Ulam's type stabilities for conformable fractional differential equations with delay

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Abstract In this paper, we investigate the existence and uniqueness of solutions and Ulam's type stabilities including the well-known Ulam-Hyers stability and the newly extended Ulam-Hyers' conformable exponential stability for two classes of fractional differential equations with the conformable fractional derivative and the time delay. The Banach contraction principle, the technique of Picard operator, the Gronwall integral inequalities and generalized iterated integral inequality in the sense of conformable fractional integral are the main tools for deriving our main results. Finally, several illustrative examples will be presented to demonstrate our work.

Keywords Conformable fractional calculus, Ulam's stability, Picard operator, Integral inequality, Delay.

1 Introduction

The conformable fractional derivative(CFD) was first introduced by the authors Khalil et al. in [1]. Since then, the conformable fractional calculus has been rapidly developed and widely applied. For details, one may refer to [2-10] and the reference therein.

The Ulam's type stability problems of various differential or integral equations have been extensively studied in the recent years. There are some commonly used methods for dealing with these problems. For example, the theory of fixed points [11-12], the approach of integral inequalities [13-15], the method of Laplace transform [16-21], the integral factors [22-23] and the technique of Picard operator [24-25]. Particularly, the authors in [24,27-28] discussed the Ulam-Hyers-Mittag-Leffler stability problem of fractional differential equations by combining the Ulam stability with the well-known Mittag-Leffler function. The Ulam stability for some nonlinear integro-differential equations were studied in [25-26].

To our knowledge, however, the Ulam's type stabilities of fractional differential equations or nonlinear integro-differential equations with CFD and delay has rarely been studied directly. Considering the special advantages of exponential stability in the linear time-invariant systems involved the theory of stability and control, it would be of interest to connect the Ulam-Hyers stability with the conformable fractional exponential function (which is defined in Definition 2.4). Besides, the author Pachpatte in [29] offered some fundamental integral inequalities with iterated integrals which play a significant role in studying the qualitative theory of integral and differential equations. In view of this, we characterize different forms of iterated integral inequalities involving the conformable fractional integral(CFI) to explore the Ulam-Hyers stability of nonlinear conformable Volterra delayed integro-differential equation. Furthermore, it is worth noting that the phenomenon of time-delay is very conventional and important in the field of engineering and practical applications. Some recent literatures published about the applicability and universality of the time-delay effect can be found in [32-36].

Motivated by the above discussion, in this paper, we will first consider the following conformable

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fractional delay differential equations

$$\begin{cases} T_{t_0}^{\alpha}x(t) = f(t, x(t), x(t-\tau)), t \in (t_0, T], \\ x(t) = \varphi(t), t \in [t_0 - \tau, t_0] \end{cases} \tag{1.1}$$

where $T_{t_0}^{\alpha}x(t)$ denotes the CFD starting from the initial time t_0 of the function f of order $\alpha \in (0,1)$, $f \in C([t_0, T] \times \mathbb{R}^2, \mathbb{R})$, $\varphi \in C([t_0 - \tau, T], \mathbb{R})$, $t_0 < T < +\infty$ and $\tau > 0$ is the time delay.

The existence and uniqueness of solutions and the Ulam-Hyers stability and Ulam-Hyers conformable exponential stability of (1.1) will be studied via different approaches including the Gronwall integral inequalities and the Picard operator.

Then we investigate the Ulam-Hyers stability of the following nonlinear Volterra delay integro-differential equation

$$\begin{cases} T_{t_0}^{\alpha}x(t) = g\Big(t, x(t), x(t-\tau), \int_{t_0}^{t}(s-t_0)^{\alpha-1}h(t, s, x(s), x(s-\tau))\Big)ds, t \in (t_0, T], \\ x(t) = \phi(t), t \in [t_0 - \tau, t_0], \end{cases} \tag{1.2}$$

where $g \in C([t_0, T] \times \mathbb{R}^3, \mathbb{R})$, $h \in C([t_0, T] \times [t_0, T] \times \mathbb{R}^2, \mathbb{R})$ and $\phi \in C([t_0 - \tau, T], \mathbb{R})$.

2 Preliminaries

This section collects some necessary definitions and lemmas.

Definition 2.1. *(see [3]) The conformable fractional derivative(CFD) starting from t_0 of the function $f : [t_0, +\infty) \to \mathbb{R}$ of order $\alpha \in (0,1]$ is defined as*

$$T_{t_0}^{\alpha}f(t) = \lim_{\delta \to 0} \frac{f(t + \delta(t-t_0)^{1-\alpha}) - f(t)}{\delta}. \tag{2.1}$$

Particularly, if f is differentiable, then

$$T_{t_0}^{\alpha}f(t) = (t-t_0)^{1-\alpha}f'(t). \tag{2.2}$$

Definition 2.2. *(see [3]) The conformable fractional integral(CFI) starting from t_0 of the function $f : [t_0, +\infty) \to \mathbb{R}$ of order $\alpha \in (0,1]$ is defined as*

$$I_{t_0}^{\alpha}f(t) = \int_{t_0}^{t}(s-t_0)^{\alpha-1}f(s)ds. \tag{2.3}$$

Lemma 2.3. *(see [3]) Let $\alpha \in (0,1]$ and $f : [t_0, +\infty) \to \mathbb{R}$ be a continuous function. Then for all $t > t_0$,*

$$T_{t_0}^{\alpha}I_{t_0}^{\alpha}f(t) = f(t). \tag{2.4}$$

Further, if f is differentiable on $(t_0, +\infty)$, then for all $t > t_0$,

$$I_{t_0}^{\alpha}T_{t_0}^{\alpha}f(t) = f(t) - f(t_0). \tag{2.5}$$

Definition 2.4. *(see [3,6,8]) The conformable fractional exponential function(CFEF) is defined by*

$$E_{\alpha}(\lambda, t-t_0) = exp(\lambda \cdot \frac{(t-t_0)^{\alpha}}{\alpha}) = \sum_{k=0}^{\infty} \frac{\lambda^k (t-t_0)^{\alpha k}}{\alpha^k k!}, \ t \geq t_0, \ 0 < \alpha \leq 1, \ \lambda \in \mathbb{R}. \tag{2.6}$$

The following two lemmas involving the Gronwall integral inequality in the sense of CFI will play a significant role in studying stability.

Lemma 2.5. *(see [3]) Assume that f is a continuous, nonnegative function on $[t_0, t_1]$ and ξ, η are two nonnegative constants such that*

$$f(t) \leq \xi + \eta \int_{t_0}^{t} (s - t_0)^{\alpha - 1} f(s) ds.$$

Then for all $t \in [t_0, t_1]$, the following inequality holds

$$f(t) \leq \xi E_\alpha(\eta, t - t_0). \tag{2.7}$$

Lemma 2.6. *(see [30]) Suppose that f is a continuous, nonnegative function on $[t_0, t_1]$ satisfying*

$$f(t) \leq \xi(t) + \eta \int_{t_0}^{t} (s - t_0)^{\alpha - 1} f(s) ds,$$

where $\eta > 0$ is a constant and $\xi(t)$ is a nonnegative differentiable function defined on $[t_0, t_1]$. Then

$$f(t) \leq \xi(t) + \eta E_\alpha(\eta, t - t_0) \int_{t_0}^{t} \xi(s) E_\alpha(-\eta, s - t_0)(s - t_0)^{\alpha - 1} ds. \tag{2.8}$$

To end this section, we introduce the following Picard operator definition and abstract Gronwall lemma which will be fairly useful to contribute to deriving our main results.

Definition 2.7. *(see [31]) Assume that (Z, d) is a metric space, the mapping $A : Z \to Z$ is called a Picard operator if there exists $z^* \in Z$ such that (i) $S_A = \{z^*\}$, where $S_A = \{z \in Z : A(z) = z\}$; (ii) the sequence $\{A^n(z)\}_{n \in \mathbb{N}}$ converges to z^* for all $z \in Z$.*

Lemma 2.8. *(see [31]) Let $(Z, d, \leq)$ be an ordered metric space, $A : Z \to Z$ is an increasing Picard operator ($S_A = \{z^*\}$). Then for any $z \in Z$, $z \leq A(z)$ implies $z \leq z^*$ and $z \geq A(z)$ implies $z \geq z^*$.*

3 Main results

3.1 Ulam-Hyers stability for (1.1)

In this subsection, we utilize the Banach contraction principle and the technique of the Gronwall inequality presented in Lemma 2.5 to investigate the existence and uniqueness of solutions and the Ulam-Hyers stability of the Equation (1.1).

Definition 3.1. *The Equation (1.1) is called Ulam-Hyers stable if there exists a constant $\Lambda > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C([t_0 - \tau, T], \mathbb{R}) \cap C^1([t_0, T], \mathbb{R})$ of the following inequality*

$$|T_{t_0}^{\alpha} y(t) - f(t, y(t), y(t - \tau))| \leq \varepsilon, \ t \in [t_0, T], \tag{3.1}$$

there exists a solution $x \in C([t_0 - \tau, T], \mathbb{R}) \cap C^1([t_0, T], \mathbb{R})$ of the Eq.(1.1) with

$$|y(t) - x(t)| \leq \Lambda \cdot \varepsilon, \ t \in [t_0 - \tau, T]. \tag{3.2}$$

Lemma 3.2. *Assume that $y \in C([t_0 - \tau, T], \mathbb{R}) \cap C^1([t_0, T], \mathbb{R})$ is the solution of (3.1). Then*

$$|y(t) - y(t_0) - I_{t_0}^{\alpha} f(t, y(t), y(t - \tau))| \leq \frac{(T - t_0)^\alpha}{\alpha} \varepsilon, \ t \in [t_0, T].$$

Proof. Obviously,

$$\begin{aligned} |y(t) - y(t_0) - I_{t_0}^{\alpha} f(t, y(t), y(t - \tau))| &= |I_{t_0}^{\alpha} \left(T_{t_0}^{\alpha} y(t) - f(t, y(t), y(t - \tau))\right)| \\ &\leq I_{t_0}^{\alpha} |T_{t_0}^{\alpha} y(t) - f(t, y(t), y(t - \tau))| \\ &\leq I_{t_0}^{\alpha} \varepsilon = \frac{(t - t_0)^\alpha}{\alpha} \varepsilon \leq \frac{(T - t_0)^\alpha}{\alpha} \varepsilon. \end{aligned}$$

□

Theorem 3.3. *For all $t \in [t_0, T]$ and $u_i, v_i \in \mathbb{R}(i = 1, 2)$, if there exists a constant $L_f > 0$ such that*

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \le L_f(|u_1 - v_1| + |u_2 - v_2|)$$

holds true and satisfying $\frac{2L_f(T-t_0)^\alpha}{\alpha} < 1$, then the Eq.(1.1) has a unique solution in $C([t_0 - \tau, T], \mathbb{R}) \cap C^1([t_0, T], \mathbb{R})$ and is Ulam-Hyers stable.

Proof. (i) Firstly, the Eq.(1.1) can be written as the following equivalent form

$$x(t) = \begin{cases} \varphi(t), \ t \in [t_0 - \tau, t_0], \\ \varphi(t_0) + \int_{t_0}^t (s - t_0)^{\alpha - 1} f(s, x(s), x(s - \tau)) ds, \ t \in [t_0, T]. \end{cases}$$

Define the space $X := C([t_0 - \tau, T], \mathbb{R}) \cap C^1([t_0, T], \mathbb{R})$ with the norm $\|\cdot\|_X = \max_{t_0 - \tau \le \theta \le T} |x(\theta)|, x \in X$ and the operator $P : X \to X$ as

$$(Px)(t) = \begin{cases} \varphi(t), \ t \in [t_0 - \tau, t_0], \\ \varphi(t_0) + \int_{t_0}^t (s - t_0)^{\alpha - 1} f(s, x(s), x(s - \tau)) ds, \ t \in [t_0, T]. \end{cases}$$

Apparently, for all $t \in [t_0 - \tau, t_0]$ and $x, z \in X$, $|(Px)(t) - (Pz)(t)| = 0$. While for $t \in [t_0, T]$,

$$\begin{aligned} |(Px)(t) - (Pz)(t)| &= |\int_{t_0}^t (s - t_0)^{\alpha - 1} f(s, x(s), x(s - \tau)) ds - \int_{t_0}^t (s - t_0)^{\alpha - 1} f(s, z(s), z(s - \tau)) ds| \\ &\le L_f \int_{t_0}^t (s - t_0)^{\alpha - 1} \big(\max_{\sigma \in [t_0 - \tau, T]} |x(\sigma) - z(\sigma)| + \max_{\upsilon \in [t_0 - \tau, T]} |x(\upsilon) - z(\upsilon)| \big) ds \\ &\le \frac{2L_f(T - t_0)^\alpha}{\alpha} \|x - z\|_X. \end{aligned}$$

Therefore, $\|Px - Pz\|_X \le \frac{2L_f(T-t_0)^\alpha}{\alpha} \|x - z\|_X$ holds for all $t \in [t_0 - \tau, T]$ and $x, z \in X$. Then P is a contraction on X, the Banach contraction principle guarantees that Eq.(1.1) has a unique solution in X.
(ii)Next, we show that the Equation (1.1) is Ulam-Hyers stable. Let $y \in C([t_0 - \tau, T], \mathbb{R}) \cap C^1([t_0, T], \mathbb{R})$ is the solution of (3.1) and $x \in C([t_0 - \tau, T], \mathbb{R}) \cap C^1([t_0, T], \mathbb{R})$ be the unique solution of (1.1) satisfying the initial condition $x(t) = y(t), t \in [t_0 - \tau, t_0]$, that is to say,

$$x(t) = \begin{cases} y(t), \ t \in [t_0 - \tau, t_0], \\ y(t_0) + \int_{t_0}^t (s - t_0)^{\alpha - 1} f(s, x(s), x(s - \tau)) ds, \ t \in [t_0, T]. \end{cases}$$

For all $t \in [t_0 - \tau, t_0]$, $|y(t) - x(t)| = 0$. For $t \in [t_0, T]$, we divide it into two intervals. When $t \in [t_0, t_0 + \tau]$, it can be deduced from Lemma 3.2 and the fact $y(t - \tau) - x(t - \tau) = 0$ for $t \in [t_0, t_0 + \tau]$ that

$$\begin{aligned} |y(t) - x(t)| =& |y(t) - y(t_0) - I_{t_0}^\alpha f(t, y(t), y(t - \tau)) + y(t_0) + I_{t_0}^\alpha f(t, y(t), y(t - \tau)) - x(t)| \\ =& |y(t) - y(t_0) - \int_{t_0}^t (s - t_0)^{\alpha - 1} f(s, y(s), y(s - \tau)) ds \\ &+ \int_{t_0}^t (s - t_0)^{\alpha - 1} f(s, y(s), y(s - \tau)) ds - \int_{t_0}^t (s - t_0)^{\alpha - 1} f(s, x(s), x(s - \tau)) ds| \\ \le& |y(t) - y(t_0) - \int_{t_0}^t (s - t_0)^{\alpha - 1} f(s, y(s), y(s - \tau)) ds| \\ &+ \int_{t_0}^t (s - t_0)^{\alpha - 1} |f(s, y(s), y(s - \tau) - f(s, x(s), x(s - \tau)| ds \\ \le& \frac{(T - t_0)^\alpha}{\alpha} \varepsilon + L_f \int_{t_0}^t (s - t_0)^{\alpha - 1} |y(s) - x(s)| ds. \end{aligned}$$

Applying the Gronwall inequality in Lemma 2.5, it follows that

$$|y(t)-x(t)| \leq \frac{(T-t_0)^\alpha \cdot \varepsilon}{\alpha} E_\alpha(L_f, t-t_0) \leq \frac{(T-t_0)^\alpha}{\alpha} E_\alpha(L_f, \tau)\varepsilon,\ t \in [t_0, t_0+\tau].$$

While for $t \in [t_0+\tau, T]$, the following inequality can be easily obtained by using the similar steps

$$\begin{aligned} |y(t)-x(t)| \leq & \frac{(T-t_0)^\alpha}{\alpha}\varepsilon + L_f \int_{t_0}^{t} (s-t_0)^{\alpha-1}|x(s)-y(s)|ds \\ & + L_f \int_{t_0+\tau}^{t} (s-t_0)^{\alpha-1}|y(s-\tau)-x(s-\tau)|ds. \end{aligned}$$

Set $u(t) = \sup_{\theta \in [-\tau, 0]} |y(t+\theta)-x(t+\theta)|$, then it yields that

$$u(t) \leq \frac{(T-t_0)^\alpha}{\alpha}\varepsilon + 2L_f \int_{t_0}^{t} (s-t_0)^{\alpha-1}u(s)ds,$$

then we get the following inequality via the Gronwall inequality in Lemma 2.5

$$|y(t)-x(t)| \leq u(t) \leq \frac{(T-t_0)^\alpha}{\alpha} E_\alpha(2L_f, T-t_0)\varepsilon,\ t \in [t_0+\tau, T].$$

From the above discussion, we can conclude that for all $t \in [t_0-\tau, T]$,

$$|y(t)-x(t)| \leq \frac{(T-t_0)^\alpha}{\alpha} E_\alpha(2L_f, T-t_0)\varepsilon,$$

which implies that the Eq.(1.1) is Ulam-Hyers stable with $\Lambda = \frac{(T-t_0)^\alpha}{\alpha} E_\alpha(2L_f, T-t_0)$. □

3.2 Ulam-Hyers' conformable exponential stability for (1.1)

In this subsection, we first define a concept of Ulam-Hyers conformable exponential stability, an extension of Ulam-Hyers stability for fractional ODEs. Then the method of Picard operator and the generalized Gronwall inequality presented in Lemma 2.6 will be used to study the Ulam-Hyers conformable exponentiable stability of the Equation (1.1).

Definition 3.4. *The Equation (1.1) is called Ulam-Hyers conformable exponentially stable with respect to the conformable fractional exponential function $E_\alpha(\lambda, t-t_0)(\lambda \neq 0)$ if there exists a constant $\Omega > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C([t_0-\tau, T], \mathbb{R}) \cap C^1([t_0, T], \mathbb{R})$ of*

$$|T_{t_0}^\alpha y(t) - f(t, y(t), y(t-\tau))| \leq \varepsilon E_\alpha(\lambda, t-t_0),\ t \in [t_0, T], \tag{3.3}$$

there exists a solution $x \in C([t_0-\tau, T], \mathbb{R}) \cap C^1([t_0, T], \mathbb{R})$ of the Eq.(1.1) with

$$|y(t)-x(t)| \leq \Omega \cdot \varepsilon E_\alpha(\lambda, t-t_0),\ t \in [t_0-\tau, T]. \tag{3.4}$$

Lemma 3.5. *If $y \in C([t_0-\tau, T], \mathbb{R}) \cap C^1([t_0, T], \mathbb{R})$ satisfies (3.3), then*

$$|y(t)-y(t_0) - I_{t_0}^\alpha f(t, y(t), y(t-\tau))| \leq \varepsilon \frac{(T-t_0)^\alpha}{\alpha} E_\alpha(\lambda, t-t_0),\ t \in [t_0, T].$$

Proof.

$$
\begin{aligned}
&|y(t)-y(t_0)-I_{t_0}^{\alpha}f(t,y(t),y(t-\tau))|\\
&\qquad\qquad = |I_{t_0}^{\alpha}\big(T_{t_0}^{\alpha}y(t)-f(t,y(t),y(t-\tau))\big)|\\
&\qquad\qquad \le I_{t_0}^{\alpha}|T_{t_0}^{\alpha}y(t)-f(t,y(t),y(t-\tau))|\\
&\qquad\qquad \le \int_{t_0}^{t}(s-t_0)^{\alpha-1}\varepsilon E_{\alpha}(\lambda,s-t_0)ds\\
&\qquad\qquad = \varepsilon\int_{t_0}^{t}(s-t_0)^{\alpha-1}\sum_{k=0}^{\infty}\frac{\lambda^k(s-t_0)^{k\alpha}}{\alpha^k\cdot k!}ds\\
&\qquad\qquad = \varepsilon\sum_{k=0}^{\infty}\frac{\lambda^k(t-t_0)^{(k+1)\alpha}}{\alpha^{k+1}\cdot(k+1)!}\\
&\qquad\qquad = \frac{\varepsilon(t-t_0)^{\alpha}}{\alpha}\sum_{k=0}^{\infty}\frac{\lambda^k(t-t_0)^{k\alpha}}{\alpha^k\cdot k!(k+1)}\\
&\qquad\qquad \le \frac{\varepsilon(T-t_0)^{\alpha}}{\alpha}\sum_{k=0}^{\infty}\frac{\lambda^k(t-t_0)^{k\alpha}}{\alpha^k\cdot k!}\\
&\qquad\qquad = \frac{\varepsilon(T-t_0)^{\alpha}}{\alpha}E_{\alpha}(\lambda,t-t_0).
\end{aligned}
$$

□

Theorem 3.6. *Under the assumptions in Theorem 3.3, the Equation (1.1) is Ulam-Hyers conformable exponentially stable with respect to $E_{\alpha}(\lambda,t-t_0)$ provided that $\lambda\neq 2L_f$.*

Proof. Let $y\in C([t_0-\tau,T],\mathbb{R})\cap C^1([t_0,T],\mathbb{R})$ be the solution of (3.3) and $x\in C([t_0-\tau,T],\mathbb{R})\cap C^1([t_0,T],\mathbb{R})$ be the unique solution of (1.1) satisfying the initial condition $x(t)=y(t),t\in[t_0-\tau,t_0]$, hence,

$$
x(t)=\begin{cases} y(t),\ t\in[t_0-\tau,t_0],\\ y(t_0)+\int_{t_0}^{t}(s-t_0)^{\alpha-1}f(s,x(s),x(s-\tau)ds,\ t\in[t_0,T]. \end{cases}
$$

For all $t\in[t_0-\tau,t_0]$, $|y(t)-x(t)|=0$. When $t\in[t_0,T]$, it follows Lemma 3.5 and the discussion in Theorem 3.3 that

$$
|y(t)-x(t)|\le\varepsilon\frac{(T-t_0)^{\alpha}}{\alpha}E_{\alpha}(\lambda,t-t_0)+L_f\int_{t_0}^{t}(s-t_0)^{\alpha-1}\big(|y(s)-x(s)|+|y(s-\tau)-x(s-\tau)|\big)ds. \tag{3.5}
$$

Consider $v\in X_+:=C([t_0-\tau,T],\mathbb{R}_+)\cap C^1([t_0,T],\mathbb{R}_+)$ and define the operator $Q:X_+\to X_+$ as

$$
(Qv)(t)=\begin{cases} 0,\ t\in[t_0-\tau,t_0],\\ \varepsilon\frac{(T-t_0)^{\alpha}}{\alpha}E_{\alpha}(\lambda,t-t_0)+L_f\int_{t_0}^{t}(s-t_0)^{\alpha-1}\big(v(s)+v(s-\tau)\big)ds,\ t\in[t_0,T]. \end{cases}
$$

Then we demonstrate that Q is a Picard operator. In fact, according to the investigation in Theorem 3.3, for all $v_1,v_2\in X_+$ and $t\in[t_0,T]$, we have

$$
\|(Qv_1)(t)-(Qv_2)(t)\|_X\le\frac{2L_f(T-t_0)^{\alpha}}{\alpha}\|v_1-v_2\|_X.
$$

Therefore, Q is a contraction on X_+. Moreover, we know that Q is a Picard operator from the Banach contraction principle and $S_Q = \{\hat{v}\}$. Then for all $t \in [t_0, T]$, we have

$$\hat{v}(t) = \varepsilon \frac{(T-t_0)^\alpha}{\alpha} E_\alpha(\lambda, t-t_0) + L_f \int_{t_0}^{t} (s-t_0)^{\alpha-1}\big(\hat{v}(s) + \hat{v}(s-\tau)\big) ds.$$

Next, we need to verify that $\hat{v}$ is increasing. For any $t_1, t_2 \in [t_0, T]$, and $t_1 < t_2$, if we let $m := \min\limits_{\theta \in [t_0, T]} \big(\hat{v}(\theta) + \hat{v}(\theta - \tau)\big) \in \mathbb{R}_+$, then

$$\begin{aligned}
\hat{v}(t_2) - \hat{v}(t_1) &= \varepsilon \frac{(T-t_0)^\alpha}{\alpha}\big(E_\alpha(\lambda, t_2-t_0) - E_\alpha(\lambda, t_1-t_0)\big) + L_f \int_{t_1}^{t_2} (s-t_0)^{\alpha-1}\big(\hat{v}(s) + \hat{v}(s-\tau)\big) ds \\
&= \varepsilon \frac{(T-t_0)^\alpha}{\alpha} E_\alpha(\lambda, t_1-t_0)\left(\frac{E_\alpha(\lambda, t_2-t_0)}{E_\alpha(\lambda, t_1-t_0)} - 1\right) + L_f \int_{t_1}^{t_2} (s-t_0)^{\alpha-1}\big(\hat{v}(s) + \hat{v}(s-\tau)\big) ds \\
&\geq \varepsilon \frac{(T-t_0)^\alpha}{\alpha} E_\alpha(\lambda, t_1-t_0)\left(exp\big(\lambda \frac{(t_2-t_0)^\alpha - (t_1-t_0)^\alpha}{\alpha}\big) - 1\right) + mL_f \frac{(t_2-t_0)^\alpha - (t_1-t_0)^\alpha}{\alpha} \\
&> 0,
\end{aligned}$$

which implies that $\hat{v}$ is increasing, and hence, $\hat{v}(t-\tau) \leq \hat{v}(t)$ since $t - \tau \leq t$, then we have

$$\hat{v}(t) \leq \varepsilon \frac{(T-t_0)^\alpha}{\alpha} E_\alpha(\lambda, t-t_0) + 2L_f \int_{t_0}^{t} (s-t_0)^{\alpha-1} \hat{v}(s) ds.$$

Gronwall inequality in Lemma 2.6 yields that

$$\begin{aligned}
\hat{v}(t) &\leq \frac{\varepsilon (T-t_0)^\alpha}{\alpha} E_\alpha(\lambda, t-t_0) + 2\frac{(T-t_0)^\alpha}{\alpha} L_f E_\alpha(2L_f, t-t_0) \int_{t_0}^{t} E_\alpha(\lambda, s-t_0) E_\alpha(-2L_f, s-t_0)(s-t_0)^{\alpha-1} ds \\
&= \frac{\varepsilon (T-t_0)^\alpha}{\alpha} E_\alpha(\lambda, t-t_0)\left[1 + \frac{2L_f E_\alpha(2L_f, t-t_0)}{E_\alpha(\lambda, t-t_0)} \int_{t_0}^{t} E_\alpha(\lambda, s-t_0) E_\alpha(-2L_f, s-t_0)(s-t_0)^{\alpha-1} ds\right] \\
&\leq \varepsilon E_\alpha(\lambda, t-t_0) \frac{(T-t_0)^\alpha}{\alpha}\left\{1 + \frac{2L_f E_\alpha(2L_f, T-t_0)}{|\lambda - 2L_f|} E_\alpha(\lambda - 2L_f, T-t_0)\right\}.
\end{aligned}$$

Particularly, if we let $v = |y - x|$, then (3.5) deduces that $v \leq Q(v)$ and Lemma 2.8 ensures that $v \leq \hat{v}$. That is to say,

$$|y(t) - x(t)| \leq \Omega \cdot E_\alpha(\lambda, t-t_0)\varepsilon.$$

Thus, the Equation (1.1) is Ulam-Hyers conformable exponentially stable with respect to $E_\alpha(\lambda, t-t_0)$ with the constant

$$\Omega = \frac{(T-t_0)^\alpha}{\alpha}\left\{1 + \frac{2L_f E_\alpha(2L_f, T-t_0)}{|\lambda - 2L_f|} E_\alpha(\lambda - 2L_f, T-t_0)\right\}.$$

□

3.3 Ulam-Hyers stability for (1.2)

In this subsection, by introducing the definition about the Ulam-Hyers stability into the nonlinear Volterra delay integro-differential problem (1.2), we propose a class of iterated integral inequality in the sense of CFI to deal with the Ulam-Hyers stability.

Definition 3.7. *The Equation (1.2) is called Ulam-Hyers stable if there exists a constant $\Delta > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C([t_0 - \tau, T], \mathbb{R}) \cap C^1([t_0, T], \mathbb{R})$ of the following inequality*

$$|T_{t_0}^{\alpha}y(t)-g\Big(t,y(t),y(t-\tau),\int_{t_0}^{t}(s-t_0)^{\alpha-1}h(t,s,y(s),y(s-\tau))\Big)|\le\varepsilon,\ t\in[t_0,T], \tag{3.6}$$

there exists a solution $x\in C([t_0-\tau,T],\mathbb{R})\cap C^1([t_0,T],\mathbb{R})$ of the Equation (1.2) with

$$|y(t)-x(t)|\le\Delta\cdot\varepsilon,\ t\in[t_0-\tau,T]. \tag{3.7}$$

Lemma 3.8. *If $y\in C([t_0-\tau,T],\mathbb{R})\cap C^1([t_0,T],\mathbb{R})$ satisfies (3.6), then for all $t\in[t_0,T]$, we have*

$$|y(t)-y(t_0)-\int_{t_0}^{t}(s-t_0)^{\alpha-1}g\Big(s,y(s),y(s-\tau),\int_{t_0}^{s}(\theta-t_0)^{\alpha-1}h(s,\theta,y(\theta),y(\theta-\tau))d\theta\Big)ds|\le\frac{(T-t_0)^{\alpha}}{\alpha}\varepsilon.$$

Proof. This proof is completely similar to Lemma 3.2. □

Now, we establish the following iterated integral inequality in the frame of CFI which will play a key role to get the main result in this subsection.

Theorem 3.9. *Let $\alpha\in(0,1)$, $u,v,w\in C([t_0,T],\mathbb{R}_+)$ and $c\ge0$ be a real constant satisfying*

$$u(t)\le c+\int_{t_0}^{t}(s-t_0)^{\alpha-1}v(s)\Big[u(s)+\int_{t_0}^{s}(\theta-t_0)^{\alpha-1}w(\theta)u(\theta)d\theta\Big]ds,\ t\in[t_0,T].$$

Then

$$u(t)\le c\cdot exp\Big(\int_{t_0}^{t}(\theta-t_0)^{\alpha-1}\big(v(\theta)+w(\theta)\big)d\theta\Big)$$

Proof. Let

$$z(t)=c+\int_{t_0}^{t}(s-t_0)^{\alpha-1}v(s)\Big[u(s)+\int_{t_0}^{s}(\theta-t_0)^{\alpha-1}w(\theta)u(\theta)d\theta\Big]ds.$$

Then $u(t)\le z(t)$, and

$$T_{t_0}^{\alpha}z(t)=v(t)\Big[u(t)+I_{t_0}^{\alpha}w(t)u(t)\Big]\le v(t)\Big[z(t)+I_{t_0}^{\alpha}w(t)z(t)\Big].$$

Let $\Phi(t)=z(t)+I_{t_0}^{\alpha}w(t)z(t)$. Then $\Phi(t_0)=z(t_0)=c$, $z(t)\le\Phi(t)$, $T_{t_0}^{\alpha}z(t)\le v(t)\Phi(t)$ and $\Phi(t)$ is nondecreasing and differentiable, hence,

$$\begin{aligned}(t-t_0)^{1-\alpha}\Phi'(t)=T_{t_0}^{\alpha}\Phi(t)&=T_{t_0}^{\alpha}z(t)+w(t)u(t)\\&\le v(t)\Phi(t)+w(t)z(t)\\&\le\big(w(t)+v(t)\big)\Phi(t),\end{aligned}$$

which implies that

$$\Phi(t)\le c\cdot exp\Big(\int_{t_0}^{t}(\theta-t_0)^{\alpha-1}\big(v(\theta)+w(\theta)\big)d\theta\Big).$$

Therefore,

$$u(t)\le z(t)\le c\cdot exp\Big(\int_{t_0}^{t}(\theta-t_0)^{\alpha-1}\big(v(\theta)+w(\theta)\big)d\theta\Big).$$

This completes the proof. □

Before deriving our main result, let us list the following assumptions:
(H_1): There exists a function $L_g\in C([t_0,T],\mathbb{R}_+)$ such that for all $t\in[t_0,T]$ and $x_i,y_i\in\mathbb{R}(i=1,2,3)$

$$|g(t,x_1,x_2,x_3)-g(t,y_1,y_2,y_3)|\leq L_g(t)\big(\sum_{i=1}^{3}|x_i(t)-y_i(t)|\big).$$

(H_2): There exists a function $L_h\in C([t_0,T],\mathbb{R}_+)$ such that for all $t\in[t_0,T]$ and $x_j,y_j\in\mathbb{R}(j=1,2)$

$$|h(t,s,x_1,x_2)-h(t,s,y_1,y_2)|\leq L_h(t)\big(\sum_{j=1}^{2}|x_j(t)-y_j(t)|\big).$$

(H_3):

$$M=2\bigg\{\int_{t_0}^{T}(s-t_0)^{\alpha-1}L_g(s)\bigg[1+\int_{t_0}^{T}(\theta-t_0)^{\alpha-1}L_h(\theta)d\theta\bigg]ds\bigg\}<1.$$

Theorem 3.10. *Suppose that (H_1), (H_2) and (H_3) are satisfied. Then the Equation (1.2) has a unique solution in $C([t_0-\tau,T],\mathbb{R})\cap C^1([t_0,T],\mathbb{R})$ and is Ulam-Hyers stable.*

Proof. (i) First of all, the problem (1.2) is equivalent to the following integral equations

$$x(t)=\begin{cases}\phi(t),\ t\in[t_0-\tau,t_0],\\ \phi(t_0)+\int_{t_0}^{t}(s-t_0)^{\alpha-1}g\Big(s,x(s),x(s-\tau),\int_{t_0}^{s}(\theta-t_0)^{\alpha-1}h(s,\theta,x(\theta),x(\theta-\tau))d\theta\Big)ds,\ t\in[t_0,T].\end{cases}$$

Consider the Banach space X defined in Theorem 3.3 and define a new operator $W:X\to X$ as

$$(Wx)(t)=\begin{cases}\phi(t),\ t\in[t_0-\tau,t_0],\\ \phi(t_0)+\int_{t_0}^{t}(s-t_0)^{\alpha-1}g\Big(s,x(s),x(s-\tau),\int_{t_0}^{s}(\theta-t_0)^{\alpha-1}h(s,\theta,x(\theta),x(\theta-\tau))d\theta\Big)ds,\ t\in[t_0,T].\end{cases}$$

Notice that for all $t\in[t_0-\tau,t_0]$ and $x,z\in X$, $|(Wx)(t)-(Wz)(t)|=0$. Then for $t\in[t_0,T]$,

$$\begin{aligned}
|(Wx)(t)-(Wz)(t)|=&\Big|\int_{t_0}^{t}(s-t_0)^{\alpha-1}g\Big(s,x(s),x(s-\tau),\int_{t_0}^{s}(\theta-t_0)^{\alpha-1}h(s,\theta,x(\theta),x(\theta-\tau))d\theta\Big)ds\\
&-\int_{t_0}^{t}(s-t_0)^{\alpha-1}g\Big(s,z(s),z(s-\tau),\int_{t_0}^{s}(\theta-t_0)^{\alpha-1}h(s,\theta,z(\theta),z(\theta-\tau))d\theta\Big)ds\\
\leq&\int_{t_0}^{t}(s-t_0)^{\alpha-1}L_g(s)\bigg\{|x(s)-z(s)|+|x(s-\tau)-z(s-\tau)|\\
&+\int_{t_0}^{s}(\theta-t_0)^{\alpha-1}L_h(\theta)\Big[|x(\theta)-z(\theta)|+|x(\theta-\tau)-z(\theta-\tau)|\Big]d\theta\bigg\}ds\\
\leq&\int_{t_0}^{t}(s-t_0)^{\alpha-1}L_g(s)\bigg\{\max_{t_0\leq\sigma\leq s}|x(\sigma)-z(\sigma)|+\max_{t_0\leq\sigma\leq s}|x(\sigma-\tau)-z(\sigma-\tau)|\\
&+\int_{t_0}^{s}(\theta-t_0)^{\alpha-1}L_h(\theta)\Big[\max_{t_0\leq\varsigma\leq\theta}|x(\varsigma)-z(\varsigma)|+\max_{t_0\leq\varsigma\leq\theta}|x(\varsigma-\tau)-z(\varsigma-\tau)|\Big]d\theta\bigg\}ds\\
\leq&\int_{t_0}^{t}(s-t_0)^{\alpha-1}L_g(s)\bigg\{\max_{t_0-\tau\leq\sigma\leq T}|x(\sigma)-z(\sigma)|+\max_{t_0-\tau\leq\sigma\leq T}|x(\sigma-\tau)-z(\sigma-\tau)|\\
&+\int_{t_0}^{s}(\theta-t_0)^{\alpha-1}L_h(\theta)\Big[\max_{t_0-\tau\leq\varsigma\leq T}|x(\varsigma)-z(\varsigma)|+\max_{t_0-\tau\leq\varsigma\leq T}|x(\varsigma-\tau)-z(\varsigma-\tau)|]d\theta\bigg\}ds\\
\leq&\int_{t_0}^{T}(s-t_0)^{\alpha-1}L_g(s)\bigg\{2\|x-z\|_X+2\int_{t_0}^{T}(\theta-t_0)^{\alpha-1}L_h(\theta)2\|x-z\|_Xd\theta\bigg\}ds\\
=&2\bigg\{\int_{t_0}^{T}(s-t_0)^{\alpha-1}L_g(s)\bigg[1+\int_{t_0}^{T}(\theta-t_0)^{\alpha-1}L_h(\theta)d\theta\bigg]ds\bigg\}\|x-z\|_X\\
=&M\|x-z\|_X.
\end{aligned}$$

Therefore, for all $x, z \in X$ and $t \in [t_0 - \tau, T]$, the following

$$\|(Wx) - (Wz)\|_X \leq M\|x - z\|_X$$

holds true. Hence, W is a contraction on X, then the Eq.(1.2) has a unique solution in X.
(ii) Let $y \in C([t_0 - \tau, T], \mathbb{R}) \cap C^1([t_0, T], \mathbb{R})$ is the solution of (3.6) and $x \in C([t_0 - \tau, T], \mathbb{R}) \cap C^1([t_0, T], \mathbb{R})$ be the unique solution of (1.2) satisfying the initial condition $x(t) = y(t), t \in [t_0 - \tau, t_0]$, that is,

$$x(t) = \begin{cases} y(t), \ t \in [t_0 - \tau, t_0], \\ y(t_0) + \int_{t_0}^{t} (s - t_0)^{\alpha - 1} g\Big(s, x(s), x(s - \tau), \int_{t_0}^{s} (\theta - t_0)^{\alpha - 1} h(s, \theta, x(\theta), x(\theta - \tau)) d\theta\Big) ds, \ t \in [t_0, T]. \end{cases}$$

Clearly, for all $t \in [t_0 - \tau, t_0]$, $|y(t) - x(t)| = 0$. For $t \in [t_0, T]$, it can be deduced from Lemma 3.8 that

$$\begin{aligned}
|y(t) - x(t)| =& |y(t) - y(t_0) - \int_{t_0}^{t} (s - t_0)^{\alpha - 1} g\Big(s, y(s), y(s - \tau), \int_{t_0}^{s} (\theta - t_0)^{\alpha - 1} h(s, \theta, y(\theta), y(\theta - \tau)) d\theta\Big) ds \\
& + y(t_0) + \int_{t_0}^{t} (s - t_0)^{\alpha - 1} g\Big(s, y(s), y(s - \tau), \int_{t_0}^{s} (\theta - t_0)^{\alpha - 1} h(s, \theta, y(\theta), y(\theta - \tau)) d\theta\Big) ds - x(t)| \\
\leq & \frac{(T - t_0)^\alpha}{\alpha} \varepsilon + \int_{t_0}^{t} (s - t_0)^{\alpha - 1} \Big| g\Big(s, y(s), y(s - \tau), \int_{t_0}^{s} (\theta - t_0)^{\alpha - 1} h(s, \theta, y(\theta), y(\theta - \tau)) d\theta\Big) \\
& - g\Big(s, x(s), x(s - \tau), \int_{t_0}^{s} (\theta - t_0)^{\alpha - 1} h(s, \theta, x(\theta), x(\theta - \tau)) d\theta\Big) \Big| ds \\
\leq & \frac{(T - t_0)^\alpha}{\alpha} \varepsilon + \int_{t_0}^{t} (s - t_0)^{\alpha - 1} L_g(s) \Big\{ |y(s) - x(s)| + |y(s - \tau) - x(s - \tau)| \\
& + \int_{t_0}^{s} (\theta - t_0)^{\alpha - 1} L_h(\theta) \Big[|y(\theta) - x(\theta)| + |y(\theta - \tau) - x(\theta - \tau)| \Big] d\theta \Big\} ds.
\end{aligned}$$

Consider $\nu \in X_+$ and define the operator $B : X_+ \to X_+$ as

$$(B\nu)(t) = \begin{cases} 0, \ t \in [t_0 - \tau, t_0], \\ \frac{(T - t_0)^\alpha \varepsilon}{\alpha} + \int_{t_0}^{t} (s - t_0)^{\alpha - 1} L_g(s) \Big\{ \nu(s) + \nu(s - \tau) \\ \qquad + \int_{t_0}^{s} (\theta - t_0)^{\alpha - 1} L_h(\theta) \Big[\nu(\theta) + \nu(\theta - \tau) \Big] d\theta \Big\} ds, \ t \in [t_0, T]. \end{cases}$$

Next, we prove that B is a Picard operator. Indeed, with the help of the discussion in (i), for all $\nu_1, \nu_2 \in X_+$ and $t \in [t_0, T]$, we have

$$\|(B\nu_1)(t) - (B\nu_2)(t)\|_X \leq M\|\nu_1 - \nu_2\|_X.$$

Therefore, B is a contraction on X_+. Moreover, we know that Q is a Picard operator from the Banach contraction principle and $S_B = \{\hat{\nu}\}$. Then for all $t \in [t_0, T]$, we have

$$\hat{\nu}(t) = \frac{(T - t_0)^\alpha \varepsilon}{\alpha} + \int_{t_0}^{t} (s - t_0)^{\alpha - 1} L_g(s) \Big\{ \nu(s) + \nu(s - \tau) + \int_{t_0}^{s} (\theta - t_0)^{\alpha - 1} L_h(\theta) \Big[\nu(\theta) + \nu(\theta - \tau) \Big] d\theta \Big\} ds.$$

Further, we need to verify that $\hat{v}$ is increasing. In fact, $\frac{d}{dt}[\hat{\nu}(t)] \geq 0$ on $[t_0, T]$ implies that $\hat{\nu}$ is increasing, and hence, $\hat{\nu}(t - \tau) \leq \hat{\nu}(t)$ since $t - \tau \leq t$, then we have

$$\hat{\nu}(t) \leq \frac{(T - t_0)^\alpha \varepsilon}{\alpha} + \int_{t_0}^{t} (s - t_0)^{\alpha - 1} 2L_g(s) \Big\{ \hat{\nu}(s) + \int_{t_0}^{s} (\theta - t_0)^{\alpha - 1} L_h(\theta) \hat{\nu}(\theta) d\theta \Big\} ds.$$

Applying the Theorem 3.9 to the above inequality with $u(t) = \hat{\nu}(t)$, $c = \frac{(T - t_0)^\alpha \varepsilon}{\alpha}$, $2L_g(t) = v(t)$ and $L_h(t) = w(t)$, it yields that

$$
\begin{aligned}
\hat{\nu}(t) &\le \frac{(T-t_0)^\alpha}{\alpha} exp\bigg(\int_{t_0}^{t}(\theta-t_0)^{\alpha-1}\big(2L_g(\theta)+L_h(\theta)\big)d\theta\bigg)\varepsilon \\
&\le \frac{(T-t_0)^\alpha}{\alpha} exp\bigg(\int_{t_0}^{T}(\theta-t_0)^{\alpha-1}\big(2L_g(\theta)+L_h(\theta)\big)d\theta\bigg)\varepsilon.
\end{aligned}
$$

Specially, if we set $\nu = |y-x|$, then $\nu \le B(\nu)$ and Lemma 2.8 guarantees that $\nu \le \hat{\nu}$. That is to say,

$$|y(t)-x(t)| \le \Delta \cdot \varepsilon.$$

Thus, the Equation (1.2) is Ulam-Hyers with the constant

$$\Delta = \frac{(T-t_0)^\alpha}{\alpha} exp\bigg(\int_{t_0}^{T}(\theta-t_0)^{\alpha-1}\big(2L_g(\theta)+L_h(\theta)\big)d\theta\bigg).$$

□

4 Illustrative examples

In this section, some examples will be shown to demonstrate our main results.

Example 4.1. *Let $\alpha = \frac{1}{2}$, $t_0 = 0$, $\tau = \frac{1}{3}$, $T = 2$, we consider the following problem*

$$
\begin{cases}
T_{0^+}^{\frac{1}{2}}x(t) = \frac{|x(t)|+|x(t-\frac{1}{3})|}{10},\ t \in (0,2], \\
x(t) = t,\ t \in [-\frac{1}{3}, 2].
\end{cases}
\tag{4.1}
$$

Note that for all $t \in [0,2]$ and $x, y \in \mathbb{R}$, we obtain

$$
\begin{aligned}
|(f(t,x(t),x(t-\frac{1}{3})) - f(t,y(t),y(t-\frac{1}{3}))| =& \frac{1}{10}\big|(|x(t)|-|y(t)|) + (|x(t-\frac{1}{3})|-|y(t-\frac{1}{3})|)\big| \\
\le& \frac{1}{10}\big(|x(t)-y(t)| + |x(t-\frac{1}{3})-y(t-\frac{1}{3})|\big).
\end{aligned}
$$

Obviously, $L_f = 0.1$ and $\frac{2L_f(T-t_0)^\alpha}{\alpha} = 0.5657 < 1$, then the conditions in Theorem 3.3 are fulfilled, hence the Equation (4.1) has a unique solution and is Ulam-Hyers stable on $[-\frac{1}{3}, 2]$ with

$$\Lambda = \frac{(T-t_0)^\alpha}{\alpha} E_\alpha(2L_f, T-t_0) = 2\sqrt{2}E_{\frac{1}{2}}(0.2, 2) = 4.9799.$$

Example 4.2. *Consider the following conformable fractional system with delay*

$$
\begin{cases}
T_{0^+}^{\frac{1}{2}}x(t) = \frac{1}{10}\frac{x^2(t-1)}{1+x^2(t-1)} + \frac{1}{10}\cos(2x(t)),\ t \in (0, \frac{3}{2}], \\
x(t) = 0,\ t \in [-1, 0],
\end{cases}
\tag{4.2}
$$

and the following inequality

$$|T_{0^+}^{\frac{1}{2}}y(t) - \frac{1}{10}\frac{y^2(t-1)}{1+y^2(t-1)} - \frac{1}{10}\cos(2y(t))| \le \varepsilon E_{\frac{1}{2}}(-1, t).$$

Obviously, $\alpha = \frac{1}{2}$, $t_0 = 0$, $T = \frac{3}{2}$, $\tau = 1$, $L_f = \frac{1}{5}$ and $\lambda = -1$. Then it is easy to verify that $\frac{2L_f(T-t_0)^\alpha}{\alpha} = 0.9798 < 1$ and $\lambda \ne L_f$. Now all the conditions in Theorem 3.6 are satisfied, then the Equation (4.2) is is Ulam-Hyers conformable exponentially stable with respect to $E_{\frac{1}{2}}(-1,t)$, that is,

$$|y(t)-x(t)| \le \Omega \cdot \varepsilon E_{\frac{1}{2}}(-1,t),\ t \in [-1, \frac{3}{2}],$$

where

$$\Omega = \frac{(T-t_0)^{\alpha}}{\alpha}\left\{1 + \frac{2L_f E_\alpha(2L_f, T-t_0)}{|\lambda - 2L_f|}E_\alpha(\lambda - 2L_f, T-t_0)\right\} = 0.6456.$$

Example 4.3. *Consider the following nonlinear delay Volterra integro-differential equations*

$$\begin{cases} T_{0^+}^{\frac{1}{2}}x(t) = \frac{x(t)}{25} + \frac{\sin(x(t-2))}{25} + \frac{1}{25}\int_0^t \frac{1}{20}[\sin^2(x(s)) - \cos(x(s-2))]s^{-\frac{1}{2}}ds, \ t \in (0,\pi], \\ x(t) = 0, \ t \in [-2,0]. \end{cases} \tag{4.3}$$

It is not difficult to verify that $L_g(t) = L_h(t) = \frac{1}{10}$, *and*

$$M = 2\left\{\int_{t_0}^{T}(s-t_0)^{\alpha-1}L_g(s)\left[1 + \int_{t_0}^{T}(\theta - t_0)^{\alpha-1}L_h(\theta)d\theta\right]ds\right\} = 0.9603 < 1.$$

Then all the conditions in Theorem 3.10 are satisfied, then the Equation (4.3) has a unique solution and is Ulam-Hyers stable with

$$\Delta = \frac{(T-t_0)^{\alpha}}{\alpha}exp\left(\int_{t_0}^{T}(\theta - t_0)^{\alpha-1}\big(2L_g(\theta) + L_h(\theta)\big)d\theta\right) = 36.8360.$$

5 Conclusion

This paper mainly investigates the Ulam's type stabilities of two classes of conformable fractional delay differential equations, including the Ulam-Hyers stability of (1.1) and (1.2) and the Ulam-Hyers' conformable exponential stability of (1.1). It is worthwhile noting that the CFEF plays an equivalent role as the well-known Mittag-Leffler function in classical fractional calculus, while the parameter λ in $E_\alpha(\lambda, t-t_0)(\lambda \neq 0)$ has a significant benefit in terms of studying the Ulam-Hyers' conformable exponential stability compared to the works [24,27-28] involved the Ulam-Hyers-Mittag-Leffler stability with respect to $E_\alpha(t^\alpha)$. Indeed, according to the definition about the Ulam-Hyers stability, we know: if y is an approximate solution of (1.1), then there exists an exact solution x of (1.1) near to y. As for the Ulam-Hyers' conformable exponential stability, the main result in Section 3.2 and the Table 5.1 in Example 4.2 may give more smaller error when $\lambda < 0$ with a larger absolute value.

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Tab. 5.1 The variation trend of the variables involved with λ when ε is chosen as 0.1 in Example 4.2

λ	1	$\frac{2}{3}$	0.1	$-\frac{2}{3}$	-1	-e	$-\pi$	-3.325	-3.675
Ω	21.3641	21.2583	6.6221	2.6285	2.5099	2.4499	2.4496	2.4495	2.4492
$E_{\frac{1}{2}}(\lambda, T-t_0)$	11.5824	5.1192	1.2776	0.2938	0.0863	0.00128	0.000455	0.00029	0.0001
$\varepsilon\Omega E_{\frac{1}{2}}(\lambda, T-t_0)$	24.7445	10.8825	0.8460	0.07722	0.002166	0.000313	0.00011	0.00007	0.00003