On the practical feedback stabilization for evolution equations in Banach spaces

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Abstract

This paper investigates the notion of practical feedback stabilization of evolution equations satisfying some relaxed conditions in infinite-dimensional Banach spaces. Moreover, sufficient conditions are presented that guarantee practical stabilizability of uncertain systems based on Lyapunov functions. These results are applied to partial differential equations.

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1 Introduction

In the literature on control theory of time-varying dynamical systems, controllability and stabilizability are the qualitative control problems that play an important role in the systems and have attracted many researchers, see [4, 10, 13, 16, 17, 18, 19]. The theory was first introduced by Kalman et al. [11] for the finite dimensional of time-invariant systems. Furthermore, the theory can first introduced by Wonham [25] which related to exponential stability. Furthermore, Lyapunov function approach and the method based on spectral decomposition are the most widely used techniques for studying stabilizability of special classes of control systems, see, for example [12, 21]. In the infinite-dimensional control systems, the investigation of practical stabilization is more complicated and require more sophisticated techniques. The practical stabilization is to find the state feedback candidate such that the solution of the closed-loop system is practically exponentially stable in the Lyapunov sense in which the origin is not necessary an equilibrium point. In this case, Damak et al. [3] proved the practical feedback stabilization of the time-varying control systems in Hilbert spaces where the nominal system is a linear time-varying control systems globally null-controllable and the perturbation term satisfies some conditions. The authors in [11, 25] have shown that in the finite-dimensional autonomous control system, if the system is null-controllable in finite time then it is stabilizable. But, it does not hold for the converse. Moreover, if the system is completely stabilizable, then it is null controllable in finite time. The results of the stabilizability for the finite-dimensional systems can be generalized into infinite-dimensional systems. For time-invariant control systems in Banach spaces, Phat and Kiet [19] defined an equivalence between solvability of the Lyapunov

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equation and exponential stability of linear system. Based on the Lyapunov theorem, a relationship between stabilizability and exact null-controllability of linear time-invariant control systems is established. Moreover, he gave the exponential stabilizability of a class of nonlinear control systems.
The purpose of this paper is to present the practical stabilization of evolution equations in Banach spaces. Based on the exact-controllability assumption of the linear control system, sufficient conditions for the stabilizability are established by solving a standard Lyapunov equation. Further, the nonlinear perturbation term is locally Lipschitz continuous and satisfy some appropriate growth conditions. Based on the null controllability of the nominal system, a stabilizing controller for nonlinear uncertain system is then synthesized. We derive a Lyapunov functional which allows us to prove and even characterize the decay rate of the trajectories.
The paper is organized as follows. In Section 2, we introduce briefly some notations and necessary preliminaries. The required assumptions and the statement of the main results in Section 3. In Section 4, we present illustrative examples showing the importance of this study. Our conclusion is given in Section 5.

2 Preliminaries

Throughout this paper we adopt the following notations: $\mathbb{R}_+$ denotes the set of all non-negative real numbers, X denotes a infinite-dimensional Banach space with the norm $\|.\|$. Let $X^\star$ the topological dual space of X and U infinite dimensional Banach space. Let $< y^\star, x >$ denote the value of $y \in X^*$ at $x \in X$. $L(X)$ (respectively, $L(X,Y)$) denotes the Banach space of all linear bounded operators T mapping X into X (respectively, X into Y) endowed with the norm $\|T\| = \sup_{x\in X} \frac{\|T(x)\|}{\|x\|}$. The domain, the image, the adjoint, and the inverse operator of an operator A are denoted by $D(A)$, ImA, A^* and A^{-1} respectively. Everywhere below A is a linear operator in X with domain $D(A)$, generating a strongly continuous semigroup $S(t)$, that is, $A = \lim\limits_{h\longrightarrow 0} \dfrac{S(h)-I}{h}$ in the strong topology. $L_2([t,s],X)$ denotes the set of all strongly measurable L_2-integrable and $X-$ valued functions on $[t,s]$. Let $Q \in L(X,X^*)$ be a duality operator. We recall that the operator Q is positive definite in X if $\langle Qx, x\rangle \geq 0$ for arbitrary $x \in X$, and $\langle Qx, x\rangle > 0$ for $x \neq 0$. In the case if $\langle Qx, x\rangle \geq c\|x\|^2$ for some $c > 0$ we say that Q is strongly positive definite. We will denote by $LPD(X,X^*)$ and $LSPD(X,X^*)$ the set of all linear bounded positive definite and strongly positive definite operators mapping X into X^*, respectively. Also, we define

- $L^p(\mathbb{R}_+,\mathbb{R}_+)$ as the set of functions positive and integrable with pth power on $\mathbb{R}_+$ where $p \geq 1$;
- $L^\infty(\mathbb{R}_+,\mathbb{R}_+)$ as the set of all measurable functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ which are essentially bounded;
- $\|\varphi\|_p = \left(\int_0^{+\infty} \varphi^p\right)^{\frac{1}{p}}$ for $\varphi \in L^p(\mathbb{R}_+,\mathbb{R}_+)$;
- $\|\varphi\|_\infty = \sup\limits_{t\in\mathbb{R}_+} \varphi(t)$ for $\varphi \in L^\infty(\mathbb{R}_+,\mathbb{R}_+)$.
- $\mathbf{1}_{[\vartheta,\zeta]} = \begin{cases} 1, & \text{si } \vartheta \leq x \leq \zeta \\ 0, & \text{elsewhere.} \end{cases}$

We consider the system:

$$\begin{cases} \dot{x} = F(t, x, u), \quad t \geq t_0 \geq 0 \\ \\ x(t_0) = x_0 \end{cases} \tag{1}$$

where $x \in X$ is the system state, $u(t) \in U$ is the control input. $F : \mathbb{R}_+ \times X \times U \to X$ is a given function.

Definition 1 *System (1) is practically stabilizable if there exists a continuous feedback control $u : X \to U$, such that system (1) with $u(t) = u(x(t))$ satisfies the following properties:*
(i) For any initial condition $x_0 \in X$, there exists a unique mild solution $x(t, x_0)$ defined on $\mathbb{R}_+$.
(ii) There exist positive scalars ω, k, r, such that the solution of the system (1) satisfies

$$\|x(t)\| \leq k\|x_0\|e^{-\omega(t-t_0)} + r, \quad \forall t \geq t_0 \geq 0.$$

When (i) and (ii) are satisfied for (3), we say that (1) with $u(t) = u(x(t))$ is globally practically uniformly exponentially stable.

Definition 2 *(See [6]) A Banach space X^* has the Radon-Nikodym property if*

$$L_2([0, T], X^*) = (L_2([0, T], X))^*.$$

In the proof of the main results, we shall use the following lemmas.

Lemma 1 *(Nonlinear generalization of Gronwall's inequality)(See [7])*
Let θ be a non-negative function on $\mathbb{R}_+$, that satisfies the following integral inequality

$$\theta(t) \leq \nu + \int_{t_0}^{t} \Big(\chi(s)\theta(s) + \sigma(s)\theta^{\alpha}(s)\Big)ds, \quad \nu \geq 0, \quad 0 \leq \alpha < 1, \quad t \geq t_0 \geq 0$$

where χ and σ are non-negative continuous functions. Then,

$$\theta(t) \leq \left[\nu^{1-\alpha}e^{(1-\alpha)\int_{t_0}^{t}\chi(s)ds} + (1-\alpha)\int_{t_0}^{t}\sigma(s)e^{(1-\alpha)\int_{s}^{t}\chi(r)dr}ds\right]^{\frac{1}{1-\alpha}}.$$

Lemma 2 *(Generalized Gronwall-Bellman Inequality)(See [24]))*
Let $\lambda, \rho : \mathbb{R}_+ \to \mathbb{R}$ be continuous functions and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function, such that

$$\dot{\varphi}(t) \leq \lambda(t)\varphi(t) + \rho(t), \quad \forall\ t \geq 0. \tag{2}$$

Then, for any $t \geq t_0 \geq 0$, we have the following inequality

$$\varphi(t) \leq \varphi(t_0)e^{\int_{t_0}^{t}\lambda(v)dv} + \int_{t_0}^{t} e^{\int_{s}^{t}\lambda(v)dv}\rho(s)ds.$$

Lemma 3 *Let $a,\ b \geq 0$ and $p \geq 1$. Then,*

$$(a + b)^p \leq 2^{p-1}(a^p + b^p).$$

3 Main Results

In this section, we shall state and prove our main results.

3.1 Practical Stabilization of infinite-dimensional evolution equations

The purpose of this section is to establish the practical stabilization of evolution equations in Banach spaces. Based on the exact null-controllability in finite time of the nominal system whose origin is an equilibrium point, a stabilizing controller for the nonlinear system is then synthesized. This leads us to address the problem of practical stability of time-varying perturbed systems.
Consider infinite-dimensional evolution equations of the form

$$\begin{cases} \dot{x} = Ax + Bu + F(t,x), \quad t \geq t_0 \geq 0 \\ \\ x(t_0) = x_0 \end{cases} \tag{3}$$

where $x \in X$ is the system state, $u \in U$ is the control input, X is a Banach space, X^* has the Radon-Nikodym property and U is a Hilbert space. Further, the operator $A : D(A) \subset X \to X$ is assumed to be the infinitesimal generator of the C_0- semigroup $S(t)$ on X, $B \in L(U,X)$ and the function $F : \mathbb{R}_+ \times X \longrightarrow X$ is continuous in t and locally Lipschitz continuous in x, uniformly in t on bounded intervals, that is, for every $t_1 \geq 0$ and constant $c \geq 0$, there is a constant $L(c,t_1)$, such that

$$\|F(t,u) - F(t,v)\| \leq L(c,t_1)\|u - v\|$$

holds for all $u, v \in X$, with $\|u\| \leq c$, $\|v\| \leq c$, and $t \in [0,t_1]$.
This system is seen as a perturbation of the nominal system

$$\begin{cases} \dot{x} = Ax + Bu, \quad t \geq 0 \\ \\ x(0) = x_0, \end{cases} \tag{4}$$

Next, we are interested in suitable feedback of the form

$$u(t) = Dx(t), \tag{5}$$

where $D \in L(X,U)$.
Let $x(t) = x(t,x_0,u)$ denote the state of a system (3) at moment $t \geq t_0 \geq 0$ associated with an initial condition $x_0 \in X$ at $t = t_0$ and input $u \in U$.
Now, we recall the definition of the generator of an exponentially stable semi-group as well as that of the exponential stability, see Curtain and Zwart [2] for details.

Definition 3 *The operator A generates an exponentially stable semigroup $S(t)$ if the initial value problem*

$$\dot{x}(t) = Ax, \quad t \geq 0, \quad x(0) = x_0 \tag{6}$$

has a unique solution $x(t) = S(t)x_0$, and $\|S(t)\| \leq Me^{-\alpha t}$, for all $t \geq 0$ with some positive numbers M and α.

Definition 4 *The linear control system (6) is exponentially stable if there exist numbers $M > 0$ and $\alpha > 0$, such that*

$$\|x(t)\| \leq Me^{-\alpha t}\|x_0\|, \quad \forall t \geq 0.$$

Definition 5 *The control system (4) is exactly null-controllable in finite time if for every $x_0 \in X$, there exist a number $T > 0$ and an admissible control $u(t) \in \mathcal{U} = \{u(.) \in L_2([0,\infty), U)\}$, such that*

$$S(T)x_0 + \int_0^T S(T-s)Bu(s)ds = 0.$$

Furthermore, if we designate by $\mathcal{C}_T$ the set of null-controllable points in time T of system (4) defined by

$$\mathcal{C}_T = \{x_0 \in X; \ \ S(T)x_0 = -\int_0^T S(T-s)Bu(s)ds; \ \ u(.) \in \mathcal{U}\}.$$

The system (4) is exactly null-controllable in time $T > 0$ if $\mathcal{C}_T = X$.
In the case A is the generator of an analytic semigroup $S(t)$, for $T > 0$, we can define the operator $W_T \in L(\mathcal{U}, X)$ by

$$W_T(u) = \int_0^T S^{-1}(s)Bu(s)ds, \quad u(.) \in \mathcal{U},$$

and we then have $C_T = ImW_T$. We state the following well-known controllability criterion for infinite-dimensional control system (4) presented in [1] for reflexive Banach spaces and then in [22] for non-reflexive Banach spaces having the Radon-Nikodym property.

Proposition 1 *(see [1] and [22]) Let X, U be Banach spaces, $S(t)$ the C_0- semigroup of A. Assume that X^*, U^* have the Radon-Nikodym property. The following conditions are equivalent.*
(i) Control system (4) is exactly null-controllable in time $T > 0$.
(ii) There exists $c > 0$, $\|W_T^* x^*\| \geq c\|x^*\|, \quad \forall x^* \in X^*$.
(iii) There exists $c > 0$, $\|B^*S^*(s)x^*\|^2 \geq c\|S^*(T)x^*\|, \quad \forall x^* \in X^*$.
(iv) If U is a Hilbert space, the operator $W_T = \int_0^T S^{-1}(s)BB^*S^{*-1}(s)ds$ is strongly positive definite.

The operator $P \in L(X, X^*)$ is called a solution of the Lyapunov equation if the following condition hold:

$$\langle PAx, x\rangle + \langle Px, Ax\rangle = -\langle Qx, x\rangle, \quad \forall x \in D(A). \tag{7}$$

Note that, if A is bounded, then the above equation (7) has the standard form

$$A^*Px + PAx = -Qx, \quad \forall x \in X.$$

Remark 1 *The author in [5] showed that if A is exponentially stable in Hilbert space, then the Lyapunov equation has a solution.*

In Proposition 2 below we present the equivalence between the solvability of the Lyapunov equation and the exponential stability of the linear system (6).

Proposition 2 *(See [19]) If for some $Q \in LSPD(X, X^*)$, $P \in LPD(X, X^*)$, the Lyapunov equation holds, then the operator A is exponentially stable. Conversely, if the generator A is exponentially stable, then for any $Q \in LSPD(X, X^*)$, there is a solution $P \in LPD(X, X^*)$ of the Lyapunov equation*

$$A^*P + PA = -Q. \tag{8}$$

Definition 6 *The linear control system (4) is completely stabilisable if for every $\alpha > 0$, there exists a linear bounded operator $D : X \to U$ and a number $M > 0$, such that the solution satisfies the condition:*

$$\|x(t)\| \leq Me^{-\alpha t}\|x_0\|, \quad \forall t \geq 0.$$

Note that, if the operator D and number M do not depend on α, then the complete stabilizability implies exponential stabilizability in usual Lyapunov sens (see Zabczyk [23]). It is known from that if the linear control system (6), where X and U are Hilbert spaces is completely stabilizable then it is exactly null-controllable in finite time (see Megan[14]). Also, Phat and Kiet [19] improved this result in Banach spaces.

Proposition 3 *If linear control system (6) is completely stabilisable then it is exactly null-controllable in finite time.*

In the sequel, Phat and Kiet [19] proved that the linear control system (6) is exponentially stabilizable by linear feedback control $D : X \to U$, if it is null-controllable in finite time.

Proposition 4 *If the linear control system (4) is exactly null-controllable in finite time, then the linear time-varying control system (4) is exponentially stabilizable.*

In what follows, we shall that $V : X \to \mathbb{R}_+$ is a Lyapunov function.

Definition 7 *The Lie derivative of V corresponding to the input u is defined by*

$$\dot{V}(x) = \overline{\lim}_{t \longrightarrow 0^+} \frac{1}{t}(V(x(t,x,u) - V(x)).$$

Now, we suppose the following assumptions.

($\mathcal{H}_1$) The linear system (4) is exactly null-controllable in finite time, there exists a constant operator $D : X \to U$, such that a sufficient condition specially related to operator $A_D = A + BD$ is presented in Phat and Kiet [19] as the following: for any $Q \in LSPD(X, X^*)$,

$$\langle Qx, x\rangle \geq b_1\|x\|^2, \quad \forall x \in X,$$

there exists $P \in LPD(X, X^*)$,

$$b_2\|x\|^2 \leq \langle Px, x\rangle \leq \|P\|\|x\|^2, \quad \forall x \in X,$$

where $b_1, b_2 > 0$, which satisfies

$$A_D^* P + PA_D = -Q. \tag{9}$$

($\mathcal{H}_2$) The perturbation term $F : \mathbb{R}_+ \times X \longrightarrow X$ verifies the following condition:

$$\|F(t,x)\| \leq \varpi(t)\|x\| + \mu(t) + \eta, \quad \forall t \geq 0, \forall x \in X, \eta \geq 0, \tag{10}$$

where ϖ and μ are non-negative continuous functions, with $\varpi \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\mu \in L^p(\mathbb{R}_+, \mathbb{R}_+)$ for some $p \in [1, +\infty)$.

Next, sufficient conditions are presented to guarantee the global existence and uniqueness of solutions of systems (3). Further, we investigate the practical stabilizability of the evolution equation using Generalized Gronwall-Bellman Inequality and Lyapunov's techniques.

Theorem 1 *Under assumptions* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$, *the closed-loop system* (3)-*(5) has a unique solution, which is globally defined for all* $t \geq t_0$ *and this system is globally practically uniformly exponentially stable.*

Proof. We break up the proof into two steps.

1. Since F is a locally Lipschitz continuous in x, uniformly in t, it follows from Pazy [15] that for every initial condition the closed-loop equation possesses a unique mild solution on some interval $[t_0, t_0 + \delta]$, with $\delta > 0$. Indeed, integrating (3), we obtain

$$x(t) = S(t - t_0)x_0 + \int_{t_0}^{t} S(t - s)[Bu(s) + F(s, x(s))]ds, \quad t_0 \leq t \leq t_0 + \delta.$$

   Since $B \in L(U, X)$, then

$$\|x(t)\| \leq M\|x_0\| + M\left(\int_{t_0}^{t} \|B\|\|D\|\|x(s)\| + M_1\|x(s)\| + M_2 + \eta\right) ds \tag{11}$$

   where $M = \sup\{\|S(t - s)\| : \ 0 \leq t_0 \leq s \leq t \leq t + \delta\}$, $M_1 = \sup\limits_{t \in [t_0, t_0+\delta]} \|\varpi(t)\|$, and $M_2 = \sup\limits_{t \in [t_0, t_0+\delta]} \|\mu(t)\|$. By applying Gronwall inequality (see [20], Lemma 2.7, p42) to inequality (11), any solution of this equation is uniformly bounded

$$\|x(t)\| \leq M(\|x_0\| + M_2 + \eta)e^{(\|B\|\|D\|+M_1)M\delta},$$

   on an arbitrary time interval $[t_0, t_0 + \delta]$. Then, using Theorem 1.4 in [15], we have $t_0 + \delta = \infty$, and so we get global existence.

2. Consider a Lyapunov function:

$$V(x) = \langle Px, x\rangle.$$

   Let us compute the Lie derivative of V with respect to system (3) in closed-loop with the controller (5). For $x \in D(A)$, we have

$$\dot{V}(x) = \langle P\dot{x}, x\rangle + \langle Px, \dot{x}\rangle$$

   From $\langle P(Ax), x\rangle = \langle Ax, Px\rangle$, and (9) with the help of Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
\dot{V}(x) &\leq -\langle Qx, x\rangle + 2\|P\|\|F(t, x)\|\|x\| \\
&\leq -b_1\|x\|^2 + 2\|P\|(\varpi(t)\|x\| + \mu(t) + \eta)\|x\| \\
&\leq \left(-\frac{b_1}{\|P\|} + \frac{2\|P\|\varpi(t)}{b_2}\right) V(x) + \frac{2\|P\|}{\sqrt{b_2}}(\mu(t) + \eta)\sqrt{V(x)}.
\end{aligned}$$

   Let,

$$z(x) = \sqrt{V(x)}.$$

   Then,

$$\dot{z}(x) \leq \left(-\frac{b_1}{2\|P\|} + \frac{\|P\|\varpi(t)}{b_2}\right) z(x) + \frac{\|P\|}{\sqrt{b_2}}(\mu(t) + \eta). \tag{12}$$

These derivations hold for $x \in D(A) \subset X$ If $x \notin D(A)$, then the solution $x(t) \in D(A)$ and $t \to z(x(t))$ is a continuously differentiable function for all $t \geq t_0$ (these properties follow from the properties of solutions $x(t)$, see Theorem 3.3.3 in [9]). Thus, by the mean value theorem we obtain that (12) holds for all $x \in X$.
Using Lemma 2, we obtain for all $t \geq t_0$

$$z(x) \leq z(x_0) e^{\frac{\|P\| M_\varpi}{b_2}} e^{-\frac{b_1}{2\|P\|}(t-t_0)} + e^{\frac{\|P\| M_\varpi}{b_2}} \int_{t_0}^{t} \frac{\|P\|}{\sqrt{b_2}} (\mu(s)+\eta) e^{-\frac{b_1}{2\|P\|}(t-s)} ds,$$

with $M_\varpi = \displaystyle\int_0^\infty \varpi(s) ds$. We discriminate three cases:

(1) if $p = 1$, we get

$$\int_{t_0}^{t} (\mu(s)+\eta) e^{-\frac{b_1}{2\|P\|}(t-s) ds} \leq \|\mu\|_1 + \frac{2\|P\|\eta}{b_1}.$$

Then, for all $t \geq t_0$,

$$\|x(t)\| \leq \sqrt{\frac{\|P\|}{b_2} e^{\frac{\|P\| M_\varpi}{b_2}} \|x_0\| e^{-\frac{b_1}{2\|P\|}(t-t_0)} + \frac{\|P\|}{b_2} e^{\frac{\|P\| M_\varpi}{b_2}} \left(\|\mu\|_1 + \frac{2\|P\|\eta}{b_1} \right)}.$$

(2) If $p \in (1, +\infty)$ and $q > 0$, such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, we have by applying Holder Inequality

$$\int_{t_0}^{t} (\mu(s)+\eta) e^{-\frac{b_1}{2\|P\|}(t-s) ds} \leq \left(\frac{2\|P\|}{b_1 q} \right)^{\frac{1}{q}} \|\mu\|_p + \frac{2\|P\|\eta}{b_1}.$$

Therefore, for all $t \geq t_0$, the solution $x(t)$ verifies the estimation

$$\|x(t)\| \leq \sqrt{\frac{\|P\|}{b_2} e^{\frac{\|P\| M_\varpi}{b_2}} \|x_0\| e^{-\frac{b_1}{2\|P\|}(t-t_0)} + \frac{\|P\|}{b_2} e^{\frac{\|P\| M_\varpi}{b_2}} \left(\left(\frac{2\|P\|}{b_1 q} \right)^{\frac{1}{q}} \|\mu\|_p + \frac{2\|P\|\eta}{b_1} \right)}.$$

(3) If $p = +\infty$. Then, we have

$$\int_{t_0}^{t} e^{-\frac{b_1}{2\|P\|}(t-s)} \mu(s) ds \leq \left(\frac{2\|P\|}{b_1} \right) \|\mu\|_\infty.$$

One can get, for all $t \geq t_0$

$$\|x(t)\| \leq \sqrt{\frac{\|P\|}{b_2} e^{\frac{\|P\| M_\varpi}{b_2}} \|x_0\| e^{-\frac{b_1}{2\|P\|}(t-t_0)} + \frac{\|P\|}{b_2} e^{\frac{\|P\| M_\varpi}{b_2}} \left(\left(\frac{2\|P\|}{b_1} \right) \|\mu\|_\infty + \frac{2\|P\|\eta}{b_1} \right)}.$$

We conclude that, the system (3) in closed-loop with the controller (5) is globally practically uniformly exponentially stable. This completes the proof. □
As a consequence of Theorem 1, we have the following corollary.

Corollary 1 *We consider the dynamical system* (3). *Assume that* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *are fulfilled, with*

$$\|F(t,x)\| \leq \mu(t), \quad \forall\, t \geq 0, \quad \forall x \in X,$$

where μ *is a non-negative continuous function on* $\mathbb{R}_+$, *such that* $\mu \in L^p(\mathbb{R}_+, \mathbb{R}_+)$ *for some* $p \in [1, +\infty)$. *Then, the system* (3) *in closed-loop with the controller* (5) *is globally practically uniformly exponentially stable.*

We can state other assumptions to obtain the global existence, uniqueness and the practical stabilizability for the evolution equation (3) under a restriction about the perturbed term bounded by the sum of Holder continuous function and a Lipschitz function.

($\mathcal{H}_3$) There exists a non-negative constant $0 < \alpha < 1$, such that the perturbation term $F : \mathbb{R}_+ \times X \longrightarrow X$ satisfies the following condition:

$$\|F(t,x)\| \leq \phi(t)\|x\|^\alpha + \sigma(t)\|x\|, \quad \forall t \geq 0, \quad \forall x \in X,$$

where ϕ, σ are non-negatives continuous functions, with $\sigma \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\phi \in L^p(\mathbb{R}_+, \mathbb{R}_+)$ for some $p \in [1, +\infty)$.

Next, one has the following theorem.

Theorem 2 *If assumptions ($\mathcal{H}_1$) and ($\mathcal{H}_3$) are fulfilled, the closed-loop system* (3)*-(5) has a unique solution, which is globally defined for all $t \geq t_0$ and this system is globally practically uniformly exponentially stable.*

Proof. We break up the proof into two steps.

1. Since F is a locally Lipschitz continuous in x, uniformly in t, it follows from Pazy [15] that for every initial condition the closed-loop equation possesses a unique mild solution on some interval $[t_0, t_0 + \delta]$, with $\delta > 0$. Indeed, integrating (3), we obtain

$$x(t) = S(t - t_0)x_0 + \int_{t_0}^{t} S(t - s)[Bu(s) + F(s, x(s))]ds, \quad t_0 \leq t \leq t_0 + \delta.$$

Since $B \in L(U, X)$, then

$$\|x(t)\| \leq M\|x_0\| + M \int_{t_0}^{t} (\|B\|\|D\|\|x(s)\| + M_1\|x(s)\|^\alpha + M_2\|x(s)\|)ds \tag{13}$$

where $M = \sup\{\|S(t - s)\| : \ 0 \leq t_0 \leq s \leq t \leq t + \delta\}$, $M_1 = \sup\limits_{t \in [t_0, t_0+\delta]} \|\phi(t)\|$, and $M_2 = \sup\limits_{t \in [t_0, t_0+\delta]} \|\sigma(t)\|$. By applying Lemmas 1 and 3 to inequality (13), any solution of this equation is uniformly bounded

$$\|x(t)\| \leq 2^{\frac{\alpha}{1-\alpha}} e^{M\delta(\|B\|\|D\|+M_2)}(M\|x_0\| + (MM_1\delta(1 - \alpha))^{\frac{1}{1-\alpha}}),$$

on an arbitrary time interval $[t_0, t_0 + \delta]$. Applying Theorem 1.4 in [15], we have $t_0 + \delta = \infty$, and so we obtain global existence.

2. Define the function $V : D(A) \to \mathbb{R}_+$ by

$$V(x) = \langle Px, x\rangle.$$

Then, the Lie derivative of V in t along the solution of the system (3) in closed-loop with the controller (5) leads to

$$\begin{array}{rcl} \dot{V}(x) & = & \langle P\dot{x}, x\rangle + \langle Px, \dot{x}\rangle \\ & \leq & -\langle Qx, x\rangle + 2\|P\|\|F(t,x)\|\|x\| \\ & \leq & -b_1\|x\|^2 + 2\|P\|(\phi(t)\|x\|^\alpha + \sigma(t)\|x\|)\|x\| \\ & \leq & \left(-\dfrac{b_1}{\|P\|} + \dfrac{2\|P\|\sigma(t)}{b_2}\right) V(x) + \dfrac{2\|P\|\phi(t)}{\sqrt{b_2}^{\alpha+1}} V(x)^{\frac{\alpha+1}{2}}. \end{array}$$

Let,

$$\vartheta(x) = V(x)^{\frac{1-\alpha}{2}},$$

which implies that

$$\dot{\vartheta}(x) \leq -\frac{(1-\alpha)}{2}\left(\frac{b_1}{\|P\|} - \frac{2\|P\|\sigma(t)}{b_2}\right)\vartheta(x) + \frac{\|P\|(1-\alpha)\phi(t)}{\sqrt{b_2}^{\alpha+1}}, \quad \forall x \in X, \quad \forall t \geq t_0.$$

Using Lemma 2, we get for all $t \geq t_0$

$$\vartheta(x) \leq e^{\frac{\|P\|(1-\alpha)N_\sigma}{b_2}}\left(\vartheta(x_0)e^{-\frac{b_1(1-\alpha)}{2\|P\|}(t-t_0)} + \frac{\|P\|(1-\alpha)}{\sqrt{b_2}^{\alpha+1}}\int_{t_0}^{t} e^{-\frac{b_1(1-\alpha)}{2\|P\|}(t-s)}\phi(s)ds\right),$$

with $N_\sigma = \displaystyle\int_0^\infty \sigma(s)ds$. We discriminate three cases:

(1) If $p = 1$, we get

$$\vartheta(x) \leq e^{\frac{\|P\|(1-\alpha)N_\sigma}{b_2}}\left(\vartheta(x_0)e^{-\frac{b_1(1-\alpha)}{2\|P\|}(t-t_0)} + \frac{\|P\|(1-\alpha)}{\sqrt{b_2}^{\alpha+1}}\|\phi\|_1 ds\right).$$

It follows that,

$$\|x(t)\|^{1-\alpha} \leq e^{\frac{\|P\|(1-\alpha)N_\sigma}{b_2}}\left(\sqrt{\frac{\|P\|}{b_2}}^{1-\alpha}\|x_0\|^{1-\alpha}e^{-\frac{b_1(1-\alpha)}{2\|P\|}(t-t_0)} + \frac{1}{\sqrt{b_2}^{1-\alpha}}\frac{\|P\|(1-\alpha)}{\sqrt{b_2}^{\alpha+1}}\|\phi\|_1 ds\right).$$

From Lemma 3, we obtain for all $t \geq t_0$

$$\|x(t)\| \leq 2^{\frac{\alpha}{1-\alpha}}e^{\frac{\|P\|N_\sigma}{b_2}}\left(\sqrt{\frac{\|P\|}{b_2}}\|x_0\|e^{-\frac{b_1}{2\|P\|}(t-t_0)} + \frac{1}{\sqrt{b_2}}\left(\frac{\|P\|(1-\alpha)}{\sqrt{b_2}^{\alpha+1}}\|\phi\|_1\right)^{\frac{1}{1-\alpha}}\right).$$

(2) If $p \in (1, +\infty)$ and $q > 0$, such that $\frac{1}{p} + \frac{1}{q} = 1$, we have by applying Holder Inequality

$$\vartheta(x) \leq e^{\frac{\|P\|(1-\alpha)N_\sigma}{b_2}}\left(\vartheta(x_0)e^{-\frac{b_1(1-\alpha)}{2\|P\|}(t-t_0)} + \frac{\|P\|(1-\alpha)}{\sqrt{b_2}^{\alpha+1}}\|\phi\|_p\left(\frac{2\|P\|}{q(1-\alpha)b_1}\right)^{\frac{1}{q}}\right).$$

By using Lemma 3, we obtain for all $t \geq t_0$

$$\begin{aligned}\|x(t)\| &\leq 2^{\frac{\alpha}{1-\alpha}}e^{\frac{\|P\|N_\sigma}{b_2}}\sqrt{\frac{\|P\|}{b_2}}\|x_0\|e^{-\frac{b_1}{2\|P\|}(t-t_0)} \\ &+ \frac{2^{\frac{\alpha}{1-\alpha}}e^{\frac{\|P\|N_\sigma}{b_2}}}{\sqrt{b_2}}\left(\frac{\|P\|(1-\alpha)}{\sqrt{b_2}^{\alpha+1}}\|\phi\|_p\right)^{\frac{1}{1-\alpha}}\left(\frac{2\|P\|}{q(1-\alpha)b_1}\right)^{\frac{1}{q(1-\alpha)}}.\end{aligned}$$

(3) If $p = +\infty$, then,

$$\|x(t)\| \leq 2^{\frac{\alpha}{1-\alpha}}e^{\frac{\|P\|N_\sigma}{b_2}}\left(\sqrt{\frac{\|P\|}{b_2}}\|x_0\|e^{-\frac{c_1}{2\|P\|}(t-t_0)} + \frac{1}{\sqrt{b_2}}\left(\frac{2\|P\|^2}{b_1\sqrt{b_2}^{\alpha+1}}\|\phi\|_\infty\right)^{\frac{1}{1-\alpha}}\right).$$

We deduce that, the system (3) in closed-loop with the controller (5) is globally practically uniformly exponentially stable. This ends the proof. $\square$

For perturbed time-varying systems (3) in finite-dimensional spaces, we also have the following consequence.

Corollary 2 *(See [8]) Assume that $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, and the assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_3)$ are satisfied, then the system* (3) *with the controller* (5) *is globally practically uniformly exponentially stable.*

4 Feedback control of uncertain systems

Let X be a Banach space, X^* has the Radon-Nikodym property and U is a Hilbert space. We consider the uncertain dynamical system:

$$\begin{cases} \dot{x} = Ax + Bu(t) + G(t,x,u), \quad t \geq t_0 \geq 0 \\ \\ x(t_0) = x_0, \end{cases} \tag{14}$$

where $x \in X$ is the system state, $u \in U$ is the control input, A is the infinitesimal generator of the C_0- semigroup $S(t)$ on a Banach space X, $B \in L(U,X)$ and $G : \mathbb{R}_+ \times X \times U \to X$ is continuous in t and locally Lipschitz continuous in x, uniformly in $u \in U$ and t on bounded intervals, that is, for every $t_1 \geq 0$ and constant $c \geq 0$, there is a constant $L(c,t_1)$, such that

$$\|G(t,x,u) - G(t,y,u)\| \leq L(c,t_1)\|x-y\|$$

holds for all $x, y \in X$, with $\|x\| \leq c$, $\|y\| \leq c$, and $t \in [0,t_1]$.

Let $x(t) = x(t,x_0,u)$ denote the state of a system (14) at moment $t \geq t_0 \geq 0$ associated with an initial condition $x_0 \in X$ at $t = t_0$ and input $u \in U$.

Before giving our syntaxes approach, we state the following standard assumption.

$(\mathcal{H}_4)$ The perturbation term $G : \mathbb{R}_+ \times X \times U \to X$ satisfies the following condition:

$$\exists a, b > 0, \|G(t,x,u)\| \leq a\|x\| + b\|u\| + \varrho(t) + \varepsilon, \quad \forall t \geq 0, \quad \forall x \in X, \quad \varepsilon \geq 0, \tag{15}$$

where ϱ is a non-negative continuous function, with $\varrho \in L^p(\mathbb{R}_+,\mathbb{R}_+)$ for some $p \in [1,+\infty)$.

The following lemma proved sufficient conditions for the global existence and uniqueness of solutions of system (14).

Lemma 4 *Under assumption $(\mathcal{H}_4)$, the closed-loop system (5)-*(14) *has a unique solution, which is globally defined for all $t \geq t_0$.*

Proof. G is a locally Lipschitz continuous in x, uniformly in $u \in U$ and t, it follows from Pazy [15] that for every initial condition the closed-loop equation possesses a unique mild solution on some interval $[t_0, t_0 + \delta]$, with $\delta > 0$. Indeed, integrating (14), we obtain

$$x(t) = S(t-t_0)x_0 + \int_{t_0}^{t} S(t-s)[Bu(s) + G(s,x(s),u(s))]ds, \quad t_0 \leq t \leq t_0 + \delta.$$

Since $B \in L(U,X)$, then by applying Gronwall inequality (see [20], Lemma 2.7, p42), we have

$$\|x(t)\| \leq M(\|x_0\| + M_1\delta + M\delta)e^{M\delta(\|B\|\|D\|+a+b\|D\|)},$$

where $M = \sup\{\|S(t-s)\| : \ 0 \leq t_0 \leq s \leq t \leq t+\delta\}$ and $M_1 = \sup_{t\in[t_0,t_0+\delta]} \|\varrho(t)\|$ on an arbitrary time interval $[t_0, t_0+\delta]$. Now, Pazy (see [15], Theorem 1.4) gives that $t_0+\delta = \infty$, and so we have global existence. The proof is completed. $\square$

The next theorem shows the practical stabilization of the system (14) using Lyapunov indirect method and Gronwall-Bellman Inequality.

Theorem 3 *Assume that A is exponentially stable and the assumption $(\mathcal{H}_4)$ is satisfied. Let $P, Q \in LPD(X, X^*)$ be the operators satisfying the Lyapunov equation (8) where $P = P^*$ and $< Qx, x > \geq \lambda\|x\|^2$, for all $x \in X, \lambda > 0$. Then, the nonlinear control system (14) is practically stabilizable by the feedback control $u(t) = -\rho B^* Px(t)$ if*

$$\rho < \frac{\lambda - 2a\|P\|}{2b\|B\|\|P\|^2}. \tag{16}$$

Proof. Let $P \in LPD(X, X^*)$ be an operator which is a solution of the Lyapunov equation (8). Define the Lyapunov function $V : D(A) \to \mathbb{R}_+$ by

$$V(x) = \langle Px, x\rangle.$$

Noting that, there exists $\alpha > 0$, such that

$$\alpha\|x\|^2 \leq V(x) \leq \|P\|\|x\|^2, \quad \alpha > 0.$$

Then, the Lie derivative of V in t along the solution of $x(t)$ of the system (14) and using the chosen feedback control and the Lyapunov equation is given by

$$\begin{aligned}\dot{V}(x) &= \langle P\dot{x}, x\rangle + \langle Px, \dot{x}\rangle \\ &= -\langle Qx, x\rangle - \rho\langle PBB^*Px, x\rangle - \rho\langle Px, BB^*Px\rangle \\ &+ \langle PG(t,x,u), x > + < Px, G(t,x,u)\rangle.\end{aligned}$$

Since P is self-adjoint, by assumption $(\mathcal{H}_4)$ and condition (16), we have for all $t \geq t_0$

$$\dot{V}(x) \leq -\kappa V(x) + \frac{2\|P\|}{\sqrt{\alpha}}(\delta(t)+\varepsilon)\sqrt{V(x)},$$

where

$$\kappa = \frac{\lambda - 2b\rho\|P\|^2\|B\| - 2a\|P\|}{\|P\|} > 0.$$

Let

$$v(x) = \sqrt{V(x)}.$$

Then,

$$\dot{v}(x) \leq -\frac{\kappa}{2}v(x) + \frac{\|P\|}{\sqrt{\alpha}}(\varrho(t)+\varepsilon), \quad \forall x \in X, \quad \forall t \geq t_0.$$

Applying Lemma 2, we obtain

$$v(x) \leq v(x_0)e^{-\frac{\kappa}{2}(t-t_0)} + \frac{\|P\|}{\sqrt{\alpha}}\int_{t_0}^{t} e^{\frac{\kappa}{2}(s-t)}(\varrho(s)+\varepsilon)ds, \quad \forall t \geq t_0.$$

We distinguish three cases:

(1) if $p=1$, we get

$$\|x(t)\| \leq \sqrt{\frac{\|P\|}{\alpha}}\|x_0\|e^{-\frac{\kappa}{2}(t-t_0)} + \frac{\|P\|}{\alpha}\left(\|\varrho\|_1 + \frac{2\varepsilon}{\kappa}\right), \quad \forall t \geq t_0.$$

(2) If $p \in (1,+\infty)$ and $q>0$, such that $\dfrac{1}{p}+\dfrac{1}{q}=1$, we get by applying Holder Inequality

$$\|x(t)\| \leq \sqrt{\frac{\|P\|}{\alpha}}\|x_0\|e^{-\frac{\kappa}{2}(t-t_0)} + \frac{\|P\|}{\alpha}\left(\left(\frac{2}{q\kappa}\right)^{\frac{1}{q}}\|\varrho\|_p + \frac{2\varepsilon}{\kappa}\right), \quad \forall t \geq t_0.$$

(3) If $p=+\infty$. Then, we have

$$\|x(t)\| \leq \sqrt{\frac{\|P\|}{\alpha}}\|x_0\|e^{-\frac{\kappa}{2}(t-t_0)} + \frac{2\|P\|}{\alpha\kappa}\left(\|\varrho\|_\infty + \varepsilon\right), \quad \forall t \geq t_0.$$

We deduce that, the system (14) is practically stabilizable. This ends the proof. $\square$
Next, we denote another sufficient condition for the practical stabilizability of system (14) in the case A is not exponentially stable and it is a generator of bounded C_0- semigroup, but the associated linear control system (4) is exactly null-controllable in finite time and the nonlinear perturbation satisfies a condition.

Theorem 4 *Assume that the linear control system (4) is exactly null-controllable in finite time, then the system (14) is practically stabilizable for some appropriate numbers $a, b > 0$, satisfying the condition (15).*

Proof. The linear control system (4) is exactly null-controllable in finite time, then from Proposition 4 there is an operator $D \in L(X,U)$, such that the operator $W_L = A + BD$ is exponentially stable. Let $P, Q \in LPD(X, X^*)$ be the operators satisfying the Lyapunov equation (8) where $P = P^*$ and $\langle Qx, x\rangle \geq \lambda\|x\|^2$, for all $x \in X$, and $\lambda > 0$. Consider the Lyapunov function:

$$V(x) = \langle Px, x\rangle.$$

We have,

$$\alpha\|x\|^2 \leq V(x) \leq \|P\|\|x\|^2, \quad \alpha > 0.$$

The Lie derivative of V along the trajectories of system (14) is given by

$$\begin{array}{rcl} \dot{V}(x) & \leq & -\lambda\|x\|^2 + 2\langle PG(t,x,Dx), x\rangle \\ & \leq & -\eta\|x\|^2 + 2\|P\|(\varrho(t)+\varepsilon), \end{array}$$

where $\eta = \lambda - 2(a\|P\| + b\|D\|)$. We take $a, b > 0$, such that $\eta > 0$, that is,

$$a\|P\| + b\|D\| < \frac{\lambda}{2}.$$

Let

$$v(x) = \sqrt{V(x)}.$$

Then,

$$\dot{v}(x) \leq -\frac{\eta}{2\|P\|}v(t) + \frac{\|P\|}{\sqrt{\alpha}}(\varrho(t)+\varepsilon), \quad \forall x \in X, \quad \forall t \geq t_0.$$

Using Lemma 2, we have

$$v(x) \leq v(x_0)e^{-\frac{\eta}{2\|P\|}(t-t_0)} + \frac{\|P\|}{\sqrt{\alpha}} \int_0^t e^{\frac{\eta}{2\|P\|}(s-t)}(\varrho(s)+\varepsilon)ds, \quad \forall t \geq t_0.$$

We distinguish three cases:

(1) if $p = 1$, we have for all $t \geq t_0$,

$$\|x(t)\| \leq \sqrt{\frac{\|P\|}{\alpha}}\|x_0\|e^{-\frac{\kappa}{2}(t-t_0)} + \frac{\|P\|}{\alpha}\left(\|\varrho\|_1 + \frac{2\|P\|\varepsilon}{\eta}\right).$$

(2) If $p \in (1, +\infty)$ and $q > 0$, such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, we obtain by applying Holder Inequality

$$\|x(t)\| \leq \sqrt{\frac{\|P\|}{\alpha}}\|x_0\|e^{-\frac{\kappa}{2}(t-t_0)} + \frac{\|P\|}{\alpha}\left(\left(\frac{2\beta}{q\eta}\right)^{\frac{1}{q}}\|\varrho\|_p + \frac{2\beta\varepsilon}{\eta}\right), \quad \forall t \geq t_0.$$

(3) If $p = +\infty$. Then, we have

$$\|x(t)\| \leq \sqrt{\frac{\|P\|}{\alpha}}\|x_0\|e^{-\frac{\kappa}{2}(t-t_0)} + \frac{2\|P\|^2}{\alpha\eta}\left(\|\varrho\|_\infty + \varepsilon\right), \quad \forall t \geq t_0.$$

We conclude that, the system (14) is practically stabilizable. This ends the proof. □

Remark 2 *The above results generalize theorems of stabilizability in [19] with $\delta(t) = 0$, and $\varepsilon = 0$.*

5 Examples

In this section, we give some illustrating examples to illustrate the effectiveness of the results obtained in the present paper.

Example 1 *We Consider the controlled metal bar:*

$$\begin{cases} \dfrac{\partial x(\zeta,t)}{\partial t} = \dfrac{\partial^2 x(\zeta,t)}{\partial \zeta^2} + \mathbf{1}_{[\frac{1}{2},1]}u(t) + \dfrac{1}{1+t^2}x(\zeta,t) + \dfrac{1+t}{(1+t^2)(1+\|x(\zeta,t)\|)}, \quad x(\zeta,0) = x_0(\zeta), \\ \dfrac{\partial x}{\partial \zeta}(0,t) = 0 = \dfrac{\partial x}{\partial \zeta}(1,t), \quad (t \geq 0) \end{cases} \tag{17}$$

$x(\zeta,t)$ represents the temperature at position ζ at time $t \geq 0$ and z_0 represents the initial temperature profile, $u(t)$ the addition of heat along the bar. The two boundary conditions state that there is no heat flow at the boundary, and thus the bar is insulated. The partial differential equation can be formulated to an abstract differential equation on $X = L^2(0,1)$ of the form

$$\dot{x} = Ax + Bu + F(t,x), \quad t \geq 0, \quad x(0) = x_0,$$

where $U = \mathbb{C}$, the operator $A = \dfrac{\partial^2}{\partial^2 \zeta}$, with

$$D(A) = \{h \in L^2(0,1), h, \frac{\partial h}{\partial \zeta} \text{ are absolutely continuous}, \frac{\partial^2 h}{\partial \zeta^2} \in L^2(0,1) \text{ and } \frac{dh}{d\zeta}(0) = 0 = \frac{dh}{d\zeta}(1)\},$$

$B = \mathbf{1}_{[\frac{1}{2},1]}$ *and* $F(t,x) = \dfrac{1}{1+t^2} x(\zeta,t) + \dfrac{1+t}{(1+t^2)(1+\|x(\zeta,t)\|)}$.
A possesses an orthonormal basis of eigenvector $\phi_0(\zeta) = 1$ and $\phi_n(\zeta) = \sqrt{2}\cos(n\pi\zeta), n \geq 1$. Furthermore, the semigroup $(S(t))_{t\geq 0}$ generated by A is given by

$$S(t)x = \sum_{n=0}^{\infty} e^{-n^2\pi^2 t} < x, \phi_n > \phi_n.$$

Using Proposition 1, it is easy to see that the nominal system of (17) is exactly null-controllable in finite time. Moreover, the assumption $(\mathcal{H}_2)$ is satisfied with $\eta = 0$ and $\varpi(t) = \dfrac{1}{1+t^2}$, $\mu(t) = \dfrac{1+t}{1+t^2}$ are non-negative functions with $\varpi \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\mu \in L^p(\mathbb{R}_+, \mathbb{R}_+)$ for some $p \in [1, +\infty)$. Then, all hypotheses of Theorem 1 are satisfied and the controlled heat equation (17) is practically stabilizable.

Example 2 *We Consider the controlled perturbed heat equation:*

$$\begin{cases} \dfrac{\partial x(\zeta,t)}{\partial t} = \dfrac{\partial^2 x(\zeta,t)}{\partial \zeta^2} + \dfrac{2+t^2}{1+t^2} u(t) + x(\zeta,t) + e^{-t}\mathbf{1}_{[0,\frac{\pi}{2}]}, \quad x(\zeta,0) = x_0(\zeta), \\ \\ x(0,t) = 0 = x(\pi,t), \quad (t \geq 0) \end{cases} \tag{18}$$

$x(\zeta,t)$ represents the temperature at position $\zeta \in [0,\pi]$ at time $t \geq 0$ and x_0 represents the initial temperature profile. The partial differential equation can be formulated to an abstract differential equation on $X = L^2(0,\pi)$ of the form

$$\dot{x} = Ax + Bu(t) + G(t,x,u), \quad t \geq 0, \quad x(0) = x_0,$$

where $U = \mathbb{C}$, the operator $A = \dfrac{\partial^2}{\partial^2 \zeta}$, with

$$D(A) = \{h \in L^2(0,\pi), \frac{\partial h}{\partial \zeta} \text{ are absolutely continuous}, \frac{\partial^2 h}{\partial \zeta^2} \in L^2(0,\pi) \text{ and } h(0) = 0 = h(\pi)\},$$

$B = I$ *and* $G(t, x(\zeta,t), u(t)) = x(\zeta,t) + \dfrac{1}{1+t^2} u(t) + e^{-t}\mathbf{1}_{[0,\frac{\pi}{2}]}$.
A possesses an orthonormal basis of eigenvector $\phi_n(\zeta) = \sqrt{\frac{2}{\pi}}\sin(n\zeta)$, $\quad n \geq 0$. Furthermore, the semigroup $(S(t))_{t\geq 0}$ generated by A is given by

$$S(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} < x, \phi_n > \phi_n.$$

Obviously, $S(t)$ is exponentially stable. Therefore, A is exponentially stable. Moreover, G satisfies the assumption $(\mathcal{H}_6)$, just take $a = 1$, $b = 1$, $\varepsilon = 0$ and $\varrho(t) = \dfrac{\pi}{2} e^{-t}$, is a non-negative continuous function, with $\varrho \in L^p(\mathbb{R}_+, \mathbb{R}_+)$ for some $p \in [1, +\infty)$. Consequently, by applying Theorem 3, the controlled heat equation (18) is practically stabilizable.

6 Conclusion

Practical stabilization of infinite-dimensional evolution equations in Banach spaces has been investigated. Moreover, sufficient conditions have been derived to guarantee the practical stabilization of a class of uncertain systems in Banach spaces. Illustrative examples are given to indicate significant improvements and the application of the results.

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