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The effective Equations for the Ultrasonic Response of Wet Cortical Bone

Mischa Blaszczyk[1] | Robert Pertsch Gilbert*[2] | Klaus Hackl[1]

[1]Institute of Mechanics and Materials, Ruhr-Universität Bochum
[2]Department of Mathematical Sciences, University of Delaware

Correspondence
*Robert Pertsch Gilbert, Email: gilbert@udel.edu

We outline the mathematical model of the ultrasonic response of wet cortical bone and its time harmonic formulation. We employ an energetic approach based on the Reuss-bound of the free energy of a porous material consisting of a piezo-electric solid and a conducting fluid part. Magnetic effects are taken into consideration. Corresponding boundary value problems are stated and associated theorems established. A conclusion is included concerning future developments of this formulation.

KEYWORDS:
wet bone, ultrasonic response, Maxwell equations

1 | INTRODUCTION

Cortical and cancellous bone respond to fractures in different ways; cancellous bone unites very rapidly because there are many points of contact which are rich in blood and cells. Fractured cortical bone, depending on whether contact of the severed portions is close and immobilized, healing takes place with very little external callus[4]. On the other hand if immobilization is not rigid then there is a build up of external callus.

Cortical bone is a composite material consisting of a piezo-elastic matrix and an interstitial fluid[8,11,12,13]. In this paper the problem of derivation of an effective model of acoustic wave propagation in a two-phase mixture of linear piezo-elastic solid and a Newtonian interstitial fluid is studied. Wave-propagation in cortical bone has been investigated in[15,20]. As the interstitial fluid is ionic; whenever the bone is stimulated ultrasonically a charge appears on the bone matrix and a streaming potential is created in the fluid. The current which occurs forms an electro-magnetic field. The solid phase occupies the region $\theta^{(s)}$ and is piezo-elastic; however, there is also ionic advective transfer in the interstitial fluid part $\theta^{(f)}$; see Figure 1 for the geometry alluded to. Such a model might be useful for studying the excitation of long bones, such as the tibia. Sending a shear wave along the tibia excites an electromagnetic field, which is measurable[9,10,11]. This procedure has two purposes, first to detect fracture and its state of healing, second as a method of promoting healing through the stimulation of osteoblast activity. As a future activity we intend to investigate this problem using the effective equations for poro-piezo-elastic equations in this work. Effective elastic and acoustic properties of cortical bone have been investigated in various settings in[14,16,19]. The idea of using the Reuss energy method is mentioned in our book[7]; however, this paper is an extension, correction and completion of this idea.

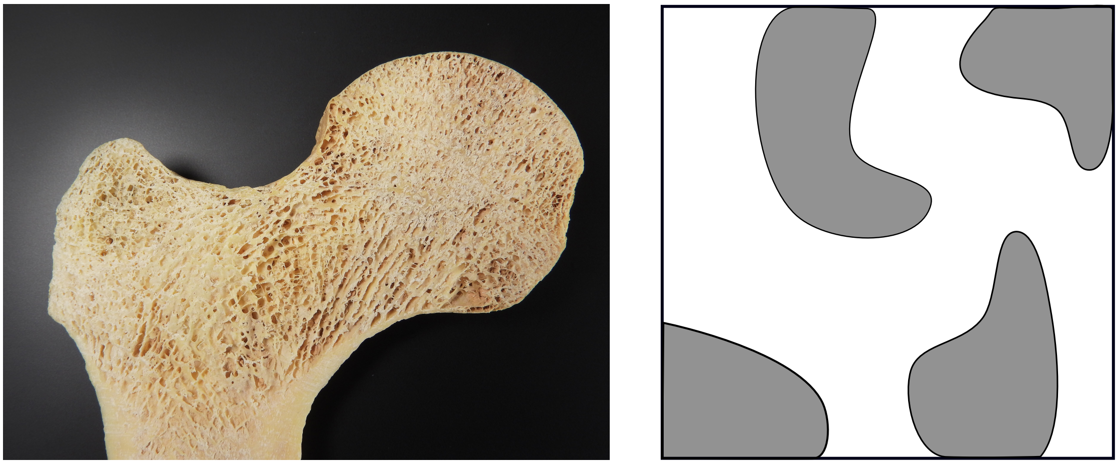

FIGURE 1 Cancellous bone, https://commons.wikimedia.org/wiki/File:Femur_(caput_femoris)_-_bone_structure_detail_(vertical_cut)_2.jpg, (left), and volume element for wet bone, white: matrix, grey: interstitial fluid.

2 | CONSTITUTIVE EQUATIONS FOR WET BONE

2.1 | Reuss bound of the energy

The Reuss bound for the energy may be found by starting with a mixture theory representation for the energy, namely

$$\Psi_{\text{mix}} := \theta^{(s)} \left[\frac{1}{2}\mathbf{e} : \mathbf{C} : \mathbf{e} - \mathbf{e} : \mathbf{\Pi} : \cdot\mathbf{E}^{(s)} - \frac{1}{2}\mathbf{E}^{(s)} \cdot \boldsymbol{\epsilon}^{(s)} \cdot \mathbf{E}^{(s)}\right] + \theta^{(f)} \left[\frac{K^{(f)}}{2}(\zeta - \operatorname{tr}\mathbf{e})^2 - \frac{1}{2}\mathbf{E}^{(f)} \cdot \boldsymbol{\epsilon}^{(f)} \cdot \mathbf{E}^{(f)}\right], \tag{2.1}$$

where $\theta^{(s)}$ and $\theta^{(f)}$ are the support of the solid and fluid phases respectively. $\mathbf{E}^{(s)}$ and $\mathbf{E}^{(f)}$ are the electric fields in the solid and fluid respectively. The dielectric tensors in the solid and fluid respectively are $\boldsymbol{\epsilon}^{(s)}$ and $\boldsymbol{\epsilon}^{(f)}$. Later on we shall consider the isotropic case where these become constants. We use the notation that vectors, second and third order tensors are represented by boldface symbols. Moreover, $\mathbf{e} = \operatorname{sym}\nabla\mathbf{u}$ and $\zeta = -\nabla \cdot \mathbf{w}$, where $\mathbf{w} = \theta\left(\mathbf{u}^{(f)} - \mathbf{u}\right)$, where $\mathbf{u}^{(f)}$ is the absolute displacement of the fluid and $\mathbf{u}$ is the overall displacement of the frame and fluid[17]. The Reuss potential is defined as

$$\Psi_{\text{Reuss}} = \min\left\{\Psi_{\text{mix}} : \theta^{(s)}\mathbf{E}^{(s)} + \theta^{(f)}\mathbf{E}^{(f)} = \mathbf{E}\right\} =$$

$$\frac{\theta^{(s)}}{2}\mathbf{e} : \mathbf{C} : \mathbf{e} + \theta^{(f)}\frac{K^{(f)}}{2}(\zeta - \operatorname{tr}\mathbf{e})^2 - \Psi_{\text{pe}}, \tag{2.2}$$

where the potential energy term Ψ_{pe} is defined as

$$\Psi_{\text{pe}} = \theta^{(s)}\left[\mathbf{e} : \mathbf{\Pi}^T \cdot \mathbf{E}^{(s)} + \mathbf{E}^{(s)} \cdot \mathbf{e}^{(s)} \cdot \mathbf{E}^{(s)}\right] + \theta^{(f)}\frac{1}{2}\mathbf{E}^{(f)} \cdot \mathbf{e}^{f} \cdot \mathbf{E}^{(f)} \tag{2.3}$$

The constitutive equations are given as

$$\mathbf{D} = \mathbf{\Pi}^T : \mathrm{e}(\mathbf{u}) + \boldsymbol{\epsilon}^{(s)} : \mathbf{E}^{(s)} = \boldsymbol{\epsilon}^{(f)} : \mathbf{E}^{(f)}, \tag{2.4}$$

where $\mathbf{D}$ is the electric displacement, which we assume is the same in the solid as fluid. $\boldsymbol{\epsilon}^{(s)}$ and $\boldsymbol{\epsilon}^{(f)}$ are the dielectric, second order tensors in the solid and fluid respectively. Using a mixture theory argument, the electric field may be expressed as

$$\mathbf{E} = \theta^{(s)}\left(\boldsymbol{\epsilon}^{(s)}\right)^{-1} \cdot \left(\mathbf{D} - \mathbf{\Pi}^T : \mathrm{e}(\mathbf{u})\right) + \theta^{(f)}\left(\boldsymbol{\epsilon}^{(f)}\right)^{-1} \cdot \mathbf{D}. \tag{2.5}$$

Then

$$\mathbf{D} = \boldsymbol{\epsilon}_{\text{eff}} \cdot \left[\mathbf{E} + \theta^{(s)}\left(\boldsymbol{\epsilon}^{(s)}\right)^{-1} \cdot (\mathbf{\Pi}^T : \mathrm{e}(\mathbf{u}))\right], \tag{2.6}$$

where back substitution into the expression for ψ_{pe} yields

$$\Psi_{\text{pe}} = \frac{1}{2}\left(\mathbf{E} + \Theta^{(s)}\left(\boldsymbol{\epsilon}^{(s)}\right)^{-1} \cdot (\mathbf{\Pi} : \mathbf{e}(\mathbf{u})\right) : \boldsymbol{\epsilon}_{\text{eff}} : \left(\mathbf{E} + \Theta^{(s)}\left(\boldsymbol{\epsilon}^{(s)}\right)^{-1} \cdot (\mathbf{\Pi} : \mathrm{e}(\mathbf{u})\right)$$

$$-\frac{1}{2}\Theta^{(s)}\left(\boldsymbol{\epsilon}^{(s)}\right)^{-1}(\boldsymbol{\Pi} : \mathrm{e}(\mathbf{u}))\cdot\left(\boldsymbol{\epsilon}_{\mathrm{eff}}\right)^{-1} : \left(\boldsymbol{\epsilon}^{(s)}\right)^{-1}(\boldsymbol{\Pi} : \mathrm{e}(\mathbf{u}))\,. \tag{2.7}$$

It follows that the Reuss-energy has the structure

$$\Psi_{\mathrm{Reuss}} = \frac{1}{2}\mathrm{e}(\mathbf{u}) : \mathbf{C}_{\mathrm{eff}} : \mathbf{e}(\mathbf{u}) - \mathbf{e}(\mathbf{u}) : \boldsymbol{\Pi}_{\mathrm{eff}}\cdot\mathbf{E} - \frac{1}{2}\mathbf{E}\cdot\boldsymbol{\epsilon}_{\mathrm{eff}}\cdot\mathbf{E} + \frac{\theta^{(f)}K^{(f)}}{2}\left(\zeta - \mathrm{tr}\left(\mathbf{e}(\mathbf{u})\right)\right)^2; \tag{2.8}$$

here

$$\mathbf{C}_{\mathrm{eff}} := \theta^{(s)}\left(\mathbf{C} - \boldsymbol{\Pi}^T\cdot\left(\boldsymbol{\epsilon}^{(s)}\right)^{-1}\cdot\boldsymbol{\Pi}\right)$$

$$\boldsymbol{\Pi}_{\mathrm{eff}} := \theta^{(s)}\epsilon_{\mathrm{eff}} : \boldsymbol{\Pi}\cdot\left(\boldsymbol{\epsilon}^{(s)}\right)^{-1}$$

2.2 | Fluid displacement

We denote spatial averages of a function F over $\theta^{(f)}$, $\theta^{(s)}$ and $\theta^{(f)} := \theta^{(f)}\cup\theta^{(s)}$ as $\langle F\rangle^{(s)}$, $\langle F\rangle^{(f)}$ and $\langle F\rangle$ respectively. The fluid displacement may be written as

$$\mathbf{v}(\mathbf{x})^{(f)} = \mathbf{u} + \mathbf{A}(\mathbf{x}) : \mathbf{e}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{w}, \tag{2.9}$$

where

$$\langle\mathbf{B}(\mathbf{x})\rangle^{(f)} = \mathbb{I},\quad \langle\mathbf{A}(\mathbf{x})\rangle^{(f)} = \mathbf{O},\quad \langle\nabla\mathbf{A}\cdot\nabla\mathbf{B}(\mathbf{x})\rangle^{(f)} = \mathbf{O},\quad \left\langle\nabla^{(s)}\mathbf{A}(\mathbf{x})\right\rangle^{(f)} = \mathbb{I}.$$

Here $\mathbf{u}$ is the movement of the composite material, and $\mathbf{w}$ is the relative fluid displacement of the composite displacement. The dissipation is given by

$$\triangle := \frac{\theta^{(f)}}{2\eta}\left\langle\|\mathrm{dev}\nabla^{(s)}\dot{\mathbf{v}}\|^2\right\rangle^{(f)}$$

$$= \frac{\theta^{(f)}}{2\eta}\left[\dot{e}(\mathbf{u}) : \left\langle(\mathrm{dev}\nabla^{(s)}\mathbf{A})^T\cdot\mathrm{dev}\nabla^{(s)}\mathbf{A})\right\rangle^{(f)} : \dot{e}(\mathbf{u}) + \dot{\mathbf{w}}\cdot\left\langle(\mathrm{dev}\nabla^{(s)}\mathbf{B})^T\cdot(\mathrm{dev}\nabla^{(s)}\mathbf{B})\right\rangle^2_{(f)}\cdot\dot{\mathbf{w}}\right]$$

$$= \frac{\kappa_1}{2}\|\mathrm{dev}\,\dot{e}(\mathbf{u})\|^2 + \frac{\kappa_2}{2}\dot{\mathbf{w}}^2. \tag{2.10}$$

2.3 | Kinetic energy

$$T = \frac{1}{2}\theta^{(s)}\rho^{(s)}\dot{\mathbf{u}}^2 + \frac{1}{2}\theta^{(f)}\rho^{(f)}\left\langle|\dot{\mathbf{u}} + \mathbf{A}(\mathbf{x}) : \mathbf{e}(\dot{\mathbf{u}}) + \mathbf{B}(\mathbf{x})\dot{\mathbf{w}}|^2\right\rangle^{(f)}$$

$$= \frac{1}{2}\rho^{(s)}\theta^{(s)}\dot{\mathbf{u}}^2 + \frac{1}{2}\theta^{(f)}\rho^{(f)}\dot{\mathbf{u}}\cdot\langle\mathbf{B}(\mathbf{x})\rangle^{(f)}\cdot\dot{\mathbf{w}} + \frac{1}{2}\theta^{(f)}\rho^{(f)}\dot{\mathbf{w}}\cdot\left\langle\mathbf{B}^T\mathbf{B}\right\rangle^{(f)}\cdot\dot{\mathbf{w}}, +\mathrm{h.o.t.}. \tag{2.11}$$

If we define

$$\rho_b := \theta^{(s)}\rho^{(s)} + \theta^{(f)}\rho^{(f)},$$

then the kinetic energy can be put in the usual form[1,2]

$$T = \frac{1}{2}\rho_b\dot{\mathbf{u}}^2 + \frac{1}{2}\theta^{(f)}\rho^{(f)}\dot{\mathbf{u}}\cdot\dot{\mathbf{w}} + \frac{1}{2}\theta^{(f)}\rho^{(f)}\dot{\mathbf{w}}^2. \tag{2.12}$$

2.4 | Constitutive equations

The state or constitutive equations may be found by differentiating the energy and the dissipation, namely

$$\boldsymbol{\sigma} = \frac{\partial\Psi_{\mathrm{Reuss}}}{\partial\mathbf{e}} + \frac{\partial\triangle}{\partial\dot{\mathbf{e}}}$$

$$= \mathbf{C}^{\mathrm{eff}} : \mathbf{e} - \boldsymbol{\Pi}_{\mathrm{eff}}\cdot\mathbf{E} + \theta^{(f)}K^{(f)}(\mathrm{tr}\mathbf{e} - \zeta) + \kappa_1\mathrm{dev}(\dot{\mathrm{e}}(\mathbf{u})). \tag{2.13}$$

$$\mathbf{D} = -\frac{\partial\Psi_{\mathrm{Reuss}}}{\partial\mathbf{E}} = \boldsymbol{\Pi}^T_{\mathrm{eff}} : \mathbf{e} + \boldsymbol{\epsilon}_{\mathrm{eff}}\cdot\mathbf{E}, \tag{2.14}$$

$$\sigma = \frac{\partial\Psi_{\mathrm{Reuss}}}{\partial\zeta} = \theta^{(f)}K^{(f)}(\zeta - \mathrm{tr}(\mathbf{e}))\,, \tag{2.15}$$

where σ is the fluid stress. The fluid stress is also the fluid pressure, denoted by $\mathfrak{p}^f$. The Lagrange equations of motion then become

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{u}}} = \rho_b \ddot{\mathbf{u}} + \theta^{(f)}\rho^f \ddot{\mathbf{w}} = \nabla \cdot \boldsymbol{\sigma}, \tag{2.16}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{w}}} + \frac{\partial \triangle}{\partial \dot{\mathbf{w}}} = \theta^{(f)}\rho^{(f)}\ddot{\mathbf{u}} + \rho_{22}\ddot{\mathbf{w}} + \kappa_2 \dot{\mathbf{w}} = \nabla s, \tag{2.17}$$

where ρ_{22} is the fluid density.

For simplicity of explanation, we assume that in the future we will restrict our discussion to time harmonic motion. In this instance, the equations of motion become

$$\nabla \cdot \left(\mathbf{C}^{\mathrm{eff}} : \mathbf{e} - \mathbf{\Pi}_{\mathrm{eff}} \cdot \mathbf{E} + p\mathbb{I}\right) + \omega^2 \left(\rho_b \mathbf{u} + \theta^{(f)}\rho^f \mathbf{w}\right) = 0; \tag{2.18}$$

whereas, the pressure equation becomes

$$\nabla\sigma + \omega^2\theta^{(f)}\rho^{(f)}\mathbf{u} + \omega^2\rho_{22}\mathbf{w} - i\omega\kappa_2\mathbf{w} = 0. \tag{2.19}$$

3 | ELECTRODYNAMICS

3.1 | Electrically isotropic solid

If the material is electrically isotropic, it means that the dielectric and permittitivity tensors are scalars, respectively ϵ and μ. We recall that from the Reuss formulation $D^{(s)} = D^{(f)} = D$ and $H^{(s)} = H^{(f)} = H$; hence, in the solid phase we construct the magnetic induction $\mathbf{B}^{(s)}$ as follows

$$\nabla \times \mathbf{H} = \dot{\mathbf{D}} = \epsilon^{(s)}\dot{\mathbf{E}} + \dot{\mathbf{e}} : \mathbf{\Pi}$$

From which we compute, as $\nabla \cdot \mathbf{H} = 0$,

$$\nabla \times \nabla \times \mathbf{H} = - \triangle \mathbf{H} = \epsilon^{(s)}\nabla \times \dot{\mathbf{E}}^{(s)} + \nabla \times (\dot{e} : \mathbf{\Pi})$$

and

$$\triangle \mathbf{B}^{(s)} = \epsilon^{(s)}\mu^{(s)}\ddot{\mathbf{B}}^{(s)} - \mu^{(s)}\nabla \times (\dot{\mathbf{e}} : \mathbf{\Pi}). \tag{3.1}$$

Next we calculate the equation of the electric force $\mathbf{E}^{(s)}$ starting with $\nabla \times \mathbf{E}^{(s)} = -\dot{\mathbf{B}}^{(s)}$ as

$$\nabla \times \nabla \times \mathbf{E}^{(s)} = \nabla\nabla \cdot \mathbf{E}^{(s)} - \triangle \mathbf{E}^{(s)} = -\nabla \times \dot{\mathbf{B}}^{(s)} = -\mu^{(s)}\nabla \times \dot{\mathbf{H}}^{(s)} = -\mu^{(s)}\ddot{\mathbf{D}}.$$

If we assume that there is no residual charge in the solid state, then

$$\nabla \cdot \mathbf{D} = 0 = \epsilon\nabla \cdot \mathbf{E}^{(s)} + \nabla \cdot (\mathbf{e}\,\mathbf{\Pi});$$

hence,

$$\nabla\nabla \cdot \mathbf{E}^{(s)} = -\frac{1}{\epsilon^{(s)}}\nabla\nabla \cdot (\mathbf{e} : \mathbf{\Pi}).$$

We arrive at

$$\triangle \mathbf{E}^{(s)} = \epsilon^{(s)}\mu^{(s)}\ddot{\mathbf{E}}^{(s)} - \frac{1}{\epsilon^{(s)}}\nabla\nabla \cdot (\mathbf{e} : \mathbf{\Pi}) + \mu^{(s)}(\ddot{\mathbf{e}} : \mathbf{\Pi}). \tag{3.2}$$

3.2 | Electro-magnetism in the fluid

The fluid is assumed isotropic, but can maintain a current $\mathbf{J}$; hence,

$$\triangle \mathbf{H}^{(f)} = -\epsilon^{(f)}\nabla \times \dot{\mathbf{E}}^{(f)} - \nabla \times \mathbf{J}^{(f)}, \quad \text{where } \mathbf{J}^{(f)} = \sigma^{(f)}\mathbf{E}^{(f)},$$

where $\sigma^{(f)}$ is the conductivity of the fluid.

Hence,

$$\triangle \mathbf{B}^{(f)} = \epsilon^{(f)}\mu^{(f)}\ddot{\mathbf{B}}^{(f)} + \mu^{(f)}\sigma^{(f)}\dot{\mathbf{B}}^{(f)}. \tag{3.3}$$

Likewise, from

$$\triangle \mathbf{E}^{(f)} = \nabla \times \dot{\mathbf{B}}^{(f)} = \mu^{(f)}\dot{\mathbf{H}}^{(f)} = \mu^{(f)}\ddot{\mathbf{D}} + \mu^{(f)}\sigma^{(f)}\dot{\mathbf{E}}^{(f)},$$

we arrive at

$$\triangle \mathbf{E}^{(f)} = \epsilon^{(f)}\mu^{(f)}\ddot{\mathbf{E}} + \mu^{(f)}\sigma^{(f)}\dot{\mathbf{E}}^{(f)}. \tag{3.4}$$

3.3 | Effective electro-magnetic equations

In this section we will assume that the permittivity and permeability are isotropic, i.e. scalars and not tensors. Summarising, in the solid we have the effective constitutive equations for the piezo-electric, poro-elastic material

$$\mathbf{B} = \theta^{(s)}\mathbf{B}^{(s)} + \theta^{(f)}\mathbf{B}^{(f)} = \left(\theta^{(s)}\mu^{(s)} + \theta^{(f)}\mu^{(f)}\right)\mathbf{H},$$
$$\mu_{\text{eff}} = \theta^{(s)}\mu^{(s)} + \theta^{(f)}\mu^{(f)},$$

$$\mathbf{D} = \epsilon_{\text{eff}} \cdot \mathbf{E} + \mathbf{\Pi}^T_{\text{eff}} : \mathbf{e}, \tag{3.5}$$
$$\nabla \cdot \mathbf{D} = 0 = \nabla \cdot \left(\epsilon_{\text{eff}} \cdot \mathbf{E} + \mathbf{\Pi}^T_{\text{eff}} : \mathbf{e}\right).$$

In addition, we have the electromagnetic constitutive equations

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \quad \nabla \times \mathbf{H} = \dot{\mathbf{D}} + \mathbf{J}^{(f)}, \quad \mathbf{B} = \mu^{(f)}\mathbf{H}, \quad \mathbf{J} = \sigma^{(f)}\mathbf{E}.$$

This leads to the effective equations for the electromagnetic field, namely

$$\Delta\mathbf{E} - \epsilon_{\text{eff}}\mu_{\text{eff}}\,\partial_t^2\mathbf{E} = -\frac{\theta^{(s)}}{\epsilon_{\text{eff}}}\nabla\nabla \cdot \left(\mathbf{e} : \mathbf{\Pi}^T_{\text{eff}}\right) + \theta^{(s)}\mu_{\text{eff}}\left(\partial_t^2\mathbf{e} : \mathbf{\Pi}^T\right), \tag{3.6}$$

and

$$\Delta\mathbf{B} - \epsilon_{\text{eff}}\mu_{\text{eff}}\partial_t^2\mathbf{B} = -\theta^{(s)}\mu^{(s)}\nabla \times \left(\partial_t\mathbf{e} : \mathbf{\Pi}^T_{\text{eff}}\right) + \theta^{(f)}\mu^{(f)}\sigma^{(f)}\partial_t\mathbf{B}. \tag{3.7}$$

in time harmonic form we consider in this paper these become

$$\Delta\mathbf{E} + \omega^2\epsilon_{\text{eff}}\mu_{\text{eff}}\mathbf{E} = -\frac{\theta^{(s)}}{\epsilon_{\text{eff}}}\nabla\nabla \cdot \left(\mathbf{e} : \mathbf{\Pi}^T_{\text{eff}}\right) - \theta^{(s)}\omega^2\mu_{\text{eff}}\left(\mathbf{e} : \mathbf{\Pi}^T_{\text{eff}}\right),$$

and

$$\Delta\mathbf{B} + \omega^2\epsilon_{\text{eff}}\mu_{\text{eff}}\mathbf{B} = -i\omega\theta^{(s)}\mu^{(s)}\nabla \times \left(\mathbf{e} : \mathbf{\Pi}^T_{\text{eff}}\right) + i\omega\theta^{(f)}\mu^{(f)}\sigma^{(f)}\mathbf{B}.$$

4 | THE TRANSMISSION PROBLEM

In this section we will present an example from bone rigidity and healing of a fracture. Usually such problems are modeled by placing the bone segment ($\theta^s \cup \theta^f$) in a water tank. The bone is assumed to consist of a composite material as studied earlier in this paper. In the present case we assume that the tank is replaced by a possibly unbounded region. More precisely, we assume that the bone specimen occupies a region denoted by $\theta^b = \theta^s \cup \theta^f$ and the exterior water region by θ^w which may be $\mathbb{R}^3 \setminus \theta^b$. In θ^w the governing equation can be reduced to the two-dimensional non-homogeneous Helmholtz equation for fluid pressure p subject to the linearized Navier-Stokes equation for compressible fluid flow for the pressure $\mathfrak{p}^{(w)}$ and fluid displacement $\mathbf{U}^w := (U_1^w, U_2^w)$. That is, we require that p and $\mathbf{U}^w$ satisfy the equations

$$-(\nabla^2\mathfrak{p}^{(w)} + k_0^2\mathfrak{p}^{(w)}) = f \quad \text{in} \quad \theta^w, \tag{4.1}$$
$$\nabla\mathfrak{p}^{(w)} - \rho^w\omega^2\mathbf{U}^w = \mathbf{f} \quad \text{in} \quad \theta^w, \tag{4.2}$$

where $\mathbf{f}$ is a given function with compact support, and $f = -div\ \mathbf{f}$. As in the simple poro-elastic case there are too many unknowns, i.e. pressure, fluid displacement and frame displacement; hence, we remove the fluid displacement $\mathbf{w}$ from (2.18) by using the substitution

$$\mathbf{w} = \frac{\nabla\mathfrak{p}^{(w)} + \omega^2\theta^{(f)}\rho^{(f)}\mathbf{u}}{-\omega^2\rho_{22} + i\omega\kappa_2} \tag{4.3}$$

and take the divergence to obtain

$$\nabla \cdot \left[\mathbb{C}^{\text{eff}} : \mathbf{e} - \mathbf{\Pi}\mathbf{E} + p\mathbb{I}\right] + \omega^2\left(\left[\rho_b + \frac{(\theta^{(f)}\rho^{(f)}\omega)^2}{-\omega^2\rho_{22} + i\omega\kappa_2}\right]\mathbf{u} - \frac{\theta^{(f)}\rho^{(f)}}{\omega^2\rho_{22} - i\omega\kappa_2}\nabla s\right) = 0. \tag{4.4}$$

Next we substitute expression (4.3) into the pressure equation (2.19) to obtain

$$\nabla\mathfrak{p}^{(w)} - \frac{\omega^2\rho_{22} - i\omega\kappa_2}{\theta^{(f)}K^{(f)}}p^f + \left(\omega^2\theta^{(f)}\rho^{(f)} - \rho_{22}\omega^2 + i\omega\kappa_2\right)e = 0, \tag{4.5}$$

where $e := \nabla\mathbf{u}$. In the region $\theta^{(w)}$ the electro-magnetic fields satisfy the equations

$$\Delta\mathbf{E}^{(w)} + \omega^2\epsilon_{(w)}\mu_{(w)}\mathbf{E}^{(w)} = 0 \tag{4.6}$$

and

$$\Delta \mathbf{B}^{(w)} + \omega^2 \epsilon_{(w)} \mu_{(w)} \mathbf{B}^{(w)} = 0. \tag{4.7}$$

Definition 1 (The non-homogeneous transition problem (TP_f)). The problem consists of finding the quintuplet $(\mathbf{u}, \mathbf{E}, \mathbf{B}, p^{(f)}, p^{(w)})$ such that

$$(E_s) \qquad \nabla \cdot \left[\mathbb{C}^{\mathrm{eff}} : \mathbf{e} - \mathbf{\Pi}\mathbf{E} + \mathfrak{p}^{(w)}\mathbb{I}\right] + \omega^2 \left(\left[\rho_b + \frac{\theta^{(f)}\rho^{(f)}\rho_b\omega^2}{-\omega^2\rho_{22} + i\omega\kappa_2}\right]\mathbf{u} + \frac{\theta^{(f)}\rho^{(f)}}{\omega^2\rho_{22} - i\omega\kappa_2}\nabla s \right) = 0, \quad \mathbf{x} \in \theta^s,$$

$$(E_f) \qquad \nabla^2 s - \frac{\omega^2\rho_{22} - i\omega\kappa_2}{\theta^{(f)}K^{(f)}} s + \left(\omega^2[\theta^{(f)}\rho^{f)} - \rho_{22}] + i\omega\kappa_2\right) e = 0, \quad \mathbf{x} \in \theta^b,$$

and the electromagnetic equations.

$$(E_E) \qquad \Delta \mathbf{E} + \omega^2 \epsilon_{\mathrm{eff}} \mu_{\mathrm{eff}} \mathbf{E} = -\frac{1}{\epsilon_{\mathrm{eff}}} \nabla\nabla \cdot \left(\mathbf{e} : \mathbf{\Pi}^T\right) - \omega^2 \mu_{\mathrm{eff}} \left(\mathbf{e} : \mathbf{\Pi}^T\right), \quad \text{in} \quad \mathbf{x} \in \theta^b,$$

$$(E_B) \qquad \Delta \mathbf{B} + \omega^2 \epsilon_{\mathrm{eff}} \mu_{\mathrm{eff}} \mathbf{B} = i\omega\theta^{(s)}\mu^{(s)} \nabla \times \left(\mathbf{e} : \mathbf{\Pi}^T\right) + i\omega\theta^{(f)}\mu^{(f)}\mathfrak{p}^{(w)}\mathbf{B}, \quad \text{in} \quad \mathbf{x} \in \theta^b$$

and with the tank-water equations

$$(E_p), \qquad -\left(\triangle \mathfrak{p}^{(w)} + k_0^2 \mathfrak{p}^{(w)}\right) = f \quad \text{in} \quad \theta^w, \quad f := -div\,\mathbf{f}$$

where f has compact support in θ^w. Moreover, the system obeys the following transmission condition

$$(B_1) \qquad \left[\mathbb{C}^{\mathrm{eff}} : \mathbf{e}(\mathbf{u}), -\mathbf{\Pi}_{\mathrm{eff}} \cdot \mathbf{E} \cdot \mathbf{u} + \mathfrak{p}^{(f)}\mathbb{I}\right] \cdot \mathbf{n} = -\mathfrak{p}^{(f)}\mathbf{n}$$

with vanishing of the tangent frame stress $\sigma_{12} = \sigma_{21} = 0$,

$$(B_2) \qquad \rho^{(w)}\omega^2 \left[1 - \theta\left(\frac{\theta^{(f)}\rho^{(f)} + \omega^2\rho_{22} - i\omega\kappa_2}{-\omega^2\rho_{22} + i\omega\kappa_2}\right)\right]\mathbf{u}\cdot\mathbf{n} + \frac{\rho^{(w)}\omega^2\theta}{-\omega^2\rho_{22} + i\omega\kappa_2}\frac{\partial \mathfrak{p}^{(f)}}{\partial n} = \frac{\partial p^{(w)}}{\partial n} - \mathbf{n}\cdot\mathbf{f} \quad \text{on } \Gamma$$

$$(B_3). \qquad \mathfrak{p}^{(w)} = -\theta^{()}\mathfrak{p}^{(f)} \quad \text{on} \quad \Gamma.$$

We now list the equations which determine the transition of electromagnetic fields from the region θ_b to the region $\theta_w := \mathbb{R}^3/\theta_b$ to[23]:

$$(B_4) \qquad \mathbf{n} \times \left(\mathbf{E}^{\mathrm{eff}} - \mathbf{E}^{(w)}\right) = \mathbf{0},$$

$$(B_5) \qquad \mathbf{n} \times \left(\mathbf{H}^{\mathrm{eff}} - \mathbf{H}^{(w)}\right) = \mathbf{J},$$

$$(B_6) \qquad \mathbf{n} \cdot \left(\mathbf{B}^{\mathrm{eff}} - \mathbf{B}^{(w)}\right) = \mathbf{0},$$

$$(B_7) \qquad \mathbf{n} \cdot \left(\mathbf{D}^{\mathrm{eff}} - \mathbf{D}^{(w)}\right) = q_S^{\partial\theta_b}.$$

In addition, when $\mathbb{R}^3/\theta^b$ is unbounded, we assume that the Sommerfeld radiation condition holds for $p^{(w)}$ when $\mathbf{x} \in \mathbb{R}^3/\theta^b$.

Definition 2 (Traction free problem[5] $\mathbf{T}P_0$). The problem for $(\mathbf{u}, p)$ in θ^b consists of the partial differential equations (E_b), (E_p), (E_E), (E_B) together with the homogeneous boundary conditions

$$(B_1) \qquad \left[\mathbb{C}^{\mathrm{eff}} : \mathbf{e}(\mathbf{u}), -\mathbf{\Pi}_{\mathrm{eff}} \cdot \mathbf{E} \cdot \mathbf{u} + \mathfrak{p}^{(f)}\mathbb{I}\right] \cdot \mathbf{n} = 0,$$

$$(B_2) \qquad \rho^{(w)}\omega^2 \left[1 - \theta\left(\frac{\theta^{(f)}\rho^{(f)} + \omega^2\rho_{22} - i\omega\kappa_2}{-\omega^2\rho_{22} + i\omega\kappa_2}\right)\right]\mathbf{u}\cdot\mathbf{n} + \frac{\rho^{(w)}\omega^2\theta}{-\omega^2\rho_{22} + i\omega\kappa_2}\frac{\partial \mathfrak{p}^{(f)}}{\partial n} = 0 \quad \text{on } \Gamma,$$

$$(B_3)_0 \qquad \mathfrak{p}^{(f)} = 0 \quad \text{on} \quad \Gamma.$$

$$(B_4) \qquad \mathbf{n} \times \left(\mathbf{E}^{\mathrm{eff}} - \mathbf{E}^{(w)}\right) = \mathbf{0},$$

$$(B_5) \qquad \mathbf{n} \times \left(\mathbf{H}^{\mathrm{eff}} - \mathbf{H}^{(w)}\right) = \mathbf{0},$$

$$(B_6) \qquad \mathbf{n} \cdot \left(\mathbf{B}^{\mathrm{eff}} - \mathbf{B}^{(w)}\right) = \mathbf{0},$$

$$(B_7) \qquad \mathbf{n}\cdot\left(\mathbf{D}^{\text{eff}} - \mathbf{D}^{(w)}\right) = \mathbf{0}.$$

are called the traction free problem for $(\mathbf{u}, \mathfrak{p}^{(f)})$, and the corresponding non-trivial solutions are referred to as the traction free solutions.

For the variational formulation, we now reduce the partial differential equation (E_p) for p to a boundary integral equation for p on Γ. We use the indirect approach for the reduction of partial differential equation by seeking a solution $p^{(w)}$ in the form of a simple-layer potential

$$\mathfrak{p}^{(w)} = -\mathbf{S}\phi + \mathfrak{p}^{\text{part}} \quad \text{in} \quad \theta^{w}, \tag{4.8}$$

where ϕ is an unknown density function and $\mathbf{S}\phi$ is the simple layer potential We note that a particular solution, $p^{(\text{part})}$, of the Helmholtz equation for the tank water pressure is given by

$$\mathfrak{p}^{(\text{part})}(\mathbf{x}) := \frac{1}{2\pi}\int_{\Gamma} \frac{e^{k_0|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\phi(y)ds_y, \quad x\in\theta^{w}.$$

Hence, if $p|_\Gamma$ is known, applying the trace operator γ_0 to (4.8), we then obtain a boundary integral equation for the known density ϕ

$$\mathfrak{p}^{(f)}(\mathbf{x})|_\Gamma = -\mathbf{V}\phi + \gamma_0 p^{\text{part}}, \tag{4.9}$$

where $\mathbf{V} = \gamma_0\mathbf{S}$ is the simple layer boundary integral operator. Then from the transmission condition (B_3), we obtain the boundary integral equation

$$(E_{pb}) \qquad \mathbf{V}\phi - \frac{1}{\theta f}\mathfrak{p}^{(f)} = \gamma_0 p^{(w)}.$$

Definition 3 (Nonlocal boundary value problem)**.** The transmission problem $\mathbf{TP}_f$ is termed a nonlocal boundary value problem for the quintet $(\mathbf{u}, \mathfrak{p}^{(f)}, \phi, \mathbf{E}, \mathbf{B})$ if the quintet satisfies equations[5] (E_b), (E_p), (E_E), (E_B) and the boundary integral equation (E_{3p}) together with the electro-magnetic transmission conditions B_4, B_5, B_6, B_7 and

$$\left\{\left[\mathbb{C}^{\text{eff}} : \mathbf{e}(\mathbf{u}) + \mathbf{\Pi}^{\text{eff}}\cdot\mathbf{E}\cdot + \left(1 - \frac{\theta^{(f)}\rho^{(f)}\omega}{\rho_{22}\omega^2 - i\kappa_2}\right)\mathfrak{p}^{(f)}\mathbb{I}\right]\cdot\mathbf{v}\right\}\cdot\mathbf{n} = \mathfrak{p}^{(w)}\mathbf{n}$$

with

$$\mathfrak{p}^{(w)} = -\mathbf{V}\phi + \gamma_0\mathfrak{p}^{(\text{part})}.$$

Here again $\sigma_{12} = \sigma_{21} = 0$, where $\sigma(\mathbf{u})$ is the stress tensor. Additionally

$$\rho^{(w)}\omega^2\left[1 - \theta^{(f)}\left(\frac{\theta^{(f)}\rho^{(f)}\kappa_2 + \omega^2\rho_{22} + i\omega\kappa_2}{-\omega^2\rho_{22} + i\omega\kappa_2}\right)\right]\mathbf{u}\cdot\mathbf{n} + \frac{\rho^{(w)}\omega^2\theta^{(f)}}{-\omega^2\rho_{22} + i\omega\kappa_2}\frac{\partial\mathfrak{p}^{(f)}}{\partial n} = \frac{\partial p^{(w)}}{\partial n} - \mathbf{n}\cdot\mathbf{f},$$

with

$$\frac{\partial\mathfrak{p}^{(w)}}{\partial n} = \frac{1}{2}\left(\phi - \mathbf{K}'\phi\right) + \frac{\partial\mathfrak{p}^{(\text{part})}}{\partial n}$$

has to be satisfied.

5 | THE VARIATIONAL FORMULATION

In this section, we formulate the variational formulation of the nonlocal boundary value problem[6,18]. As usual, multiplying (E_s) by the conjugate of the test function $\mathbf{v}$

$$\int_{\theta^b}\left\{\nabla\cdot\left[\mathbb{C}^{\text{eff}} : \mathbf{e}(\mathbf{u}) - \mathbf{\Pi}_{\text{eff}}\cdot\mathbf{E} + s\mathbb{I}\right] + \omega^2\left(\rho_b\mathbf{u} - \theta^{(f)}\rho^{(f)}\left[\nabla p^{(f)} + \frac{\omega^2\theta^{(f)}\rho^{(f)}}{\omega^2\rho_{22} + i\omega\kappa_2}\right]\right)\cdot\overline{\mathbf{v}}\right\}d\mathbf{x} = 0$$

and then integrating by parts, we obtain from the above

$$\int_{\theta^b}\left\{-\mathbb{C}^{\text{eff}} : \mathrm{e}(\overline{\mathbf{v}}) + \mathbf{\Pi}^{\text{eff}} : \mathrm{e}(\overline{\mathbf{v}})\cdot\mathbf{E} - \mathfrak{p}^{(f)}\nabla\cdot\overline{\mathbf{v}} + \omega^2\left(\rho_b\mathbf{u} + \theta^{(f)}\rho^{(f)}\left[\frac{\nabla p + (\omega\theta^{(f)}\rho^{(f)})^2\mathbf{u}}{-\omega^2 + i\omega\kappa_2}\right]\right)\cdot\overline{\mathbf{v}}\,d\mathbf{x}\right.$$

$$+\int_{\Gamma^b}\left(\left[\mathbb{C}^{\text{eff}} : \mathbf{e}(\mathbf{u}) - \mathbf{\Pi}_{\text{eff}}\cdot\mathbf{E}\mathbf{u} + s\mathbb{I}\right]\cdot\overline{\mathbf{v}}\right)\cdot\mathbf{n}\,ds_\Gamma = 0. \tag{5.1}$$

We define the sesquilinear bilinear form

$$a(\mathbf{u},\mathbf{v}) \;:=\; \int\limits_{\theta^b} \left\{ \mathbb{C}^{\text{eff}} \,:\, \mathbf{e}(\mathbf{u}) \,:\, \mathbf{e}(\overline{\mathbf{v}}) \right\} d\mathbf{x}, \tag{5.2}$$

and by rewriting the boundary term in (5.1) we see that

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) \,+\, &\int\limits_{\theta^b} \left[\boldsymbol{\Pi}_{\text{eff}} \cdot \mathbf{E} \cdot \overline{\mathbf{v}} \,+\, \frac{\theta^{(f)}\rho^{(f)}\mathfrak{p}^{(f)}\,\nabla\,\overline{\mathbf{v}}}{\omega^2\rho_{22} - i\omega\kappa_2} \right] d\mathbf{x} \,+\, \int\limits_{\theta^b} \omega^2 \left[\rho_b - \frac{\theta^{(f)}\rho^{f)}}{\omega^2\rho_{22} - i\omega\kappa_2} \right] \mathbf{u}\cdot\overline{\mathbf{v}}\, d\mathbf{x} \\ &+ \omega^2 \int\limits_{\theta^b} \theta^{(f)} \frac{\rho^{(f)}\mathfrak{p}^{(f)}\nabla\cdot\mathbf{v}}{\omega^2\rho_{22} - i\omega\kappa_2}\, d\mathbf{x} \,=\, \int\limits_{\Gamma^b} \left[\mathbb{C}^{\text{eff}} \,:\, \mathbf{e}(\mathbf{u}) \,-\, \boldsymbol{\Pi}_{\text{eff}}\cdot\mathbf{E} + \,\mathfrak{p}^{(f)}\mathbb{I} \right] \overline{\mathbf{v}}\cdot\mathbf{n}\, ds \end{aligned} \tag{5.3}$$

We repeat this process for the $\mathfrak{p}^{(f)}$ equation by multiplying equation (E_p) by the test function π and integrating by parts yields

$$\int\limits_{\theta_b} \nabla s \cdot \nabla\overline{\pi} d\mathbf{x} \,=\, \int\limits_{\Gamma_b} \nabla\mathfrak{p}^{(f)}\cdot\mathbf{n}\overline{\pi} d s_\Gamma \,-\, \int\limits_{\theta_b} \left(\omega^2 \left[\theta^f\rho^f - \rho_{22} \right] + i\omega\kappa_2 \right) e\,\overline{\pi} d\mathbf{x} \,-\, \int\limits_{\theta_b} \left(\frac{\omega^2\rho_{22} - i\omega\kappa_2\omega}{\theta^f K^f} \right) \mathfrak{p}^{(f)}\overline{\pi}\, d\mathbf{x}\,.$$

By introducing the sesquilinear form

$$b(s,\pi) \,=\, \int\limits_{\theta^b} \nabla\, s \cdot \nabla\overline{\pi}\, d\mathbf{x} \tag{5.4}$$

using the condition (B_{2b}) and replacing θ_b by its measure, the variational form of the $\mathfrak{p}^{(f)}$ equation now may be written as

$$b(s,\pi) \,+\, \omega^2 \int\limits_{\theta^b} s\, e(\mathbf{u})\overline{\pi}\; d\mathbf{x} - \frac{\omega^2\rho_{22} + i\omega\kappa_2}{\omega^2\rho^{(w)}} \left[\left\langle \mathbf{n}\cdot\mathbf{f} \,-\, \frac{\partial\mathfrak{p}^{\text{part}}}{\partial n}, \overline{\pi} \right\rangle - \left\langle \frac{1}{2}\phi - \mathbf{K}', \overline{\pi} \right\rangle \right] \quad \forall\,\pi \in H^1(\theta_b) \tag{5.5}$$

Finally, we multiply the boundary integral equation (E_{pb}) by the test function ψ, and integrate it. This yields the variational equation for (E_{pb})

$$\frac{p_{22}}{2\rho^w\omega^2} \left\langle \mathbf{V}\phi, \overline{\psi} \right\rangle \,-\, \frac{p_{22}}{2\rho^w\omega^2\theta} \left\langle \mathfrak{p}^{(f)}, \overline{\psi} \right\rangle \,=\, \frac{p_{22}}{2\rho^w\omega^2} \left\langle \gamma_0 p_p, \overline{\psi} \right\rangle, \quad \forall\,\psi \in H^{-1/2}(\Gamma). \tag{5.6}$$

Recalling the time harmonic versions of the electromagnet equations are for the electric field

$$\Delta\mathbf{E} \,+\, \omega^2\epsilon_{\text{eff}}\mu_{\text{eff}}\mathbf{E} \,+\, \frac{1}{\epsilon_{\text{eff}}}\nabla\nabla\cdot\mathbf{e}(\mathbf{u}) \,:\, \boldsymbol{\Pi}^T \,+\, \omega^2\mu_{\text{eff}} e(\mathbf{u}) \,:\, \boldsymbol{\Pi}^T \,=\, 0;$$

likewise, we have

$$\Delta\mathbf{B} \,+\, \omega^2\epsilon_{\text{eff}}\mu_{\text{eff}}\mathbf{B} \,+\, i\omega\theta^{(s)}\mu^{(s)}\nabla\times\left(\mathbf{e}(\mathbf{u}) \,:\, \boldsymbol{\Pi}^T\right) + i\omega\theta^{(f)}\mu^{(f)}\sigma^{(f)}\mathbf{B} \,=\, 0.$$

As previously done we identify two sesquilinear bilinear forms:

$$c(\mathbf{E},\mathbf{F}) \;:=\; \int\limits_{\theta_b} \left[\nabla\mathbf{E} \,:\, \nabla\mathbf{F} \,-\, \omega^2\epsilon_{\text{eff}}\mu_{\text{eff}}\mathbf{E}\cdot\mathbf{F} \right] d\mathbf{x}$$

and

$$d(\mathbf{B},\mathbf{b}) \;:=\; \int\limits_{\theta_b} \left[\nabla\mathbf{B} \,:\, \nabla\mathbf{b} \,-\, \omega^2\epsilon_{\text{eff}}\mu_{\text{eff}}\mathbf{E}\cdot\mathbf{F} \right] d\mathbf{x}.$$

The variational forms of the electro-magnetic equations become

$$c(\mathbf{E},\mathbf{F}) \,+\, \int\limits_{\theta_b} \frac{1}{\epsilon_{\text{eff}}}\nabla\cdot\mathbf{e}(\mathbf{u})\boldsymbol{\Pi}^T\nabla\cdot\overline{\mathbf{F}}\, d\mathbf{x} \,+\, \int\limits_{\theta_b} \mu_{\text{eff}}\omega^2\mathbf{e}(\mathbf{u}) \,:\, \boldsymbol{\Pi}^T\cdot\overline{\mathbf{F}}\, d\mathbf{x} \,-\, \int\limits_{\Gamma_b} \left[(\nabla\mathbf{E}\cdot\mathbf{F})\cdot\mathbf{n} \,+\, \frac{1}{\epsilon_{\text{eff}}}\nabla\cdot\mathbf{e}(\mathbf{u})\boldsymbol{\Pi}^T\overline{\mathbf{F}} \right] ds \,=\, 0 \tag{5.7}$$

$$d(\mathbf{B},\mathbf{b}) \,+\, i\omega\theta \int\limits_{\theta_b} \mu^{(f)}\sigma^{(f)}\mathbf{B}\cdot\overline{\mathbf{b}}\, d\mathbf{x} \,-\, \int\limits_{\theta_b} (\mathbf{e}(\mathbf{u}) \,:\, \boldsymbol{\Pi}^T)\cdot(\nabla\times\mathbf{b})\, d\mathbf{x} \,+\, i\omega\mu^{(f)} \int\limits_{\Gamma_b} \left(\mathbf{e} \,:\, \boldsymbol{\Pi}^T \times \mathbf{b} \right) d s_\Gamma \,=\, 0 \tag{5.8}$$

Collecting (5.8), (5.7), and (5.8), we have the variational formulation for the nonlocal boundary value problem:

Definition 4 (Variational formulation). Given $\mathbf{f}$, find the quintuplet $(\mathbf{u},\mathbf{E},\mathbf{B},\mathfrak{p}^{(f)},\phi) \in \left(H^1(\theta^b)\right)^3 \times \left(H^1(\theta^b)\right)^3 \times \left(H^1(\theta^b)\right)^3 \times H^1(\theta^b) \times H^{-1/2}(\Gamma)$ such that

$$\mathcal{A}\Big((\mathbf{u},\mathbf{E},\mathbf{B},\mathfrak{p}^{(f)},\phi), (\mathbf{v},\mathbf{F},\mathbf{b},\pi,\psi) \Big) = \boldsymbol{\ell}_{\boldsymbol{f}}\;(\mathbf{v},\mathbf{F},\mathbf{b},\pi,\psi) \tag{5.9}$$

for all $(\mathbf{v}, \mathbf{F}, \mathbf{b}, \pi, \psi) \in \left((H^1(\theta^b))^3 \times (H^1(\theta^b))^3 \times (H^1(\theta^b))^3 \times H^1(\theta^b) \times (H^{-1/2}(\Gamma)^2)\right)$ where $\mathcal{A}$ and ℓ_f are respectively the sesqulinear form and linear functional defined by

$$\mathcal{A}\Big(\mathbf{u}, \mathbf{E}, \mathbf{B}, \mathfrak{p}^{(f)}, \phi), (\mathbf{v}, \pi, \psi)\Big) := a(\mathbf{u}, \mathbf{v}) + b(\mathfrak{p}^{(f)}, \pi) + c(\mathbf{E}, \mathbf{F}) + d(\mathbf{B}, \mathbf{b})$$

$$+ \int_{\theta^b} \frac{\theta^{(f)}\rho^{(f)}}{\nabla \overline{\mathbf{v}}} \omega^2 \rho_{22} - i\omega\kappa_2 d\mathbf{x} + \int_{\theta^b} \omega^2 \left[\rho_b - \frac{\theta^{(f)}\rho^{(f)}}{\omega^2\rho_{22} - i\omega\kappa_2}\right] \mathbf{u} \cdot \overline{\mathbf{v}}\, d\mathbf{x} + \omega^2 \int_{\theta^b} \rho^{(f)} e(\mathbf{u}) \overline{\pi}\; d\mathbf{x} + \int_{\theta_b} \frac{1}{\epsilon_{\text{eff}}} \nabla \cdot \mathbf{e}(\mathbf{u}) \mathbf{\Pi}^T \nabla \cdot \overline{\mathbf{F}}\, d\mathbf{x}$$

$$+ \int_{\theta_b} \mu_{\text{eff}} \omega^2 \mathbf{e}(\mathbf{u}) : \mathbf{\Pi}^T \cdot \overline{\mathbf{F}}\, d\mathbf{x} - \int_{\Gamma_b} \left[(\nabla \mathbf{E} \cdot \mathbf{F}) \cdot \mathbf{n} + \frac{1}{\epsilon_{\text{eff}}} \nabla \cdot \mathbf{e}(\mathbf{u}) \mathbf{\Pi}^T \overline{\mathbf{F}}\right] ds_\Gamma + \frac{\theta}{1-\theta} \frac{\omega^2\rho_{22} + i\omega\kappa_2}{\omega^2\rho^{(w)}} \left\langle \frac{1}{2}\phi - \mathbf{K}'\phi, \overline{\pi} \right\rangle$$

$$+ i\omega \int_{\theta_b} \mu^{(f)}\sigma^{(f)} \mathbf{B} \cdot \overline{\mathbf{b}}\, d\mathbf{x} - \int_{\theta_b} (\mathbf{e}(\mathbf{u}) : \mathbf{\Pi}^T) \cdot (\nabla \times \mathbf{b})\, d\mathbf{x} + \int_{\Gamma^b} \left[\mathbb{C}^{\text{eff}} : \mathbf{e}(\mathbf{u}) - \mathbf{\Pi}^{\text{eff}} \cdot \mathbf{E} - \phi \mathbb{I}\right] \cdot \overline{\mathbf{v}} \cdot \mathbf{n}\, ds_\Gamma. \tag{5.10}$$

$$\ell(\mathbf{v}, \pi, \psi) := -\frac{\omega^2\rho_{22} + i\omega\kappa_2}{\omega^2\rho^{(w)}} \left\langle \mathbf{n} \cdot \mathbf{f} - \frac{\partial \mathfrak{p}^{\text{part}}}{\partial n} \pi \right\rangle - i\omega\mu^{(f)} \int_{\Gamma_b} \left(\mathbf{e} : \mathbf{\Pi}^T \times \mathbf{b}\right) ds_\Gamma + \int_{\Gamma_b} \gamma_0 \mathfrak{p}^{\text{part}} \overline{\mathbf{v}} \cdot \mathbf{n} \tag{5.11}$$

6 | EXISTENCE AND UNIQUENESS

From the definition of the sesquilinear form $\mathcal{A}$ in (6.1), it is clear that $\mathcal{A}$ satisfies a Gårding inequality. Substituting $(\mathbf{u}, p, \mathbf{E}, \mathbf{B}, \phi)$ for $(\mathbf{v}, \mathbf{F}, \mathbf{b}, \pi, \psi)$ we see that

$$\mathcal{A}\Big((\mathbf{u}, \mathfrak{p}^{(f)}, \mathbf{E}, \mathbf{B}, \phi), ((\mathbf{u}, \mathfrak{p}^{(f)}, \mathbf{E}, \mathbf{B}, \phi))\Big) := a(\mathbf{u}, \mathbf{u}) + b(\mathfrak{p}^{(f)}, \mathfrak{p}^{(f)}) + c(\mathbf{E}, \mathbf{E}) + d(\mathbf{B}, \mathbf{B})$$

$$+ \int_{\theta^b} \frac{\theta^{(f)}\rho^{(f)} s\, \nabla \overline{\mathbf{u}}}{\omega^2\rho_{22} - i\omega\kappa_2} d\mathbf{x} + \int_{\theta^b} \omega^2 \left[\rho_b - \frac{\theta^{(f)}\rho^{(f)}}{\omega^2\rho_{22} - i\omega\kappa_2}\right] \mathbf{u} \cdot \overline{\mathbf{u}}\, d\mathbf{x} + \omega^2\theta \int_{\theta^b} \rho^{(f)} e(\mathbf{u}) \overline{\pi}\; d\mathbf{x}$$

$$+ \int_{\theta_b} \frac{1}{\epsilon_{\text{eff}}} \nabla \cdot \mathbf{e}(\mathbf{u}) \mathbf{\Pi}^T \nabla \cdot \overline{\mathbf{E}}\, d\mathbf{x} + \int_{\theta_b} \mu_{\text{eff}} \omega^2 \mathbf{e}(\mathbf{u}) : \mathbf{\Pi}^T \cdot \overline{\mathbf{E}}\, d\mathbf{x}$$

$$- \int_{\Gamma_b} \left[(\nabla \mathbf{E} \cdot \mathbf{E}) \cdot \mathbf{n} + \frac{1}{\epsilon_{\text{eff}}} \nabla \cdot \mathbf{e}(\mathbf{u}) \mathbf{\Pi}^T \overline{\mathbf{E}}\right] ds_\Gamma + \frac{\theta}{1-\theta} \frac{\omega^2\rho_{22} + i\omega\kappa_2}{\omega^2\rho^{(w)}} \left\langle \frac{1}{2}\phi - \mathbf{K}'\phi, \overline{\pi} \right\rangle$$

$$+ i\omega\theta \int_{\theta_b} \mu^{(f)}\sigma^{(f)} \mathbf{B} \cdot \overline{\mathbf{B}}\, d\mathbf{x} - \int_{\theta_b} (\mathbf{e}(\mathbf{u}) : \mathbf{\Pi}^T) \cdot (\nabla \times \mathbf{B})\, d\mathbf{x} + \int_{\Gamma^b} \left[\mathbb{C}^{\text{eff}} : \mathbf{e}(\mathbf{u}) - \mathbf{\Pi}^{\text{eff}} \cdot \mathbf{E} - \mathbf{v}\phi \mathbb{I}\right] \cdot \overline{\mathbf{u}} \cdot \mathbf{n}\, ds_\Gamma. \tag{6.1}$$

$$\ell\Big((\mathbf{u}, \mathfrak{p}^{(f)}, \mathbf{E}, \mathbf{B}, \phi)\Big) := -\frac{\theta}{1-\theta} \frac{\omega^2\rho_{22} + i\omega\kappa_2}{\omega^2\rho^{(w)}} \left\langle \mathbf{n} \cdot \mathbf{f} - \frac{\partial \mathfrak{p}^{\text{part}}}{\partial n} \pi \right\rangle -$$

$$i\omega(1-\theta)\mu^{(f)} \int_{\Gamma_b} \left(\mathbf{e} : \mathbf{\Pi}^T \times \mathbf{B}\right) ds_\Gamma + \int_{\Gamma_b} \gamma_0 p^{\text{part}} \overline{\mathbf{v}} \cdot \mathbf{n} \tag{6.2}$$

We can show, using the methodology of[6,5], that

$$\mathfrak{Re}\left\{\mathcal{A}\Big((\mathbf{u}, p, \mathbf{E}, \mathbf{B}, \phi), (\mathbf{u}, p, \mathbf{E}, \mathbf{B}, \phi)\Big)\right\} = a(\mathbf{u}, \mathbf{u}) + b(s, s) + c(\mathbf{E}, \mathbf{E}) + d(\mathbf{B}, \mathbf{B}) + C\Big((\mathbf{u}, p, \mathbf{E}, \mathbf{B}, \phi)\Big),$$

where C is compact on $(H^1(\theta^b))^3 \times H^1(\theta^b) \times (H^1(\theta^b))^3 \times (H^1(\theta^b))^3 \times (H^{-1/2}(\Gamma)^2)$; furthermore, it follows that

Theorem 1. The sesqulinear form in (6.1) satisfies the Gårding's inequality in the form

$$\mathfrak{Re}\mathcal{A}\Big(\left(\mathbf{u}, \mathbf{E}, \mathbf{B}, \mathfrak{p}^{(f)}, \phi)\right), \left(\mathbf{u}, \mathfrak{p}^{(f)}, \mathbf{E}, \mathbf{B}, \phi\right)\Big) \geq \alpha \left\{\|\mathbf{u}\|^2_{(H^1(\theta^b))^2} + \|s\|^2_{H^1(\theta^b)} + \|s\mathfrak{p}^{(f)}\|^2_{H^{-\frac{1}{2}}(\Gamma)}\right\}$$
$$- \delta \left\{\|\mathbf{u}\|^2_{(H^{1-\epsilon}(\theta^b))^2} + \|\mathfrak{p}^{(f)}\|^2_{H^{1-\epsilon}(\theta^b)} + \|\mathfrak{p}^{(f)}\|^2_{H^{-\frac{1}{2}-\epsilon}(\Gamma)}\right\},$$

where $\alpha > 0$ and $\delta \geq 0$ are constant and $\epsilon > 0$ is a small parameter.

As is well known, Gårding's inequality implies the validity of the Fredholm alternative. Hence uniqueness implies the existence. For this purpose, we now consider the the homogeneous transmission problem $\mathbf{TP}_f$ with $f = 0$, since the uniqueness of the solution of the variational equation (5.9) will be depending upon that of $\mathbf{TP}_f$.

Theorem 2. If the quintiplet $(\mathbf{u}, \mathbf{E}, \mathbf{B}, s, p)$ is a classical solution of homogeneous transmission problem $\mathbf{TP}_0$ with $Im\, k_0 = 0$, then $p = 0$.

Proof. The proof mirrors the standard uniqueness proof used for the scattering transition problem and is, moreover, well known. We repeat it here only to make this section translucent. An application of the divergence theorem together with the radiation condition permits

$$\int_{S_R} \left| \frac{\partial \mathfrak{p}^{(w)}}{\partial r} - ik_0 p \right|^2 ds = \int_{S_R} \left(\left| \frac{\partial \mathfrak{p}^{(w)}}{\partial r} \right|^2 + \left| k_0\, p \right|^2 \right) ds + 2k_0 Im \int_{\partial\theta^b} \mathfrak{p}^{(w)} \frac{\partial \bar{p}}{\partial n} ds = o(1)$$

as $R \to \infty$, where S_R is the surface of a ball of radius R, enclosing the body θ^b. The main idea here is to show that $\int_{\partial\theta^b} \mathfrak{p}^{(w)} \frac{\partial \bar{\mathfrak{p}}^{(w)}}{\partial n} ds$ is real. Then

$$\int_{S_R} \left(\left| \frac{\partial p}{\partial r} \right|^2 + \left| k_0\, p \right|^2 \right) ds = o(1)$$

and from the Rellich-Vekua lemma, $p = 0$. To show this, we now compute $\int_{\partial\theta^b} \mathfrak{p} \frac{\partial \bar{p}}{\partial n} ds$ by using the variational forms for (E_b) and (E_s) in Section 4.

We consider $(\mathbf{T}P_0)$ and show $(\mathbf{u}_0, \mathbf{E}_0, \mathbf{B}_0, s_0, \phi_0)$ is the trivial solution. We assume $(\mathbf{u}_0, s_0) \neq (0, 0)$ and $(\mathbf{u}, \mathbf{E}, \mathbf{B}, s, \phi)$ is a solution of $\mathbf{T}P_0$. If so then it satisfies the transmission conditions

$$s_0|_\Gamma = \theta^{(f)} \mathbf{V} \phi_0, \quad \left[\mathbb{C}^{\text{eff}} : \mathbf{e}(\mathbf{u}), \, -\mathbf{\Pi}_{\text{eff}} \cdot \mathbf{E} \cdot \mathbf{u} + s\mathbb{I} \right] \cdot \mathbf{n} = \mathbf{V}\phi_0$$

$$\left[\mathbb{C}^{\text{eff}} : \mathbf{e}(\mathbf{u}), \, -\mathbf{\Pi}_{\text{eff}} \cdot \mathbf{E} \cdot \mathbf{u} + s_0\mathbb{I} \right] \cdot \mathbf{n} = \left(\frac{1}{2}\mathbb{I} - \mathbf{K}' \right) \phi_0$$

Since $p_0 = -\mathbf{s}\phi_0, \ \mathbf{x} \in \theta^w, \ \ p_0$ is a solution of the homogeneous Helmholtz in θ^w, i.e.

$$\Delta p_0 + k_0^2 p_0 = 0, \quad \mathbf{x} \in \theta^w.$$

Hence,

$$p_0\Big|_\Gamma = -\mathbf{V}\phi_0, \ \ \frac{\partial p_0}{\partial n}\Big|_\Gamma = \left(\frac{1}{2}\mathbb{I} - \mathbf{K}' \right) \phi_0.$$

□

We remark that Theorem 2 does not imply that the components $(\mathbf{u}, s)$ of the quintuple $(\mathbf{u}_0, \mathbf{E}_0, \mathbf{B}_0, s_0, \phi_0)$ considered in $\mathbf{T}P_0$ are trivial solutions, since they may be solutions of the traction free problem defined in Section 3. Hence in order to ensure the existence of a solution of the variational equation (5.9), we make the following assumptions:

(I) There is no traction free solution.

(II) The square of the wave number, k_0^2, is not an eigenvalue of the Dirichlet problem for the negative Laplacian in θ^b.

We remark that Assumption (II) is a guarantee for the invertibility of the simple-layer operator $\mathbf{V}$ (see[18], [p. 30]). We now summarize our results in the following theorem.

Theorem 3. Under Assumptions (I) and (II) , there exists an unique solution of the problem $\mathbf{TP}_f$ in $\left(H^1(\theta^b)\right)^3 \times H^1(\theta^b)^3 \times H^1(\theta^b)^3 \times H^1(\theta^b) \times H^{-\frac{1}{2}}(\Gamma)$.

7 | FUTURE DEVELOPMENTS

For many applications, such as the growth of bone, it is necessary to consider ionic transfer in the fluid phase. As the mass of the ions is insignificant with respect to the density of the fluid, we may treat this as diffusion dominated transport.

If $D_{\pm}$ is the binary water-ion diffusion coefficient, T the absolute temperature and R the universal ideal gas constant, the convection-diffusion equations governing ion, $c^{\pm}$ transport are given by[22]

$$\partial_t c^{\pm} + \nabla \cdot \left(\frac{D_{\pm} c^D_{\pm}}{RT} \nabla \mathfrak{m}^{\pm} \right) = 0, \tag{7.1}$$

where $\mathfrak{m}^{\pm}$ are molar electrochemical potentials of cations and anions according to the dilute solution approximation are given by[3]

$$\mathfrak{m}^{\pm} := \pm F \phi_f + RT \log c^{\pm}. \tag{7.2}$$

Following[22] we introduce a dimensionless electric potential $\overline{\phi_f} = \frac{F\phi_f}{RT}$ and assume mono-valent ions (i.e. $Z = 1$) we have

$$\frac{1}{RT} \nabla \mathfrak{m}^{\pm} = \frac{\nabla c^{\pm}}{c^{\pm}} \pm \nabla \phi_f, \tag{7.3}$$

and from which the Nernst-Plank, concentration-transport equation follows[22]

$$\partial_t c^{\pm} + \nabla \cdot \left(c^{\pm} \mathbf{v} \right) = \nabla \cdot \left[D_{\pm} \left(\nabla c^{\pm} \pm c^{\pm} \nabla \overline{\phi_f} \right) \right] = \nabla \cdot \left[D_{\pm} \exp(\mp \overline{\phi_f}) \nabla \left(c^{\pm} \exp(\pm \overline{\phi_f}) \right) \right]. \tag{7.4}$$

In the fluid there will be a flow of ions and hence, a current which induces a magnetic field. This leads to the use of non-homogeneous, time-dependent equations in the fluid phase.

$$\epsilon^{(f)} \nabla \cdot \mathbf{E}^{(f)} = q^{\nu}. \tag{7.5}$$

The induced electromagnetic field must satisfy the Maxwell-Gauss equation to obtain

$$\mathbf{E}^{(f)} = -\nabla \phi_f - \frac{\partial \mathbf{A}^{(f)}}{\partial t} \tag{7.6}$$

where $\mathbf{A}^{(f)}$ is a vector potential related to $\mathbf{B}^{(f)}$ by $\mathbf{B}^{(f)} = \nabla \times \mathbf{A}^{(f)}$, ϕ_f is the electric potential in the fluid and q^{ν}_f the volume charge density.

The Debye length characterizes the thickness of the diffuse ionic layer compensating for negative surface charge.

7.1 | Electrolyte movement

Here we take s, the fluid stress, to be a tensor of the form

$$\sigma^{(f)} := -p\mathbb{I} + \underline{\mathcal{T}}_M \tag{7.7}$$

where

$$\underline{\mathcal{T}}_M = \frac{\epsilon_f}{2} \left(2\mathbf{E} \otimes \mathbf{E} - E^2 \mathbb{I} \right) + \frac{1}{2\mu_f} \left(2\mathbf{B} \otimes \mathbf{B} - B^2 \mathbb{I} \right), \tag{7.8}$$

and $\underline{\mathcal{T}}_M$ is the electro-magnetic stress tensor[23]. Our equation for fluid motion takes the form

$$\nabla \cdot \left(-p\mathbb{I} + \underline{\mathcal{T}}_M \right) + \omega^2 \theta^{(f)} \rho^{(f)} \mathbf{u} + \omega^2 \rho_{22} \mathbf{w} - i\omega\kappa_2 \mathbf{w} = 0. \tag{7.9}$$

The pressure varies over the fluid region in the equilibrium state and its magnitude is a sum of the bulk phase pressure p_b and the osmatic, dipolar effect on the pressure close to the boundary between fluid and solid. For dilute solutions is defined as $\pi := RT(c^+ + c^- - 2c_b)$[22]; hence,

$$p_b := p - RT \left(c^+ + c^- - 2c_b \right).$$

and the equation of fluid motion becomes

$$-\nabla p_b - 2RT \left(\cosh \bar{\phi} - 1 \right) \nabla c_b + 2RTc_b \left(\sinh \bar{\phi} \nabla \bar{\phi} + \underline{\mathcal{T}}_M \right) + \omega^2 \theta^{(f)} \rho^{(f)} \mathbf{u} + \omega^2 \rho_{22} \mathbf{w} - i\omega\kappa_2 \mathbf{w} = 0. \tag{7.10}$$

To handle the electromagnetic equations, using[21,22] we split ϕ_f into a streaming potential and the dipole potential[22] one obtains

$$\nabla \cdot \left[\nabla \left(\overline{\psi}_b + \overline{\phi} \right) + \frac{\partial \mathbf{A}^{(f)}}{\partial t} \right] = \frac{1}{L_D^2} \sinh(\overline{\phi}). \tag{7.11}$$

where L_D is the Debye length given by $L_D = \sqrt{\tilde{\epsilon}_f RT/(2F^2 c_b)}$.

$$\triangle \mathbf{A}^{(f)} = \tilde{\mu}^f \tilde{\epsilon}^f \frac{\partial^2 \mathbf{A}^{(f)}}{\partial t^2} - \tilde{\mu}_f \mathbf{J},$$

where the current $\mathbf{J}$ is given by

$$\mathbf{J} = 2\mathbf{v}^f c_b \sinh \overline{\phi}. \tag{7.12}$$

In general, the procedure for healing bone uses time-harmonic waves. If we assume that the frequency is ω we have

$$\omega^2 \rho_f \mathbf{u}^f + \nabla \left(\rho_f a_f^2 \nabla \cdot \mathbf{u} \right) + i\omega \tilde{\mu}_f \mathrm{e}(\mathbf{u}) + q \left(\nabla \phi + i\omega \mathbf{A_f} \right) = 0, \tag{7.13}$$

where the vector potential $\mathbf{A}^{(f)}$ now satisfies a non-homogeneous, reduced wave equation

$$\triangle \mathbf{A}^{(f)} + \frac{1}{a_f^2} \tilde{\mu}_f \tilde{\epsilon}_f \omega^2 \mathbf{A}^{(f)} = -2i \frac{\tilde{\mu}_f \omega}{a_f^2} \mathbf{u}^f c_b, \tag{7.14}$$

where, as pointed out by Moyne and Murad[22] the relation of volume to surface electric charge densities q_f^v and q_f^s are related by $q_f^s = q_f^v / \ell$.

8 | CONCLUSION

Modeling of bone has important applications, for example to detect fractures or promote healing through stimulation. In this paper, cortical bone is studied as a composite material. A mixture theory is applied to obtain effective equations for a poro-elastic, piezo-electric bone matrix, assuming a linear piezo-elastic solid and a Newtonian interstitial fluid. The formulation includes mechanical, electrical and magnetic effects, as a mechanical excitation will create an electromagnetic field.

For the transmission problem, the bone segment is modeled surrounded by water and different problems depending on the transition are defined. The variational formulation of the equations is introduced. Existence and uniqueness of the solution is studied for the different boundary problems.

Finally, an outlook on future developments of bone modeling is given, where the ionic transfer in the fluid phase is also taken into account. For further future activities numerical simulations are intended. Here, fracture and healing will be investigated by sending a shear wave through a long bone, such as the tibia, which results in a measurable electromagnetic field. Additionally, an alternative model using the FE^2 method instead of the mixture argument is possible, thus not only including the supports of the single phases, but also the exact geometry using an RVE.

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