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Controllability and Stability of Hilfer Fractional Evolution Equations

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Summary

In this article, we discuss the controllability and stability of Hilfer fractional evolution equations in Banach space. We derive these results by first proving the existence and uniqueness of mild solutions for proposed system of equations. Existence and uniqueness results are obtained with the help of theory of propagation family, techniques of measure of non-compactness and fixed point theorem. An example is also given for the demonstration of obtained results.

KEYWORDS:
Hilfer Fractional Derivative, Controllability, Ulam-Hyers Stability, Fixed point theory

MSC CLASSIFICATION: 26A33, 34K35, 47H10, 47J35, 93B05, 93D99

1 | INTRODUCTION

In last few decades, fractional calculus has emerged as an active branch of both pure and applied mathematics. This branch deals with the differential and integral operators of arbitrary real or complex order[1,2,3,4]. Fractional order derivatives being non-locally distributed, well described the memory and hereditrary effects of complex processes and materials. In fractional calculus, study of fractional evolution equations (FEE) is very significant[5]. These equations appeared in the models of arterial geometries, viscoelastic behaviours, ferromagnetic materials, advection diffusion equations and many other phenomenas of physics and engineering[6,7,8,9]. Nowadays, researchers show their great interest in investigating several aspects of FEE such as an existence and uniqueness results, stability results, approximate controllability and exact controllability[10,11,12,13]. The concept of controllability of FEE has drawn a lot of attention of mathematicians and engineers since it plays an important role in control theory and engineering. There are mainly three types of controllability of fractional dynamical systems: exact controllability, approximate controllability, and null controllability[14]. For an appropriate choice of the admissible control inputs, exact controllability steers the system to arbitrary final state; approximate controllability steers the system to the small neighbourhood of arbitrary final state and null controllability steers the system to its state of origin. Many contributions have been made on the study of exact and approximate controllability of fractional differential equations. We refer to readers[15,16,14] and references there in. Wang et al.[17] etablished controllability results for a class of semilinear FEE with two classes of control sets in separable Banach space. These results are derived with the help of theory of propagation family and measure of non-compactness.
The stability theory for FEE was first proposed by Ulam[18] and then Hyers[19]. After that it is popular as Ulam-Hyers stability. Rassias[20] generalized these results by using a dominant function to control the estimate instead of a positive constant, and this is

usually called Hyers-Ulam-Rassias stability problem, or generalized Hyers-Ulam stability problem. In published works, there is wide variety of articles focussing on Hyers-Ulam and generalized Hyers-Ulam stability of fractional differential equations [21,4,22]. In fractional calculus, though the literature on definitions of fractional derivatives and integrals is increasing, the most commonly used derivatives are classical derivatives. The range of applicability of classical derivatives is limited due to their incapacities to correctly address some unsolved issues in dynamics thermal media, in electromagnetic waves and in viscoelastic processes [23]. This motivates several researchers to propose new generalizations of existing derivatives. For instance, Almeida [24] and Sousa et al. [25] proposed new generalizations of Caputo and Hilfer fractional derivatives w.r.t function ψ. These generalizations offer higher accuracy of models for the suitable choice of function ψ. In recent works, Suechoei et al. [26] established the existence and stability results for semilinear FEE with ψ-Caputo derivative. Borisut et al. [27] discussed the existence and stability of ψâĂŘHilfer FEE with non-local conditions. Motivated by these works, in this paper we aim to establish the controllabilty and stability results for FEE involving generalized Hilfer fractional derivative. Consider the following $\zeta-$Hilfer fractional evolution system in Banach space Ω.

$$\begin{cases} D_{0^+}^{\rho,\sigma;\zeta}\mathcal{E}\omega(t) & = \mathcal{A}\omega(t) + \mathcal{E}\big(\Psi(t,\omega(t))\big) + \mathcal{E}\big(\Upsilon u(t)\big), \quad t \in \mathfrak{J} = [0,a], \\ \\ \mathcal{E}I_{0^+}^{(1-\rho)(1-\sigma);\zeta}\omega(0) & = \mathcal{E}\omega_0, \ \omega_0 \in D(\mathcal{E}), \end{cases} \tag{1}$$

where $D_{0^+}^{\rho,\sigma;\zeta}$ represents the $\zeta-$Hilfer fractional derivative of order $0 < \rho < 1$ and type $0 < \sigma \leq 1$. The state $\omega(\cdot)$ takes value in Banach space Ω and control function $u(\cdot)$ is defined in $\mathcal{U} = L^\infty(\mathfrak{J}, U)$, the Banach space of admissible control functions. $\Upsilon : U \longrightarrow D(\mathcal{E})$ is bounded linear operator and $\Psi : \mathfrak{J} \times \Omega \longrightarrow D(\mathcal{E}) \subset \Omega$ will be specified later. The pair of closed linear operators $(\mathcal{A}, \mathcal{E})$ generates an exponentially bounded propagation family $\{T(t), t \geq 0\}$ from $D(\mathcal{E})$ to Ω, $\mathcal{A} : D(\mathcal{A}) \subset \Omega \longrightarrow \Omega$ and $\mathcal{E} : D(\mathcal{E}) \subset \Omega \longrightarrow \Omega$. $I_{0^+}^{(1-\rho)(1-\sigma);\zeta}$ is the $\zeta-$Riemann-Liouville fractional integral of order $(1-\rho)(1-\sigma)$. This article is structured as:

In section 2, we mention some basic definitions and fundamental concepts of $\zeta-$Hilfer fractional derivative. Here we also construct the mild solutions for the system of Eqs.(1) using theory of propagation family. In next section, we derive the existence and uniqueness results by choosing suitable control function first and hence discuss the controllabilty of system (1). In section 4, we establish stability results for the given system of equations. In section 5, we give an example to confirm the applicability of obtained results.

2 | PRELIMINARIES

Consider $C(\mathfrak{J}, \Omega)$ as the space of continuous functions from $\mathfrak{J}$ to Ω. $C(\mathfrak{J}, \Omega)$ is a complete normed linear space with norm $\|\omega\| = \sup\limits_{t\in\mathfrak{J}} \|\omega(t)\|$.

Definition 1. [28] Let $\zeta \in C^1([a,b])$ be an increasing function with $\zeta'(t) \neq 0$ for all $t \in [a,b]$ and Ψ be an integrable function defined on $[a,b]$. The $\zeta-$Riemann-Liouville fractional integral operator of function Ψ of order $\rho > 0$ is given by:

$$I_{a^+}^{\rho;\zeta}\Psi(t) = \frac{1}{\Gamma(\rho)} \int_a^t (\zeta(t) - \zeta(s))^{\rho-1} \Psi(s)\zeta'(s) ds.$$

Definition 2. [28] Let $n - 1 < \rho < n$ and $\zeta \in C^1([a,b])$ be an increasing function with $\zeta'(t) \neq 0$ for all $t \in [a,b]$. The $\zeta-$Riemann-Liouville fractional derivative of order $\rho > 0$ of an integrable function Ψ defined on $[a,b]$ is given by:

$$D_{a^+}^{\rho;\zeta}\Psi(t) = \left(\frac{1}{\zeta'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\rho;\zeta}\Psi(t), \ n = [\rho] + 1.$$

Definition 3. Let $0 < \rho < 1$ and $\zeta \in C^1([a,b])$ be such that $\zeta'(t)$ is increasing and $\zeta'(t) \neq 0$ for all $t \in [a,b]$.The $\zeta-$Hilfer fractional derivative of function $\Psi \in C^1([a,b])$ of order $0 < \rho < 1$ and type $0 < \sigma \leq 1$ is defined as:

$$D_{a^+}^{\rho,\sigma;\zeta}\Psi(t) = I_{a^+}^{\sigma(1-\rho);\zeta}\left(\frac{1}{\zeta'(t)}\frac{d}{dt}\right)I_{a^+}^{(1-\sigma)(1-\rho);\zeta}\Psi(t), \ t > a.$$

It can also be expressed as

$$D_{a^+}^{\rho,\sigma;\zeta}\Psi(t) = \frac{1}{\Gamma(\gamma-\rho)}\int_a^t (\zeta(t)-\zeta(s))^{\gamma-1}\left(\frac{1}{\zeta'(s)}\frac{d}{ds}\right)I_{a^+}^{(1-\rho)(1-\sigma);\zeta}\Psi(s)ds, \text{ for } \gamma = \rho+\sigma-\rho\sigma.$$

Lemma 1. [29,25] If $\Psi \in C^n[a,b]$, $n-1 < \rho < n$, $0 < \sigma \leq 1$ and $\gamma = \rho+\sigma-\rho\sigma$. Then

$$I_{a^+}^{\rho;\zeta}D_{a^+}^{\rho,\sigma;\zeta}\Psi(t) = \Psi(t) - \sum_{i=1}^{n}\frac{\big(\zeta(t)-\zeta(a)\big)^{\gamma-i}}{\Gamma(\gamma-i+1)}\left(\frac{1}{\zeta'(s)}\frac{d}{ds}\right)^{n-i} I_{a^+}^{(1-\sigma)(n-\rho);g}\Psi(a), \ \ \forall\ t \in (a,b].$$

In particular if $0 < \rho < 1$, then

$$I_{a^+}^{\rho;\zeta}D_{a^+}^{\rho,\sigma;\zeta}\Psi(t) = \Psi(t) - \frac{\big(\zeta(t)-\zeta(a)\big)^{\gamma-1}}{\Gamma(\gamma)}I_{a^+}^{(1-\gamma);g}\Psi(a), \ \forall\ t \in (a,b].$$

Definition 4. [28] Let $\zeta : [a,\infty) \longrightarrow \mathbb{R}$ be such that $\zeta(t)$ is continuous and $\zeta'(t) > 0$ on $[a,\infty)$. The generalized Laplace transform of function $\Psi : [a,\infty) \longrightarrow \mathbb{R}$ w.r.t function ζ is defined as:

$$L_\zeta\{\Psi(t)\}(s) = \int_a^\infty e^{-s(\zeta(t)-\zeta(a))}\Psi(t)\zeta'(t)dt, \ \forall s.$$

Definition 5. [28] Let Ψ and Φ be two piecewise continuous functions of exponential order defined on interval $[a,t]$. The generalized convolution of functions Ψ and Φ is defined as:

$$\big(\Psi *_\zeta \Phi\big)(t) = \int_a^t \Psi(\tau)\Phi\Big(\zeta^{-1}\big(\zeta(t)+\zeta(a)-\zeta(\tau)\big)\Big)\zeta'(\tau)d\tau.$$

Definition 6. [30] Let Ω be complete normed linear space and $\mathcal{G}$ be a bounded subset of Ω. Kuratowskii measure of non-compactness is a map $\eta : \mathcal{G} \longrightarrow [0,\infty)$ defined as:

$$\eta(\mathcal{G}) = \inf\left\{\epsilon > 0 : \mathcal{G} \subset \cup\mathcal{G}_j, \ \text{diam}(\mathcal{G}_j) < \epsilon, \ j = 1,2,\cdots,n\right\} \text{ where, } \text{diam}(\mathcal{G}_j) = \sup\left\{|x-y| : x,y \in \mathcal{G}_j\right\}.$$

Lemma 2. [30] For bounded subsets $\mathcal{G}$ and $\mathcal{H}$ of Banach space Ω, the measure of non-compactness η has following properties :

1. $\eta(\mathcal{G}) = 0$ iff $\bar{\mathcal{G}}$ is compact, $\bar{\mathcal{G}}$ denotes the convex hull of $\mathcal{G}$.
2. $\eta(\mathcal{G}) = \eta(\bar{\mathcal{G}})$.
3. $\eta(\mathcal{G} \cup \mathcal{H}) = \max\left\{\eta(\mathcal{G}), \eta(\mathcal{H})\right\}$.
4. $\eta(\mathcal{G}) \leq \eta(\mathcal{H})$ if $\mathcal{G} \subset \mathcal{H}$.
5. $\eta(\mathcal{G}+\mathcal{H}) \leq \eta(\mathcal{G}) + \eta(\mathcal{H})$.

Lemma 3. [31] Let $\mathcal{D}$ be a bounded subset of Banach space Ω. Then there exists a countable set $\mathcal{D}_o \subset \mathcal{D}$ such that $\eta(\mathcal{D}) \leq 2\eta(\mathcal{D}_o)$.

Lemma 4. [30,32] If $\mathcal{G} \subset C(\mathfrak{J}, \Omega)$ is bounded and equicontinuous then $\eta(\mathcal{G}(t))$ is continuous on $\mathfrak{J}$ and

1. $\eta\big(\mathcal{G}(\mathfrak{J})\big) = \max_{t\in\mathfrak{J}}\left\{\eta(\mathcal{G}(t))\right\}$.
2. $\eta\left(\int_{t\in\mathfrak{J}}\mathcal{G}(t)dt\right) \leq \int_{t\in\mathfrak{J}}\eta\big(\mathcal{G}(t)\big)dt$.

Lemma 5. [33] If $\mathcal{Q} = \left\{q_n\right\}_{n=1}^{\infty} \subset C(\mathfrak{J}, \Omega)$ is bounded and countable then $\eta(\mathcal{Q}(t))$ is Lebesgue integrable on $\mathfrak{J}$ and

$$\eta\Big(\int\limits_{t\in\mathfrak{J}} \mathcal{Q}(t)dt\Big) \leq 2\int\limits_{t\in\mathfrak{J}} \eta\big(\mathcal{Q}(t)\big)dt.$$

Lemma 6. [30] If $\mathcal{G} \subset C(\mathfrak{J}, \Omega)$ is bounded and equicontinuous then closure of the convex hull of $\mathcal{G}$ is also bounded and equicontinuous.

Definition 7. [34] Let Ω be a Banach space and η be the measure of non-compactness defined in Ω. A continuous mapping $\mathcal{T} : \Omega \longrightarrow \Omega$ is called condensing mapping if for any bounded set $\mathcal{C} \subset \Omega$, $\mathcal{T}(\mathcal{C})$ is bounded and $\eta(\mathcal{T}(\mathcal{C})) < \eta(\mathcal{C})$, $\eta(\mathcal{C}) > 0$.

Lemma 7. [34] Let η be the measure of non-compactness defined on Banach space Ω and $\mathcal{C}$ be a nonempty bounded, closed and convex subset of Ω. If $\mathcal{T} : \mathcal{C} \longrightarrow \mathcal{C}$ is a condensing mapping then $\mathcal{T}$ has atleast one fixed point in $\mathcal{C}$.

Definition 8. [35,17] Consider the following Cauchy problem:

$$\begin{cases} \big(\mathcal{E}\omega(t)\big)' = \mathcal{A}\omega(t), \ t \in \mathfrak{J}, \\ \mathcal{E}\omega(0) = \mathcal{E}\omega_0, \ \omega_0 \in D(\mathcal{E}). \end{cases}$$

An exponentially bounded propagation family generated by pair $(\mathcal{A}, \mathcal{E})$, is the strongly continuous and exponentially bounded operator family $\{T(t) : t \geq 0\}$ of $D(\mathcal{E})$ to Banach space Ω satisfying

$$(\lambda\mathcal{E} - \mathcal{A})^{-1}\mathcal{E}\omega = \int\limits_0^{\infty} e^{-\lambda t}T(t)\omega dt, \ \text{ for } \lambda > 0 \text{ and } \omega \in D(\mathcal{E}).$$

2.1 | Representation of mild solutions using theory of propagation family

By Lemma 2.1. the equivalent integral form of Eq.(1) is as follows:

$$\begin{aligned} \mathcal{E}\omega(t) =& \frac{(\zeta(t)-\zeta(0))^{\rho-1}}{\Gamma(\rho)}\ \omega_0 + \frac{1}{\Gamma(\rho)}\int\limits_0^t (\zeta(t)-\zeta(s))^{\rho-1}\mathcal{A}\omega(s)\zeta'(s)ds \\ &+ \frac{1}{\Gamma(\rho)}\int\limits_0^t (\zeta(t)-\zeta(s))^{\rho-1}\left\{\mathcal{E}\Psi\big(s,\omega(s)\big) + \mathcal{E}\Upsilon u(s)\right\}\zeta'(s)ds. \end{aligned} \tag{2}$$

Applying generalized Laplace transform on both sides of Eq.(2), we have

$$\begin{aligned} \omega(\lambda) =& \frac{\lambda^{\rho(\sigma-1)}\mathcal{E}\omega_0}{(\lambda^{\rho}\mathcal{E}-\mathcal{A})} + \frac{\mathcal{E}\Psi(\lambda)}{(\lambda^{\rho}\mathcal{E}-\mathcal{A})} + \frac{\mathcal{E}\Upsilon u(\lambda)}{(\lambda^{\rho}\mathcal{E}-\mathcal{A})}, \ \text{ where} \\ \omega(\lambda) =& \int\limits_0^{\infty} e^{-\lambda(\zeta(t)-\zeta(0))}\omega(t)\zeta'(t)dt, \ \Psi(\lambda) = \int\limits_0^{\infty} e^{-\lambda(\zeta(t)-\zeta(0))}\Psi\big(t,\omega(t)\big)\zeta'(t)dt, u(\lambda) = \int\limits_0^{\infty} e^{-\lambda(\zeta(t)-\zeta(0))}u(t)\zeta'(t)dt. \end{aligned} \tag{3}$$

We have

$$\int\limits_0^{\infty} e^{-\lambda^{\rho}s}\ T(s)x\ ds = \frac{\mathcal{E}x}{(\lambda^{\rho}\mathcal{E}-\mathcal{A})}.$$

Using this expression in Eq.(3), we have

$$\omega(\lambda) = \lambda^{\rho(\sigma-1)}\int_0^\infty e^{-\lambda^\rho s}\, T(s)\omega_0 ds + \int_0^\infty e^{-\lambda^\rho s}\, T(s)\Psi(\lambda) ds + \int_0^\infty e^{-\lambda^\rho s} \Upsilon u(\lambda) ds.$$

$$\omega(\lambda) = \lambda^{\rho(\sigma-1)}\int_0^\infty \rho t^{\rho-1} e^{-(\lambda t)^\rho} T(t^\rho)\omega_0 dt + \int_0^\infty \rho t^{\rho-1} e^{-(\lambda t)^\rho} T(t^\rho)\Psi(\lambda) dt + \int_0^\infty \rho t^{\rho-1} e^{-(\lambda t)^\rho} T(t^\rho)\Upsilon u(\lambda) dt.$$

Consider the following probability density function

$$\vartheta_\rho(\theta) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\theta^{-\rho k-1}\frac{\Gamma(\rho k+1)}{k!}\sin(k\pi\rho),\ \theta\in(0,\infty),\ \text{whose integration is given by}$$

$$\int_0^\infty e^{-\lambda\theta}\vartheta_\rho(\theta)d\theta = e^{-\lambda^\rho},\ \rho\in(0,1). \tag{4}$$

$$\omega(\lambda) = I_1 + I_2 + I_3. \tag{5}$$

$$I_1 = \lambda^{\rho(\sigma-1)}\int_0^\infty \rho t^{\rho-1} e^{-(\lambda t)^\rho} T(t^\rho)\omega_0 dt.$$

Taking $t = \zeta(t) - \zeta(0)$.

$$\begin{aligned}
I_1 &= \lambda^{\rho(\sigma-1)}\int_0^\infty \rho(\zeta(t)-\zeta(0))^{\rho-1} e^{-(\lambda(\zeta(t)-\zeta(0))^\rho} T((\zeta(t)-\zeta(0))^\rho)\omega_0\zeta'(t)dt.\\
&= \lambda^{\rho(\sigma-1)}\int_0^\infty\int_0^\infty \rho(\zeta(t)-\zeta(0))^{\rho-1} e^{-\lambda(\zeta(t)-\zeta(0))\theta}\,\vartheta_\rho(\theta)T\big((\zeta(t)-\zeta(0))^\rho\big)\zeta'(t)\omega_0 d\theta dt.\\
&= \lambda^{\rho(\sigma-1)}\int_0^\infty e^{-\lambda(\zeta(t)-\zeta(0))}\left(\int_0^\infty \rho\,\vartheta_\rho(\theta)T\Big(\frac{(\zeta(t)-\zeta(0))^\rho}{\theta^\rho}\Big)\frac{1}{\theta^\rho}d\theta\right)(\zeta(t)-\zeta(0))^{\rho-1}\omega_0\zeta'(t)dt.\\
&= \lambda^{\rho(\sigma-1)}\int_0^\infty e^{-\lambda(\zeta(t)-\zeta(0))}\left(\int_0^\infty \rho\theta\phi_\rho(\theta)T\big((\zeta(t)-\zeta(0))^\rho\theta\big)d\theta\right)(\zeta(t)-\zeta(0))^{\rho-1}\omega_0\zeta'(t)dt,
\end{aligned}$$

where $\phi_\rho(\theta) = \frac{-1}{\rho}\,\vartheta_\rho\big(\theta^{\frac{-1}{\rho}}\big)\theta^{-1-\frac{-1}{\rho}}$. Here $\phi_\rho(\theta)$represents the Wright function satisfying $\int_0^\infty \theta^\delta\phi_\rho(\theta)d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\rho\delta)},\ \delta>0.$

$$I_1 = \lambda^{\rho(\sigma-1)}\int_0^\infty e^{-\lambda(\zeta(t)-\zeta(0))}\mathbb{P}_\rho\big(\zeta(t)-\zeta(0)\big)(\zeta(t)-\zeta(0))^{\rho-1}\omega_0\zeta'(t)dt,\ \text{for}$$

$$\mathbb{P}_\rho\big(\zeta(t)-\zeta(0)\big) = \int_0^\infty \rho\theta\phi_\rho(\theta)T\big((\zeta(t)-\zeta(0))^\rho\theta\big)d\theta.$$

$$= \lambda^{\rho(\sigma-1)}\int_0^\infty e^{-\lambda(\zeta(t)-\zeta(0))}(\zeta(t)-\zeta(0))^{\rho-1}\mathbb{P}_\rho\big(\zeta(t)-\zeta(0)\big)\zeta'(t)\omega_0 dt.$$

$$\Rightarrow\ I_1 = L_\zeta\left[\frac{(\zeta(t)-\zeta(0))^{\rho(1-\sigma)}}{\Gamma(\rho(1-\sigma))}\right]\cdot L_\zeta\Big[(\zeta(t)-\zeta(0))^{\rho-1}\mathbb{P}_\rho\big(\zeta(t)-\zeta(0)\big)\Big]\omega_0. \tag{6}$$

In the following

$$I_2 = \int_0^\infty \rho t^{\rho-1} e^{-(\lambda t)^\rho} T(t^\rho)\Psi(\lambda)dt.$$

$$= \int_0^\infty \int_0^\infty \rho(\zeta(t)-\zeta(0))^{\rho-1} e^{-(\lambda(\zeta(t)-\zeta(0))^\rho} \vartheta_\rho(\theta) T\big((\zeta(t)-\zeta(0))^\rho\big) e^{-\lambda(\zeta(s)-\zeta(0))}\Psi\big(s,\omega(s)\big)\zeta'(s)\zeta'(t)ds\, dt.$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \rho(\zeta(t)-\zeta(0))^{\rho-1} e^{-\lambda(\zeta(t)-\zeta(0))\theta} \vartheta_\rho(\theta) T\big((\zeta(t)-\zeta(0))^\rho\big) e^{-\lambda(\zeta(s)-\zeta(0))}\Psi\big(s,\omega(s)\big)\zeta'(s)\zeta'(t)d\theta\, ds\, dt.$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \rho\frac{(\zeta(t)-\zeta(0))^{\rho-1}}{\theta^\rho} e^{-\lambda(\zeta(t)-\zeta(0))} \vartheta_\rho(\theta) T\Big(\frac{(\zeta(t)-\zeta(0))^\rho}{\theta^\rho}\Big) e^{-\lambda(\zeta(s)-\zeta(0))}\Psi\big(s,\omega(s)\big)\zeta'(s)\zeta'(t)d\theta\, ds\, dt.$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \rho\frac{(\zeta(t)-\zeta(0))^{\rho-1}}{\theta^\rho} e^{-\lambda(\zeta(t)+\zeta(s)-2\zeta(0))} \vartheta_\rho(\theta) T\Big(\frac{(\zeta(t)-\zeta(0))^\rho}{\theta^\rho}\Big)\Psi\big(s,\omega(s)\big)\zeta'(s)\zeta'(t)d\theta\, ds\, dt.$$

$$= \int_0^\infty \int_t^\infty \int_0^\infty \rho\frac{(\zeta(t)-\zeta(0))^{\rho-1}}{\theta^\rho} e^{-\lambda(\zeta(\tau)-\zeta(0))} \vartheta_\rho(\theta) T\Big(\frac{(\zeta(t)-\zeta(0))^\rho}{\theta^\rho}\Big)$$
$$\Psi\Big(\zeta^{-1}\big(\zeta(\tau)-\zeta(t)+\zeta(0)\big), x\big(\zeta^{-1}\big(\zeta(\tau)-\zeta(t)+\zeta(0)\big)\big)\Big)\zeta'(s)\zeta'(t)d\theta\, d\tau\, dt.$$

$$= \int_0^\infty \int_0^\tau \int_0^\infty \rho\frac{(\zeta(t)-\zeta(0))^{\rho-1}}{\theta^\rho} e^{-\lambda(\zeta(\tau)-\zeta(0))} \vartheta_\rho(\theta) T\Big(\frac{(\zeta(t)-\zeta(0))^\rho}{\theta^\rho}\Big)$$
$$\Psi\Big(\zeta^{-1}\big(\zeta(\tau)-\zeta(t)+\zeta(0)\big), x\big(\zeta^{-1}\big(\zeta(\tau)-\zeta(t)+\zeta(0)\big)\big)\Big)\zeta'(s)\zeta'(t)d\theta\, dt\, d\tau.$$

$$= \int_0^\infty e^{-\lambda(\zeta(\tau)-\zeta(0))}\Bigg[\int_0^\tau \int_0^\infty \rho\theta\phi_\rho(\theta)(\zeta(\tau)-\zeta(s))^{\rho-1} T\big((\zeta(\tau)-\zeta(s))^\rho\theta\big)\Psi\big(s,\omega(s)\big)\zeta'(s)d\theta ds\Bigg]\zeta'(\tau)d\tau.$$
$$= \int_0^\infty e^{-\lambda(\zeta(\tau)-\zeta(0))}\Bigg[\int_0^\tau (\zeta(\tau)-\zeta(s))^{\rho-1}\mathbb{P}_\rho\big(\zeta(\tau)-\zeta(s)\big)\Psi\big(s,\omega(s)\big)\zeta'(s)ds\Bigg]\zeta'(\tau)d\tau.$$
$$= \int_0^\infty e^{-\lambda(\zeta(\tau)-\zeta(0))}\Bigg[\int_0^\tau \mathbb{K}_\rho\big(\zeta(\tau)-\zeta(s)\big)\Psi\big(s,\omega(s)\big)\zeta'(s)ds\Bigg]\zeta'(\tau)d\tau,\ \ \mathbb{K}_\rho(t) = t^{\rho-1}\mathbb{P}_\rho(t).$$

$$\Rightarrow\ I_2 = L_\zeta\Bigg[\int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)\Psi\big(s,\omega(s)\big)\zeta'(s)ds\Bigg]. \tag{7}$$

In the similar way, we get

$$\Rightarrow\ I_3 = L_\zeta\Bigg[\int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)Yu(s)\zeta'(s)ds\Bigg]. \tag{8}$$

Using (6), (7) and (8) in (5), we have

$$\omega(\lambda) = L_\zeta\left[\frac{(\zeta(t)-\zeta(0))^{\rho(1-\sigma)}}{\Gamma(\rho(1-\sigma))}\right]\cdot L_\zeta\left[(\zeta(t)-\zeta(0))^{\rho-1}\mathbb{P}_\rho\big(\zeta(t)-\zeta(0)\big)\right]\omega_0. \\ + L_\zeta\left[\int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)\left\{\Psi\big(s,\omega(s)\big)+\Upsilon u(s)\right\}g'(s)ds\right]. \tag{9}$$

Taking inverse generalized Laplace transform on both sides, we have

$$\omega(t) = I_{a^+}^{\rho(1-\sigma);\zeta}\mathbb{K}_\rho(t)\omega_0 + \int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)\Psi\big(s,\omega(s)\big)\zeta'(s)ds + \int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)\Upsilon u(s)\zeta'(s)ds,$$

$$\omega(t) = \mathbb{S}_{\rho,\sigma}(t)\omega_0 + \int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)\Psi\big(s,\omega(s)\big)\zeta'(s)ds + \int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)\Upsilon u(s)\zeta'(s)ds,$$

$$\mathbb{S}_{\rho,\sigma}(t) = I_{a^+}^{\rho(1-\sigma);\zeta}\mathbb{K}_\rho(t),\ \mathbb{K}_\rho(t) = t^{\rho-1}\mathbb{P}_\rho(t),\ \mathbb{P}_\rho(t) = \int_0^\infty \rho\theta\phi_\rho(\theta)T(t^\rho\theta)d\theta.$$

Lemma 8. [17] Suppose that the exponentially bounded propagation family $\{T(t), t \geq 0\}$ generated by pair $(\mathcal{A}, \mathcal{E})$ is norm continuous and uniformly bounded. Then the following properties hold:

(i) $\|T(t)\| < \hat{\mathfrak{M}}$, for some $\hat{\mathfrak{M}} > 0$.

(ii) $\left\|\mathbb{P}_\rho(t)\right\| = \left\|\int_0^\infty \rho\theta\phi_\rho(\theta)T(t^\rho\theta)d\theta\right\| \leq \hat{\mathfrak{M}}\rho\int_0^\infty \theta\phi_\rho(\theta)d\theta = \dfrac{\hat{\mathfrak{M}}}{\Gamma(\rho)}$.

$\Rightarrow \quad \left\|\mathbb{P}_\rho(t)\omega\right\| \leq \dfrac{\hat{\mathfrak{M}}}{\Gamma(\rho)}\|\omega\|,\ \forall\omega\in\Omega.$

(iii) $\left\|\mathbb{K}_\rho(t)\omega\right\| = \left\|t^{\rho-1}\mathbb{P}_\rho(t)\omega\right\| \leq \dfrac{\hat{\mathfrak{M}}t^{\rho-1}}{\Gamma(\rho)}\|\omega\|,\ \forall\omega\in\Omega.$

(iv) $\left\|\mathbb{S}_{\rho,\sigma}(t)\omega\right\| = \left\|I_{a^+}^{\rho(1-\sigma);\zeta}\mathbb{K}_\rho(t)\omega\right\| \leq \mathscr{M}_{\mathbb{S}}\|\omega\|$, for some $\mathscr{M}_{\mathbb{S}} > 0$.

(v) $\left\{\mathbb{S}_{\rho,\sigma}(t),\ t\geq 0\right\}$ and $\left\{\mathbb{K}_\rho(t),\ t\geq 0\right\}$ are norm continuous families in sense of uniform topology.

Definition 9. For each $u \in \mathcal{U}$ and $\omega_0 \in D(\mathcal{E})$, $\omega \in C(\mathfrak{J}, \Omega)$ is called mild solutions of Eq. (1) if it satisfies

$$\omega(t) = \mathbb{S}_{\rho,\sigma}(t)\omega_0 + \int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)\Psi\big(s,\omega(s)\big)\zeta'(s)ds + \int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)\Upsilon u(s)\zeta'(s)ds,\ t\in\mathfrak{J}. \tag{10}$$

3 | CONTROLLABILITY RESULTS

Here we investigate controllability of the system of Eqs.(1).

Definition 3.1 [17] The system of Eqs.(1) is controllable on interval $\mathfrak{J} = [0, a]$ if for $\omega(0) \in D(\mathcal{E})$ and $\omega_1 \in D(\mathcal{E})$ there exists a control function $u \in \mathcal{U}$ such that the mild solution $\omega(t)$ of system (1) satisfies $\omega(a) = \omega_1$.

For obtaining proposed results, let us introduce the following hypothesis:

C_1) For each $\omega \in \Omega$, $\Psi(\cdot, \omega) : \mathfrak{J} \longrightarrow D(\mathcal{E}) \subset \Omega$ is strongly measurable and for each $t \in \mathfrak{J}$, the function $\Psi(t, \cdot) : \Omega \longrightarrow D(\mathcal{E})$

is continuous.

C_2) For any $k > 0$, there exists a measurable function h_k such that

$$\sup_{\|\omega\|\le k} \|\Psi(t,\omega)\| \le h_k(t) \text{ with } \|h_k\|_\infty = \sup_{t\in\mathfrak{J}} \{h_k(t)\} < \infty \text{ and}$$

$$\sup_{t\in\mathfrak{J}} \int_0^t (\zeta(t)-\zeta(s))^{\rho-1} h_k(s)\zeta'(s)ds \le N_k(t),$$

for some measurable function N_k such that $\lim_{k\to\infty} \sup \dfrac{\|N_k(t)\|}{k} = L$ for some $L > 0$.

C_3) There exists a constant $K^* > 0$ such that for any bounded set $\mathcal{D} \subset \Omega$ and $t \in \mathfrak{J}$,

$$\eta\big(\Psi(t,\mathcal{D})\big) \le K^*\eta(\mathcal{D}).$$

C_4) $\Upsilon : U \longrightarrow D(\mathcal{E})$ is a bounded linear operator. Define an invertible linear operator $\Delta : U \longrightarrow \Omega$ as:

$$\Delta u = \int_0^a \mathbb{K}_\rho\big(\zeta(a)-\zeta(s)\big)\Upsilon u(s)\zeta'(s)ds, \quad \Delta \text{ is well defined as}$$

$$\|\Delta u\| = \left\|\int_0^a \mathbb{K}_\rho\big(\zeta(a)-\zeta(s)\big)\Upsilon u(s)\zeta'(s)ds\right\| \le \frac{\hat{\mathfrak{M}}\,\|\Upsilon\|\,(\zeta(a)-\zeta(0))^\rho}{\Gamma(\rho)}\,\|u\|_{\mathcal{U}}.$$

For an arbitrary function $\omega(\cdot)$, let us define the control function as follows:

$$u(t) = \Delta^{-1}\left[\omega_1 - \mathbb{S}_{\rho,\sigma}(t)\omega_0 - \int_0^a \mathbb{K}_\rho\big(\zeta(a)-\zeta(s)\big)\Psi\big(s,\omega(s)\big)\zeta'(s)ds\right]. \tag{11}$$

$$\|u(t)\| \le \left\|\Delta^{-1}\right\| \left[\left\|v_1\right\| + \mathscr{M}_{\mathbb{S}}\left\|\omega_0\right\| + \frac{\hat{\mathfrak{M}}N_k(a)}{\Gamma(\rho)}\right]. \tag{12}$$

Define an operator $\Theta : C(\mathfrak{J},\Omega) \longrightarrow C(\mathfrak{J},\Omega)$ as follows:

$$\Theta\omega(t) = \mathbb{S}_{\rho,\sigma}(t)\omega_0 + \int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)\Psi\big(s,\omega(s)\big)\zeta'(s)ds + \int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)\Upsilon u(s)\zeta'(s)ds, \; t\in\mathfrak{J}. \tag{13}$$

Now we show that operator $\Theta : C(\mathfrak{J},\Omega) \longrightarrow C(\mathfrak{J},\Omega)$ has a unique fixed point w.r.t control function $u(t)$ defined in (11).

3.1 | Existence and Uniqueness of Mild Solutions

Theorem 1. The systems of Eqs. (1) has at least one mild solution if hypothesis $(C_1 - C_4)$ and following conditions hold true

$$\frac{\hat{\mathfrak{M}}L}{\Gamma(\rho)}\left[1 + \frac{\hat{\mathfrak{M}}\,\|\Upsilon\|\,(\zeta(a)-\zeta(0))^\rho\,\left\|\Delta^{-1}\right\|}{\Gamma(1+\rho)}\right] < 1,$$

$$\frac{\hat{\mathfrak{M}}K^*(\zeta(a)-\zeta(0))^\rho}{\Gamma(1+\rho)}\left[1 + \frac{\hat{\mathfrak{M}}\,\|\Upsilon\|\,(\zeta(a)-\zeta(0))^\rho\,\left\|\Delta^{-1}\right\|}{\Gamma(1+\rho)}\right] < \frac{1}{4}.$$

Proof. For any $k > 0$, let us define $B_k = \{v \in C(\mathfrak{J}, \Omega) : \|\omega(t)\| \le k\}$. B_k is bounded closed and convex subset of $C(\mathfrak{J}, \Omega)$. We divide the proof into following steps:

Step 1: To show that $\Theta : B_k \longrightarrow B_k$. Let us assume that for each $k > 0$ there exists $\omega \in B_k$ such that $\|\Theta\omega(t)\| > k$.

For $t \in \mathfrak{J}$.

$$\|\Theta\omega(t)\| \le \mathscr{M}_{\mathbb{S}} \|\omega_0\| + \frac{\hat{\mathfrak{M}}}{\Gamma(\rho)} \int_0^t (\zeta(t) - \zeta(s))^{\rho-1} \left\|\Psi\left(s, \omega(s)\right)\right\| \zeta'(s)\, ds + \frac{\hat{\mathfrak{M}}}{\Gamma(\rho)} \int_0^t (\zeta(t) - \zeta(s))^{\rho-1} \|\Upsilon\| \|u(s)\| \zeta'(s)\, ds.$$

$$\le \mathscr{M}_{\mathbb{S}} \|\omega_0\| + \frac{\hat{\mathfrak{M}}}{\Gamma(\rho)} \int_0^t (\zeta(t) - \zeta(s))^{\rho-1} h_k(s)\zeta'(s) ds$$

$$+ \frac{\hat{\mathfrak{M}} \|\Upsilon\|}{\Gamma(\rho)} \left\|\Delta^{-1}\right\| \left[\|\omega_1\| + \mathscr{M}_{\mathbb{S}} \|\omega_0\| + \frac{\hat{\mathfrak{M}} N_k(t)}{\Gamma(\rho)} \right] \int_0^t (\zeta(t) - \zeta(s))^{\rho-1} \zeta'(s) ds$$

$$\le \mathscr{M}_{\mathbb{S}} \|\omega_0\| + \frac{\hat{\mathfrak{M}} N_k(t)}{\Gamma(\rho)} + \frac{\|\omega_0\| \hat{\mathfrak{M}} \mathscr{M}_{\mathbb{S}} \|\Delta^{-1}\| \|\Upsilon\| (\zeta(a) - \zeta(0))^\rho}{\Gamma(1+\rho)}$$

$$+ \frac{\|\omega_1\| \hat{\mathfrak{M}} \|\Delta^{-1}\| \|\Upsilon\| (\zeta(a) - \zeta(0))^\rho}{\Gamma(1+\rho)} + \frac{\hat{\mathfrak{M}}^2 N_k(t) \|\Delta^{-1}\| \|\Upsilon\| (\zeta(a) - \zeta(0))^\rho}{\Gamma(\rho)\Gamma(1+\rho)}.$$

As $\|\Theta\omega(t)\| > k$.

$$\Rightarrow \frac{\mathscr{M}_{\mathbb{S}}\|\omega_0\|}{k} + \frac{\|\omega_0\| \hat{\mathfrak{M}} \mathscr{M}_{\mathbb{S}} \|\Delta^{-1}\| \|\Upsilon\| (\zeta(a) - \zeta(0))^\rho}{k\Gamma(1+\rho)} + \frac{\|\omega_1\| \hat{\mathfrak{M}} \|\Delta^{-1}\| \|\Upsilon\| (\zeta(a) - \zeta(0))^\rho}{k\Gamma(1+\rho)}$$

$$+ \frac{\hat{\mathfrak{M}} N_k(t)}{k\Gamma(\rho)} \left[1 + \frac{\hat{\mathfrak{M}} \|\Delta^{-1}\| \|\Upsilon\| (\zeta(a) - \zeta(0))^\rho}{\Gamma(1+\rho)} \right] > 1.$$

Taking $\lim\limits_{k\to\infty}$ on both sides, we have

$$\begin{aligned} &\lim_{k\to\infty} \frac{\hat{\mathfrak{M}} N_k(t)}{k\Gamma(\rho)} \left[1 + \frac{\hat{\mathfrak{M}} \|\Delta^{-1}\| \|\Upsilon\| (\zeta(a) - \zeta(0))^\rho}{\Gamma(1+\rho)} \right] > 1. \\ &\Rightarrow \frac{\hat{\mathfrak{M}} L}{\Gamma(\rho)} \left[1 + \frac{\hat{\mathfrak{M}} \|\Delta^{-1}\| \|\Upsilon\| (\zeta(a) - \zeta(0))^\rho}{\Gamma(1+\rho)} \right] > 1, \end{aligned} \tag{14}$$

which is a contradiction to given assumption. Therefore our suppoistion is wrong. This proves $\Theta : B_k \longrightarrow B_k$.

Step 2: To prove that $\Theta : B_k \longrightarrow B_k$ is continuous.

Let $\{\omega_n\} \subset B_k$ with $\omega_n \longrightarrow \omega \in B_k$ as $n \longrightarrow \infty$. By assumption $(\mathcal{C}_1 - \mathcal{C}_2)$.

For $t \in J$,

$$\Psi\left(t, \omega_n(t)\right) \longrightarrow \Psi\left(t, \omega(t)\right) \text{ as } n \longrightarrow \infty \text{ and}$$

$$\left\|\Psi\left(t, \omega_n(t)\right) - \Psi\left(t, \omega(t)\right)\right\| \le 2h_k(t), \ \ \forall n \in \mathbb{N}.$$

We have

$$\begin{aligned}\left\|\Theta\omega_n(t)-\Theta\omega(t)\right\| \leq &\left\|\int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)\left\{\Psi\big(s,\omega_n(s)\big)-\Psi\big(s,\omega(s)\big)\right\}\zeta'(s)ds\right\|\\
&+\left\|\int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)B\Delta^{-1}\left[\int_0^a \mathbb{K}_\rho\big(\zeta(a)-\zeta(\tau)\big)\left\{\Psi\big(\tau,\omega_n(\tau)\big)-\Psi\big(\tau,\omega(\tau)\big)\right\}\zeta'(\tau)d\tau\right]\zeta'(s)\,ds\right\|\\
\leq&\frac{\hat{\mathfrak{M}}}{\Gamma(\rho)}\int_0^t(\zeta(t)-\zeta(s))^{\rho-1}\left\|\Psi\big(s,\omega_n(s)\big)-\Psi\big(s,\omega(s)\big)\right\|\zeta'(s)\,ds\\
&+\frac{\hat{\mathfrak{M}}^2\left\|\Delta^{-1}\right\|\left\|\Upsilon\right\|}{\Gamma^2(\rho)}\int_0^t(\zeta(t)-\zeta(s))^{\rho-1}\left[\int_0^a(\zeta(a)-\zeta(\tau))^{\rho-1}\left\|\Psi\big(\tau,\omega_n(\tau)\big)-\Psi\big(\tau,\omega(\tau)\big)\right\|\zeta'(\tau)d\tau\right]\zeta'(s)\,ds.\end{aligned}$$

By dominated convergence theorem and continuity of function $\Psi(t,.)$,

$$\left\|\Theta\omega_n(t)-\Theta\omega(t)\right\|\longrightarrow 0 \text{ as } n\longrightarrow\infty.$$

This proves the continuity of operator Θ.

Step 3: To prove $\Theta : B_k \longrightarrow B_k$ is equicontinuous operator.

For $0\leq t_1\leq t_2\leq a$,

$$\begin{aligned}\left\|\Theta\omega(t_2)-\Theta\omega(t_1)\right\|\leq&\left\|\mathbb{S}_{\rho,\sigma}(t_2)-\mathbb{S}_{\rho,\sigma}(t_1)\right\|\left\|\omega_0\right\|\\
&+\left\|\int_0^{t_1}\left\{\mathbb{K}_\rho\big(\zeta(t_2)-\zeta(s)\big)-\mathbb{K}_\rho\big(\zeta(t_1)-\zeta(s)\big)\right\}\Psi\big(s,\omega(s)\big)\zeta'(s)ds\right\|\\
&+\left\|\int_{t_1}^{t_2}\mathbb{K}_\rho\big(\zeta(t_2)-\zeta(s)\big)\Psi\big(s,\omega(s)\big)\zeta'(s)ds\right\|\\
&+\left\|\int_0^{t_1}\left\{\mathbb{K}_\rho\big(\zeta(t_2)-\zeta(s)\big)-\mathbb{K}_\rho\big(\zeta(t_1)-\zeta(s)\big)\right\}\Upsilon u(s)\zeta'(s)ds\right\|\\
&+\left\|\int_{t_1}^{t_2}\mathbb{K}_\rho\big(\zeta(t_2)-\zeta(s)\big)\Upsilon u(s)\zeta'(s)ds\right\|\\
=&I_1+I_2+I_3+I_4+I_5.\end{aligned}\tag{15}$$

By the norm continuity of $\left\{\mathbb{S}_{\rho,\sigma}(t) : t\geq 0\right\}$ in uniform operator topology

$$I_1=\left\|\mathbb{S}_{\rho,\sigma}(t_2)-\mathbb{S}_{\rho,\sigma}(t_1)\right\|\longrightarrow 0 \text{ as } t_1\longrightarrow t_2.\tag{16}$$

$$\begin{aligned}I_2=&\left\|\int_0^{t_1}\left\{\mathbb{K}_\rho\big(\zeta(t_2)-\zeta(s)\big)-\mathbb{K}_\rho\big(\zeta(t_1)-\zeta(s)\big)\right\}\Psi\big(s,\omega(s)\big)\zeta'(s)ds\right\|\\
\leq&\int_{t_1}^{t_2}\left\|\left\{\mathbb{K}_\rho\big(\zeta(t_2)-\zeta(s)\big)-\mathbb{K}_\rho\big(\zeta(t_1)-\zeta(s)\big)\right\}\right\|h_k(s)\zeta'(s)ds.\end{aligned}$$

Since $\left\{K_\rho(t) : t \geq 0\right\}$ is continuous in uniform operator topology.

$$\Rightarrow \quad I_2 \longrightarrow 0 \text{ as } t_1 \longrightarrow t_2. \tag{17}$$

$$\text{Similarly } I_4 \longrightarrow 0 \text{ as } t_1 \longrightarrow t_2. \tag{18}$$

$$I_3 = \left\| \int_{t_1}^{t_2} \mathbb{K}_\rho\big(\zeta(t_2)-\zeta(s)\big)\Psi\big(s,\omega(s)\big)\zeta'(s)ds \right\| \leq \frac{\hat{\mathfrak{M}}}{\Gamma(\rho)} \int_{t_1}^{t_2} \big(\zeta(t_2)-\zeta(s)\big)^{\rho-1} \|h_k(s)\| \zeta'(s)ds \leq \frac{\hat{\mathfrak{M}}\|h_k\|_\infty}{\Gamma(1+\rho)}\big(\zeta(t_2)-\zeta(t_1)\big)^\rho.$$

$$\text{Clearly } I_3 \longrightarrow 0 \text{ as } t_1 \longrightarrow t_2. \tag{19}$$

$$\begin{aligned} I_5 = \left\| \int_{t_1}^{t_2} \mathbb{K}_\rho\big(\zeta(t_2)-\zeta(s)\big)\Upsilon u(s)\zeta'(s)ds \right\| &\leq \frac{\hat{\mathfrak{M}}}{\Gamma(\rho)} \int_{t_1}^{t_2} \big(\zeta(t_2)-\zeta(s)\big)^{\rho-1} \|\Upsilon\| \|u(s)\| \zeta'(s)ds. \\ &\leq \frac{\hat{\mathfrak{M}}\|\Delta^{-1}\|\|\Upsilon\|}{\Gamma(1+\rho)}\left[\|\omega_1\| + \mathscr{M}_{\mathbb{S}}\|\omega_0\| + \frac{MN}{\Gamma(\rho)}\right]\big(\zeta(t_2)-\zeta(t_1)\big)^\rho. \end{aligned}$$

$$\text{Clearly } I_5 \longrightarrow 0 \text{ as } t_1 \longrightarrow t_2. \tag{20}$$

Using $(16-20)$ in (15). We have

$$\|\Theta\omega(t_2) - \Theta\omega(t_1)\| \longrightarrow 0 \text{ as } t_2 \longrightarrow t_1.$$

This proves the equicontinuity of operator Θ.

Step 4: Let $\mathcal{D} = \bar{Co}\big(\Theta(B_k)\big)$, where $\bar{Co}$ denotes the closure of convex hull. By Lemma 2.6. $\bar{Co}\big(\Theta(B_k)\big) \subset B_k$ is bounded and equicontinuous. We show that $\Theta : \mathcal{D} \longrightarrow \mathcal{D}$ is a condensing mapping. For $\mathcal{D} = \bar{Co}\big(\Theta(B_k)\big)$, there exists a countable set $\mathcal{D}_0 = \left\{\omega_n\right\}_{n=1}^\infty \subset \mathcal{D}$ such that $\eta\big(\Theta(\mathcal{D})\big) \leq 2\eta\big(\Theta(\mathcal{D}_0)\big)$.

$$\begin{aligned} \eta\big(\Theta(\mathcal{D}_0)(t)\big) =& \eta\left[\mathbb{S}_{\rho,\sigma}(t)\omega_0 + \int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)\left\{\Psi\big(s,\omega_n(s)\big) + \Upsilon u(s)\right\}\zeta'(s)ds\right]. \\ \leq& \eta\left[\int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)\left(\Psi\big(s,\omega_n(s)\big) + \Upsilon\Delta^{-1}\left(\omega_1 - \mathbb{S}_{\rho,\sigma}(t)\omega_0 - \int_0^a \mathbb{K}_\rho\big(\zeta(a)-\zeta(\tau)\big)\Psi\big(\tau,\omega_n(\tau)\big)\zeta'(\tau)d\tau\right)\right)\zeta'(s)ds\right]. \\ \leq& \frac{2\hat{\mathfrak{M}}}{\Gamma(\rho)}\int_0^t \big(\zeta(t)-\zeta(s)\big)^{\rho-1}\left[\eta\big(\Psi(s,\omega_n(s))\big) + \frac{\hat{\mathfrak{M}}\|\Upsilon\|\|\Delta^{-1}\|}{\Gamma(\rho)}\int_0^a \big(\zeta(a)-\zeta(\tau)\big)^{\rho-1}\eta\big(\Psi(\tau,\omega_n(\tau))\big)\zeta'(\tau)d\tau\right]\zeta'(s)ds. \\ \leq& \frac{2\hat{\mathfrak{M}}}{\Gamma(\rho)}\int_0^t \big(\zeta(t)-\zeta(s)\big)^{\rho-1}\left[K^*\eta\big(\omega_n(s)\big) + \frac{\hat{\mathfrak{M}}\|\Upsilon\|\|\Delta^{-1}\|}{\Gamma(\rho)}\int_0^a \big(\zeta(a)-\zeta(\tau)\big)^{\rho-1}K^*\eta\big(\omega_n(\tau)\big)\zeta'(\tau)d\tau\right]\zeta'(s)ds. \\ \leq& \frac{2\hat{\mathfrak{M}}K^*}{\Gamma(1+\rho)}\left[1 + \frac{\hat{\mathfrak{M}}\|\Upsilon\|\|\Delta^{-1}\|\big(\zeta(a)-\zeta(0)\big)^\rho}{\Gamma(1+\rho)}\right]\big(\zeta(a)-\zeta(0)\big)^\rho\eta(\mathcal{D}_0). \\ \leq& \frac{2\hat{\mathfrak{M}}K^*}{\Gamma(1+\rho)}\left[1 + \frac{\hat{\mathfrak{M}}\|\Upsilon\|\|\Delta^{-1}\|\big(\zeta(a)-\zeta(0)\big)^\rho}{\Gamma(1+\rho)}\right]\big(\zeta(a)-\zeta(0)\big)^\rho\eta(\mathcal{D}). \end{aligned}$$

As $\eta\big(\Theta(\mathcal{D})\big) \leq 2\eta\big(\Theta(\mathcal{D}_0)\big)$.

$$\Rightarrow \ \eta\big(\Theta(\mathcal{D})\big) \leq \frac{4\hat{\mathfrak{M}}K^*}{\Gamma(1+\rho)}\left[1 + \frac{\hat{\mathfrak{M}}\,\|\Upsilon\|\,\|\Delta^{-1}\|\,\big(\zeta(a)-\zeta(0)\big)^{\rho}}{\Gamma(1+\rho)}\right]\big(\zeta(a)-\zeta(0)\big)^{\rho}\eta(\mathcal{D}).$$
$$< 1.\eta(\mathcal{D}).$$

This proves that Θ is a condensing mapping.
By Lemma 2.8. Θ has at least one fixed point which is a mild solution of the system of Eqs.(1). Hence the result. □

Theorem 2. If hypothesis $(\mathcal{C}_1 - \mathcal{C}_4)$ hold and there exists constant $L_\Psi > 0$ such that $\big\|\Psi\big(t,\omega_1(t)\big) - \Psi\big(t,\omega_2(t)\big)\big\| \leq L_\Psi \|\omega_1 - \omega_2\|$, $\forall \omega_1, \omega_2 \in B_k$. Then system of Eqs.(1) has unique mild solution if following condition is satisfied:

$$\frac{\hat{\mathfrak{M}}L_\Psi\big(\zeta(a)-\zeta(0)\big)^{\rho}}{\Gamma(1+\rho)}\left[1 + \frac{\hat{\mathfrak{M}}\,\|\Delta^{-1}\|\,\|\Upsilon\|\,\big(\zeta(a)-\zeta(0)\big)^{\rho}}{\Gamma(1+\rho)}\right] < 1.$$

Proof. Let ω_1 and ω_2 be the two solutions of Eq.(1) in B_k. Each $\omega_i,\ i \in \{1,2\}$ satisfies:

$$\Theta\omega_i(t) = \mathbb{S}_{\rho,\sigma}(t)\omega_0 + \int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)\Psi\big(s,\omega(s)\big)\zeta'(s)ds + \int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)\Upsilon u(s)\zeta'(s)ds,\ t \in \mathfrak{J}. \tag{21}$$

For any $t \in \mathfrak{J}$,

$$\begin{aligned}
\|\omega_1 - \omega_2\| &= \|\Theta\omega_1(t) - \Theta\omega_2(t)\| \\
&\leq \left\|\int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)\left\{\Psi\big(s,\omega_1(s)\big) - \Psi\big(s,\omega_2(s)\big)\right\}\zeta'(s)ds\right\| \\
&+ \left\|\int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)B\Delta^{-1}\left[\int_0^a \mathbb{K}_\rho\big(\zeta(a)-\zeta(\tau)\big)\left\{\Psi\big(\tau,\omega_1(\tau)\big) - \Psi\big(\tau,\omega_2(\tau)\big)\right\}\zeta'(\tau)d\tau\right]\zeta'(s)ds\right\|. \\
&\leq \frac{\hat{\mathfrak{M}}}{\Gamma(\rho)}\int_0^t \big(\zeta(t)-\zeta(s)\big)^{\rho-1}\big\|\Psi\big(s,\omega_1(s)\big) - \Psi\big(s,\omega_2(s)\big)\big\|\zeta'(s)ds \\
&+ \frac{\hat{\mathfrak{M}}\,\|\Delta^{-1}\|\,\|\Upsilon\|}{\Gamma(\rho)}\int_0^t \big(\zeta(t)-\zeta(s)\big)^{\rho-1}\left[\frac{\hat{\mathfrak{M}}}{\Gamma(\rho)}\int_0^a \big(\zeta(a)-\zeta(\tau)\big)^{\rho-1}\big\|\Psi\big(\tau,\omega_1(\tau)\big) - \Psi\big(\tau,\omega_2(\tau)\big)\big\|d\tau\right]\zeta'(s)ds. \\
&\leq \frac{\hat{\mathfrak{M}}L_\Psi}{\Gamma(\rho)}\int_0^t \big(\zeta(t)-\zeta(s)\big)^{\rho-1}\|\omega_1-\omega_2\|\zeta'(s)ds \\
&+ \frac{\hat{\mathfrak{M}}\,\|\Delta^{-1}\|\,\|\Upsilon\|}{\Gamma(\rho)}\int_0^t \big(\zeta(t)-\zeta(s)\big)^{\rho-1}\left[\frac{\hat{\mathfrak{M}}L_\Psi\big(\zeta(a)-\zeta(0)\big)^{\rho}\|\omega_1-\omega_2\|}{\Gamma(1+\rho)}\right]\zeta'(s)ds. \\
&\leq \frac{\hat{\mathfrak{M}}L_\Psi\big(\zeta(a)-\zeta(0)\big)^{\rho}}{\Gamma(1+\rho)}\left[1 + \frac{\hat{\mathfrak{M}}\,\|\Delta^{-1}\|\,\|\Upsilon\|\,\big(\zeta(a)-\zeta(0)\big)^{\rho}}{\Gamma(1+\rho)}\right]\|\omega_1-\omega_2\|. \\
&< 1\cdot\|\omega_1-\omega_2\| \qquad \text{(By stated condition).}
\end{aligned}$$

$\Rightarrow \quad \|\omega_1 - \omega_2\| = 0, \forall\ t \in\ J$. Therefore $\omega_1 \equiv \omega_2$. Hence the result.

□

3.2 | Controllability

Theorem 3. The system of Eqs.(1) is controllable if it posesses unique mild solution and hypothesis $(C_1 - C_4)$ hold true.

Proof. Here we all need to show that is w.r.t control function $u(t)$ defined in (11), mild solution $\omega(t)$ of (1) satisfies $\omega(a) = \omega_1$.

$$\begin{aligned}
\omega(a) =& \Theta\big(\omega(a)\big) \\
&= \mathbb{S}_{\rho,\sigma}(t)\omega_0 + \int_0^a \mathbb{K}_\rho\big(\zeta(a)-\zeta(s)\big)\Psi\big(s,\omega(s)\big)\zeta'(s)ds \\
&+ \int_0^a \mathbb{K}_\rho\big(\zeta(a)-\zeta(s)\big)\Upsilon u(s)\zeta'(s)ds. \\
&= \mathbb{S}_{\rho,\sigma}(t)\omega_0 + \int_0^a \mathbb{K}_\rho\big(\zeta(a)-\zeta(s)\big)\Psi\big(s,\omega(s)\big)\zeta'(s)ds + \Delta u(t). \qquad \text{By (H4)} \\
&= \mathbb{S}_{\rho,\sigma}(t)\omega_0 + \int_0^a \mathbb{K}_\rho\big(\zeta(a)-\zeta(s)\big)\Psi\big(s,\omega(s)\big)\zeta'(s)ds + \omega_1 \\
&\quad - \mathbb{S}_{\rho,\sigma}(t)\omega_0 - \int_0^a \mathbb{K}_\rho\big(\zeta(a)-\zeta(s)\big)\Psi\big(s,\omega(s)\big)\zeta'(s)ds. \\
=& \omega_1.
\end{aligned}$$

This proves that control function $u(t)$ defined in (11) steers the fractional system (1) from $\omega(0)$ to ω_1 in finite time a. Hence the result. □

4 | ULAM-HYERS AND ULAM-HYERS-RASSIAS STABILITY

For $\epsilon > 0, \Psi : \mathfrak{J} \times \Omega \longrightarrow D(\mathcal{E}) \subset \Omega$, $\zeta \in C(J, \mathbb{R}^+)$, $\mu \in C(\mathfrak{J}, \Omega)$ and non-decreasing function $\nu(t) \in C(\mathfrak{J}, \mathbb{R}^+)$ consider equation

$$\begin{cases} D_{0^+}^{\rho,\sigma;\zeta}\mathcal{E}\omega(t) = \mathcal{A}\omega(t) + \mathcal{E}\Psi(t,\omega(t)) + \mathcal{E}\Upsilon u(t), \quad t \in \mathfrak{J} = [0,a], \\ \mathcal{E}I_{0^+}^{(1-\rho)(1-\sigma);\zeta}\omega(0) = \mathcal{E}I_{0^+}^{(1-\rho)(1-\sigma);\zeta}\mu(0) = \mathcal{E}\omega_0, \ \omega_0 \in D(\mathcal{E}), \end{cases} \tag{22}$$

Definition 4.1. The system of Eqs.(1) is Ulam-Hyers stable if there exists real number $\beta > 0$ such that for every $\epsilon > 0$ and $\mu \in C^1(\mathfrak{J}, \Omega)$ satisfying

$$\left| D_{0^+}^{\rho,\sigma;\zeta}\mathcal{E}\mu(t) - \mathcal{A}\mu(t) - \mathcal{E}\Psi(t,\mu(t)) - \mathcal{E}\Upsilon u(t) \right| \leq \epsilon, \tag{23}$$

there exists a unique solution $\omega(t)$ of Eq.(22) with

$$|\omega(t) - \mu(t)| \leq \beta\epsilon, \ t \in \mathfrak{J}.$$

Definition 4.2.? The system of Eqs.(1) is Ulam-Hyers-Rassias stable w.r.t function ν if there exists real number $\beta_\nu > 0$ such that for every $\epsilon > 0$ and $\mu \in C^1(\mathfrak{J}, \Omega)$ satisfying

$$\left| D_{0^+}^{\rho,\sigma;\zeta}\mathcal{E}\mu(t) - \mathcal{A}\mu(t) - \mathcal{E}\Psi(t,\mu(t)) - \mathcal{E}\Upsilon u(t) \right| \leq \epsilon\nu(t), \ t \in \mathfrak{J}. \tag{24}$$

there exists a unique solution $\omega(t)$ of Eq.(22) with

$$|\omega(t) - z(t)| \leq \beta_\nu\epsilon\nu(t), \ t \in \mathfrak{J}.$$

Remark 4.1. A function $z \in C^1(\mathfrak{J}, \Omega)$ satisfies inequality (23) iff there exists a function $Q_1(t) \in C^1(\mathfrak{J}, \Omega)$ such that following holds:

1. $|Q_1(t)| \leq \epsilon$ for $t \in \mathfrak{J}$.

2. $D_{0^+}^{\rho,\sigma;\zeta} \mathcal{E}\mu(t) = \mathcal{A}\mu(t) + \mathcal{E}\Psi(t, \mu(t)) + \mathcal{E}\Upsilon u(t) + Q_1(t), \quad t \in \mathfrak{J} = [0, a]$.

Remark 4.2. A function $z \in C^1(\mathfrak{J}, \Omega)$ satisfies inequality (24) iff there exists a function $Q_2(t) \in C^1(\mathfrak{J}, \Omega)$ such that following holds:

1. $|Q_2(t)| \leq \epsilon\nu(t)$ for $t \in \mathfrak{J}$.

2. $D_{0^+}^{\rho,\sigma;\zeta} \mathcal{E}\mu(t) = \mathcal{A}\mu(t) + \mathcal{E}\Psi(t, \mu(t)) + \mathcal{E}\Upsilon u(t) + Q_2(t), \quad t \in \mathfrak{J} = [0, a]$.

Theorem 4. Assume that hypothesis $(C_1 - C_4)$ hold and $\Psi\big(t, \omega(t)\big)$ is Lipchitz continuous function with Lipschitz constant L_Ψ. Then system of Eqs.(1) is Ulam-Hyers stable provided that

$$\mathcal{L} = \frac{\hat{\mathfrak{M}} L_\Psi \big(\zeta(a) - \zeta(0)\big)^\rho}{\Gamma(1+\rho)} \left[1 + \frac{\hat{\mathfrak{M}} \left\|\Delta^{-1}\right\| \|\Upsilon\| \big(\zeta(a) - \zeta(0)\big)^\rho}{\Gamma(1+\rho)}\right] < 1.$$

Proof. Let $\mu \in C^1(\mathfrak{J}, \Omega)$ be the solution of inequality (23). Then $\mu(t)$ satisfies.

$$\mu(t) = \mathbb{S}_{\rho,\sigma}(t)\omega_0 + \int_0^t \mathbb{K}_\rho\big(\zeta(t) - \zeta(s)\big) \left\{\Psi\big(s, \mu(s)\big) + \Upsilon u(s)\right\} \zeta'(s)ds + \int_0^t \mathbb{K}_\rho\big(\zeta(t) - \zeta(s)\big) Q_1(s)\zeta'(s)ds, \ t \in \mathfrak{J}.$$

Let $\omega \in C(\mathfrak{J}, \Omega)$ be the unique mild solution of Cauchy problem (22). $\omega(t)$ is defined as

$$\omega(t) = \mathbb{S}_{\rho,\sigma}(t)\omega_0 + \int_0^t \mathbb{K}_\rho\big(\zeta(t) - \zeta(s)\big)\Psi\big(s, \omega(s)\big)\zeta'(s)ds + \int_0^t \mathbb{K}_\rho\big(\zeta(t) - \zeta(s)\big)\Upsilon u(s)\zeta'(s)ds.$$

For $t \in \mathfrak{J}$.

$$|\omega(t)-\mu(t)| \leq \int_0^t |\mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)|\,|\Psi\big(s,\omega(s)\big)-\Psi\big(s,\mu(s)\big)|\zeta'(s)ds + \int_0^t |\mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)|\,|Q_1(s)|\zeta'(s)ds$$
$$+\int_0^t |\mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)|\,\Upsilon\Delta^{-1}\left[\int_0^a |\mathbb{K}_\rho\big(\zeta(a)-\zeta(\tau)\big)|\,|\Psi\big(\tau,\omega(\tau)\big)-\Psi\big(\tau,\mu(\tau)\big)|\,\zeta'(\tau)d\tau\right]\zeta'(s)ds.$$

$$\leq \frac{\hat{\mathfrak{M}}L_\Psi\big(\zeta(a)-\zeta(0)\big)^\rho}{\Gamma(1+\rho)}\left[1+\frac{\hat{\mathfrak{M}}\,\|\Delta^{-1}\|\,\|\Upsilon\|\,\big(\zeta(a)-\zeta(0)\big)^\rho}{\Gamma(1+\rho)}\right]|\omega(t)-\mu(t)| + \frac{\hat{\mathfrak{M}}\epsilon\big(\zeta(a)-\zeta(0)\big)^\rho}{\Gamma(1+\rho)}.$$
$$=\mathcal{L}|\mu(t)-\omega(t)| + \frac{\hat{\mathfrak{M}}\epsilon\big(\zeta(a)-\zeta(0)\big)^\rho}{\Gamma(1+\rho)}.$$

$$\Rightarrow\ (1-\mathcal{L})\,|\mu(t)-\omega(t)| \leq \frac{\hat{\mathfrak{M}}\epsilon\big(\zeta(a)-\zeta(0)\big)^\rho}{\Gamma(1+\rho)}.$$
$$\Rightarrow\ |\mu(t)-\omega(t)| \leq \frac{\hat{\mathfrak{M}}\epsilon\big(\zeta(a)-\zeta(0)\big)^\rho}{\Gamma(1+\rho)(1-\mathcal{L})} \leq \frac{\hat{\mathfrak{M}}\epsilon\big(\zeta(a)-\zeta(0)\big)^\rho}{\Gamma(1+\rho)},\ \mathcal{L}<1.$$
$$\Rightarrow\ |\mu(t)-\omega(t)| \leq \beta\epsilon,\ \text{for } \beta = \frac{\hat{\mathfrak{M}}\big(\zeta(a)-\zeta(0)\big)^\rho}{\Gamma(1+\rho)}. \tag{25}$$

□

Theorem 5. The system of Eqs.(1) is Ulam-Hyers-Rassias stable if hypothesis (C_1-C_4) hold, $\mathcal{L}<1$ and continuous non-decreasing function $\nu(t)$ satisfies

$$\int_0^t \big(\zeta(t)-\zeta(s)\big)^{\rho-1}\nu(s)\zeta'(s)ds \leq \kappa_\nu\nu(t),\ \text{for } \kappa_\nu>0.$$

Proof. Let $\mu \in C^1(\mathfrak{J},\Omega)$ be the solution of inequality (24). Then $\mu(t)$ satisfies.

$$\mu(t)=\mathbb{S}_{\rho,\sigma}(t)\omega_0+\int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)\left\{\Psi\big(s,\mu(s)\big)+\Upsilon u(s)\right\}\zeta'(s)ds+\int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)Q_2(s)\zeta'(s)ds,\ t\in\mathfrak{J}.$$

Let us denote by $\omega \in C(\mathfrak{J},\Omega)$ the unique mild solution of Cauchy problem (22). $\omega(t)$ is defined as

$$\omega(t)=\mathbb{S}_{\rho,\sigma}(t)\omega_0+\int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)\Psi\big(s,\omega(s)\big)\zeta'(s)ds+\int_0^t \mathbb{K}_\rho\big(\zeta(t)-\zeta(s)\big)\Upsilon u(s)\zeta'(s)ds,\ t\in\mathfrak{J}. \tag{26}$$

For $t \in \mathfrak{J}$,

$$|\omega(t)-\mu(t)| \le \int_0^t |\mathbb{K}_\rho(\zeta(t)-\zeta(s))|\, |\Psi(s,\omega(s)) - \Psi(s,\mu(s))|\zeta'(s)ds + \int_0^t |\mathbb{K}_\rho(\zeta(t)-\zeta(s))|\, |Q_2(s)|\zeta'(s)ds$$
$$+ \int_0^t |\mathbb{K}_\rho(\zeta(t)-\zeta(s))|\, \Upsilon \Delta^{-1}\left[\int_0^a |\mathbb{K}_\rho(\zeta(a)-\zeta(\tau))||f(\tau,\omega(\tau)) - f(\tau,\mu(\tau))|\,\zeta'(\tau)d\tau\right]\zeta'(s)ds.$$

$$\le \frac{\hat{\mathfrak{M}} L_\Psi (\zeta(a)-\zeta(0))^\rho}{\Gamma(1+\rho)}\left[1 + \frac{\hat{\mathfrak{M}}\, \|\Delta^{-1}\|\, \|\Upsilon\|\, (\zeta(a)-\zeta(0))^\rho}{\Gamma(1+\rho)}\right]|\omega(t)-\mu(t)| + \frac{\hat{\mathfrak{M}}\epsilon\; \kappa_\nu \nu(t)(\zeta(a)-\zeta(0))^\rho}{\Gamma(1+\rho)}.$$
$$= \mathcal{L}|\omega(t)-\mu(t)| + \frac{\hat{\mathfrak{M}}\epsilon(\zeta(a)-\zeta(0))^\rho}{\Gamma(1+\rho)}.$$

$$\Rightarrow\ (1-\mathcal{L})\,|\omega(t)-\mu(t)| \le \frac{\hat{\mathfrak{M}}\epsilon\kappa_\nu \nu(t)(\zeta(a)-\zeta(0))^\rho}{\Gamma(1+\rho)}$$
$$\Rightarrow\ |\omega(t)-\mu(t)| \le \frac{\hat{\mathfrak{M}}\epsilon\kappa_\nu \nu(t)(\zeta(a)-\zeta(0))^\rho}{\Gamma(\rho)(1-\mathcal{L})} \le \frac{\hat{\mathfrak{M}}\epsilon\kappa_\nu \nu(t)(\zeta(a)-\zeta(0))^\rho}{\Gamma(1+\rho)},\ \mathcal{L}<1.$$
$$\Rightarrow\ |\omega(t)-\mu(t)| \le \beta_\nu \epsilon \nu(t),\ \text{for } \beta_\nu = \frac{\hat{\mathfrak{M}}\kappa_\nu(\zeta(a)-\zeta(0))^\rho}{\Gamma(1+\rho)}. \tag{27}$$

□

5 | EXAMPLE

Take $\Omega = U = L^2[0,\pi]$. Consider the following partial differential equation:

$$\begin{cases} D^{\frac{1}{5},\frac{1}{7};\zeta}\left(\omega(t,y) - \dfrac{\partial^2\omega(t,y)}{\partial y^2}\right) = \dfrac{\partial^2\omega(t,y)}{\partial y^2} + \dfrac{t^2}{20}\left[\dfrac{\cos\omega(t,y)}{t} - \dfrac{\partial^2}{\partial y^2}\left(\dfrac{\cos\omega(t,y)}{t}\right)\right] + \Upsilon u(t), \\ \qquad\qquad y\in[0,\pi],\ t\in[0,1]=J. \\ I_{0^+}^{(1-\frac{1}{5})(1-\frac{1}{7})}\left(\omega(0,y) - \dfrac{\partial^2\omega(0,y)}{\partial y^2}\right) = \omega_0(y),\ 0\le y\le\pi. \\ \omega(t,0) = \omega(t,\pi) = 0, \end{cases} \tag{28}$$

where

$$\rho = \frac{1}{5},\ \sigma = \frac{1}{7},\ \theta = \rho+\sigma-\rho\sigma = \frac{11}{35},\ \zeta(t) = \sqrt{t}.$$

Define $\mathcal{A} : D(\mathcal{A}) \subset \Omega \longrightarrow \Omega$ by $\mathcal{A}\omega = \dfrac{\partial^2\omega(t,y)}{\partial y^2}$ and $\mathcal{E} : D(\mathcal{E}) \subset \Omega \longrightarrow \Omega$ by $\mathcal{E}\omega = \left(\omega - \dfrac{\partial^2\omega}{\partial y^2}\right)$, where $D(\mathcal{A}) = D(\mathcal{E}) = \left\{\omega\in\Omega :\ \omega, \dfrac{\partial\omega}{\partial y} \text{ are absolutely continuous },\ \dfrac{\partial^2\omega}{\partial y^2}\in\Omega, \omega(0)=\omega(\pi)=0\right\}$. It follows from reference[35], that pair $(\mathcal{A},\mathcal{E})$ generates a propagation family $\{T(t),\ t\ge 0\}$ which is norm continuous and uniformly bounded. It follows from[36] that $\mathcal{A}$ and $\mathcal{E}$ can be written as

$$\mathcal{A}\omega = -\sum_{q=1}^{\infty} q^2 < \omega, \omega_q >,\ \omega\in D(\mathcal{A}),$$
$$\mathcal{E}\omega = \sum_{q=1}^{\infty}(1+q^2) < \omega,\omega_q > \omega_q,\ \omega\in D(\mathcal{E}),\ \text{where}$$

$\omega_q(y) = \sqrt{\frac{2}{\pi}} \sin qy,\ n \in \mathbb{N}$ is the orthonormal set of eigen functions of $\mathcal{A}$. For any $\omega \in D(\mathcal{E}),\ \lambda > 0$, we obtain

$$(\lambda\mathcal{E} - \mathcal{A})^{-1}\mathcal{E}\omega = \sum_{q=1}^{\infty} \frac{1+q^2}{\lambda(1+q^2)+q^2} < \omega, \omega_q > \omega_q = \sum_{q=1}^{\infty} \int_0^{\infty} e^{-\lambda t} e^{-\frac{q^2 t}{1+q^2}} dt \ < \omega, \omega_q > \omega_q.$$

Therefore $\{T(t), t \geq 0\}$ generated by $-\mathcal{A}\mathcal{E}^{-1}$ can be written as

$$T(t)\omega = \sum_{q=1}^{\infty} \int_0^{\infty} e^{-\lambda t} e^{-\frac{q^2 t}{1+q^2}} < \omega, \omega_q > \omega_q \text{ with } \|T(t)\| \leq 1.$$

This implies

$$\left\|\mathbb{P}_\rho(t)\right\| \leq \frac{1}{\Gamma(\frac{1}{5})}, \ \left\|\mathbb{K}_\rho(t)\right\| \leq \frac{1}{\Gamma(\frac{1}{5})} \text{ and } \left\|\mathbb{S}_{\rho,\sigma}(t)\right\| \leq 1.$$

Define $Y : U \longrightarrow D(\mathcal{E})$ as $Y = bI$ for $b > 0$.

$$\Delta u(t,y) = \int_0^1 K_\rho\big(\zeta(1) - \zeta(s)\big)Yu\big(s, y(s)\big)\zeta'(s)ds.$$

$$\|\Delta u\| \leq \frac{b}{\Gamma(\frac{1}{5})} \int_0^1 \big(\zeta(1) - \zeta(s)\big)^{-\frac{4}{5}} \|u\| \, \zeta'(s)ds = \frac{b\,\|u\|}{\Gamma(\frac{6}{5})}.$$

$\Delta^{-1} : \Omega \longrightarrow U$ with $\left\|\Delta^{-1}\right\| \leq \dfrac{\Gamma(\frac{6}{5})}{b}$.

$$\Psi\big(t, \omega(t,y)\big) = \frac{t^2}{20}\left[\cos\left(\frac{\omega(t,y)}{t}\right)\right].$$

$$\sup_{\|\omega\|\leq k} \left\|\Psi\big(t, \omega(t,y)\big)\right\| \leq \frac{t^2}{20}\left\|\frac{\upsilon(t,y)}{t}\right\| \leq \frac{k}{20} = h_k(t).$$

It is easy to see that Ψ is caratheodary function and

$$\left\|\Psi(t,\omega_1) - \Psi(t,\omega_2)\right\| \leq \frac{1}{20}\left\|\omega_1 - \upsilon_2\right\|.$$

$\Rightarrow\ \Psi$ is Lipschitz continuous function with $L_\Psi = \dfrac{1}{20}$.

$$\int_0^t \big(\zeta(t) - \zeta(s)\big)^{\rho-1} h_k(s)\zeta'(s)ds \leq \frac{k}{4}\big(\zeta(1) - \zeta(0)\big)^{\frac{1}{5}} \leq \frac{k}{4} = N_k(t).$$

$$L = \lim_{k\longrightarrow\infty} \frac{\left\|N_k(t)\right\|}{k} = \frac{1}{4}.$$

With all the parameters discussed above, it is easy to check that conditions stated in Theorem 3.1., Theorem 3.2., Theorem 4.1. and Theorem 4.2. hold true as

$$\frac{\widehat{\mathfrak{M}}L}{\Gamma(\rho)}\left[1 + \frac{\widehat{\mathfrak{M}}\,\|Y\|\,(\zeta(a)-\zeta(0))^\rho \left\|\Delta^{-1}\right\|}{\Gamma(1+\rho)}\right] \leq \frac{1}{2\Gamma(\frac{1}{5})} = \frac{1}{2\times 4.590843} = 0.108908 < 1.$$

$$\frac{\widehat{\mathfrak{M}}L_\Psi\big(\zeta(a)-\zeta(0)\big)^\rho}{\Gamma(1+\rho)}\left[1 + \frac{\widehat{\mathfrak{M}}\left\|\Delta^{-1}\right\|\|Y\|\big(\zeta(a)-\zeta(0)\big)^\rho}{\Gamma(1+\rho)}\right] \leq \frac{1}{20\Gamma(\frac{6}{5})} = \frac{1}{20\times 0.9181687} = 0.05444 < 1.$$

$$\frac{\widehat{\mathfrak{M}}L}{\Gamma(\rho)}\left[1 + \frac{\widehat{\mathfrak{M}}\,\|Y\|\,(\zeta(a)-\zeta(0))^\rho \left\|\Delta^{-1}\right\|}{\Gamma(1+\rho)}\right] \leq \frac{1}{2\Gamma(\frac{1}{5})} = \frac{1}{2\times 4.590843} = 0.108908 < 1.$$

$$\frac{\hat{\mathfrak{M}} L_{\Psi}\left(\zeta(a)-\zeta(0)\right)^{\rho}}{\Gamma(1+\rho)}\left[1+\frac{\hat{\mathfrak{M}}\left\|\Delta^{-1}\right\|\|\Upsilon\|\left(\zeta(a)-\zeta(0)\right)^{\rho}}{\Gamma(1+\rho)}\right] \leq \frac{1}{20\Gamma(\frac{6}{5})}=\frac{1}{20\times 0.9181687}=0.05444<1.$$

This proves that system (28) is controllable and stable on $\mathfrak{J}=[0,1]$.

CONCLUSION

The main aim of this paper is to study the controllability results for FEE with generalized Hilfer fractional derivatives. To the best of our knowledge, existence results for different forms of these equations have obtained commonly however controllability and stability results have discussed rarely. These results are obtained with the help of theory of propagation family and measure of non-compactness. In our future work, we aim to work on the applications of obtained results in modeling theory.

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The authors declare no potential conflict of interests.

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