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ARTICLE TYPE

Elastic Order Reduction Transformation and Its Application in Solving a Class of Third-order Nonlinear Differential Equations†

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Abstract

Based on the theoretical knowledge of elasticity, this paper introduces elasticity (i.e. elastic transformation) as a tool to solve differential equations for the first time, and takes a class of third-order nonlinear ordinary differential equations as an example. First, elastic transformation is used to reduce the order to second-order solvable differential equation. Second, by using the elastic inverse transformation, the analytical solutions of the original equation are obtained. Finally, the steps of the elastic reduction transformation method are summarized. The examples show that some nonlinear equations can be transferred into linear equations, and this method has theoretical and applied value which is extensive and in-depth.

KEYWORDS:
third-order nonlinear differential equation; elastic transformation;elastic reduction transformation method; A class of solvable differential equations.

1 | INTRODUCTION

The elasticity has a wide range of applications in the analysis of economic problems, and the solution of nonlinear differential equations is a hot spot of modern research and application.

In the field of economics, Marshall first proposed the concept of elasticity, which refers to the sensitivity of consumers and producers to price changes [1]. Moreover, elasticity is widely used in many economic analysis fields, such as price and demand issues[2], the quality of economic growth[3], the production of consumers[4].In mathematics, the concept of elasticity can be applied to all variables that have a causal relationship. In 1997, Woods put forward the concept of elasticity, which reflects the influence of change range of the dependent variable with the independent variable[5].

In practical applications, most of the problems are essentially nonlinear, and linear system is only a linear abstraction and description of the approximate problems, therefore, it is difficult to truly and accurately reflect the actual problems. Nonlinear equations and models can reveal their internal relations and essential laws more accurately and in detail, so it is important for us to solve nonlinear differential equations. Nowadays, some methods are proposed by scholars such as constant variation method and integral method to solve differential equations, but there are few solvable classes of differential equations.[7].

With the help of the theoretical knowledge of elasticity, In this paper, elastic transformation is introduced into the solution of differential Equations.The following is a class of third-order nonlinear ordinary differential equations $xy^2y''' + [(2+ax)y - 3xy'^3]yy'' - (2+ax)yy'^2 + (a+by)y^2y' = 0$, (is constant) as an example. It is difficult to find their analytical solutions by using existing methods, so can they be reduced to the familiar equations by some transformations [8,9]? This paper presents a new method—elastic reduction transformation method that can reduce the order of such nonlinear differential equations to solvable

†This is an example for title footnote.

equations. Firstly, the elastic transformation is used to reduce the order to a second-order solvable differential equation [10], then the inverse elastic transformation is used to obtain the analytical solution of the original equation.

2 | PRELIMINARY KNOWLEDGE

2.1 | Definition of elasticity

The differentiable function $y = f(x)$ is ,and $f(x) \neq 0$, and the elastic relationship between the independent variable x and the dependent variable y is expressed as:

$$u = \frac{x}{y} \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{x}{y} \frac{dy}{dx}, \tag{1}$$

Where u is function of x , and u is also the elasticity of the function y with respect to x.

The inverse elastic transformation is

$$y = e^{\int \frac{u}{x} dx}, \tag{2}$$

2.2 | Economic significance of u

$u$reflects the influence of the change range $\frac{\Delta y}{y}$ of dependent variable y with the change range $\frac{\Delta x}{x}$ of independent variable x .In other words, u means the intensity or sensitivity of y the relative change ofto the relative change of x.Specifically,u means that y changes approximately $u\%$ when x changes 1% at $x = x$.

2.3 | The elastic representation of the derivative

According to Eq. (1), we have:

$$y' = \frac{dy}{dx} = \frac{y}{x} \cdot u, \tag{3}$$

$$y'' = \frac{y}{x} \left[u' + \frac{1}{x} u (u - 1) \right], \tag{4}$$

$$y''' = \frac{d^3 y}{dx^3} = \frac{y}{x} \left[u'' + \frac{1}{x} (3u - 2) u' + \frac{1}{x^2} u (u - 1)(u - 2) \right], \tag{5}$$

3 | ANALYTICAL SOLUTIONS OF A CLASS OF THIRD ORDER NONLINEAR DIFFERENTIAL EQUATIONS THEOREM

We consider:

$$xy^2 y''' + \left[(2 + ax) y - 3xy'\right] yy'' + 2x\left(y'\right)^3 - (2 + ax) y\left(y'\right)^2 + (a + bx) y^2 y' = 0, \tag{6}$$

Where a, b are all real constants.

The general solutions can be expressed as follows:

$$y = \begin{cases} C_3 e^{C_1 \int \frac{e^{r_1 x}}{x} dx + C_2 \int \frac{e^{r_2 x}}{x} dx}, \ \Delta > 0; \\ C_3 e^{C_1 \int \frac{e^{rx}}{x} dx + \frac{C_2}{r} e^{rx}}, \ \Delta = 0; \\ C_3 e^{C_1 \int \frac{e^{\alpha x} \cos \beta x}{x} dx + C_2 \int \frac{e^{\alpha x} \sin \beta x}{x} dx}, \ \Delta < 0. \end{cases} \tag{7}$$

Where:$C_1, C_2, C_3$are arbitrary constants.

Proof: According to Eqs. (1),(3),(4),(5) and (6), We can reduce the order of the third-order ordinary differential equation about $y \sim x$ to the second-order solvable ordinary differential equation about $u \sim x$. (i.e. the equation satisfied by the elasticity of y

with respect to x):

$$u'' + au' + bu = 0, a, b \in R, \tag{8}$$

Hence, the characteristic equation of differential Eq. (8) is $r^2 + ar + b = 0$, the discriminant is $\Delta = a^2 - 4b$, then the corresponding eigenvalue (root) is $r_1 = \frac{-a+\sqrt{a^2-4b}}{2}, r_2 = \frac{-a-\sqrt{a^2-4b}}{2}$.

Thus, the general solution of Eq (6) can be obtained as follows:

$$u = \begin{cases} C_1 e^{r_1 x} + C_2 e^{r_2 x},\ \Delta > 0; \\ \left(C_1 + C_2 x\right) e^{rx},\ \Delta = 0; \\ e^{\alpha x}\left(C_1 \cos\beta x + C_2 \sin\beta x\right),\ \Delta < 0. \end{cases} \tag{9}$$

Where $C_1, C_2$is an arbitrary constant, and $r = -\frac{a}{2}, \alpha = -\frac{a}{2}, \beta = \frac{\sqrt{4b-a^2}}{2}$.

By substituting Eq (9) into Eq (2), the general solution of differential Eq (6) can be obtained as follows:

a.If$\Delta > 0, u = C_1 e^{r_1 x} + C_2 e^{r_2 x}$,

$$y = e^{\int \frac{u}{x} dx} = e^{\int \frac{C_1 e^{r_1 x} + C_2 e^{r_2 x}}{x} dx} = C_3 e^{C_1 \int \frac{e^{r_1 x}}{x} dx + C_2 \int \frac{e^{r_2 x}}{x} dx}$$

b.If$\Delta = 0, u = \left(C_1 + C_2 x\right) e^{rx}$,

$$y = e^{\int \frac{u}{x} dx} = e^{\int \frac{(C_1 + C_2 x) e^{rx}}{x} dx} = C_3 e^{C_1 \int \frac{e^{rx}}{x} dx + \frac{C_2}{r} e^{rx}}$$

c.If$\Delta < 0, u = e^{\alpha x}\left(C_1 \cos\beta x + C_2 \sin\beta x\right)$,

$$y = e^{\int \frac{u}{x} dx} = e^{\int \frac{e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)}{x} dx} = C_3 e^{C_1 \int \frac{e^{\alpha x}\cos\beta x}{x} dx + C_2 \int \frac{e^{\alpha x}\sin\beta x}{x} dx}$$

On the contrary, the solution of y is substituted into the original Eq(6), which is verified as the solution of Eq.(6).

4 | ELASTIC REDUCTION TRANSFORMATION METHOD

According to the theorem, We can summarize the specific steps of the elastic reduction transformation method:

Step1:Perform the elastic reduction transformation on y of the equation to be solved, that is Eq.(1).

Step2:The elastic reduction transformation Eq.(1) and the elastic expression of the derivative Eq.(3), Eq. (4) and Eq. (5) are substituted into the equation to be solved, and the order of the equation can be reduced to a solvable differential equation.

Step3:According to the inverse elastic transformation Eq. (2), the analytical solution of the equation can be obtained.

5 | APPLICATION IN THE INSTANCE

Based on the above elastic reduction transformation, the ordinary differential equation of the third order is solved as follows:

Example1

$$xy^2y''' + [(2+3x)y - 3xy'^3]yy'' - (2+3x)yy'^2 + (3+2y)y^2y' = 0, \tag{10}$$

Solving steps:

Step 1:The elastic order reduction transformation of the function y is made, namely, Eq (1).

Step 2:The elastic reduction transformation Eq.(1) and the elastic expression Eq.(3), Eq. (4) and Eq. (5) of derivatives are substituted into Eq (10) to reduce the order of the third-order ordinary differential Eq (10) with respect to $y \sim x$ to the second-order solvable ordinary differential equation with respect to $u \sim x$:

$$u'' + 3u' + 2u = 0, \tag{11}$$

According to the existing results, the general solution of the second order ordinary differential Eq (11) can be expressed as

$$u = C_1 e^{-x} + C_2 e^{-2x}, \tag{12}$$

Step 3:By substituting Eq (12) into the inverse elastic transform Eq (2), the general solution of the third-order nonlinear differential Eq (10) can be obtained, that is

$$y = C_3 e^{C_1 \int \frac{e^{-x}}{x} dx + C_2 \int \frac{e^{-2x}}{x} dx}$$

Let $C_1 = 1, C_2 = 0, C_3 = 1$, then the graph of the general solution is Figure 1;
Let $C_1 = 0, C_2 = 1, C_3 = 1$, then the graph of the general solution is Figure 2;
Let $C_1 = 1, C_2 = 1, C_3 = 1$, then the graph of the general solution is Figure 3.

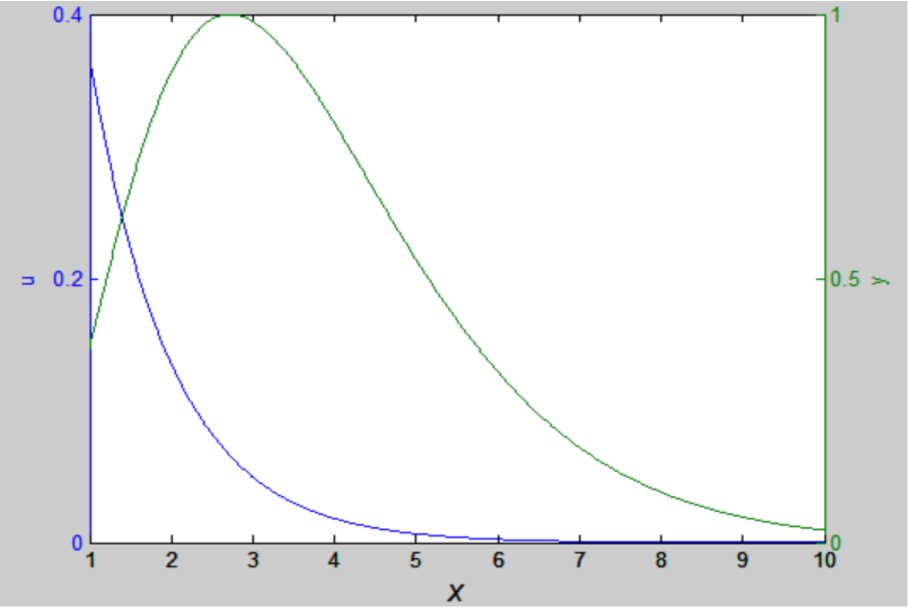

FIGURE 1

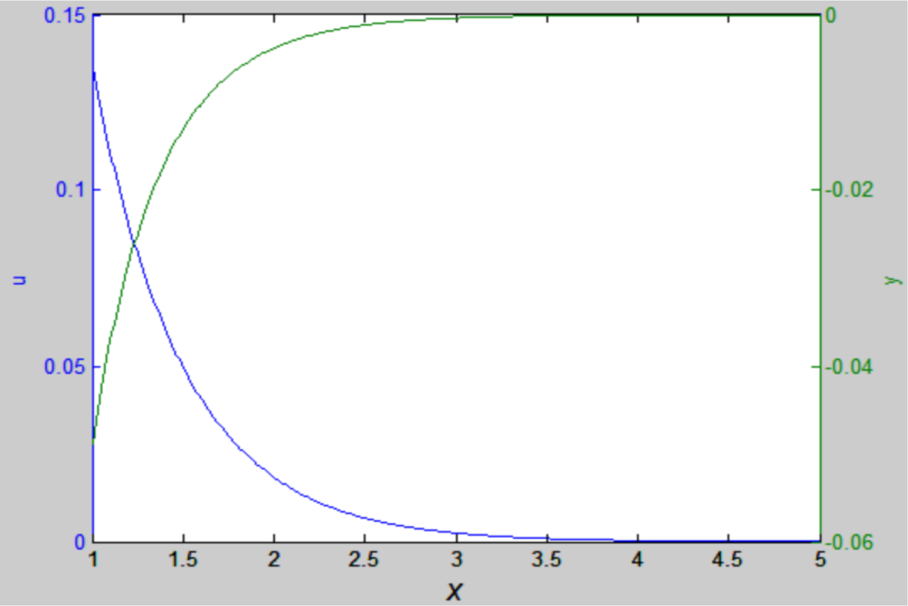

FIGURE 2

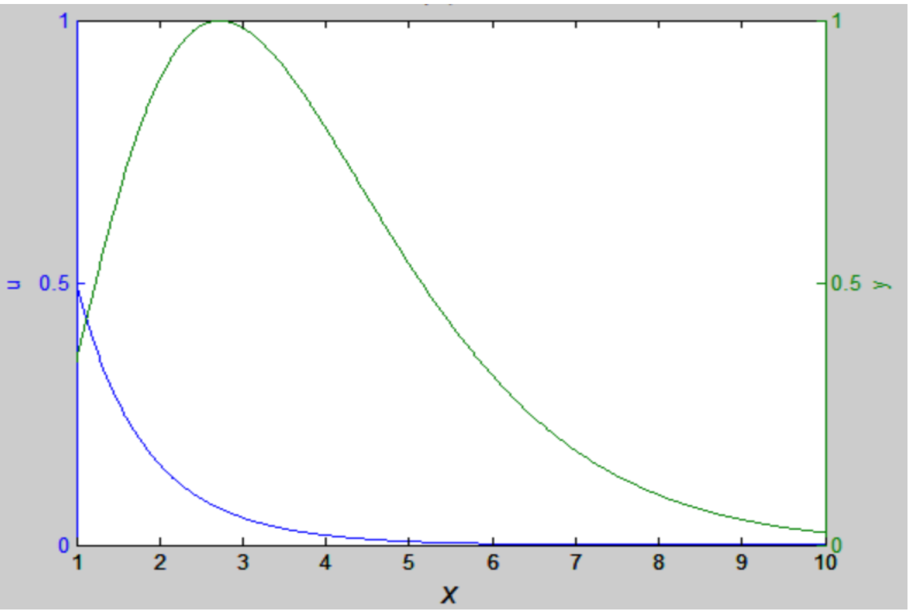

FIGURE 3

Example2

$$xy^2y''' + \left[(2+4x)\,y - 3xy'\right] yy'' + 2x\left(y'\right)^3 - (2+4x)\,y\left(y'\right)^2 + (4+4x)\,y^2y' = 0, \tag{13}$$

Solving steps:

Step 1:The elastic order reduction transformation of the function y is made, namely, Eq (1);

Step 2:The elastic expression (1) and the elastic expression (3), (4) and (5) of derivatives are substituted into Eq (13) to reduce the order of the third-order ordinary differential Eq (13) with respect to $y \sim x$ to the second-order solvable ordinary differential equation with respect to $u \sim x$:

$$u'' + 4u' + 4u = 0, \tag{14}$$

According to the existing results, the general solution of the second order ordinary differential Eq (14) can be expressed as

$$u = \left(C_1 + C_2x\right) e^{-2x}, \tag{15}$$

Step 3:By substituting Eq (15) into the inverse elastic transform Eq (2), the general solution of the third-order nonlinear differential Eq (13) can be obtained.

$$y = C_3 e^{C_1 \int \frac{e^{-2x}}{x} dx + \frac{C_2}{-2} e^{-2x}}$$

Let $C_1 = 1, C_2 = 0, C_3 = 1$, then the graph of the general solution is Figure 4;

Let $C_1 = 0, C_2 = 1, C_3 = 1$, then the graph of the general solution is Figure 5;

Let $C_1 = 1, C_2 = 1, C_3 = 1$, then the graph of the general solution is Figure 6.

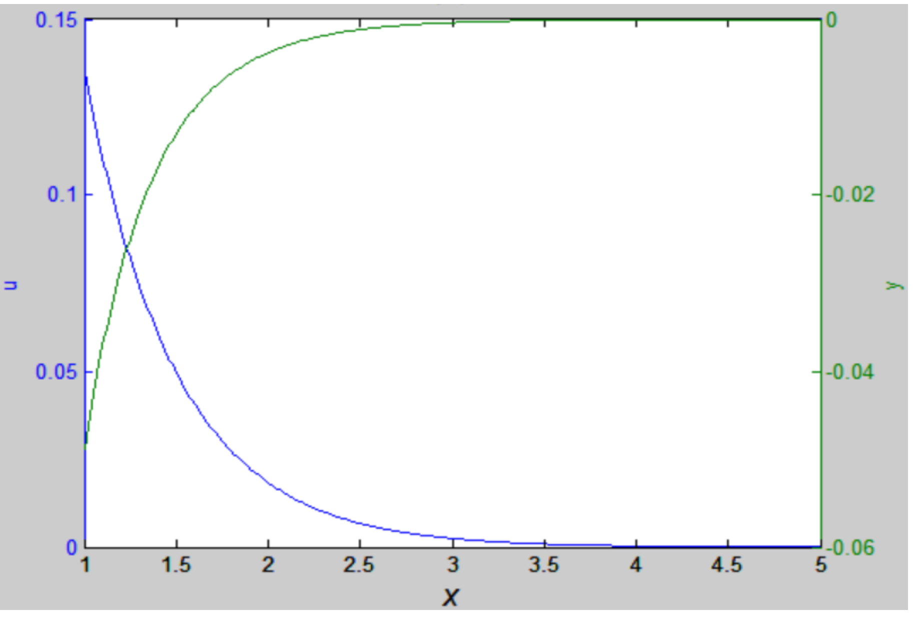

FIGURE 4

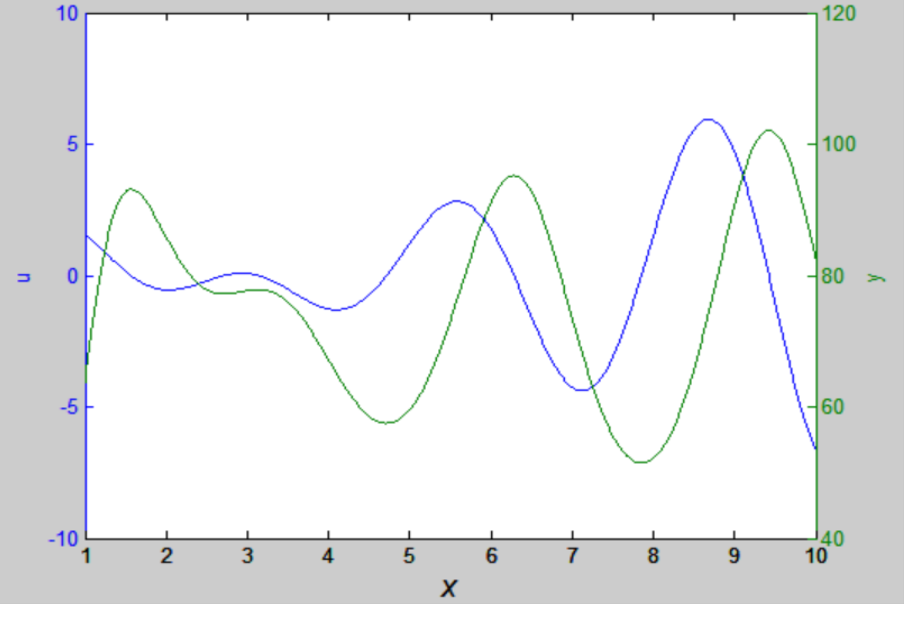

FIGURE 5

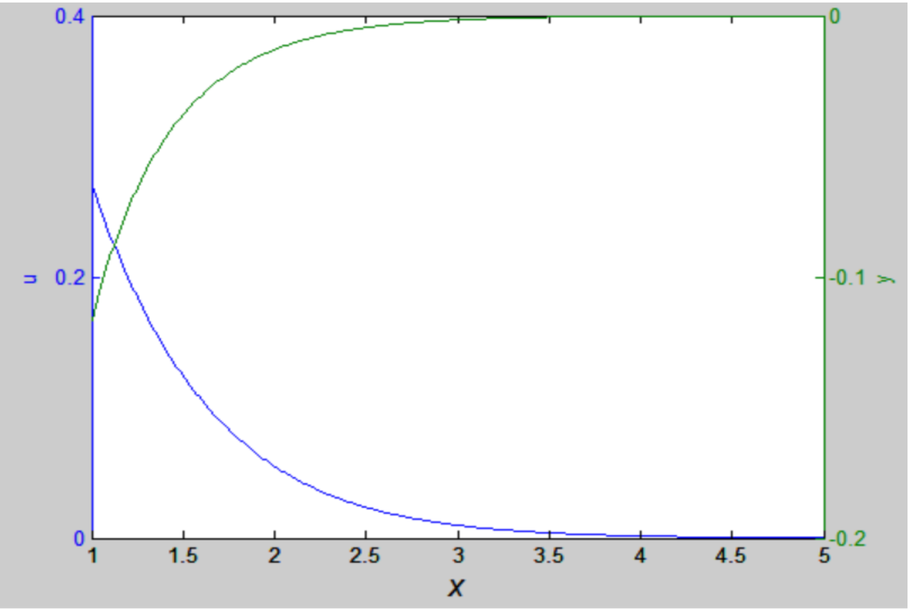

FIGURE 6

Example3

$$xy^2y''' + \left[(2+2x)\,y - 3xy'\right] yy'' + 2x\left(y'\right)^3 - (2+2x)\,y\left(y'\right)^2 + (2+5x)\,y^2y' = 0, \tag{16}$$

Solving steps:

Step 1:The elastic order reduction transformation of the function y is made, namely, Eq (1);

Step 2:The elastic expression (1) and the elastic expression (3), (4) and (5) of derivatives are substituted into Eq (16) to reduce the order of the third-order ordinary differential Eq (16) with respect to $y \sim x$ to the second-order solvable ordinary differential equation with respect to $u \sim x$:

$$u'' + 2u' + 5u = 0, \tag{17}$$

According to the existing results, the general solution of the second order ordinary differential Eq (17) can be expressed as

$$u = e^{\alpha x}\left(C_1 \cos\beta x + C_2 \sin\beta x\right)$$

Step 3:By substituting Eq (18) into the inverse elastic transform Eq (2), the general solution of the third-order nonlinear differential Eq (16) can be obtained.

$$y = C_3 e^{C_1 \int \frac{e^{\alpha x}\cos\beta x}{x}dx + C_2 \int \frac{e^{\alpha x}\sin\beta x}{x}dx}$$

Let $C_1 = 1, C_2 = 0, C_3 = 1$, then the graph of the general solution is Figure 7;

Let $C_1 = 0, C_2 = 1, C_3 = 1$, then the graph of the general solution is Figure 8;

Let $C_1 = 1, C_2 = 1, C_3 = 1$, then the graph of the general solution is Figure 9.

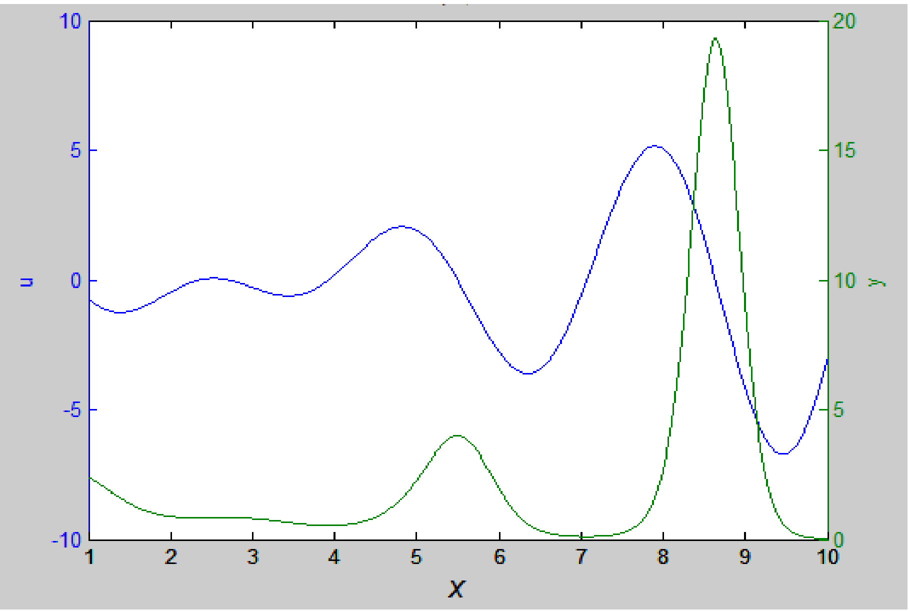

FIGURE 7

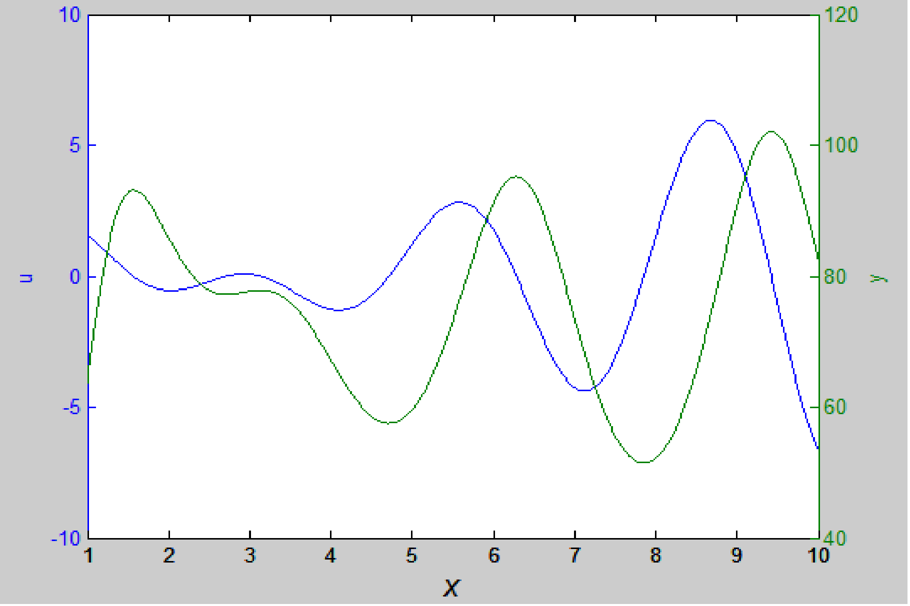

FIGURE 8

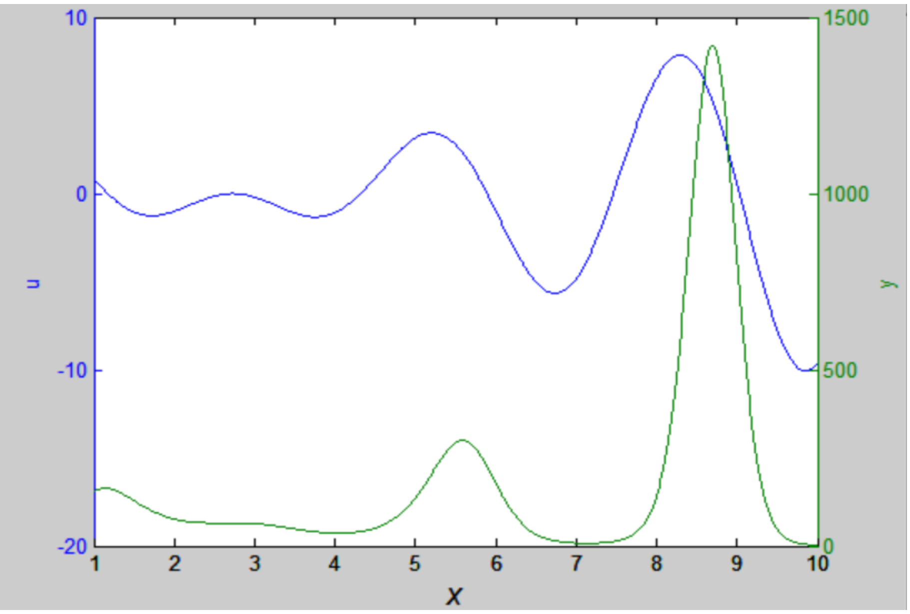

FIGURE 9

6 | CONCLUSIONS

(1) This paper first introduces elasticity in economics as a tool for solving differential equations, which is an innovation for the study of elasticity;

(2)The elastic reduction transformation method explored in this article has the outstanding characteristics of transforming some nonlinear equations into solvable equations, expanding the solvable categories of differential equations. And the method, which has a broad and deep theoretical and application value, allows some unsolvable differential equations to be solved analytically.

6.1 | Acknowledgements

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6.2 | Bibliography

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