Fuzzy Stochastic Capacitated Vehicle Routing Problem and Its Applications

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Abstract

This paper considers Capacitated Vehicle Routing Problem(CVRP) in an imprecise and random environment. The deterministic version of the problem deals with finding a set of routes in such a way that the demand of all the customers present in the network are satisfied and the cost incurred in performing these operations comes out to be a minimum. In practical life situations, problems are not always defined in crisp form. Phenomenons like randomness and impreciseness are quite natural to arise in real life. This work presents CVRP in such a mixed environment. In this work, the demands of the customers are assumed to be stochastic in nature and are revealed only upon the arrival of the salesman. Moreover, the edge weights are representing the time required to traverse the edge and hence are both imprecise and random in nature. Different traffic conditions, weather conditions and many other factors corresponds to the random nature of edge weights and varying speed of the vehicle corresponds to the impreciseness. Thus, in this work, edge weights are represented by discrete fuzzy random variables. In this paper, an expectation based approach has been used to deal with randomness. A procedure based on Branch and Bound algorithm has been used to find routes with minimum cost. A numerical example has also been presented to explain the working of the method proposed.

Keywords: Transportation and logistics, Capacitated Vehicle Routing Problem, Branch and Bound Algorithm, Network, Triangular Fuzzy Number.

1. Introduction

In a network, the problem of finding the shortest possible tour which starts and ends at origin node and every node(*except origin node*) is visited exactly once is known as Traveling Salesman Problem (TSP)[3]. Vehicle routing problem(VRP) was introduced in 1959, approximately 60 years before, by Dantzig and Ramser [4]. Earlier this was known as Truck Dispatching Problem. Capacitated Vehicle Routing Problem (CVRP) [27] is one of the most studied version of VRP. In CVRP, a fleet of vehicles with finite carrying capacity is provided at the origin node. A network is provided in which the customers with some specific demands are present at the nodes. The motive is to find a set of vehicle routes to perform all transportation requests with the given fleet at a minimum cost. Finding optimal routes[3] in CVRP is one of the basic real life problems. Because of its abundant use in route planning, communication networks, school bus scheduling etc., it has attracted lots of academic and industrial researchers in last few decades. The deterministic version of CVRP is simply solvable. Several effective algorithms such as Branch cut and price algorithm[9], Genetic Algorithm [17] [21], Clark and Wright algorithm [23] [16] etc. are used by various researchers to solve the classical version of the problem. Apart from few exact solution approaches, there are various algorithm which includes the use of heuristics and meta heuristics to solve VRP faster and reduce the time complexity. One of the very early works in heuristics was performed by Clark and Wright in the year 1964 where a savings heuristics was designed to calculate the path to be followed. Later on, various algorithms such as Christofides algorithm [8], an improvised version of nearest neighbour algorithm[18] and several other heuristics were also introduced by various researchers.

The parameters of VRP includes the structure of network, the customers present at the nodes, the demands of various customers, the cost matrix(*time matrix*) and many more. In real life problems, these parameters of VRP may be uncertain as well as imprecise. Demands of customers, travel time, presence of customers or service time of the customers are examples of few parameters which can be responsible for uncertainty and impreciseness of the network. Impreciseness and randomness can be dealt by using fuzzy set theory[31] and probability theory[20] respectively. In this work,we are considering a network where only the demands of the customers and the edge weights of the network are random in nature. The exact demands of the customer is revealed only upon the arrival of the travelling salesman but a probability distribution function of customers' demand is very well known to the salesman in advance. The edge weights denoting the approximate time required to cover a particular edge in a random traffic condition includes both randomness and impreciseness, and hence are represented by using fuzzy random variables[19]. In this work, the random traffic conditions on each arc of the network are represented by using fuzzy linguistic variables[31]such as high, low and medium.

If one or more parameters of VRP are stochastic in nature, then such a VRP is termed as Stochastic Vehlicle Routing Problem ($SVRP$)[27]. Under Vehicle routing problem with stochastic demands[6], a planned route may fail when upon reaching a certain customer, it is realized that the demand of observed customer is more than the residual capacity of the vehicle. This situation is termed as route failure and this can be handled by using recourse actions[22]. Recourse actions are usually of two types, reactive recourse action, when the vehicle executes the return trip to the depot at failure location and refill the vehicle and start the journey with the remaining customers which are yet to be served; preventive recourse action, when the vehicle execute a preventive return trip to depot when the residual capacity of the vehicle falls below a certain threshold. Reactive recourse policy was introduced by Dror and Trudeau [6] and implemented by Gendreau et. al [10], whereas the preventive recourse policy was introduced by Yee et. al [29].

VRP itself is a NP hard problem [3], and the introduction of randomness and impreciseness [25, 26, 24] only increases the intractability of the problem. So, it is not much surprising that the very first numerical methods used for solving CVRP included the use of heuristics and meta heuristics. The purpose of this paper is to propose an algorithm based on Branch and Bound[3] to find routes for CVRP in a mixed environment. The uncertainty of the network related to edge weights has been handled by using expectation of fuzzy random variables and uncertainty of demands of customers is handled by taking optimal recourse action. In this paper, reactive recourse action policy has been used.

This paper is organized as follows: In section 2, some basic definitions regarding fuzzy set theory have been reviewed. This section also comprises of the definition of fuzzy random variables and the concept of expectation of random variable has been extended to the concept of expectation of fuzzy random variable. In this section, a brief review about Branch and Bound algorithm which is used to find routes for TSP is also provided. In section 3, various assumptions for solving CVRP have been discussed and a mathematical model for solving CVRP in a mixed environment has also been presented. Section 4 proposes the descriptive algorithms for the methodology presented in the paper. In this section, a flow chart of the method has also been presented. In section 5, a numerical example with 5 customers has been presented to explain the working of the method. The later part of the section compares the results of the method used with other methods and provides useful insights. The last section comprises of the concluding remarks.

2. Preliminaries

2.1. Fuzzy Set [31]

Definition 1. *If X is a universe of discourse and x is a particular element of X, then a fuzzy set $\tilde{A}$ defined on X can be defined as a set of ordered pairs, i.e.*

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)), x \in X\}$$

where, $\mu_{\tilde{A}}(x) : X \to [0, 1]$ is known as membership function.

2.2. Fuzzy Number [13]

Definition 2. *A fuzzy set $\tilde{A}$ on $\mathbb{R}$ is said to be a fuzzy number, if the following three properties are satisfied:*

1. *Fuzzy set $\tilde{A}$ must be normal, i.e. $\exists\ x$ such that $\sup \mu_{\tilde{A}}(x) = 1$, where sup stands for supremum.*
2. *The support of fuzzy set i.e. the set of all the elements with non zero degree of membership must be bounded.*
3. *α level set, A_α, i.e. the set of all the elements with membership degree greater than α must be a closed interval for $\alpha \in [0, 1]$.*

2.3. Triangular Fuzzy Number [13]

Definition 3. *A fuzzy number $\tilde{A}$ is said to be a triangular fuzzy number if the graph of its membership function is triangular in shape.*
The membership function of a triangular fuzzy number $\tilde{A} = (a, b, c)$ is defined by 1 and its graph is represented by figure 1.

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & if\, x < a \\ \dfrac{x-a}{b-a} & if\, a \leq x \leq b \\ \dfrac{c-x}{c-b} & if\, b \leq x \leq c \\ 0 & if\, x \geq c \end{cases} \tag{1}$$

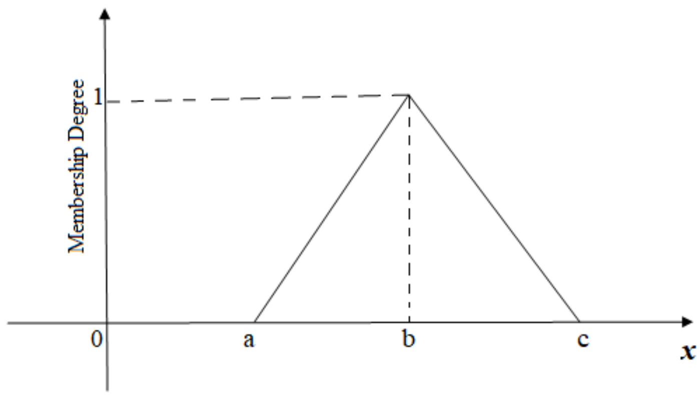

Figure 1: Triangular Fuzzy Number $\tilde{A} = (a, b, c)$

2.4. Operations on Triangular Fuzzy numbers [13]

Definition 4. *Let $\tilde{A} = (a, b, c)$ and $\tilde{B} = (p, q, r)$ are two triangular fuzzy numbers, then the arithmetic operations on them are defined in the following way:*

1. ***Addition:***

$$(a, b, c) \oplus (p, q, r) = (a + p, b + q, c + r)$$

2. ***Subtraction:***

$$(a, b, c) \ominus (p, q, r) = (a - r, b - q, c - p)$$

3. ***Scalar multiplication:***

$$k(a, b, c) = (ka, kb, kc) \qquad when\, k > 0$$

2.5. Graded Mean Integration Representation Method [12]

Definition 5. *Let L^{-1} and R^{-1} are the inverse functions of $L(left)$ and $R(right)$ respectively, then the graded mean integration representation of a generalized triangular fuzzy number $\tilde{A}$ is given by:*

$$G(\tilde{A}) = \frac{\int_0^1 h(\frac{L^{-1}(h)+R^{-1}(h)}{2})dh}{\int_0^1 hdh}.$$

Here,

$$L(x) = \frac{x-a}{b-a}, \qquad a \leq x \leq b,$$

and

$$R(x) = \frac{c-x}{c-b}, \qquad b \leq x \leq c,$$

then

$$L^{-1}(h) = a + (b-a)h, \qquad 0 \leq h \leq 1,$$

$$R^{-1}(h) = c - (c-b)h, \qquad 0 \leq h \leq 1,$$

Thus, the graded mean integration representation[12] of $\tilde{A}$ is given by

$$G(\tilde{A}) = \frac{a+4b+c}{6} \tag{2}$$

2.6. Comparison of Two Triangular Fuzzy Numbers

Definition 6. *Based on the graded mean integration representation of a triangular fuzzy number, two TFNs $\tilde{A}$ and $\tilde{B}$ can be compared in the following manner:*

$$\begin{aligned} &\textit{If } G(\tilde{A}) > G(\tilde{B}), \textit{ then } \tilde{A} > \tilde{B} \\ &\textit{If } G(\tilde{A}) < G(\tilde{B}), \textit{ then } \tilde{A} < \tilde{B} \\ &\textit{If } G(\tilde{A}) = G(\tilde{B}), \textit{ then } \tilde{A} = \tilde{B} \end{aligned} \tag{3}$$

2.7. Fuzzy Linguistic Variable [30]

Definition 7. *There are linguistic variables whose states are fuzzy numbers. Fuzzy variables representing linguistic concepts such as young, middle aged, old etc. with respect to age are known as fuzzy linguistic variables. The figure 2 represents few fuzzy linguistic variables corresponding to age.*

Example 1. *Consider a survey where individuals are questioned about weather conditions of a specific area. The responses of individuals are classified into three categories, namely:*

1. *Hot*
2. *Cold*
3. *Pleasant*

In the above example, the anonymity about the opinion of individual gives rise to randomness. Once the opinion of an individual is known, there is still some uncertainty about accurate meaning of the response. This lack of preciseness denotes the fuzziness of random variable weather conditions [15].

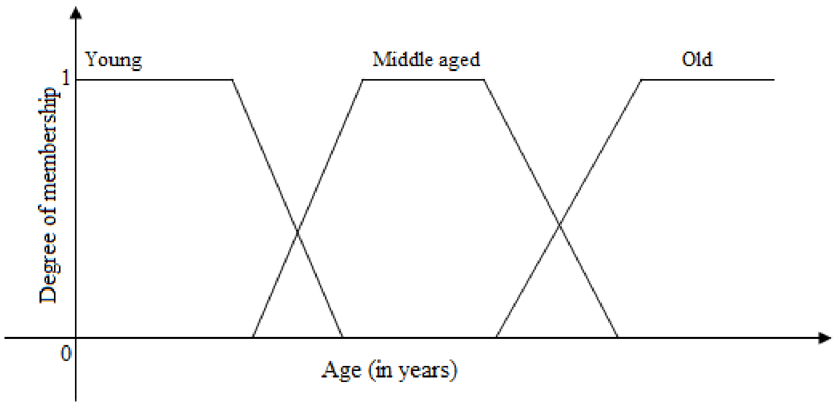

Figure 2: Fuzzy Linguistic Variable

2.8. Expectation of a Fuzzy Random Variable [19]

Definition 8. *Let $\tilde{X}$ be a fuzzy random variable. Then the expectation of $\tilde{X}$ is denoted by $E[\tilde{X}]$, can formally be defined as*

$$E[\tilde{X}] = \sum_{\tilde{x}} \tilde{x} \odot p(\tilde{x}) \tag{4}$$

where $p(\tilde{x})$ denotes the probability of fuzzy random variable $\tilde{X}$ taking the value $\tilde{x}$

2.9. Branch and Bound Algorithm [3]

In order to solve combinatorial optimization problems, one of the algorithm used is Branch and Bound algorithm. Typically, the time complexity of combinatorial optimization problems is exponential and in worst cases, exploration of all possible nodes may be required. Branch and Bound technique solve such problems relatively quickly. In Branch and Bound method, for a current node in the tree, a lower bound is calculated which gives the best possible solution that can be obtained if a path through that node is traversed. If the bound on best possible solution is worse than current best, then the sub tree obtained with that node is ignored and the nodes giving better lower bounds are only traversed.

In Traveling Salesman Problem, the bound on best possible solution can be obtained by using the given formula:

$$\text{lower bound} = \left\lceil \frac{\text{Sum of two minimum weighted edges adjacent to every vertex}}{2} \right\rceil \tag{5}$$

Without the loss of generality, we assume that the starting vertex is 0(depot node) and a lower bound for the solution can be obtained by the formula given by (5). After this, we compute the lower bound when a particular vertex i is traversed after 0 for each i. In calculation of that lower bound, cost of $0 - i$ and $i - 0$ is considered (even if they are not the minimum). The vertex which gives minimum lower bound will be traversed next until and unless the best solution down the node become worse than any other solution which have been obtained in another part of the sub tree. This process is continued until all the nodes of the network are traversed.

3. CVRP in an imprecise and random environment

Deterministic version of CVRP generally focuses on the distribution of goods from a single source node to a given set of n other nodes. The source node is usually known as depot node and other n nodes are known as customer nodes. These customers have some predefined demands. On the depot node, a fleet of vehicle is present which is used to fulfil the demands of the customers located at different nodes. In CVRP, the motive is to find a set of routes in a way such that every customer is visited exactly once and by one vehicle only and their demands are fulfilled in that visit only and the cost incurred in executing such routes is minimum.

Impreciseness and randomness are two of the major phenomenons occurring in real life problems. In order to bridge the gap between real world problems and their corresponding mathematical problem, the mathematical model of such problems should also consider about how to tackle the issues like impreciseness and randomness. In this work, the demands of customers have been assumed to be stochastic in nature i.e. the demand of customers can only be known upon the arrival of the vehicle. Here, the cost matrix stores the time required to cover the path. Time taken to cover the path is imprecise and random in nature, since it is affected by traffic conditions, weather conditions, nature of road etc.. Here, in this work, we are considering traffic conditions to be the only reason of stochastic nature of edge weights. In addition to randomness, the edge weights are also imprecise in nature, the varying speed of the vehicle corresponds to the impreciseness of edge weights. Thus, the edge weights in the network are given by fuzzy random variables[14] whose value is the approximate time required, to cover the edge in a random traffic condition. The random traffic conditions in the network are given by discrete fuzzy linguistic variables.

A function $\tilde{f}_{ij}$ denotes the fuzzy random variable for each edge e_{ij} joining the nodes i and j. The domain of this fuzzy random variable is the different types of traffic conditions and the value of this fuzzy random variable is the approximate time taken to traverse the edge e_{ij} in the specific traffic condition. This introduces both randomness and impreciseness in the network.
i.e.

$$\tilde{f}_{ij}(D) \rightarrow \tilde{F}_k(\mathbb{R})$$

The domain of this function is $D = \{High, Low, Medium\}$ which is the set of different traffic conditions on the edge. The value of the fuzzy random variable is a fuzzy number which informs about the approximate time taken to cover that edge in the respective traffic conditions. The customers present at various nodes have uncertain demands and are denoted by using discrete random variables. The uncertain demands of the customers are also a reason of randomness.

3.1. Service Policies in CVRP

There are two major service policies in CVRP, first one is full delivery, which means that if a vehicle arrives at a customer, then the demands of the customer is fully accomplished in that trip only and in such case, the assumption of demands of the customers to be lesser than the carrying capacity of the vehicle is mandatory for the feasibility of the problem, This service policy is mostly widely used in the literature. Second service policy is split delivery where demands of one customer can either be fulfilled by several vehicles or by several visits of the same vehicle. In the case of split delivery, the assumption that a customer is visited exactly once and by only one vehicle no longer holds. The problem of Vehicle Routing under split delivery has been very less studied.

3.2. Stages of CVRP

In addition to the service policies, one of the important attribute of the SVRP is the time at which demand becomes known. On the basis of demand revelation, there are two extreme cases. First is advanced information, in which the demands of the customers are known even before the routes are planned and this leads to the classical case of CVRP. The second case is about the late information in which the demand of the customers are revealed only after the arrival of the vehicle. In between these 2 extreme cases, there is a whole spectrum of possibilities, i.e. when the demand becomes known one, two, three steps ahead, etc.

Several policy and availability of information determine the number of stages in which SVRP may be solved. In SVRP with advanced information with full delivery, there is only one stage. In SVRP with late information and full delivery, there are n stages, where each stage corresponds to delivery of goods at one customer and at each stage decisions are to be taken regarding either going to next customer node or returning back to the depot on the basis of residual capacity of the vehicle. In the case of split delivery with late information, there can be more than n stages, where n is the number of customers in the network.

3.3. Assumptions of the model

Before formulating the mathematical model of CVRP in a mixed environment, we state few assumptions that will be used throughout the paper. The very first assumption is regarding the non-divisibility of goods, i.e. the goods to be delivered to customers are already in the smallest possible unit available. The journey of fleet of vehicle is assumed to originate and terminate only at the depot node and the fleet of vehicles is also assumed to be homogeneous i.e. operational costs and capacity of vehicles are assumed to be same. The service policy has been assumed to be full delivery policy in which the customer is to be visited exactly once and by one vehicle only. The stochastic demand D_i of the customer follows a discrete probability distribution with a finite support, defined as $\{{D_i}^1, {D_i}^2, \dots {D_i}^n\}$ where ${D_i}^k \leq Q$, where Q is the carrying capacity of the vehicle, i.e. only those customers are considered whose demands are lesser than the capacity of vehicle. The demands of customers are also assumed to be positive and independent, i.e. demand of a customer is not influenced by the demand of other customers. ${p_i}^k$ represents the probability of i^{th} customer observing the k^{th} demand. i.e. $P[D_i = {D_i}^k] = {p_i}^k$.

The traveling cost along an arc $(v_i, v_j) \in E$ is denoted by a fuzzy random variable, $\tilde{f}_{ij}$, where cost matrix $C = [\tilde{f}_{ij}]$ is symmetric and satisfies triangle inequality. An expectation based approach 4 has been used in this work to compare two fuzzy random variables and 2 and 3 are used to compare two fuzzy numbers. Since the demands of customers are only revealed upon the arrival, so there are chances of failure and recourse actions are adopted to handle those failures. In this work, the possibility of occurrence of more than one failure has been assumed while executing the planned route. In this work, reactive recourse policy has been used, i.e. a return trip is performed only upon the failure.

3.4. Real life applications of CVRP in mixed environment

Real world applications of the SVRP include among others the planning of cash collection from various branches of a bank in a city [2], and, in this case, the amount of the cash to be collected from various branches is a random variable and there is a bound on the amount of the cash that can be carried in a vehicle. The nature of the network, which is stored by the cost matrix(storing the time) is imprecise as well as random because of varying speed of vehicle and different traffic conditions respectively. Another applications include distribution of cash to different automatic teller machines (ATMs) in the city, where the nature of the network and the problem remains the same. This situation is represented by the use of Figure 3.Other examples include the delivery of essential commodity (milk, oil) where daily customer consumption is random in nature but can be predicted with the use of discrete random variable. The objective of the task is to find the traveling salesman tour in a mixed (imprecise and random) network when the demand of the customer is stochastic in such a way that the cost incurred comes out to be a minimum.

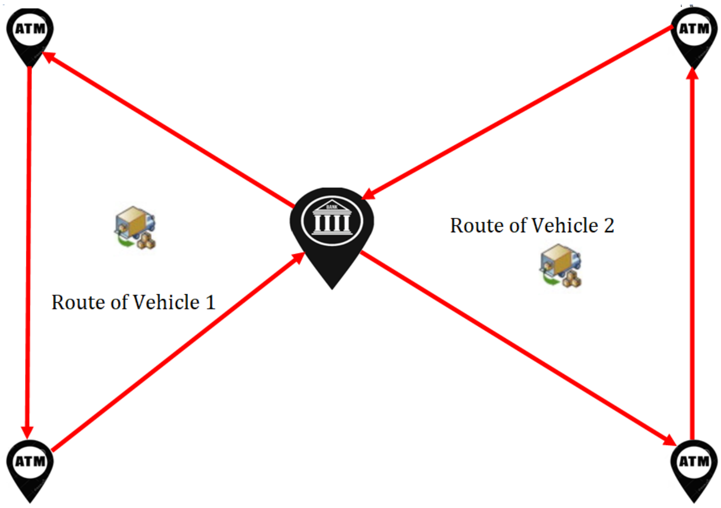

Figure 3: Money distribution to various ATM machines

3.5. Properties of the problem

It has already been observed by various researchers that a number of properties of optimal TSP does not hold true for VRP with stochastic demands even if the first stage of the problem corresponds to finding an optimal solution of TSP.

Property 1: The route designed by using Branch and Bound algorithm is the least cost route.

For the SVRP, the *a priori* route designed by using Branch and Bound algorithm does not, always, corresponds to a least cost route. This can be illustrated with the help of the Figure 4.

Consider a network of customers in which there are 3 customers and the coordinates for depot and customers 1, 2 and 3 are given by (0,0), (-4,0), (0,6) and (0,-7) respectively. Suppose there is only one vehicle of capacity 10 units. Distance between the points are calculated by using Euclidean distance norm. Then the *a priori* route designed by using Branch and Bound algorithm is 0-3-1-2-0. Suppose, while executing the routes the demands at the customers 3, 1 and 2 are realized as 3 units, 8 units and 5 units respectively and the demands are realized only when the delivery vehicle arrives at the corresponding customer. Then, while executing the route, when vehicle arrives at customer 3, it fulfils its demand and then move to customer 1 and then a route failure happens and hence the vehicle is bound to return to depot node and perform the refilling. After refilling, the execution of route starts from customer 1 and after fulfilling the demand of customer 1, the vehicle moves to customer 2, where again a failure of route is realized and thus routing to depot and refilling is performed again. After fulfilling the demands of all the customers, the vehicle returns to depot node. Thus, while executing the route 0-3-1-2-0 with given stochastic demands, the cost of the tour comes out to be 48.2732, whereas executing the 0-3-2-1-0, with the given stochastic demands the cost of the tour comes out to be 39.211; of course lower than the cost

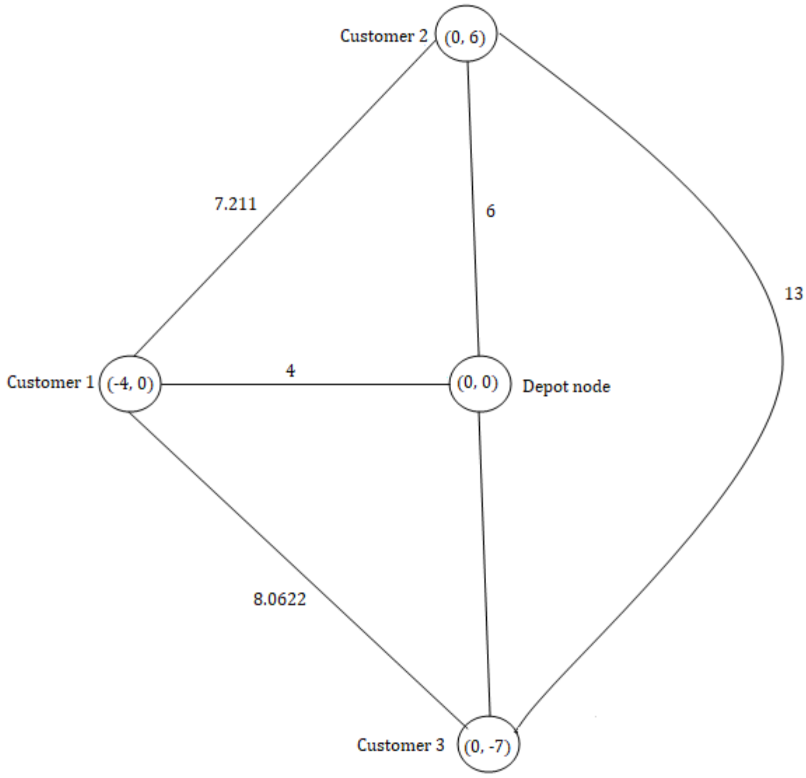

Figure 4: A network with 3 customers

of least cost tour obtained by using Branch and Bound algorithm.

Property 2 : The route direction has no impact on the cost of the route.

For the SVRP, the cost of the route designed by using the Branch and Bound algorithm is not always independent of the direction of the route. This can also be illustrated with the help of the above figure. In the above example, the cost incurred while executing the route 0-3–1-2-0 is 48.2732 and the cost incurred while executing the route 0-2-1-3-0(the same route in opposite direction) is 42.2732.

Thus, with the help of above two properties, we can clearly say that an optimal *a priori* route might not always result in a least cost tour in the presence of stochastic demands and the direction of traversal of route also comes out as an important factor while deciding the cost of the route under consideration.

3.6. Mathematical Model

A CVRP in an imprecise and random environment is represented by a complete weighted graph $G = (V, E)$ where V is the set of vertices and E is the set of edges. $V = \{v_0\} \cup \{v_1, v_2, \ldots v_n\}$, where $\{v_0\}$ is the depot node and $\{v_1, v_2, \ldots v_n\}$ are the customers with stochastic demands. D_i is the random variable representing the demand of customer located at node i and its expected value and variance are given by $E[D_i]$ and $Var[D_i]$ respectively.

A route is defined as a path of the form $r = (i_1, i_2, \ldots, i_{|r|})$ where $i_1 = i_{|r|}$=Depot node with $i_k \in V$ for $k \in \{2, 3, \cdots \mid r \mid -1\}$. Given such a route, we let $TD(\mu_{i_k}, {\sigma^2}_{i_k}) = \sum_{l=1}^{k} D_{i_l}$ denote the random variable indicating the total actual cumulative demand at customer i_k for $k \in \{2, 3, \cdots \mid r \mid\}$. Since the demands of customers are independent, thus we have $\mu_{i_k} = \sum_{l=1}^{k} E[D_{i_l}]$ and $\sigma^2_{i_k} = \sum_{l=1}^{k} Var[D_{i_l}]$. It can be easily seen that failures are separated by vehicles and that all the vehicles are identical. Given a route $r = (i_1, i_2, \ldots, i_{|r|})$, we let $EFC_{i_k}(\mu_{i_k}, \sigma^2_{i_k})$ denote expected failure cost at customer i_k. So, we can write

$$EFC_{i_k}(\mu_{i_k}, \sigma^2_{i_k}) = 2c_{0i_k} \sum_{u=1}^{\infty} (P\{TD(\mu_{i_{k-1}}, \sigma^2_{i_{k-1}}) \leq uQ\} - P\{TD(\mu_{i_k}, \sigma^2_{i_k}) \leq uQ\}) \tag{6}$$

where $P\{E\}$ denotes the probability of occurrence of event E. $P\{TD(\mu_{i_{k-1}}, \sigma^2_{i_{k-1}}) \leq uQ\} - P\{TD(\mu_{i_k}, \sigma^2_{i_k}) \leq uQ\}$ can therefore be interpreted as probability of having u^{th} failure at i_k given that it has not occurred on any previously visited customer along the route. Thus, an *a priori* model for CVRP with fuzzy stochastic travel times and stochastic demands is given by:

$$\text{Minimize} \qquad \sum E[\tilde{f}_{ij}].x_{ij} + \sum EFC_{i_{h+1}}(\mu_{i_{h+1}}, \sigma^2_{i_{h+1}})$$

subject to

$$\begin{aligned}
\sum_{j=2}^{n} x_{ij} &= 2m && && (7)\\
\sum_{i<k} x_{ik} + \sum_{k<j} x_{kj} &= 2 && && (8)\\
\sum_{v_i, v_j \in S} x_{ij} &\leq \mid S \mid - \frac{\sum_{v_i \in S} E[D_i]}{Q} && S \subset V - \{v_0\}; 2 \leq \mid S \mid \leq n-2 && (9)\\
x_{ij} &= \{0, 1\} && j = 2, \ldots, n && (10)\\
x_{0j} &= \{0, 1, 2\} && \forall \{0, j\} \in E && (11)\\
x &= x_{ij} && \text{an integer array} && (12)
\end{aligned}$$

In the mathematical model represented above, constraint 7 ensures that exactly m vehicles start their tour from depot node and end their tour at depot node. Constraint 8 ensures that every customer is connected to 2 other vertices. Constraint 9 eliminates the infeasible routes with excessive capacity demand. First stage of *a priori* approach deals with finding the routes for traveling salesman/vehicle without considering the stochastic demands of the customers. Since the edge weights in the network are represented by fuzzy random variables. Hence, for the comparison of edge weights, expected value of fuzzy random variables have been compared. The first stage deterministic cost for travelling all the customers exactly once is given by $\sum E[\tilde{f}_{ij}].x_{ij}$ where x_{ij} is a binary decision variable whose value is 1 if edge ij is traversed and 0 otherwise. (11) entails that the variable x_{0j} takes the value 1 if the edge $0 - j$ is traversed only once, i.e. j is the vertex from where the service starts. x_{0j} takes the value 2 if a failure at node j occurs because in such a case the path $0 - j$ is traversed twice, once for replenishment of goods and then for the resumption of the delivery and x_{0j} takes the value 0 if edge $0 - j$ is never traversed. The first stage of the *a priori* approach corresponds to finding the minimum weighted traveling salesman tour visiting all the customers exactly once and the first component of the objective function corresponds to this deterministic cost. In first stage, the demands of customers are ignored.

The first stage solution was obtained without considering the demands of the customers. However, in the presence of stochastic demands of the customers, a route designed earlier may fail because the observed demands of the customer may exceed the residual capacity of the vehicle and in such cases, recourse actions need to be taken. Thus, the total cost of operation increases. Hence, the total cost of operation is given by the sum of deterministic cost, which is obtained in the first stage of *a priori* approach and expected cost of recourse actions, which is calculated in second stage and is denoted by second component of the objective function.

4. Flowchart of the Method

The flowchart of the method discussed is given by Figure 5

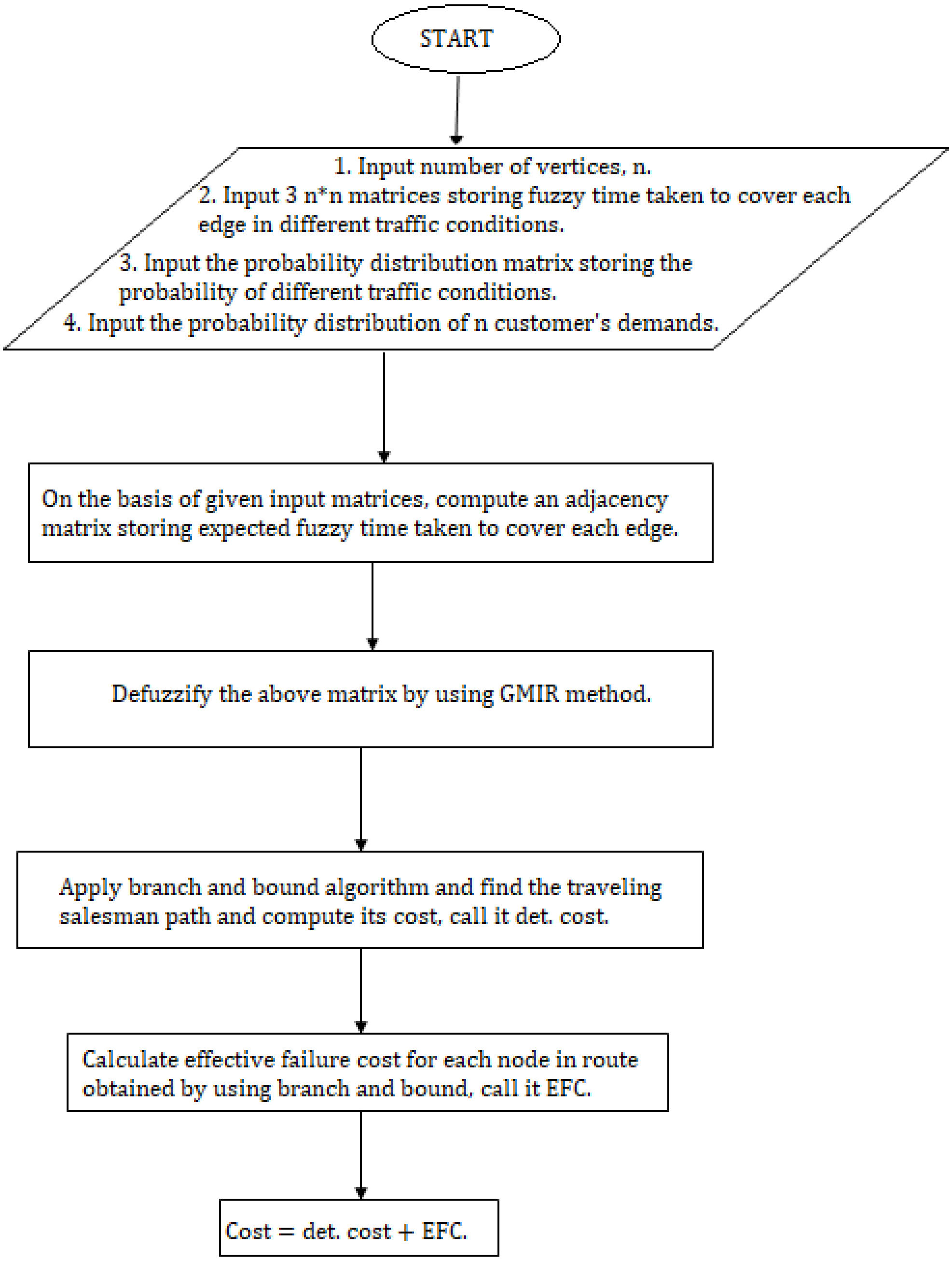

Figure 5: Flowchart of the model

5. Algorithm and Methodology of the Method

In the given algorithm, the symbols used along with their descriptions are provided in Table 1.

Table 1: Symbols used in the algorithm with their descriptions

Sr. no.	Symbols	Description
1.	$\tilde{A}[\,][\,]$	Adjacency matrix with low traffic conditions
2.	$\tilde{B}[\,][\,]$	Adjacency matrix with usual traffic conditions
3.	$\tilde{C}[\,][\,]$	Adjacency matrix with high traffic conditions
4.	p[ ][ ][ ]	Matrix storing pmf values for different traffic conditions
5.	n	Number of Vertices
6.	Cost[ ][ ]	Adjacency matrix storing expected time taken to cover every edge.
7.	cost	Deterministic cost of tour in stage 1.
8.	TEFC	Total effective failure cost
9.	Tcost	Total cost
10.	Final-path	Route

The methodology to solve such a problem has been divided into two parts, part A and part B and part A is further subdivided into 2 parts.

In part A, we find the *a priori* sequence of nodes which are to be followed while execution. The sequence of nodes is obtained by using Branch and Bound algorithm, which can only be applied when the edge weights are deterministic. So to reduce the edge weights from fuzzy random variables to deterministic weights, we find the expected edge weights first by using expectation of a fuzzy random variable by using (4) and then convert them into a crisp form by using GMIR method (3). This part of the methodology has been represented by Algorithm 1

Algorithm 1 An algorithm to handle the cost matrix

Input: $\tilde{A}[\,][\,]$, $\tilde{B}[\,][\,]$, $\tilde{C}[\,][\,]$, p[ ][ ][ ], number of vertices

Output: Cost[ ][ ].

Start.
Declare variables i, j
$i \leftarrow 0, j \leftarrow 0;$
for all $i \leq n$ **do**
  for all $j \leq n$ **do**
    $Fuzzy-Cost[i][j] = p[i][j][1] \odot \tilde{A}[i][j] \oplus p[i][j][2] \odot \tilde{B}[i][j] \oplus p[i][j][3] \odot \tilde{C}[i][j];$
    $Cost[i][j] = G(\tilde{C}[i][j]);$
  end for
end for

We then use the cost matrix obtained by Algorithm 1 and find a sequence of nodes which is to be followed by using the Branch and Bound Algorithm. This part of the methodology has been represented by Algorithm 2

Algorithm 2 An algorithm to find the *a priori* route

Input: Cost[ ][ ], number of vertices

Output: A minimum cost tour and its cost.

```
def RecTour(Cost[][], curr-bound, curr-cost, l, curr-path, visited):
    Declare global variable cost;
     if level==n:
         if (Cost[curr-path[level-1]][curr-path[0]]!=0)
             curr-res=curr-cost+Cost[curr-path[level-1]][curr-path[0]]
             if (curr-res < cost)
                 Final-path= curr-path.append(curr-path[0]);
                 cost = curr-res;
     for all i ≤ n do
         if (Cost[curr-path[level-1]][i]!=0 and visited[i]==False)
             temp=curr-bound
             curr-weight= curr-weight+Cost[curr-path[level-1]][i]
             if level==1:
                 curr-bound = curr-bound - ((firstmin(Cost,curr-path[level-1])+firstmin(Cost,i))/2)
             else
                 curr-bound = curr-bound - ((secondmin(Cost,curr-path[level-1])+(firstmin(Cost,i)))/2)
             RecTour(Cost, curr-bound, curr-weight, level+1, curr-path, visited)
         else:
             curr-weight = curr-weight - Cost[curr-path[level-1]][i]
             curr-bound =temp, visited = [False]*len(visited)
     for all j ≤ level
         if (curr-path)[j]!=-1
             visited[curr-path[j]]=True
def Tour(C[ ][ ])
     curr-bound ← 0, curr-path ← [-1]* N+1, visited ← [False]*N
     for all i ≤ n do
        curr-bound = curr-bound+(firstmin(Cost,i)+secondmin(Cost,i)
        curr-bound = ceil(curr-bound/2)
    visited[0]← True, curr-path[0]←0;
    RecTour(Cost[][], curr-bound, 0, 1, curr-path, visited)
return  cost, Final-path
```

After finding the sequence of routes, the execution of routes starts and in this procedure, there is the possibility of failure, the failure occurs because the vehicle may have lesser quantity than the demand revealed by the customer and that will be realized only when the vehicle arrives at that customer. In such a case, the vehicle is bound to return to the depot and after replenishing, the vehicle resumes its service. This cost is named as the effective failure cost and the sum of effective failure cost and the cost of tour obtained in 2 gives us the total cost of the operation. The calculation of effective failure cost and total cost is represented in Algorithm 3

Algorithm 3 An algorithm for calculating total cost of the tour

Input: Final-path and cost from 31 of Algorithm 2.

Output: Total cost of the tour.

Start
$i \leftarrow 1$;
for all $i \leq n$ **do**
  EFC[Fpath[i]]=2cost[0][Fpath[i]]($\sum_{u=1}^{\infty}$ P(TD at Fpath[i-1] $\leq$ 60u - P(TD at Fpath[i] $\leq$ 60u))
  TEFC $\leftarrow$ TEFC + EFC[Fpath[i]]
end for
Tcost $\leftarrow$ cost + TEFC
return Tcost

6. Numerical Example

Let us consider an instance when there are 5 customers and there is a single depot. Suppose that the customers in the network have stochastic demands, i.e. the demand of a customer is revealed only when that customer is visited. In this example, the cost matrix store the information about the time required to traverse the edge, which usually depends upon the random traffic conditions and is also given imprecisely (here, impreciseness has been handled by using Triangular Fuzzy Numbers). The matrix $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ stores the information about time required to traverse the edges when traffic conditions are low, medium and high respectively.

$$\tilde{A} = \begin{bmatrix} \infty & (0,2,4) & (4,6,8) & (0,1,3) & (5,7,9) & (2,4,6) \\ (0,2,4) & \infty & (5,7,9) & (2,4,6) & (6,8,10) & (0,1,3) \\ (4,6,8) & (5,7,9) & \infty & (4,6,8) & (0,2,4) & (4,6,8) \\ (0,1,3) & (2,4,6) & (4,6,8) & \infty & (5,7,9) & (4,6,8) \\ (5,7,9) & (6,8,10) & (0,2,4) & (5,7,9) & \infty & (7,9,11) \\ (2,4,6) & (0,1,3) & (4,6,8) & (4,6,8) & (7,9,11) & \infty \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} \infty & (2,4,6) & (6,8,10) & (1,3,5) & (7,9,11) & (4,6,8) \\ (2,4,6) & \infty & (7,9,11) & (4,6,8) & (8,10,12) & (1,3,5) \\ (6,8,10) & (7,9,11) & \infty & (6,8,10) & (2,4,6) & (6,8,10) \\ (1,3,5) & (4,6,8) & (6,8,10) & \infty & (7,9,11) & (6,8,10) \\ (7,9,11) & (8,10,12) & (2,4,6) & (7,9,11) & \infty & (9,11,13) \\ (4,6,8) & (1,3,5) & (6,8,10) & (6,8,10) & (9,11,13) & \infty \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} \infty & (4,6,8) & (8,10,12) & (3,5,7) & (9,11,13) & (6,8,10) \\ (4,6,8) & \infty & (9,11,13) & (6,8,10) & (10,12,14) & (3,5,7) \\ (8,10,12) & (9,11,13) & \infty & (8,10,12) & (4,6,8) & (8,10,12) \\ (3,5,7) & (6,8,10) & (8,10,12) & \infty & (9,11,13) & (8,10,12) \\ (9,11,13) & (10,12,14) & (4,6,8) & (9,11,13) & \infty & (11,13,15) \\ (6,8,10) & (3,5,7) & (8,10,12) & (8,10,12) & (11,13,15) & \infty \end{bmatrix}$$

The matrix p stores the information about the random traffic conditions. It stores the probability of different traffic conditions. For example, the element in the fourth row and third column (which stores the information about the edge 3-2) i.e. (0.2, 0.5, 0.3) represents that the probability of traffic conditions being low, medium and high on the edge 3-2 are 0.2, 0.5 and 0.3 respectively.

$$p = \begin{bmatrix} \infty & (0.4,0.2,0.4) & (0.3,0.4,0.3) & (0.2,0.5,0.3) & (0.6,0.3,0.1) & (0.6,0.1,0.3) \\ (0.4,0.2,0.4) & \infty & (0.5,0.2,0.3) & (0.4,0.3,0.3) & (0.2,0.6,0.2) & (0.3,0.4,0.3) \\ (0.3,0.4,0.3) & (0.5,0.2,0.3) & \infty & (0.2,0.5,0.3) & (0.7,0.2,0.1) & (0.3,0.5,0.2) \\ (0.2,0.5,0.3) & (0.4,0.3,0.3) & (0.2,0.5,0.3) & \infty & (0.4,0.4,0.2) & (0.3,0.3,0.4) \\ (0.6,0.3,0.1) & (0.2,0.6,0.2) & (0.7,0.2,0.1) & (0.4,0.4,0.2) & \infty & (0.7,0.1,0.2) \\ (0.6,0.1,0.3) & (0.3,0.4,0.3) & (0.3,0.5,0.2) & (0.3,0.3,0.4) & (0.7,0.1,0.2) & \infty \end{bmatrix}$$

An expectation based approach has been used to handle the randomness of edge weights, i.e. expected time to cover every edge is calculated first and then a shortest route traversing every vertex exactly once is calculated by the use of Branch and Bound algorithm in the first stage of *a priori* approach. For example, Consider the edge 0-2, the matrix p tells that the probability of having low, medium and high traffic conditions is 0.3, 0.4 and 0.3 respectively. The matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ stores the time required to traverse the edges in these traffic conditions. Thus the expected time to cover the edge 0-2 is given by:

$$E_{02}[T] = 0.3 \odot (4,6,8) + 0.4 \odot (6,8,10) + 0.3 \odot (8,10,12) = (6,8,10)$$

Reducing it to a crisp number by the use of Graded mean integration representation method gives the expected time to cover the edge 0-2 is 8 units.

After finding the expected time to cover every edge in a crisp form, Branch and Bound algorithm can be used to find out the shortest route that visits every vertex exactly once. A schematic diagram representing the Branch and Bound algorithm for the network in the consideration is given by 6 and the route that should be taken is 0-1-5-2-4-3-0 and the cost of traversal of this route is 29.4 units.

In the first stage of *a priori* approach, the demands of the customers were not considered. In the second stage, the demands of the customers are also considered. The stochastic demands of the customers are given by Table 2. For example, the third row of the table informs that the demand of third customer is 15, 20 with the probability 0.2 and 0.8 respectively. The demand of depot node has been considered to be 0 units.

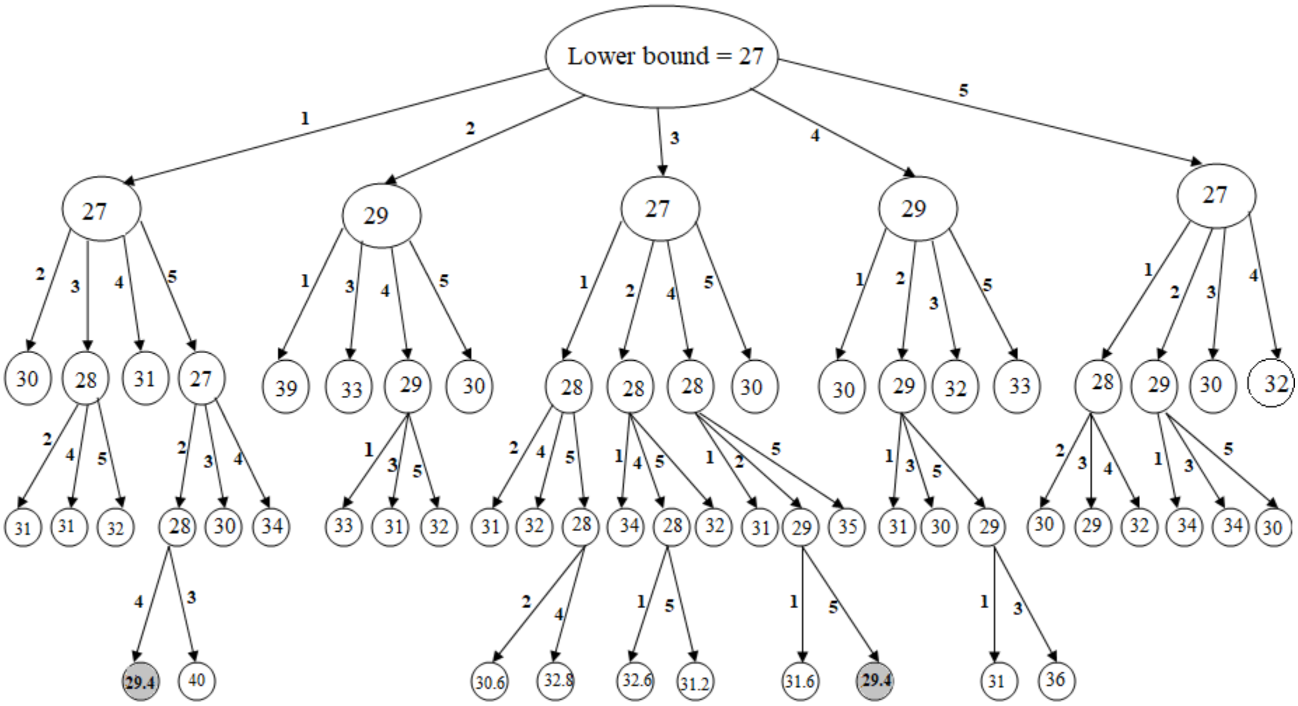

Figure 6: The path obtained in stage 1

Table 2: Demands of the customers

Node	Demand	Probability
1	(20, 25)	(0.4, 0.6)
2	(30, 35)	(0.3, 0.7)
3	(15, 20)	(0.2, 0.8)
4	(25, 30)	(0.7, 0.3)
5	(10, 15)	(0.8, 0.2)

While dealing with the stochastic demands, the route obtained in first stage is traversed and whenever a failure occurs, a trip to depot node is made to refill the vehicle and continue the service. In such case, an effective failure cost gets associated with every vertex which represents the cost of route if a failure failure occurs at that specific node, assuming that the failure has yet not occurred on any other previous node of the route. The formula to calculate the effective failure cost is given by 6. The carrying capacity of the vehicle is 60 units. Calculating the effective failure cost at every node gives

For the customer waiting at vertex 1:

$$EFC_1 = 2c_{01}\{\sum_{u=1}^{\infty} P(\text{Total Demand at } 0) \leq 60u - P(\text{Total Demand at 0 and 1}) \leq 60u\}$$

$$= 0$$

For the customer waiting at vertex 5:

$$EFC_5 = 2c_{05}\{\sum_{u=1}^{\infty} P(\text{Total Demand at 0 and 1}) \leq 60u - P(\text{Total Demand 0, 1 and 5}) \leq 60u\}$$

$$= 0$$

For the customer waiting at vertex 2:

$$EFC_2 = 2c_{02}\{\sum_{u=1}^{\infty} P(\text{Total Demand at 1 and 5}) \leq 60u - P(\text{Total Demand at 1, 5 and 2}) \leq 60u\}$$

$$= 14.464$$

For the customer waiting at vertex 4:

$$EFC_4 = 2c_{04}\{\sum_{u=1}^{\infty} P(\text{Total Demand at 0, 1, 5 and 2}) \leq 60u - P(\text{Total Demand at 0, 1, 5, 2 and 4}) \leq 60u\}$$

$$= 1.536$$

For the customer waiting at vertex 3:

$$EFC_3 = 2c_{03}\{\sum_{u=1}^{\infty} P(\text{Total Demand at 0, 1, 5, 2 and 4}) \leq 60u - P(\text{Total Demand at 0, 1, 5, 2, 4 and 3}) \leq 60u\}$$

$$= 0.129204$$

$$\text{Total effective failure cost} = \sum_{i=1}^{n} EFC_i$$

$$= 16.129024$$

Then the sum of effective failure cost of every vertex gives the total effective failure cost and the sum of total effective failure cost and cost obtained in stage 1 gives the total cost of traversal in the given mixed environment.

$$\text{Total cost} = \text{Total effective failure cost} + \text{Deterministic Cost obtained in stage 1}$$

$$= 45.52$$

Thus, the total cost of traversal is 45.52 units and and the route that should be taken is 0-1-5-2-4-3-0.

7. Discussion and Analysis

In the presented work, the traveling salesman path in first stage is obtained by using Branch and Bound algorithm, which is an optimal solution method for combinatorial optimization problems. The time complexity of the algorithm is O($n^2 2^n$), which is exponential in nature, but no algorithm with lesser time complexity than Branch and Bound [11] gives optimal solution for TSP. The major advantage of the algorithm is that we can control the quality of the solution to be expected, even if it is not yet found. Only in worst case scenario, the exploration of all possible permutations is required. In other classical methods of solving VRP, random nature of the network has never been clubbed with impreciseness. In this work, a mixed environment has been considered where expectation of fuzzy random variables and GMIR method has been used to deal with randomness and impreciseness respectively.

Apart from the exact methods like Dynamic programming methods [1], Branch and Bound[11], several approximation algorithms such as Christofides algorithm[8] and algorithms based on heuristics like Nearest Neighbour Algorithm[18], Clark and Wright algorithm[23] can also be used to find out the traveling salesman path. Several methods based on meta-heuristic like Genetic algorithm [17], tabu search methods [28], Simulated annealing methods [7], Particle Swarm optimization[7] and Ant colony optimization[5] can also be used to solve the stage 1 of the problem considered. Table 3 comprises of the comparison of the method presented in this paper with several other algorithms. The comparison has been done on the network presented in Numerical Example.

Table 3: Comparison of Various Methods

Method	Path	Det Cost	EFC	Total Cost	Time Complexity
Brute Force [11]	0-3-4-2-5-1-0	29.4	16.129024	45.52	O(n!)
	0-1-5-2-4-3-0	29.4	16.1728	45.57	
Bellman Held Karp Algorithm [1]	0-3-4-2-5-1-0	29.4	16.129024	45.52	O($n^2 2^n$)
	0-1-5-2-4-3-0	29.4	16.1728	45.57	
Christofides Algorithm [8]	0-2-4-3-5-1-0	34.6	8.57728	43.17728	O(n^4)
Clark and Wright Algorithm [16]	0-1-5-2-4-3-0	29.4	16.129024	45.52	O($n^2 logn$)
Nearest Neighbour [18]	0-3-1-5-2-4-0	30.6	16.3456	46.9456	O(n^2)
Genetic Algorithm [17]	0-3-4-2-1-5-0	31.6	16.217728	47.817728	O($n^2 logn$)
Proposed method	0-1-5-2-4-3-0	29.4	16.129024	45.52	O($n^2 2^n$)

The algorithm like Bellman Held Karp algorithm gives the optimal solutions but the method has exponential time complexity (same as that of Branch and Bound) and it explores all possible permutations in all cases, thus requiring more space in the memory as compared to Branch and Bound Algorithm. The Brute force approach gives optimal solution by exploring all possible permutations and thus have a time complexity of O(n!), which is worse than that of Branch and Bound. Other algorithms such as Christofides algorithm, nearest neighbour algorithm, Clark and Wright algorithm and Genetic algorithm have polynomial time complexity but the solution obtained by these methods is not always guaranteed to be optimal.

8. Conclusion

In this work, the mathematical model of fuzzy stochastic vehicle routing problem and the algorithm to solve such a problem has been presented. In practical life, to traverse a route, different paths may be taken which may have different traffic conditions. In mathematical modelling of CVRP in mixed environment, the objective is to find a minimum weighted travelling salesman tour starting as well as terminating at the origin node, in such a way that the demands of all the customers present in the network are fulfilled. In this work, different traffic conditions occur on different edges with certain probabilities and in those traffic conditions, time taken to cover every edge is given by triangular fuzzy number. The demands of the customers present at the nodes are stochastic in nature and here are given by probability mass function. Randomness and Impreciseness in this work are dealt by using expectation approach and GMIR method respectively.

The approach used in this paper is *a priori* i.e. the route designing has been done first and while serving the customers on that route,an effective failure cost has been calculated. Branch and Bound algorithm has been used to find the route that the traveling salesman should follow when the demands are not known such that cost of traversal is minimum. The recourse policy used is reactive in nature, i.e. the return trip to the depot are only performed upon the occurrence of the failure of the route planned by using Branch and Bound. A numerical example is also been solved using proposed approach. This result may be useful to find the minimum weighted tour for any commodity delivery problem in a network, when the demands of the customers are stochastic in nature and the nature of the network under consideration is imprecise and random.

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