Remarks on Inverse Resonance Problem on the Line Missing Bound States Information

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Abstract

We consider the inverse resonance problem in scattering theory in one dimension. The signal is in form of Fourier transform, which has various sorts of representation theorem via its zero set. We are interested at the question if certain bound states information are disregarded, then how much more information on the potential V is needed to recover the potential? If partial knowledge of the potential function is given, certain amount of zeros or bound states can be removed to locally recover a representation theorem of the Fourier transform. Once the representation form is recovered, we compare to conclude the inverse uniqueness.
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1 Introduction

In this note, we consider the scattering theory of one-dimensional Schrödinger equation.

$$\psi''(k,x) + k^2\psi(k,x) = V(x)\psi(k,x),\ x \in I := [0,1], \tag{1.1}$$

which we assume that

$$V(x) = V_1(x) + V_2(x) \in L^1_1(\mathbb{R}) := \{V \mid \int_{\mathbb{R}} |V(x)|(1+|x|)dx < \infty\}, \tag{1.2}$$

$$I \text{ is the convex hull of the support of } V, \tag{1.3}$$

$$I = I_1 \cup I_2 := [0,a] \cup [a,1], 0 < a < 1, \text{ where } I_j \text{ is the convex hull of the support of } V_j, \tag{1.4}$$

and $V_1(x)$ is a known functional parameter placed on interval I_1 a priori.

The scattering solution of (1.1) are asymptotic to the linear combinations of $e^{\pm ikx}$ as $x \to \mp\infty$, for all $k \in \mathbb{R} \setminus \{0\}$. Among all such solutions [6, 8, 10, 17, 19], we consider the Jost solutions of (1.1) from the left and right satisfy

$$\phi_+(x,k) = \begin{cases} e^{ikx}, & x \gg 1; \\ \frac{\hat{X}(k)}{ik}e^{ikx} + \frac{\hat{Y}(-k)}{ik}e^{-ikx}, & x \ll 0; \end{cases} \tag{1.5}$$

$$\phi_-(x,k) = \begin{cases} \frac{\hat{X}(k)}{ik}e^{-ikx} + \frac{\hat{Y}(k)}{ik}e^{ikx}, & x \gg 1; \\ e^{-ikx}, & x \ll 0, \end{cases} \tag{1.6}$$

where $\hat{X}(k)$ and $\hat{Y}(k)$ are certain Fourier transform that are entire in $\mathbb{C}$, and there is the unitary identity for real k:

$$\hat{X}(k)\overline{\hat{X}}(k) = k^2 + \hat{Y}(k)\overline{\hat{Y}}(k), \tag{1.7}$$

in which $\overline{\hat{X}}(k) = \overline{\hat{X}(\overline{k})} = \hat{X}(-k)$ for real k.

In this paper, the scattering matrix $S(x)$ is defined to be

$$S(k)=\begin{pmatrix} \frac{ik}{\hat{X}(k)} & \frac{\hat{Y}(k)}{\hat{X}(k)} \\ \frac{\hat{Y}(-k)}{\hat{X}(k)} & \frac{ik}{\hat{X}(k)} \end{pmatrix} := \begin{pmatrix} T(k) & R(k) \\ L(k) & T(k) \end{pmatrix}, \tag{1.8}$$

in which

$$T(k):=\frac{ik}{\hat{X}(k)} \tag{1.9}$$

is the transmission coefficient, and $R(k)$ and $L(k)$ are the reflection coefficients from the right and left respectively. In particular, the zeros of $\hat{X}$ are the resonances of (1.1), which we refer to [19, 21, 24, 25], and we denote the resonant set of potential function V in $\mathbb{C}^-$ as Σ. The scattering matrix $S(k)$ is meromorphic in $\mathbb{C}$, and its poles in $\{\Im k > 0\}$ are the square roots of the bound-states of (1.1), say, $\{i\kappa_1, i\kappa_2, \cdots, i\kappa_N\}$. It is well-known that $T(k)$ in (1.9) is understood via $\hat{X}(k)$ which is constructed through the one-dimensional wave equation

$$\begin{cases} \big(D_x^2 - D_y^2 + V(x)\big)A_\pm(x,y)=0; \\ A_\pm(x,y)=\delta(x-y),\ \pm x \gg 0, \end{cases} \tag{1.10}$$

where $D_yA_-(x,y)=X(y-x)+Y(y+x)$. In particular [19, p. 727],

$$X(x)-\delta'(x)+\frac{\int V(t)dt}{2}\delta(x)\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R}); \tag{1.11}$$

$$Y(y)-\frac{V(y/2)}{4}\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R}). \tag{1.12}$$

Thus, $A_\pm(x,y)$ satisfies the wave equation with x taking the place of time (this choice is dictated by the forcing condition imposed). The uniqueness part then follows from the energy estimates of the wave equation [21].

In this paper, we consider the complex analysis of entire function $\hat{X}(k)$ and $\hat{Y}(k)$ which are represented in the form of

$$\hat{X}(k)=\int_{-2}^{0}X(x)e^{-ikx}dx; \tag{1.13}$$

$$\hat{Y}(k)=\int_{0}^{2}Y(y)e^{-iky}dy,\ k\in\mathbb{C}, \tag{1.14}$$

in which Fourier transform

$$\hat{X}(0)=-\hat{Y}(0), \tag{1.15}$$

and we refer the details to Mellin [19, p. 734] and [21, 24, 25]. The upper and lower limits in integral (1.13) and (1.14) are proved by Zworski in [24].

The bound-state solutions decay exponentially as $x\to\pm\infty$, and they occur only at the bound-states, $\{i\kappa_1, i\kappa_2, \cdots, i\kappa_N\}$, with $0<\kappa_1<\kappa_2<\cdots<\kappa_N$. Each bound-state at $k=i\kappa_j$, $j=1,\cdots,N$, is simple. The bound-state norming constants $\{c_{lj}\}$ from the left and $\{c_{rj}\}$ from the right respectively are defined as

$$c_{lj}=[\int_{-\infty}^{\infty}f_l(i\kappa_j,x)^2dx]^{-\frac{1}{2}}; \tag{1.16}$$

$$c_{rj}=[\int_{-\infty}^{\infty}f_r(i\kappa_j,x)^2dx]^{-\frac{1}{2}} \tag{1.17}$$

The coefficients of the scattering matrix plays a role in determining the norming constant [4].

In literature [6, 8, 10, 17, 19], the potential function is determined by either by the left scattering data $\{R,\{\kappa_j\},\{c_{lj}\}\}$ or by the right scattering data $\{L,\{\kappa_j\},\{c_{rj}\}\}$. In inverse resonance problem, we consider to determine the potential V from the resonances of (1.1) which includes the square root of L^2-eigenvalues. That is, we consider the meromorphic structure of $R(k)=\hat{Y}(k)/\hat{X}(k)$ in the context of certain sorts of zero representation theorem in complex analysis. Such an inverse process to identify the knowledge of the emitting source by measuring all sorts of respects of the emittance or perturbed wave-field in observational area. The inverse spectral-scattering problem of Schrödinger operator on the line with various conditions have been studied in [2, 3, 7, 12, 13, 18, 21, 23, 24, 25] and in many other references. For the half line case, the unique recovery of the potential from the resonances is justified in [12]. Moreover, if the potential is known a priori on a larger interval, then infinitely many resonances can be removed from the unique determination of potential on the interval [23]. However, in the full line case, the inverse resonance problems mainly remain open for a long time. It is known that the potential cannot be solely determined by the eigenvalues and resonances. Specifically, Zworski [25] proved the uniqueness theorem for the symmetric potentials along with certain isopolar results. Furthermore, Korotyaev [12, 13] applied the value distribution theory in complex analysis to prove that all eigenvalues and resonances, and a signed sequence condition can uniquely determine the potential V. In this paper, we consider the inverse problem without full knowledge of neither $\{\{\kappa_j\},\{c_{lj}\}\}$ nor $\{\{\kappa_j\},\{c_{rj}\}\}$ [3, 11, 22].

In photonics, we consider the example of Fourier optics in the Saleh and Teich's book [20, Ch. 4] when the Fresnel number is small, that is, $N_F = B^2/\lambda d \ll 1$, where B is the radius confining the object in the object plane, d is the distance between the object and the measured intensity plane. The λ is the wavelength of light. The relationship between the measured intensity I_{out} and the intensity of wavefield at the object plane E_{in} is given proportionally as

$$I_{out}(x,y) \propto \left|\hat{E}_{in}(x/\lambda d', y/\lambda d)\right|^2,$$

in which $\hat{E}_{in} = \mathcal{F}\{E_{in}\}$ and $\mathcal{F}$ denoting the Fourier transform. We refer the details to [20]. Once the far-field intensity is measured, the goal is to recover E_{in}, which is equivalent to recovering the object from I_{out}. In this case, the phase information is not provided. This is a typical phaseless problem in mathematical physics. Therefore, we are interested to study the inverse problem of (1.1) in the context of phaseless setting. In our case, that is to determine the potential V by the measurements partially or fully on functional modulus $\{|R(k)|,|L(k)|,|T(k)|\}$ in (1.8). When there are no bound states, the potential is determined by the reflection coefficient $R(k)$ by Levinson's theorem [8, 16, 17].

We state the following result in this paper.

Theorem 1.1. *If the Lebesgue measurement $|I_1| > 0$, then the non-trivial potential V_2 is uniquely determined by scattering data $\{|T(k)|, V_1\}$, $k \in \mathbb{R}$.*

Furthermore, we ask the question if certain bound states information are lost, then how much more information on the potential V is needed to recover the potential [3, 11, 22]?

Theorem 1.2. *Let $\{i\kappa_1, i\kappa_2, \cdots, i\kappa_N\}$ be the bound states of potential function $V = V^1 + V^2$, and $\Sigma' := \Sigma \setminus \{i\kappa_1, i\kappa_2, \cdots, i\kappa_N\}$. If the Lebesgue measurement $|I_1| > 0$, then the non-trivial potential V_2 is uniquely determined by data $\{\Sigma', V^1\}$.*

We may compare our results to [3, 7, 11, 22, 23].

2 Lemmata

To count the zero set of $\Lambda(k)$, we review some results from complex analysis [5, 14, 15].

Definition 2.1. Let $F(z)$ be an entire function. Let

$$M_F(r) := \max_{|z|=r} |F(z)|.$$

An entire function of $F(z)$ is said to be a function of finite order if there exists a positive constant k such that the inequality

$$M_F(r) < e^{r^k}$$

is valid for all sufficiently large values of r. The greatest lower bound of such numbers k is called the order of the entire function $F(z)$. By the type σ of an entire function $F(z)$ of order ρ, we mean the greatest lower bound of positive number A for which asymptotically we have

$$M_F(r) < e^{Ar^\rho}.$$

That is,

$$\sigma_F := \limsup_{r\to\infty} \frac{\ln M_F(r)}{r^\rho}.$$

If $0 < \sigma_F < \infty$, then we say $F(z)$ is of normal type or mean type. For $\sigma_F = 0$, we say $F(z)$ is of minimal type.

Definition 2.2. If an entire function $F(z)$ is of order one and of normal type, then we say it is an entire function of exponential type (EFET).

Definition 2.3. Let $F(z)$ be an integral function of finite order ρ in the angle $[\theta_1, \theta_2]$. We call the following quantity as the indicator function of function $F(z)$.

$$h_F(\theta) := \lim_{r\to\infty} \frac{\ln |F(re^{i\theta})|}{r^\rho},\ \theta_1 \le \theta \le \theta_2.$$

The type of a function is connected to the maximal value of indicator function.

Lemma 2.4 (Levin [14], p.72)**.** *The maximal value of indicator function $h_F(\theta)$ of $F(z)$ on the interval $\alpha \le \theta \le \beta$ is equal to the type σ_F of this function inside the angle $\alpha \le \arg z \le \beta$.*

Definition 2.5. Let $f(z)$ be an integral function of order 1, and let $N(f, \alpha, \beta, r)$ denote the number of the zeros of $f(z)$ inside the angle $[\alpha, \beta]$ and $|z| \le r$. We define the density function as

$$\Delta_f(\alpha, \beta) := \lim_{r\to\infty} \frac{N(f, \alpha, \beta, r)}{r},$$

and

$$\Delta_f(\beta) := \Delta_f(\alpha_0, \beta),$$

with some fixed $\alpha_0 \notin E$ such that E is at most a countable set [5, 14, 15]. In particular, we denote the density function of f on the open right/left half complex plane as Δ_f^+/Δ_f^- respectively. Similarly, we can define the set density of a zero set S. Let $N(S, r)$ be the number of the discrete elements of S in $\{|z| < r\}$. We define

$$\Delta_S := \lim_{r\to\infty} \frac{N(S, r)}{r}, \tag{2.1}$$

The class of function of completely regular growth is important is in this paper. We refer the definition to Levin [14, p. 139]. The definition seems complicated, but this will be compensated by all sorts of integral conditions provided by Cartwright's theory.

Definition 2.6. A function $F(z)$ that is holomorphic and of proximate order $\rho(r)$ within some angle (θ_1, θ_2) will be called a function of completely regular growth on the ray $\arg z = \theta$ if the limit

$$h_F(\theta) = \lim_{r\to\infty} \frac{\ln |F(re^{i\theta})|}{r^{\rho(r)}}$$

exists under the condition that r goes to infinity by taking on all positive values except possibly those of a set of zero relative measure (an E^0-set). We will say that a function is of completely regular growth on some set of rays $R_{\mathcal{M}}(\mathcal{M}$ is the set of values of $\theta)$ if the function

$$h_{F,r}(\theta) = \frac{\ln |F(re^{i\theta})|}{r^{\rho(r)}}$$

converges uniformly to $h_F(\theta)$ for $\theta \in \mathcal{M}$, when r goes to infinity by taking on all positive values except possibly for a set $E_{\mathcal{M}}$ of zero relative measure, this set being the same for all rays $R_{\mathcal{M}}$. The set $E_{\mathcal{M}}$ will be called the exceptional set for the given function. We shall say that $F(z)$ is a function of completely regular growth within the angle (θ_1, θ_2) if this is true for every closed interior angle, and we simply say that $F(z)$ is of completely regular growth if it is an entire function and is of completely regular growth in the entire plane.

Lemma 2.7. *Let f, g be two entire functions. Then the following two inequalities hold.*

$$h_{fg}(\theta) \leq h_f(\theta) + h_g(\theta), \text{ if one limit exists}; \tag{2.2}$$

$$h_{f+g}(\theta) \leq \max_\theta\{h_f(\theta), h_g(\theta)\}, \tag{2.3}$$

where the equality in (2.2) holds if one of the functions is of completely regular growth, and secondly the equality (2.3) holds if the indicator of the two summands are not equal at some θ_0.

Proof. We can find the details in [14]. □

Definition 2.8. The following quantity is called the width of the indicator diagram of entire function f:

$$d = h_f(\frac{\pi}{2}) + h_f(-\frac{\pi}{2}).$$

Theorem 2.9 (Cartwright)**.** *Let f be an entire function of exponential type with zero set $\{a_k\}$. We assume f satisfies one of the following conditions:*

$$\text{the integral } \int_{-\infty}^{\infty} \frac{\ln^+|f(x)|}{1+x^2} dx \text{ exists.}$$

$$|f(x)| \text{ is bounded on the real axis.}$$

Then

1. *all of the zeros of the function $f(z)$, except possibly those of a set of zero density, lie inside arbitrarily small angles $|\arg z| < \epsilon$ and $|\arg z - \pi| < \epsilon$, where the density*

   $$\Delta_f(-\epsilon, \epsilon) = \Delta_f(\pi - \epsilon, \pi + \epsilon) = \lim_{r\to\infty} \frac{N(f, -\epsilon, \epsilon, r)}{r} = \lim_{r\to\infty} \frac{N(f, \pi-\epsilon, \pi+\epsilon, r)}{r}, \tag{2.4}$$

   is equal to $\frac{d}{2\pi}$, where d is the width of the indicator diagram in (2.8). Furthermore, the limit $\delta = \lim_{r\to\infty} \delta(r)$ exists, where

   $$\delta(r) := \sum_{\{|a_k|<r\}} \frac{1}{a_k};$$

2. *moreover,*

   $$\Delta_f(\epsilon, \pi - \epsilon) = \Delta_f(\pi + \epsilon, -\epsilon) = 0;$$

3. *the function $f(z)$ can be represented in the form*

   $$f(z) = cz^m e^{i\kappa z} \lim_{r\to\infty} \prod_{\{|a_k|<r\}} (1 - \frac{z}{a_k}),$$

   where c, m, κ are constants and κ is real;

4. *the indicator function of f is of the form*

   $$h_f(\theta) = \sigma|\sin\theta|. \tag{2.5}$$

We refer the last statement to Levin [15, p. 126].

Lemma 2.10. *The Fourier transform $\hat{X}(z)$ as in (1.13) is of Cartwright class, and the function can be represented in form*

$$\hat{X}(z) = cz^m e^{i\delta z} \lim_{R\to\infty} \prod_{|\sigma_n|<R} (1 - \frac{z}{\sigma_n}),\ z = x + iy,$$

where $\delta \in \mathbb{R}$, and the following integral converges:

$$\int_{-\infty}^{\infty} \frac{\ln^+|\hat{X}(x)|}{1+x^2} dx < \infty. \tag{2.6}$$

Similar results hold for $\hat{Y}(k)$.

Proof. We refer the definition of Cartwright class to [14, 15]. ☐

Theorem 2.11 (Nevanlinna-Levin)**.** *If the function $F(z)$ is holomorphic and of exponential type in the half-plane $\Im z \geq 0$, and if (2.6) holds, then*

1.
$$F(z)\prod_{k=1}^{\infty}\frac{1-\frac{z}{\bar{a}_k}}{1-\frac{z}{a_k}} = e^{i\gamma}e^{u(z)+iv(z)},$$
where
$$u(z) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\ln|F(t)|}{(t-x)^2+y^2}dt + \sigma_F^+ y,$$
$\sigma_F^+ = h_F(\frac{\pi}{2})$, *$v(z)$ is the harmonic conjugate of $u(z)$, and $\{a_k\}$ are the zeros of the function $F(z)$ in the half-plane $\Im z > 0$;*

2.
$$\ln|F(z)| = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\ln|F(t)|}{(t-x)^2+y^2}dt + \sigma_F^+ y + \ln|\chi(z)|,\ z = x+iy,$$
where
$$\chi(z) = \prod_{k=1}^{\infty}\frac{1-\frac{z}{a_k}}{1-\frac{z}{\bar{a}_k}}.$$

Proof. We refer the proof to [14, p. 240]. ☐

Lemma 2.12. *Let $\hat{X}(z) = \int_{-2}^{0} X(x)e^{-izx}dx$, $z = |z|e^{i\arg z}$. Then,*

1. *the width of indicator diagram of $\hat{X}(z)$ and $\hat{Y}(z)$ is* 2*;*
2.
$$\begin{aligned}\Delta_{\hat{X}}(-\epsilon,\epsilon) = \Delta_{\hat{X}}(\pi-\epsilon,\pi+\epsilon) = \frac{\operatorname{ch}\operatorname{supp}(V)}{\pi} = \frac{I}{\pi};\\ \Delta_{\hat{Y}}(-\epsilon,\epsilon) = \Delta_{\hat{Y}}(\pi-\epsilon,\pi+\epsilon) = \frac{\operatorname{ch}\operatorname{supp}(V)}{\pi} = \frac{I}{\pi};\end{aligned} \tag{2.7}$$

Proof. It is straightforward from Boas [5, p. 109] that the indicator function at $\arg z = \frac{\pi}{2}$ and $\arg z = -\frac{\pi}{2}$ can be computed respectively as

$$\begin{aligned}h_{\hat{X}}(\frac{\pi}{2}) = 0;\\ h_{\hat{X}}(-\frac{\pi}{2}) = 2.\end{aligned}$$

Using Definition 2.8, we deduce the width of the indicator function, and the second statement is deduced from the Cartwright-Levinson theorem [14, p. 251]. The effective support of $X(x)$ is $[-2,0]$, and $Y(y)$ is $[0,2]$. Similarly,

$$\hat{Y}(k) = \int_0^2 Y(x)e^{-izx}dx,$$

and, then

$$\begin{aligned}h_{\hat{Y}}(\frac{\pi}{2}) = 0;\\ h_{\hat{Y}}(-\frac{\pi}{2}) = 2.\end{aligned}$$

☐

3 Proof of Theorem 1.1

Proof. We start with the assumption of Theorem 1.1, which leads to

$$|\hat{X}^1_{12}(k)| = |\hat{X}^2_{12}(k)|,$$

and then apply Theorem 2.11 to deduce that

$$\hat{X}^1_{12}(k)\prod_{n=1}^{N^1}\frac{1-\frac{k}{\overline{a}^1_n}}{1-\frac{k}{a^1_n}} = e^{i\gamma}\hat{X}^2_{12}(k)\prod_{n=1}^{N^2}\frac{1-\frac{k}{\overline{a}^2_n}}{1-\frac{k}{a^2_n}}, \tag{3.1}$$

where $\{a^j_n\}$ are the zeros of $\hat{X}^j_{12}(k)$ in $\mathbb{C}^+$. Let

$$B^j(k) := \prod_{n=1}^{N^j}\frac{1-\frac{k}{\overline{a}^j_n}}{1-\frac{z}{a^j_n}}.$$

We note that both sides of (3.1) are analytic in $\mathbb{C}$. Now we apply Theorem 2.9 to obtain that there is a non-empty set S such that

$$\hat{X}^1_{12}(k)B^1(k) = \hat{X}^2_{12}(k)B^2(k),\ k\in S. \tag{3.2}$$

Therefore, $e^{i\gamma}=1$, and then $\gamma=0$. Then, we deduce from (3.1) that

$$\hat{X}^1_{12}(k)B^1(k) = \hat{X}^2_{12}(k)B^2(k). \tag{3.3}$$

Moreover, we compute the zero density of the zero set S' for the equation

$$\hat{X}^1_{12}(k) = \hat{X}^2_{12}(k). \tag{3.4}$$

Using Theorem 2.9 and Lemma 2.12, the zero density of S' is

$$\Delta_{S'} = \frac{|I|}{\pi}. \tag{3.5}$$

We then plug all of element of S' into (3.3) and deduce that

$$B^1(k) = B^2(k),\ k\in S'. \tag{3.6}$$

This is not possible unless $B^1(k)=B^2(k)$. Therefore,

$$\hat{X}^1_{12}(k) = \hat{X}^2_{12}(k),\ k\in\mathbb{C}. \tag{3.7}$$

Now we consider the transition matrix [1, 9]

$$\Lambda^j_{12}(k) = \begin{pmatrix} \frac{\hat{X}^j_{12}(k)}{ik} & -\frac{\hat{Y}^j_{12}(k)}{ik} \\ \frac{\hat{Y}^j_{12}(-k)}{ik} & \frac{\hat{X}^j_{12}(-k)}{-ik} \end{pmatrix} \tag{3.8}$$

of potential $V^j_1+V^j_2$, and

$$\Lambda^j_1(k)\Lambda^j_2(k) = \begin{pmatrix} \frac{\hat{X}^j_1(k)}{ik} & -\frac{\hat{Y}^j_1(k)}{ik} \\ \frac{\hat{Y}^j_1(-k)}{ik} & \frac{\hat{X}^j_1(-k)}{-ik} \end{pmatrix}\begin{pmatrix} \frac{\hat{X}^j_2(k)}{ik} & -\frac{\hat{Y}^j_2(k)}{ik} \\ \frac{\hat{Y}^j_2(-k)}{ik} & \frac{\hat{X}^j_2(-k)}{-ik} \end{pmatrix} \tag{3.9}$$

be the product of transition matrices of V^j_1 and V^j_2. Then we use Aktosun's identity [1]:

$$\begin{aligned} \Lambda^j_{12}(k) &= \Lambda^j_1(k)\Lambda^j_2(k) \\ &= \begin{pmatrix} \frac{\hat{X}^j_1(k)}{ik} & -\frac{\hat{Y}^j_1(k)}{ik} \\ \frac{\hat{Y}^j_1(-k)}{ik} & \frac{\hat{X}^j_1(-k)}{-ik} \end{pmatrix}\begin{pmatrix} \frac{\hat{X}^j_2(k)}{ik} & -\frac{\hat{Y}^j_2(k)}{ik} \\ \frac{\hat{Y}^j_2(-k)}{ik} & \frac{\hat{X}^j_2(-k)}{-ik} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\hat{X}^j_1(k)\hat{X}^j_2(k)}{-k^2} + \frac{\hat{Y}^j_1(k)\hat{Y}^j_2(-k)}{k^2} & \cdots \\ \frac{-\hat{Y}^j_1(-k)\hat{X}^j_2(k)+\hat{X}^j_1(-k)\hat{Y}^j_2(-k)}{k^2} & \cdots \end{pmatrix}. \end{aligned} \tag{3.10}$$

Here,

$$\Lambda_{12}^j(k) = \begin{pmatrix} \frac{\hat{X}_{12}^j(k)}{ik} & -\frac{\hat{Y}_{12}^j(k)}{ik} \\ \frac{\hat{Y}_{12}^j(-k)}{ik} & \frac{\hat{X}_{12}^j(-k)}{-ik} \end{pmatrix}. \tag{3.11}$$

Moreover, we observe that

$$\hat{X}_{j'}^j(k) \sim ik - \frac{\int_{\mathbb{R}} V_{j'}^j(x)dx}{2} + o(\frac{1}{k}); \tag{3.12}$$

$$\hat{Y}_{j'}^j(k) \sim o(1),\ j=1,2;\ j'=1,2. \tag{3.13}$$

Hence, we use (3.7), (3.10) and (3.11) to deduce that

$$\hat{X}_1^1(k)\hat{X}_2^1(k) - \hat{Y}_1^1(k)\hat{Y}_2^1(-k) = \hat{X}_1^2(k)\hat{X}_2^2(k) - \hat{Y}_1^2(k)\hat{Y}_2^2(-k). \tag{3.14}$$

Let us look into the zero set S'' of

$$\hat{Y}_1^1(k)\hat{Y}_2^1(-k) = \hat{Y}_1^2(k)\hat{Y}_2^2(-k). \tag{3.15}$$

Using Theorem 2.9, we deduce the zero density of this zero set S'' is

$$\frac{|I|}{\pi}. \tag{3.16}$$

We plug in all of elements in S'' into (3.14), and deduce

$$\hat{X}_1^1(k)\hat{X}_2^1(k) = \hat{X}_1^2(k)\hat{X}_2^2(k),\ k \in S''. \tag{3.17}$$

Applying $V_1^1 \equiv V_2^1$ by the assumption of the theorem, we obtain from (3.17) that

$$\hat{X}_2^1(k) = \hat{X}_2^2(k),\ k \in S''. \tag{3.18}$$

This contradicts Theorem 2.9 and the computation of Lemma 2.12. Hence,

$$\hat{X}_2^1(k) = \hat{X}_2^2(k). \tag{3.19}$$

Therefore, we conclude that

$$\hat{Y}_1^1(k)\hat{Y}_2^1(-k) = \hat{Y}_1^2(k)\hat{Y}_2^2(-k).$$

Applying $V_1^1 \equiv V_2^1$ again, we obtain

$$\hat{Y}_2^1(k) = \hat{Y}_2^2(k). \tag{3.20}$$

Now we apply Zworski [24, Proposition 8], and Korotyaev [13] with the knowledge of the zeros of $\hat{Y}_2^j(z)$, $j = 1,2$, which implies the potential function with compact support is determined by the scattering matrix, we deduce that

$$V_2^1(x) \equiv V_2^2(x). \tag{3.21}$$

The theorem is thus proven.

□

4 Proof of Theorem 1.2

Proof. We begin with Lemma 2.10, and write

$$\hat{X}_{12}^j(k) = c^j k^{m^j} e^{i\delta^j k} \lim_{R\to\infty} \prod_{|\sigma_n^j|<R} (1 - \frac{k}{\sigma_n^j}),\ k = x+iy,$$

in its zero set, and $\delta^j \in \mathbb{R}$, $j = 1,2$. Hence,

$$\hat{X}_{12}^j(k) = c^j k^{m^j} e^{i\delta^j k} \lim_{R\to\infty} \prod_{\{\Im\sigma_n^j>0,\,|\sigma_n^j|<R\}} \prod_{\{\Im\sigma_n^j<0,\,|\sigma_n^j|<R\}} (1 - \frac{k}{\sigma_n^j}).$$

Now we consider the fraction

$$\begin{aligned}\frac{\hat{X}^1_{12}(k)}{\hat{X}^2_{12}(k)} &= \frac{c^1 k^{m^1} e^{i\delta^1 k} \lim_{R\to\infty} \prod_{\Im\sigma^1_n>0,\, |\sigma^1_n|<R}(1-\frac{k}{\sigma^1_n})}{c^2 z^{m^2} e^{i\delta^2 k} \lim_{R\to\infty} \prod_{\Im\sigma^2_n>0,\, |\sigma^2_n|<R}(1-\frac{k}{\sigma^2_n})} \\ &= \frac{c^1 k^{m^1} e^{i\delta^1 k} \prod_{\Im\sigma^j_n>0}(1-\frac{k}{\sigma^j_n})}{c^2 k^{m^2} e^{i\delta^2 k} \prod_{\Im\sigma^j_n>0}(1-\frac{k}{\sigma^j_n})},\end{aligned} \tag{4.1}$$

which is a rational function due to the finite number of bound states in the upper half complex plane, and the assumption of Theorem 1.2. Therefore, we write

$$\frac{\hat{X}^1_{12}(k)}{\hat{X}^2_{12}(k)} = R(k),$$

where $R(k) = \frac{p(k)}{q(k)}$ represents a rational function in $\mathbb{C}$, and p, q are polynomials. In this case, we obtain

$$\hat{X}^1_{12}(k)q(k) = \hat{X}^2_{12}(k)p(k). \tag{4.2}$$

Now, we let $D \subset \mathbb{C}$ the zero set for the equation

$$\hat{X}^1_{12}(k) = \hat{X}^2_{12}(k).$$

According to Theorem 2.9, we compute the zero density of D as

$$\Delta_D = \frac{|I|}{\pi}. \tag{4.3}$$

We then plug all elements of D into (4.2), and deduce

$$q(k) = p(k),\ k \in D.$$

This is impossible unless $p(k) \equiv q(k)$. Therefore, we deduce from (4.2) such that

$$\hat{X}^1_{12}(k) \equiv \hat{X}^2_{12}(k). \tag{4.4}$$

Now we repeat the argument ever since (3.14) to deduce that

$$\hat{X}^1_2(k) \equiv \hat{X}^2_2(k) \tag{4.5}$$

and

$$\hat{Y}^1_2(k) \equiv \hat{Y}^2_2(k). \tag{4.6}$$

Using again Zworski [24, Proposition 8] and Korotyaev [13], we deduce that

$$V^1_2(x) \equiv V^2_2(x). \tag{4.7}$$

This proves the theorem. □

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