle\\ &\quad+\theta_n\|x_n-x_{n-1}\|\left(2\|x_n-p\|+\theta_n\|x_n-x_{n-1}\|^2\right)\\ &\le\left(1-(1-\kappa)\alpha_n\right)\|x_n-p\|^2\\ &+(1-\kappa)\alpha_n\cdot\frac{2}{1-\kappa}\left\langle f(p)-p,x_{n+1}-p\right\rangle+3M\theta_n\|x_n-x_{n-1}\|\\ &\le\left(1-(1-\kappa)\alpha_n\right)\|x_n-p\|^2\\ &+(1-\kappa)\alpha_n\cdot\left[\frac{2}{1-\kappa}\left\langle f(p)-p,x_{n+1}-p\right\rangle+\frac{3M}{1-\kappa}\cdot\frac{\theta_n}{\alpha_n}\cdot\|x_n-x_{n-1}\|\right].\end{aligned}$$

where $\sup_{n\in N}\left\{\|x_n-p\|,\theta_n\|x_n-x_{n-1}\|\right\}\le M$ for some $M > 0$.

Step 4 We show that $\left\{\|x_n-p\|\right\}$ converges strongly to zero. Indeed, suppose that $\left\{\|x_{n_k}-p\|\right\}$ is a subsequence of $\left\{\|x_n-p\|\right\}$ satisfying

$\liminf_{k\to\infty}\left(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|\right)\quad 0$. Then

$$\begin{aligned}&\liminf_{k\to\infty}\left(\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2\right)\\ &=\liminf_{k\to\infty}\left[\left(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|\right)\left(\|x_{n_k+1}-p\|+\|x_{n_k}-p\|\right)\right]\quad 0.\end{aligned}$$

By **Step 2** we obtain

$$\begin{aligned}
&\limsup_{k\to\infty}[\beta_n\gamma_n\|Uz_n - z_n\|^2 + \left(1-\alpha_{n_k}\right)(1-\delta)\|w_n - z_n\|^2 \\
&+\left(1-\alpha_{n_k}\right)\left(1-\frac{\lambda_{n_k}}{\lambda_{n_k+1}}\mu\right)\left(\|w_{n_k} - y_{n_k}\|^2 + \|z_{n_k} - y_{n_k}\|^2\right)] \\
&\le \limsup_{k\to\infty}\left(\|x_{n_k} - p\|^2 - \|x_{n_k+1} - p\|^2 + \alpha_{n_k}M_4\right) \\
&\le \limsup_{k\to\infty}\left(\|x_{n_k} - p\|^2 - \|x_{n_k+1} - p\|^2\right) + \limsup_{k\to\infty}\alpha_{n_k}M_4 \\
&= -\liminf_{k\to\infty}\left(\|x_{n_k+1} - p\|^2 - \|x_{n_k} - p\|^2\right) \\
&\le 0.
\end{aligned}$$

This implies that

$$\lim_{k\to\infty}\|y_{n_k} - w_{n_k}\| = \lim_{k\to\infty}\|z_{n_k} - y_{n_k}\| = \lim_{k\to\infty}\|z_{n_k} - w_{n_k}\| = \lim_{k\to\infty}\|Uz_{n_k} - z_{n_k}\| = 0. \tag{3.20}$$

It is clear that

$$\|x_{n_k} - w_{n_k}\| = \alpha_{n_k}\|x_{n_k} - x_{n_k-1}\| = \beta_{n_k}\cdot\frac{\alpha_{n_k}}{\beta_{n_k}}\|x_{n_k} - x_{n_k-1}\| \to 0, k\to\infty. \tag{3.21}$$

and

$$\|x_{n+1} - z_{n_k}\| \le \alpha_{n_k}\|f(x_{n_k}) - z_{n_k}\| + \beta_n\|z_{n_k} - z_{n_k}\| + \gamma_n\|Uz_{n_k} - z_{n_k}\| \to 0, k\to\infty. \tag{3.22}$$

From (3.20), (3.21) and (3.22), we get

$$\|x_{n_k+1} - x_{n_k}\| \le \|x_{n_k+1} - z_{n_k}\| + \|z_{n_k} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\| \to 0, k\to\infty. \tag{3.23}$$

Since the sequence $\{x_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$, which converges weakly to some $z\in H$, such that

$$\limsup_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle = \lim_{k\to\infty}\langle f(p) - p, x_{n_{k_j}} - p\rangle = \langle f(p) - p, z - p\rangle. \tag{3.24}$$

Using (3.20), (3.21) and Lemma 3.3, we have $z\in\Omega$. Hence, from (3.24) and the definition of $p\in P_\Omega\circ f(p)$, we get

$$\limsup_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle = \langle f(p) - p, z - p\rangle \le 0. \tag{3.25}$$

Combining (3.23) and (3.25), we have

$$\begin{aligned}
&\limsup_{k\to\infty}\langle f(p) - p, x_{n_k+1} - p\rangle \\
&= \limsup_{k\to\infty}\left[\langle f(p) - p, x_{n_k+1} - x_{n_k}\rangle + \langle f(p) - p, x_{n_k} - p\rangle\right] \\
&\le \limsup_{k\to\infty}\|f(p) - p\|\|x_{n_k+1} - x_{n_k}\| + \limsup_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle \le 0.
\end{aligned} \tag{3.26}$$

Hence, by (3.26), $\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| = 0$, Step 3 and Lemma 2.6, we have $\lim_{n\to\infty}\|x_n - p\| = 0$. This completes the proof.

4 Numerical experiments

In this section, we present some numerical examples to illustrate the convergence and the efficiency of the proposed algorithm in comparison with other existing algorithms. First, We compare Algorithm 3.1 with other Algorithms. [14,18] Next, we compare Algorithm 3.1 with the Algorithms. [23,24] We take $\alpha = 200$ and $e_n = 5/(t+1)^2$ for our algorithms. We report the number of iterations (iter.) and computing time (time) measured in seconds for all the tests.

To terminate the algorithms, we use the condition $\|x_{n+1} - x_n\| \le \varepsilon$ and $\varepsilon = 10^{-3}$ for all the algorithms. The followings are the examples in details.

Problem 1. We consider the equilibrium problem for the following bifunction $f: H\times H\to \mathbb{R}$ which comes from the Nash-Cournot equilibrium model.[15]

$$f(x,y) = \langle Px + Qy + q, y - x\rangle$$

where $q \in \mathbb{R}^m$ is the zero vector, and the matrices P and Q are two square matrices of order m such that Q is symmetric positive semidefinite and $Q-P$ is negative semidefinite. In this case, the bifunction f satisfies (A1) - (A4) with the Lipschitz-type constants $c_1 = c_2 = \frac{\|P-Q\|}{2}$. [12, Lemma 6.2] We take $\lambda_0 = \frac{1}{4c_1}$ and $x_0 = x_1 = (1,\ldots,1)$ for all Algorithm. For Algorithm 3.1 and Algorithm 3.1, [18] we choose $\mu = 0.9$, $\alpha_n = 1/(n+1)$ and $\beta_n = (n+1)/3n$. For Algorithm 3.1, [14] we use $\alpha_n = 0.2$. For Algorithm 3.1, we choose $Ux = 0.5x$ and $\delta = 0.2$.

For numerical experiments: we suppose that the feasible set $C \subset \mathbb{R}^m$ has the form of

$$C = \left\{ x \in \mathbb{R}^m : Ax \le b \right\},$$

where A is a matrix of the size $k \times m$ ($m = 10,\ 100,\ 200$ and $k = 100$) with its entries generated randomly in [–2, 0] and $b \in \mathbb{R}^k$ is a vector with its elements generated randomly. [1, 3] The numerical results are showed in Table 1.

Problem 2 The second problem is HpHard problem, we consider a linear operator $F(x) = Mx + q$, where $M = NN^T + S + D$, every entry of the $n \times n$ matrix N and of the $n \times n$ skew-symmetric matrix S is uniformly generated from [–2, 2], every diagonal entry of the $n \times n$ diagonal D is uniformly generated from (0, 2) (so M is positive definite), and q is equal to zero vector. The feasible set is $C = R_n^+$. For all tests, we take $\lambda_0 = 1/8\|M\|$ and $x_0 = x_1 = (1,\ldots,1)$. For Algorithm 3.1, we choose $Ux = 0.25x$, $\mu = 0.3$, $\delta = 0.8$, $\alpha_n = 1/(n+1)$ and $\beta_n = (2n-1)/3(n+1)$. For Algorithm 2, [23] we choose $\alpha_n = (n+2)/(n+3)$ and $\beta_n = \alpha_n/3$. For Algorithm 3.1, [24] we take $\alpha = 0.6$ and $\beta_n = 1/(n+2)$. The results are presented in Table 2.

From the numerical results, we see that the proposed algorithms are effective.

5 Conclusions

In this paper, we consider an improved subgradient extragradient method, which combines the inertial method and the viscosity method, and uses a new step calculation method. It is proved that the sequence generated by the algorithm strongly converges to a common solution of an equilibrium problem and a fixed point problem in a real Hilbert space. Numerical experiments verify the effectiveness of the proposed algorithm.

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Table 1. Problem 1.

m	Algorithm 3.1		Algorithm 3.1 [14]		Algorithm 3.1 [18]	
	Iter.	Time	Iter.	Time	Iter.	Time
10	16	0.076	149	0.717	47	1.490
100	18	0.111	148	1.066	85	2.551
200	18	0.194	182	1.920	103	4.645

Table 2. Problem 2.

m	Algorithm 3.1		Algorithm 2 [23]		Algorithm 3.1 [24]	
	Iter.	Time	Iter.	Time	Iter.	Time
50	9	0.001	97	0.001	148	0.006
500	11	0.002	154	0.013	57	0.058
1000	11	0.004	173	0.073	885	0.606