

Quasi-Laplacian energy of graphs based on R -graphs

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Abstract. Graph energy is a measurement of determining the structural information content of graphs. In this paper we determine the Quasi-Laplacian energy of classes of composite graphs based on R -graphs.

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1. Introduction

All graphs considered in this paper are simple and undirected. Let $G=(V(G),E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. If $|V(G)|=n$ and $|E(G)|=q$, we say that G is an (n,q) -graph. Let $\deg(v_i)$ be the degree of vertex v_i in G and $D(G)=diag(\deg(v_1),\deg(v_2),\dots,\deg(v_{|V(G)|}))$ the diagonal matrix with all vertex degrees of G as its diagonal entries. Let $A(G)$ denote the adjacency matrix of G . The Quasi-Laplacian matrix of G is defined as $Q(G)=D(G)+A(G)$ [4]. The Quasi-Laplacian matrix $Q(G)$ of G is a real symmetric positive semi-definite matrix [9], and we use $\mu_1(G)\geq\mu_2(G)\geq\dots\geq\mu_{|V(G)|}(G)\geq 0$ to denote the eigenvalues of $Q(G)$. The Quasi-Laplacian energy of G is defined as [11]

$$E_Q(G)=\sum_{i=1}^{|V(G)|}\mu_i^2,$$

and also in [11], it is pointed out that the Quasi-Laplacian energy of G can also be expressed as

$$E_Q(G)=\sum_{i=1}^{|V(G)|}\mu_i^2=\sum_{i=1}^{|V(G)|}\deg_{v_i}^2+\sum_{i=1}^{|V(G)|}\deg(v_i).$$

The first Zagreb index of a graph were first introduced by Gutman and Trinajstić in [6] defined as,

$$M_1(G) = \sum_{v \in V(G)} \deg(v)^2,$$

The first Zagreb index can be also expressed as a sum over edges of G ,

$$M_1(G) = \sum_{uv \in E(G)} (\deg_G(u) + \deg_G(v)).$$

We encourage the reader to consult [1-3,5,7] for historical background, computational techniques and mathematical properties of the Zagreb indices.

We will omit the subscript G when the graph is clear from the context.

It is well known that composite graph classes arise from simpler graphs via various graph operations. Therefore, it is important and of interest to understand how certain invariants of those composite graphs are related to the corresponding invariants of the simpler graphs.

The R -graph [4,8] of a graph G , denoted by $R(G)$, is the graph obtained from G by adding a vertex v_e and then joining v_e to the end vertices of e for each $e \in E(G)$. Let $I(G)$ be the set of newly added vertices, that is $I(G) = V(R(G)) \setminus V(G)$. In [10] and in [8], new graph operations based on R -graphs are defined as follows. Let G_1 and G_2 be two vertex-disjoint graphs. The R -vertex join of G_1 and G_2 , denoted by $G_1 \langle v \rangle G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and G_2 by joining every vertex of $V(G_1)$ with every vertex of $V(G_2)$. The R -edge join of G_1 and G_2 , denoted by $G_1 \langle e \rangle G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and G_2 by joining every vertex of $I(G_1)$ with every vertex of $V(G_2)$. The R -vertex corona of G_1 and G_2 , denoted by $G_1 \cdot G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|V(G_1)|$ copies of G_2 by joining the i th vertex of $V(G_1)$ to every vertex in the i th copy of G_2 . The R -edge corona of G_1 and G_2 , denoted by $G_1 \times G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|I(G_1)|$ copies of G_2 by joining the i th vertex of $I(G_1)$ to every vertex in the i th copy of G_2 . The R -vertex neighborhood corona of G_1 and G_2 , denoted by $G_1 \odot G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|V(G_1)|$ copies of G_2 by joining the neighbors of the i th vertex of G_1 in $R(G_1)$ to every vertex in the i th copy of G_2 . The R -edge neighborhood corona of G_1 and G_2 , denoted by $G_1 \otimes G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|I(G_1)|$ copies of G_2 by joining the neighbors of the i th vertex of $I(G_1)$ in $R(G_1)$ to every vertex in the i th copy of G_2 .

In this paper we derive closed-form formulas for the Quasi-Laplacian energy of R -graphs, R -vertex and edge join, R -vertex and edge corona, R -vertex and edge neighborhood graphs in terms of the corresponding energy (and some other quantities) of G_1 and G_2 . The paper ends with a short summary and conclusion.

2. Quasi-Laplacian energy of graphs based on R -graphs

2.1 Quasi-Laplacian energy of R -graphs

The degree of a vertex $v \in V(R(G))$ is given by

$$\deg_{R(G)}(v) = \begin{cases} 2\deg_G(v), & \text{if } v \in V(G); \\ 2, & \text{if } v \in I(G). \end{cases}$$

Theorem 2.1 *Let G be (n, m) -graph. Then,*

$$E_Q(R(G)) = 4(E_Q(G)) + 2m.$$

Proof. By the definition of Quasi-Laplacian energy of a graph, we start with

$$E_Q(R(G)) = \sum_{i=1}^{|V(R(G))|} \deg_{R(G)}^2(v_i) + \sum_{i=1}^{|V(R(G))|} \deg_{R(G)}(v_i).$$

Since $|V(R(G))| = n + m$,

$$\begin{aligned} E_Q(R(G)) &= \sum_{i=1}^{n+m} \deg_{R(G)}^2(v_i) + \sum_{i=1}^{n+m} \deg_{R(G)}(v_i) \\ &= \sum_{v \in V(G)} \deg_{R(G)}^2(v) + \sum_{v \in I(G)} \deg_{R(G)}^2(v) + \sum_{v \in V(G)} \deg_{R(G)}(v) + \sum_{v \in I(G)} \deg_{R(G)}(v). \end{aligned}$$

By substituting the values of parameters due to the degrees of the vertices of $R(G)$, we compute

$$\begin{aligned} E_Q(R(G)) &= \left(\sum_{i=1}^n (2\deg_G(v_i))^2 + \sum_{i=1}^m (2)^2 \right) + \left(\sum_{i=1}^n (2\deg_G(v_i)) + \sum_{i=1}^m (2) \right) \\ &= \left(4 \sum_{i=1}^n \deg_G^2(v_i) + 4m \right) + \left(2 \sum_{i=1}^n \deg_G(v_i) + 2m \right). \end{aligned}$$

By handshaking lemma, we compute each summation as follows:

$$E_Q(R(G)) = 4 \sum_{i=1}^n \deg_G^2(v_i) + 4m + 2(2m) + 2m$$

$$= 4 \sum_{i=1}^n \deg_G^2(v_i) + 4 \sum_{i=1}^n \deg_G(v_i) + 2m.$$

By the definition of Quasi-Laplacian energy of a graph, we receive

$$E_Q(R(G)) = 4 \left(\sum_{i=1}^n \deg_G^2(v_i) + \sum_{i=1}^n \deg_G(v_i) \right) + 2m = 4(E_Q(G)) + 2m.$$

Hence, the desired result holds. \square

2.2 Quasi-Laplacian energy of R -vertex join graphs

The degree of a vertex $v \in V(G_1 \langle v \rangle G_2)$ is given by

$$\deg_{G_1 \langle v \rangle G_2}(v) = \begin{cases} 2 \deg_{G_1}(v) + n_2, & \text{if } v \in V(G_1); \\ 2, & \text{if } v \in I(G_1); \\ \deg_{G_2}(v) + n_1, & \text{if } v \in V(G_2). \end{cases}$$

Theorem 2.2 Let G_1 be (n_1, m_1) -graph and G_2 be (n_2, m_2) -graph. Then,

$$E_Q(G_1 \langle v \rangle G_2) = 4E_Q(G_1) + E_Q(G_2) + n_1 n_2 (n_1 + n_2 + 2) + 2m_1 (4n_2 + 1) + 4n_1 m_2.$$

Proof. By the definition of Quasi-Laplacian energy of a graph, we start with

$$E_Q(G_1 \langle v \rangle G_2) = \sum_{i=1}^{|V(G_1 \langle v \rangle G_2)|} \deg_{G_1 \langle v \rangle G_2}^2(v_i) + \sum_{i=1}^{|V(G_1 \langle v \rangle G_2)|} \deg_{G_1 \langle v \rangle G_2}(v_i).$$

Since $|V(G_1 \langle v \rangle G_2)| = n_1 + m_1 + n_2$,

$$\begin{aligned} E_Q(G_1 \langle v \rangle G_2) &= \sum_{i=1}^{n_1+m_1+n_2} \deg_{G_1 \langle v \rangle G_2}^2(v_i) + \sum_{i=1}^{n_1+m_1+n_2} \deg_{G_1 \langle v \rangle G_2}(v_i) \\ &= \sum_{v \in V(G_1)} \deg_{G_1 \langle v \rangle G_2}^2(v) + \sum_{v \in I(G_1)} \deg_{G_1 \langle v \rangle G_2}^2(v) + \sum_{v \in V(G_2)} \deg_{G_1 \langle v \rangle G_2}^2(v) + \sum_{v \in V(G_1)} \deg_{G_1 \langle v \rangle G_2}(v) + \\ &\quad \sum_{v \in I(G_1)} \deg_{G_1 \langle v \rangle G_2}(v) + \sum_{v \in V(G_2)} \deg_{G_1 \langle v \rangle G_2}(v). \end{aligned}$$

By substituting the values of parameters due to the degrees of the vertices of $G_1 \langle v \rangle G_2$, we compute

$$\begin{aligned} E_Q(G_1 \langle v \rangle G_2) &= \sum_{i=1}^{n_1} (2 \deg_{G_1}(v_i) + n_2)^2 + \sum_{i=1}^{m_1} (2)^2 + \sum_{i=1}^{n_2} (\deg_{G_2}(v_i) + n_1)^2 + \sum_{i=1}^{n_1} (2 \deg_{G_1}(v_i) + n_2) + \\ &\quad \sum_{i=1}^{m_1} (2) + \sum_{i=1}^{n_2} (\deg_{G_2}(v_i) + n_1) \\ &= 4 \sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + 4n_2 \sum_{i=1}^{n_1} \deg_{G_1}(v_i) + n_1 n_2^2 + 4m_1 + \sum_{i=1}^{n_2} \deg_{G_2}^2(v_i) + 2n_1 \sum_{i=1}^{n_2} \deg_{G_2}(v_i) + \end{aligned}$$

$$n_2 n_1^2 + 2 \sum_{i=1}^{n_1} \deg_{G_1}(v_i) + n_1 n_2 + 2m_1 + \sum_{i=1}^{n_2} \deg_{G_2}(v_i) + n_1 n_2.$$

By handshaking lemma, we compute each summation as follows:

$$\begin{aligned} E_Q(G_1 \langle v \rangle G_2) &= 4 \sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + 4n_2(2m_1) + n_1 n_2^2 + 4m_1 + \sum_{i=1}^{n_2} \deg_{G_2}^2(v_i) + 2n_1(2m_2) + \\ & n_2 n_1^2 + 2(2m_1) + n_1 n_2 + 2m_1 + 2m_2 + n_1 n_2. \end{aligned}$$

By the definition of Quasi-Laplacian energy of a graph, we receive

$$\begin{aligned} E_Q(G_1 \langle v \rangle G_2) &= 4 \left(\sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + 2m_1 \right) + \left(\sum_{i=1}^{n_2} \deg_{G_2}^2(v_i) + 2m_2 \right) + 4n_1 m_2 + 2m_1(4n_2 + 1) + \\ & + n_1 n_2(n_1 + n_2 + 2) \\ &= 4E_Q(G_1) + E_Q(G_2) + 4n_1 m_2 + 2m_1(4n_2 + 1) + n_1 n_2(n_1 + n_2 + 2). \end{aligned}$$

The theorem is thus proved. \square

2.3 Quasi-Laplacian energy of R -edge join graphs

The degree of a vertex $v \in V(G_1 \langle e \rangle G_2)$ is given by

$$\deg_{G_1 \langle e \rangle G_2}(v) = \begin{cases} 2 \deg_{G_1}(v), & \text{if } v \in V(G_1); \\ n_2 + 2, & \text{if } v \in I(G_1); \\ \deg_{G_2}(v) + m_1, & \text{if } v \in V(G_2). \end{cases}$$

Theorem 2.3 Let G_1 be (n_1, m_1) -graph and G_2 be (n_2, m_2) -graph. Then,

$$E_Q(G_1 \langle e \rangle G_2) = 4(E_Q(G_1)) + E_Q(G_2) + m_1(n_2(n_2 + m_1 + 6) + 4m_2).$$

Proof. By the definition of Quasi-Laplacian energy of a graph, we have that

$$E_Q(G_1 \langle e \rangle G_2) = \sum_{i=1}^{|V(G_1 \langle e \rangle G_2)|} \deg_{G_1 \langle e \rangle G_2}^2(v_i) + \sum_{i=1}^{|V(G_1 \langle e \rangle G_2)|} \deg_{G_1 \langle e \rangle G_2}(v_i).$$

Since $|V(G_1 \langle e \rangle G_2)| = n_1 + m_1 + n_2$,

$$\begin{aligned} E_Q(G_1 \langle e \rangle G_2) &= \sum_{i=1}^{n_1+m_1+n_2} \deg_{G_1 \langle e \rangle G_2}^2(v_i) + \sum_{i=1}^{n_1+m_1+n_2} \deg_{G_1 \langle e \rangle G_2}(v_i) \\ &= \sum_{v \in V(G_1)} \deg_{G_1 \langle e \rangle G_2}^2(v) + \sum_{v \in I(G_1)} \deg_{G_1 \langle e \rangle G_2}^2(v) + \sum_{v \in V(G_2)} \deg_{G_1 \langle e \rangle G_2}^2(v) + \sum_{v \in V(G_1)} \deg_{G_1 \langle e \rangle G_2}(v) + \\ & \sum_{v \in I(G_1)} \deg_{G_1 \langle e \rangle G_2}(v) + \sum_{v \in V(G_2)} \deg_{G_1 \langle e \rangle G_2}(v). \end{aligned}$$

By substituting the values of parameters due to the degrees of the vertices of $G_1 \langle e \rangle G_2$, we compute

$$\begin{aligned}
E_Q(G_1 \langle e \rangle G_2) &= \sum_{i=1}^{n_1} (2 \deg_{G_1}(v_i))^2 + \sum_{i=1}^{m_1} (n_2 + 2)^2 + \sum_{i=1}^{n_2} (\deg_{G_2}(v_i) + m_1)^2 + \sum_{i=1}^{n_1} (2 \deg_{G_1}(v_i)) + \\
&\quad \sum_{i=1}^{m_1} (n_2 + 2) + \sum_{i=1}^{n_2} (\deg_{G_2}(v_i) + m_1) \\
&= 4 \sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + m_1 (n_2 + 2)^2 + \sum_{i=1}^{n_2} \deg_{G_2}^2(v_i) + 2m_1 \sum_{i=1}^{n_2} \deg_{G_2}(v_i) + n_2 m_1^2 + 2 \sum_{i=1}^{n_1} \deg_{G_1}(v_i) + \\
&\quad m_1 (n_2 + 2) + \sum_{i=1}^{n_2} \deg_{G_2}(v_i) + n_2 m_1.
\end{aligned}$$

By handshaking lemma, we compute each summation as follows:

$$\begin{aligned}
E_Q(G_1 \langle e \rangle G_2) &= 4 \sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + m_1 (n_2 + 2)^2 + \sum_{i=1}^{n_2} \deg_{G_2}^2(v_i) + 2m_1 (2m_2) + n_2 m_1^2 + 2(m_1) + \\
&\quad m_1 (n_2 + 2) + 2m_2 + n_2 m_1.
\end{aligned}$$

By the definition of Quasi-Laplacian energy of a graph, we receive

$$\begin{aligned}
E_Q(G_1 \langle e \rangle G_2) &= 4 \left(\sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + 2m_1 \right) + \left(\sum_{i=1}^{n_2} \deg_{G_2}^2(v_i) + 2m_2 \right) + m_1 (n_2 (n_2 + m_1 + 6) + 4m_2) \\
&= 4(E_Q(G_1)) + E_Q(G_2) + m_1 (n_2 (n_2 + m_1 + 6) + 4m_2).
\end{aligned}$$

Thus, the proof is complete. \square

2.4 Quasi-Laplacian energy of R -vertex corona graphs

The degree of a vertex $v \in V(G_1 \cdot G_2)$ is given by

$$\deg_{G_1 \cdot G_2}(v) = \begin{cases} 2 \deg_{G_1}(v) + n_2, & \text{if } v \in V(G_1); \\ 2, & \text{if } v \in I(G_1); \\ \deg_{G_2}(v) + 1, & \text{if } v \in V(G_2). \end{cases}$$

Theorem 2.4 Let G_1 be (n_1, m_1) -graph and G_2 be (n_2, m_2) -graph. Then,

$$E_Q(G_1 \cdot G_2) = 4E_Q(G_1) + n_1 E_Q(G_2) + n_1 (4m_2 + n_2 (n_2 + 3)) + 2m_1 (4n_2 + 1).$$

Proof. By the definition of Quasi-Laplacian energy of a graph, we have that

$$E_Q(G_1 \cdot G_2) = \sum_{i=1}^{|V(G_1 \cdot G_2)|} \deg_{G_1 \cdot G_2}^2(v_i) + \sum_{i=1}^{|V(G_1 \cdot G_2)|} \deg_{G_1 \cdot G_2}(v_i).$$

Since $|V(G_1 \cdot G_2)| = n_1 (n_2 + 1) + m_1$,

$$\begin{aligned}
E_Q(G_1 \cdot G_2) &= \sum_{i=1}^{n_1(n_2+1)+m_1} \deg_{G_1 \cdot G_2}^2(v_i) + \sum_{i=1}^{n_1(n_2+1)+m_1} \deg_{G_1 \cdot G_2}(v_i) \\
&= \sum_{v \in V(G_1)} \deg_{G_1 \cdot G_2}^2(v) + \sum_{v \in I(G_1)} \deg_{G_1 \cdot G_2}^2(v) + \sum_{i=1}^{n_1} \sum_{v \in V(G_2)} \deg_{G_1 \cdot G_2}^2(v) + \sum_{v \in V(G_1)} \deg_{G_1 \cdot G_2}(v) + \\
&\quad \sum_{v \in I(G_1)} \deg_{G_1 \cdot G_2}(v) + \sum_{i=1}^{n_1} \sum_{v \in V(G_2)} \deg_{G_1 \cdot G_2}(v).
\end{aligned}$$

By substituting the values of parameters due to the degrees of the vertices of $G_1 \cdot G_2$, we compute

$$\begin{aligned}
E_Q(G_1 \cdot G_2) &= \sum_{i=1}^{n_1} (2 \deg_{G_1}(v_i) + n_2)^2 + \sum_{i=1}^{m_1} (2)^2 + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (\deg_{G_2}(v_j) + 1)^2 + \sum_{i=1}^{n_1} (2 \deg_{G_1}(v_i) + n_2) + \\
&\quad \sum_{i=1}^{m_1} (2) + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (\deg_{G_2}(v_j) + 1) \\
&= 4 \sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + 4n_2 \sum_{i=1}^{n_1} \deg_{G_1}(v_i) + n_1 n_2^2 + 4m_1 + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \deg_{G_2}^2(v_j) + 2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \deg_{G_2}(v_j) + n_1 n_2 + \\
&\quad 2 \sum_{i=1}^{n_1} \deg_{G_1}(v_i) + n_1 n_2 + 2m_1 + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (\deg_{G_2}(v_j)) + n_1 n_2.
\end{aligned}$$

By handshaking lemma, we compute each summation as follows:

$$\begin{aligned}
E_Q(G_1 \cdot G_2) &= 4 \sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + 4n_2 (2m_1) + n_1 n_2^2 + 4m_1 + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \deg_{G_2}^2(v_j) + 2 \sum_{i=1}^{n_1} (2m_2) + n_1 n_2 + \\
&\quad 2(2m_1) + n_1 n_2 + 2m_1 + \sum_{i=1}^{n_1} (2m_2) + n_1 n_2 \\
&= 4 \sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + 4n_2 (2m_1) + n_1 n_2^2 + 4m_1 + n_1 \sum_{j=1}^{n_2} \deg_{G_2}^2(v_j) + 4n_1 m_2 + n_1 n_2 + 2(2m_1) + n_1 n_2 + \\
&\quad 2m_1 + 2n_1 m_2 + n_1 n_2.
\end{aligned}$$

By the definition of Quasi-Laplacian energy of a graph, we receive

$$\begin{aligned}
E_Q(G_1 \cdot G_2) &= 4 \left(\sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + 2m_1 \right) + n_1 \left(\sum_{j=1}^{n_2} \deg_{G_2}^2(v_j) + 2m_2 \right) + n_1 (4m_2 + n_2(n_2 + 3)) + \\
&\quad 2m_1 (4n_2 + 1) \\
&= 4E_Q(G_1) + n_1 E_Q(G_2) + n_1 (4m_2 + n_2(n_2 + 3)) + 2m_1 (4n_2 + 1).
\end{aligned}$$

The theorem is thus proved. \square

2.5 Quasi-Laplacian energy of R -edge corona graphs

The degree of a vertex $v \in V(G_1 \times G_2)$ is given by

$$\deg_{G_1 \times G_2}(v) = \begin{cases} 2\deg_{G_1}(v), & \text{if } v \in V(G_1); \\ n_2 + 2, & \text{if } v \in I(G_1); \\ \deg_{G_2}(v) + 1, & \text{if } v \in V(G_2). \end{cases}$$

Theorem 2.5 Let G_1 be (n_1, m_1) -graph and G_2 be (n_2, m_2) -graph. Then,

$$E_Q(G_1 \times G_2) = 4E_Q(G_1) + m_1(E_Q(G_2) + n_2(n_2 + 6) + 4m_2 + 2).$$

Proof. By the definition of Quasi-Laplacian energy of a graph, we have that

$$E_Q(G_1 \times G_2) = \sum_{i=1}^{|V(G_1 \times G_2)|} \deg_{G_1 \times G_2}^2(v_i) + \sum_{i=1}^{|V(G_1 \times G_2)|} \deg_{G_1 \times G_2}(v_i).$$

Since $|V(G_1 \times G_2)| = n_1 + m_1(n_2 + 1)$,

$$\begin{aligned} E_Q(G_1 \times G_2) &= \sum_{i=1}^{n_1+m_1(n_2+1)} \deg_{G_1 \times G_2}^2(v_i) + \sum_{i=1}^{n_1+m_1(n_2+1)} \deg_{G_1 \times G_2}(v_i) \\ &= \sum_{v \in V(G_1)} \deg_{G_1 \times G_2}^2(v) + \sum_{v \in I(G_1)} \deg_{G_1 \times G_2}^2(v) + \sum_{i=1}^{m_1} \sum_{v \in V(G_2)} \deg_{G_1 \times G_2}^2(v) + \sum_{v \in V(G_1)} \deg_{G_1 \times G_2}(v) + \\ &\quad \sum_{v \in I(G_1)} \deg_{G_1 \times G_2}(v) + \sum_{i=1}^{m_1} \sum_{v \in V(G_2)} \deg_{G_1 \times G_2}(v). \end{aligned}$$

By substituting the values of parameters due to the degrees of the vertices of $G_1 \times G_2$, we compute

$$\begin{aligned} E_Q(G_1 \times G_2) &= \sum_{i=1}^{n_1} (2\deg_{G_1}(v_i))^2 + \sum_{i=1}^{m_1} (n_2 + 2)^2 + \sum_{i=1}^{m_1} \sum_{j=1}^{n_2} (\deg_{G_2}(v_j) + 1)^2 + \sum_{i=1}^{n_1} (2\deg_{G_1}(v_i)) + \\ &\quad \sum_{i=1}^{m_1} (n_2 + 2) + \sum_{i=1}^{m_1} \sum_{j=1}^{n_2} (\deg_{G_2}(v_j) + 1) \\ &= 4 \sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + m_1(n_2 + 2)^2 + \sum_{i=1}^{m_1} \sum_{j=1}^{n_2} \deg_{G_2}^2(v_j) + 2 \sum_{i=1}^{m_1} \sum_{j=1}^{n_2} \deg_{G_2}(v_j) + m_1 n_2 + 2 \sum_{i=1}^{n_1} \deg_{G_1}(v_i) + \\ &\quad m_1(n_2 + 2) + \sum_{i=1}^{m_1} \sum_{j=1}^{n_2} \deg_{G_2}(v_j) + m_1 n_2. \end{aligned}$$

By handshaking lemma, we compute each summation as follows:

$$\begin{aligned} E_Q(G_1 \times G_2) &= 4 \sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + m_1(n_2 + 2)^2 + \sum_{i=1}^{m_1} \sum_{j=1}^{n_2} \deg_{G_2}^2(v_j) + 2 \sum_{i=1}^{m_1} (2m_2) + m_1 n_2 + 2(2m_1) + \\ &\quad m_1(n_2 + 2) + \sum_{i=1}^{m_1} (2m_2) + m_1 n_2 \end{aligned}$$

$$= 4 \sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + m_1(n_2 + 2)^2 + \sum_{i=1}^{m_1} \sum_{j=1}^{n_2} \deg_{G_2}^2(v_j) + 4m_1m_2 + m_1n_2 + 2(2m_1) + m_1(n_2 + 2) + 2m_1m_2 + m_1n_2.$$

By the definition of Quasi-Laplacian energy of a graph, we receive that

$$\begin{aligned} E_Q(G_1 \times G_2) &= 4 \left(\sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + 2m_1 \right) + m_1 \left(\sum_{j=1}^{n_2} \deg_{G_2}^2(v_j) + 2m_2 \right) + m_1(n_2(n_2 + 6) + 4m_2 + 2) \\ &= 4E_Q(G_1) + m_1(E_Q(G_2) + n_2(n_2 + 6) + 4m_2 + 2) \end{aligned}$$

After simple computation, the desired result of $E_Q(G_1 \times G_2)$ holds. \square

2.6 Quasi-Laplacian energy of R -vertex neighborhood corona graphs

The degree of a vertex $v \in V(G_1 \odot G_2)$ is given by

$$\deg_{G_1 \odot G_2}(v) = \begin{cases} (n_2 + 2)\deg_{G_1}(v), & \text{if } v \in V(G_1); \\ 2(n_2 + 1), & \text{if } v \in I(G_1); \\ \deg_{G_2}(v) + 2\deg_{G_1}(u), & \text{if } v \in V(G_2) \text{ where } u \text{ is the } i\text{th vertex of } G_1 \text{ in } R(G_1) \\ & \text{and } v \text{ is a vertex in the } i\text{th copy of } G_2. \end{cases}$$

Theorem 2.6 Let G_1 be (n_1, m_1) -graph and G_2 be (n_2, m_2) -graph. Then,

$$E_Q(G_1 \odot G_2) = ((n_2 + 2)^2 + 2n_2)E_Q(G_1) + n_1(E_Q(G_2)) + 2(n_2M_1(G_1) + m_1(8m_2 + 1)).$$

Proof. By the definition of Quasi-Laplacian energy of a graph, we have that

$$E_Q(G_1 \odot G_2) = \sum_{i=1}^{|V(G_1 \odot G_2)|} \deg_{G_1 \odot G_2}^2(v_i) + \sum_{i=1}^{|V(G_1 \odot G_2)|} \deg_{G_1 \odot G_2}(v_i).$$

Since $|V(G_1 \odot G_2)| = n_1(n_2 + 1) + m_1$,

$$\begin{aligned} E_Q(G_1 \odot G_2) &= \sum_{i=1}^{n_1(n_2+1)+m_1} \deg_{G_1 \odot G_2}^2(v_i) + \sum_{i=1}^{n_1(n_2+1)+m_1} \deg_{G_1 \odot G_2}(v_i) \\ &= \sum_{v \in V(G_1)} \deg_{G_1 \odot G_2}^2(v) + \sum_{v \in I(G_1)} \deg_{G_1 \odot G_2}^2(v) + \sum_{i=1}^{n_1} \sum_{v \in V(G_2)} \deg_{G_1 \odot G_2}^2(v) + \sum_{v \in V(G_1)} \deg_{G_1 \odot G_2}(v) + \\ &\quad \sum_{v \in I(G_1)} \deg_{G_1 \odot G_2}(v) + \sum_{i=1}^{n_1} \sum_{v \in V(G_2)} \deg_{G_1 \odot G_2}(v). \end{aligned}$$

By substituting the values of parameters due to the degrees of the vertices of $G_1 \odot G_2$, we get,

$$E_Q(G_1 \odot G_2) = \sum_{i=1}^{n_1} ((n_2 + 2)\deg_{G_1}(v_i))^2 + \sum_{i=1}^{m_1} (2(n_2 + 1))^2 + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (\deg_{G_2}(v_j) + 2\deg_{G_1}(u_i))^2 +$$

$$\begin{aligned}
& \sum_{i=1}^{n_1} \left((n_2 + 2) \deg_{G_1}(v_i) \right) + \sum_{i=1}^{m_1} (2(n_2 + 2)) + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left(\deg_{G_2}(v_j) + 2 \deg_{G_1}(u_i) \right) \\
&= (n_2 + 2)^2 \sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + m_1 (2(n_2 + 1))^2 + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \deg_{G_2}^2(v_j) + 4 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \deg_{G_2}(v_j) \deg_{G_1}(u_i) + \\
& 4 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \deg_{G_1}^2(u_i) + (n_2 + 2) \sum_{i=1}^{n_1} \deg_{G_1}(v_i) + m_1 (2(n_2 + 2)) + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \deg_{G_2}(v_j) + 2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \deg_{G_1}(u_i).
\end{aligned}$$

By handshaking lemma, we compute each summation as follows:

$$\begin{aligned}
E_Q(G_1 \odot G_2) &= (n_2 + 2)^2 \sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + m_1 (2(n_2 + 1))^2 + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \deg_{G_2}^2(v_j) + \\
& 4 \left(\sum_{i=1}^{n_1} \deg_{G_1}(u_i) \right) \left(\sum_{i=1}^{n_2} \deg_{G_2}(v_i) \right) + 4 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \deg_{G_1}^2(u_i) + (n_2 + 2)(2m_1) + m_1 (2(n_2 + 2)) + \\
& \sum_{i=1}^{n_1} (2m_2) + 2n_2 \sum_{i=1}^{n_1} \deg_{G_1}(u_i) \\
&= (n_2 + 2)^2 \sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + m_1 (2(n_2 + 1))^2 + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \deg_{G_2}^2(v_j) + 4(2m_1)(2m_2) + \\
& 4 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \deg_{G_1}^2(u_i) + (n_2 + 2)(2m_1) + m_1 (2(n_2 + 2)) + n_1 (2m_2) + 2n_2 (2m_1).
\end{aligned}$$

By the definition of Quasi-Laplacian energy of a graph, we have that

$$\begin{aligned}
E_Q(G_1 \odot G_2) &= \left((n_2 + 2)^2 + 2n_2 \right) \left(\sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + 2m_1 \right) + 2 \left(n_2 \sum_{i=1}^{n_1} \deg_{G_1}^2(u_i) + m_1 (8m_2 + 1) \right) + \\
& + n_1 \left(\sum_{j=1}^{n_2} \deg_{G_2}^2(v_j) + 2m_2 \right) \\
&= \left((n_2 + 2)^2 + 2n_2 \right) E_Q(G_1) + n_1 (E_Q(G_2)) + 2 \left(n_2 \sum_{i=1}^{n_1} \deg_{G_1}^2(u_i) + m_1 (8m_2 + 1) \right).
\end{aligned}$$

By the definition of the first Zagreb index of a graph, we conclude that

$$E_Q(G_1 \odot G_2) = \left((n_2 + 2)^2 + 2n_2 \right) E_Q(G_1) + n_1 (E_Q(G_2)) + 2(n_2 M_1(G_1) + m_1 (8m_2 + 1)).$$

Hence, the desired result of $E_Q(G_1 \odot G_2)$ holds. \square

2.7 Quasi-Laplacian energy of R -edge neighborhood corona graphs

The degree of a vertex $v \in V(G_1 \otimes G_2)$ is given by

$$\deg_{G_1 \otimes G_2}(v) = \begin{cases} (n_2 + 2)\deg_{G_1}(v), & \text{if } v \in V(G_1); \\ \deg_{G_2}(v) + 2, & \text{if } v \in V(G_2); \\ 2, & \text{if } v \in I(G_1). \end{cases}$$

Theorem 2.7 Let G_1 be (n_1, m_1) -graph and G_2 be (n_2, m_2) -graph. Then,

$$E_Q(G_1 \otimes G_2) = 4(n_2 + 1)E_Q(G_1) + n_2^2 M_1(G_1) + m_1(E_Q(G_2) + 2(4m_2 + 1)).$$

Proof. By the definition of Quasi-Laplacian energy of a graph, we have that

$$E_Q(G_1 \otimes G_2) = \sum_{i=1}^{|V(G_1 \otimes G_2)|} \deg_{G_1 \otimes G_2}^2(v_i) + \sum_{i=1}^{|V(G_1 \otimes G_2)|} \deg_{G_1 \otimes G_2}(v_i).$$

Since $|V(G_1 \otimes G_2)| = n_1 + m_1(n_2 + 1)$,

$$\begin{aligned} E_Q(G_1 \otimes G_2) &= \sum_{i=1}^{n_1+m_1(n_2+1)} \deg_{G_1 \otimes G_2}^2(v_i) + \sum_{i=1}^{n_1+m_1(n_2+1)} \deg_{G_1 \otimes G_2}(v_i) \\ &= \sum_{v \in V(G_1)} \deg_{G_1 \otimes G_2}^2(v) + \sum_{v \in I(G_1)} \deg_{G_1 \otimes G_2}^2(v) + \sum_{v \in V(G_2)} \deg_{G_1 \otimes G_2}^2(v) + \sum_{v \in V(G_1)} \deg_{G_1 \otimes G_2}(v) + \\ &\quad \sum_{v \in I(G_1)} \deg_{G_1 \otimes G_2}(v) + \sum_{v \in V(G_2)} \deg_{G_1 \otimes G_2}(v). \end{aligned}$$

By substituting the values of parameters due to the degrees of the vertices of $G_1 \otimes G_2$, we get,

$$\begin{aligned} E_Q(G_1 \otimes G_2) &= \sum_{i=1}^{n_1} ((n_2 + 2)\deg_{G_1}(v_i))^2 + \sum_{i=1}^{m_1} (2)^2 + \sum_{i=1}^{m_1} \sum_{j=1}^{n_2} (\deg_{G_2}(v_j) + 2)^2 + \\ &\quad \sum_{i=1}^{n_1} ((n_2 + 2)\deg_{G_1}(v_i)) + \sum_{i=1}^{m_1} (2) + \sum_{i=1}^{m_1} \sum_{j=1}^{n_2} (\deg_{G_2}(v_j) + 2) \\ &= (n_2 + 2)^2 \sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + 4m_1 + \sum_{i=1}^{m_1} \sum_{j=1}^{n_2} \deg_{G_2}^2(v_j) + 4 \sum_{i=1}^{m_1} \sum_{j=1}^{n_2} \deg_{G_2}(v_j) + 4m_1 n_2 + \\ &\quad (n_2 + 2) \sum_{i=1}^{n_1} \deg_{G_1}(v_i) + 2m_1 + \sum_{i=1}^{m_1} \sum_{j=1}^{n_2} \deg_{G_2}(v_j) + 2m_1 n_2. \end{aligned}$$

By handshaking lemma, we compute each summation as follows:

$$\begin{aligned} E_Q(G_1 \otimes G_2) &= (n_2 + 2)^2 \sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + 4m_1 + \sum_{i=1}^{m_1} \sum_{j=1}^{n_2} \deg_{G_2}^2(v_j) + 4 \sum_{i=1}^{m_1} (2m_2) + 4m_1 n_2 + \\ &\quad (n_2 + 2)(2m_1) + 2m_1 + \sum_{i=1}^{m_1} (2m_2) + 2m_1 n_2 \\ &= (n_2 + 2)^2 \sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + 4m_1 + \sum_{i=1}^{m_1} \sum_{j=1}^{n_2} \deg_{G_2}^2(v_j) + 8m_1 m_2 + 4m_1 n_2 + (n_2 + 2)(2m_1) + 2m_1 + \\ &\quad 2m_1 m_2 + 2m_1 n_2. \end{aligned}$$

By the definition of Quasi-Laplacian energy of a graph, we have that

$$\begin{aligned} E_Q(G_1 \otimes G_2) &= 4(n_2 + 1) \left(\sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + 2m_1 \right) + n_2^2 \sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + m_1 \left(\sum_{j=1}^{n_2} \deg_{G_2}^2(v_j) + 2m_2 \right) + \\ &2m_1(4m_2 + 1) \\ &= 4(n_2 + 1)E_Q(G_1) + n_2^2 \sum_{i=1}^{n_1} \deg_{G_1}^2(v_i) + m_1(E_Q(G_2) + 2(4m_2 + 1)). \end{aligned}$$

By the definition of the first Zagreb index of a graph, we conclude that

$$E_Q(G_1 \otimes G_2) = 4(n_2 + 1)E_Q(G_1) + n_2^2 M_1(G_1) + m_1(E_Q(G_2) + 2(4m_2 + 1)).$$

Thus, the desired result of $E_Q(G_1 \otimes G_2)$ holds. \square

3. Summary and conclusion

Graph energy has so many applications in the field of chemistry, physics, biology, mathematics and sociology. By the approach presented in [11], the relation between Quasi-Laplacian energy and the vertex degrees of a graph was envisaged. In this paper, it is also viewed that the first Zagreb index can be handled with its connection to graph energy. A new and significant application of the first Zagreb index to composite graphs based on R -graphs is revealed, and exact formulae for Quasi-Laplacian energy are derived in terms of the corresponding energies, the first Zagreb indices, number of vertices and edges of the underlying graphs of those composite graph types.

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