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RESEARCH ARTICLE

Generalized approximate boundary synchronization for a coupled system of wave equations

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In this paper, we consider the generalized approximate boundary synchronization for a coupled system of wave equations with Dirichlet boundary controls. We analyse the relationship between the generalized approximate boundary synchronization and the generalized exact boundary synchronization, give a sufficient condition to realize the generalized approximate boundary synchronization and a necessary condition in terms of Kalman's matrix, and show the meaning of the number of total controls. Besides, by the generalized synchronization decomposition, we define the generalized approximately synchronizable state, and obtain its properties and a sufficient condition for it to be independent of applied boundary controls.

KEYWORDS:
coupled system of wave equations, generalized approximate boundary synchronization, generalized approximately synchronizable state, Kalman's matrix, synchronization decomposition

1 | INTRODUCTION

Based on the exact boundary synchronization for a coupled system of wave equations by Li and Rao [3,4,6,7,9,11], the corresponding generalized exact boundary synchronization was established in [16,15,13,14]. Since there will always be errors in applications, the approximate boundary synchronization was then delivered in [5,8], which does not demand the geometrical control condition, and can be realized by much fewer boundary controls. The aim of this paper is to consider the generalized approximate boundary synchronization and the corresponding generalized approximately synchronizable state.

Consider the following coupled system of wave equations with Dirichlet boundary controls:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } (0,+\infty)\times\Omega, \\ U = 0 & \text{on } (0,+\infty)\times\Gamma_0, \\ U = DH & \text{on } (0,+\infty)\times\Gamma_1 \end{cases} \tag{1}$$

with the initial data

$$t = 0: \ (U, U') = (\hat{U}_0, \hat{U}_1) \ \text{ in } \Omega, \tag{2}$$

in which, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, satisfying $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and mes $(\Gamma_1) > 0$; $U = (u^{(1)}, \dots, u^{(N)})^T$ $(N > 1)$ denotes the state variables, coupled by a given matrix $A = (a_{ij}) \in \mathbb{M}^{N\times N}(\mathbb{R})$; $H = (h^{(1)}, \dots, h^{(M)})^T$ $(M \le N)$ stands for the boundary controls acting on Γ_1, and $D \in \mathbb{M}^{N\times M}(\mathbb{R})$ is the boundary control matrix of full column-rank. Both A and D are with constant components.

Let

$$\mathcal{H}_0 = L^2(\Omega),\ \mathcal{H}_1 = H_0^1(\Omega),\ \mathcal{L} = L^2_{\text{loc}}(0,+\infty; L^2(\Gamma_1)), \tag{3}$$

and let the dual of $\mathcal{H}_1$ be $\mathcal{H}_{-1} = H^{-1}(\Omega)$. For a full row-rank matrix $\Theta_p \in \mathbb{M}^{(N-p)\times N}(\mathbb{R})$ $(0 \le p < N)$, called the generalized synchronization matrix, we define the corresponding generalized approximate boundary synchronization as follows.

Definition 1. System (1) is generalized approximately synchronizable with respect to Θ_p at time $T > 0$, if for any given initial data $(\hat{U}_0, \hat{U}_1) \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$, there exists a sequence $\{H_n\}$ of boundary controls, $H_n \in \mathcal{L}^M$ with compact support in $[0, T]$, such that the corresponding sequence $\{U_n\}$ of solutions to problem (1)–(2) satisfies

$$\Theta_p U_n \to 0 \ (n \to +\infty) \text{ in } C^0_{\text{loc}}([T, +\infty); (\mathcal{H}_0)^{N-p}) \cap C^1_{\text{loc}}([T, +\infty); (\mathcal{H}_{-1})^{N-p}). \tag{4}$$

Similarly to the generalized exact boundary synchronization, by taking different generalized synchronization matrices Θ_p, we can get many kinds of approximate boundary synchronization, such as the approximate boundary synchronization (by groups) etc. (see the details in [16]).

Remark 1. Like in [16], any given generalized synchronization matrix Θ_p with the same kernel space $\text{Ker}(\Theta_p)$ can be regarded as the same generalized synchronization matrix. In fact, noting the generalized approximate boundary synchronization (4), since $\text{Ker}(\tilde{\Theta}_p) = \text{Ker}(\Theta_p)$ is equivalent to $\tilde{\Theta}_p = X\Theta_p$, where X is an invertible matrix, the generalized approximate boundary synchronization with respect to Θ_p is actually the generalized approximate boundary synchronization with respect to $\tilde{\Theta}_p$. □

Remark 2. When $p = 0$, Θ_0 is reversible, the generalized approximate boundary synchronization with respect to Θ_p of system (1) is in fact its approximate boundary null controllability.

□

The first part of this paper (Section 2) is devoted to investigate the generalized approximate boundary synchronization. In Section 2.1, we reconsider the approximate boundary null controllability of system (1), and show that it is equivalent to the exact boundary null controllability of system (1) in a dense subspace of $(\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$. Then in Section 2.2, we give the relationship between the generalized approximate boundary synchronization and the generalized exact boundary synchronization for system (1). In Section 2.3, we show some properties on the number of total controls.

In the second part of this paper (Section 3), we define the generalized synchronizable state when system (1) possesses the generalized approximate boundary synchronization, and give its properties and a sufficient condition to guarantee that the generalized synchronizable state dose not depend on applied boundary controls.

2 | GENERALIZED APPROXIMATE BOUNDARY SYNCHRONIZATION

2.1 | Approximate boundary null controllability

Let $\mathcal{W} = \mathcal{A} \times \mathcal{B} \subseteq (\mathcal{H}_0)^N \times (\mathcal{H}_0)^N$ and let its dual space $\mathcal{W}'$ be $\mathcal{V} = \mathcal{B}' \times \mathcal{A}'$ given by

$$\langle (f_1, f_2), (g_1, g_2) \rangle_{\mathcal{A}\times\mathcal{B}, \mathcal{B}'\times\mathcal{A}'} = (f_1, g_2)_{(\mathcal{H}_0)^N} + (f_2, g_1)_{(\mathcal{H}_0)^N}.$$

Let

$$\mathcal{W} \subseteq (\mathcal{H}_0)^N \times (\mathcal{H}_0)^N \subseteq \mathcal{V}$$

with dense embedding, and $\mathcal{V}$ be a Hilbert space such that problem (1)–(2) is well-posed for (U, U') in $\mathcal{V}$. For instance, $\mathcal{V} = (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$, while $\mathcal{W} = (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$.

Definition 2. System (1) is approximately null controllable in $\mathcal{V}$ at time $T > 0$, if for any given initial data $(\hat{U}_0, \hat{U}_1) \in \mathcal{V}$, there exists a sequence $\{H_n\}$ of boundary controls $H_n \in \mathcal{L}^M$ with compact support in $[0, T]$, such that the sequence $\{U_n\}$ of solutions to problem (1)–(2) satisfies

$$(U_n, U_n')(T) \to (0, 0) \text{ in } \mathcal{V}, \ n \to +\infty. \tag{5}$$

Remark 3. By [5, Remark 3.1], for the sequence $\{U_n\}$ of solutions to problem (1)–(2), its convergence (5) at $t = T$ is equivalent to its local convergence at $t \ge T$:

$$(U_n, U_n') \to (0, 0) \text{ in } C^0_{\text{loc}}([T, +\infty); \mathcal{V}), \ n \to +\infty.$$

□

Consider the following backward problem

$$\begin{cases} \mho'' - \Delta \mho + A\mho = 0 & \text{in } (0,T)\times\Omega, \\ \mho = 0 & \text{on } (0,T)\times\Gamma_0, \\ \mho = DH & \text{on } (0,T)\times\Gamma_1, \\ t = T:\ (\mho,\mho') = (0,0) & \text{in } \Omega. \end{cases} \tag{6}$$

Let $\mathcal{C}$ be the set of all the initial data $(\mho,\mho')(0)$ of problem (6) as H varies in $\mathcal{L}^M$. From[5, Lemma 3.1], we have

Lemma 1. System (1) is approximately null controllable in $\mathcal{V}$ if and only if $\bar{\mathcal{C}} = \mathcal{V}$.

Thus, we have the following relationship between the approximate and exact boundary null controllabilities.

Theorem 1. System (1) is approximately null controllable in $\mathcal{V}$ if and only if there exists a dense subspace $\mathcal{V}_0$ of $\mathcal{V}$: $\bar{\mathcal{V}}_0 = \mathcal{V}$, such that system (1) is exactly null controllable in $\mathcal{V}_0$.

Proof. By the definition of $\mathcal{C}$, for any given initial data $(\hat{U}_0,\hat{U}_1) \in \mathcal{C}$, there exists a boundary control $H \in \mathcal{L}^M$ with compact support in $[0,T]$, such that the corresponding solution U of the original system (1) satisfies

$$t = T:\ \ (U,U') = (0,0), \tag{7}$$

therefore

$$t \geq T:\ \ \ U \equiv 0, \tag{8}$$

hence system (1) is exactly null controllable in $\mathcal{C}$.

By Lemma 1, if system (1) is approximately null controllable in $\mathcal{V}$, then $\bar{\mathcal{C}} = \mathcal{V}$. Thus, taking $\mathcal{V}_0 = \mathcal{C}$, we have that $\bar{\mathcal{V}}_0 = \mathcal{V}$, and system (1) is exactly null controllable in $\mathcal{V}_0$.

On the other hand, if there exists a $\mathcal{V}_0$ satisfying $\bar{\mathcal{V}}_0 = \mathcal{V}$, such that system (1) is exactly null controllable in $\mathcal{V}_0$, namely, for any given $(\mho_0,\mho_1) \in \mathcal{V}_0$, there exists an $H \in \mathcal{L}^M$ with compact support in $[0,T]$, such that the corresponding initial data of problem (6) are given by $(\mho,\mho')(0) = (\mho_0,\mho_1)$.

Since $\bar{\mathcal{V}}_0 = \mathcal{V}$, for any given $(\hat{U}_0,\hat{U}_1) \in \mathcal{V}$, there exists a sequence $\{H_n\}$ of boundary controls, $H_n \in \mathcal{L}^M$ with compact support in $[0,T]$, such that the corresponding sequence of initial data of problem (6) satisfies

$$(\mho_n,\mho_n')(0) \to (\hat{U}_0,\hat{U}_1) \text{ in } \mathcal{V},\ \ n \to +\infty. \tag{9}$$

By well-posedness, the sequence $\{U_n\}$ of solutions to problem (1)–(2) with $\{H_n\}$ as the sequence of boundary controls satisfies

$$||(U_n,U_n')(T)||_{\mathcal{V}} = ||(U_n,U_n')(T) - (0,0)||_{\mathcal{V}} \leq c||(\hat{U}_0,\hat{U}_1) - (\mho_n,\mho_n')(0)||_{\mathcal{V}} \to 0,\ \ n \to +\infty, \tag{10}$$

where $c > 0$ is a constant. Therefore, system (1) is approximately null controllable in $\mathcal{V}$. □

By Lemma 1, similarly to[5, Theorem 3.1], we can get the following equivalence between the approximate boundary null controllability of system (1) and the D-observation of the corresponding adjoint system.

Lemma 2. System (1) is approximately null controllable in $\mathcal{V} = \mathcal{W}'$ at time $T > 0$ if and only if its adjoint problem

$$\begin{cases} \Phi'' - \Delta\Phi + A^T\Phi = 0 & \text{in } (0,+\infty)\times\Omega, \\ \Phi = 0 & \text{on } (0,+\infty)\times\Gamma, \\ t = 0:\ (\Phi,\Phi') = (\Phi_0,\Phi_1) & \text{in } \Omega \end{cases} \tag{11}$$

with $(\Phi_0,\Phi_1) \in \mathcal{W}$ is D-observable, namely, the partial observation

$$D^T\partial_\nu\Phi \equiv 0 \ \text{ on } [0,T]\times\Gamma_1 \tag{12}$$

implies $(\Phi_0,\Phi_1) \equiv (0,0)$, then $\Phi \equiv 0$.

Next, we consider the relationship for the approximate boundary null controllability in different spaces. Let $\mathcal{V}_0$ be a dense subspace of $\mathcal{V}$:

$$\bar{\mathcal{V}}_0 = \mathcal{V}. \tag{13}$$

We have

Lemma 3. System (1) is approximately null controllable in $\mathcal{V}$ at time $T > 0$ if and only if (5) holds for system (1) with any given initial data $(\hat{U}_0, \hat{U}_1) \in \mathcal{V}_0$.

Proof. We only need to prove the sufficiency. For any given $(\hat{U}_0, \hat{U}_1) \in \mathcal{V}$, noting (13), there exist $(\hat{U}_0^n, \hat{U}_1^n) \in \mathcal{V}_0$, such that

$$(\hat{U}_0^n, \hat{U}_1^n) \to (\hat{U}_0, \hat{U}_1) \text{ in } \mathcal{V},\ n \to +\infty. \tag{14}$$

For the initial data $(\hat{U}_0^n, \hat{U}_1^n)$ corresponding to every n, by (5), there exist $H_k^n \in \mathcal{L}^M$ with compact support in $[0, T]$, such that the corresponding solutions U_k^n to system (1) satisfy

$$(U_k^n, U_k^{n\prime})(T) \to (0, 0) \text{ in } \mathcal{V},\ k \to +\infty. \tag{15}$$

Then there exists $k = k(n)$ such that

$$(U_{k(n)}^n, U_{k(n)}^{n\prime})(T) \to (0, 0) \text{ in } \mathcal{V},\ n \to +\infty. \tag{16}$$

Noting (14), by well-posedness, when system (1) possesses $(\hat{U}_0, \hat{U}_1)$ as the initial data, taking $H_n = H_{k(n)}^n$ as the boundary controls, and denoting the corresponding solution as U_n, we have

$$(U_n, U_n')(T) - (U_{k(n)}^n, U_{k(n)}^{n\prime})(T) \to (0, 0) \text{ in } \mathcal{V},\ n \to +\infty. \tag{17}$$

Noting (16), we get (5), that is, system (1) is approximately null controllable in $\mathcal{V}$. The proof is complete. □

Theorem 2. System (1) is approximately null controllable in $(\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$ if and only if it is approximately null controllable in a Hilbert space $\mathcal{V}$ with $(\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$ being its dense subspace.

Proof. First, assume that system (1) is approximately null controllable in $(\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$, by Lemma 2, the adjoint problem (11) is D-observable in $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$.

For the Hilbert space $\mathcal{V}$ with $(\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$ being its dense subspace, denote it as the dual space of a Hilbert space $\mathcal{W}$: $\mathcal{V} = \mathcal{W}'$. By $(\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N \subseteq \mathcal{V}$, we have $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N \supseteq \mathcal{W}$. Therefore, the adjoint problem (11) is D-observable in $\mathcal{W}$. Then, by Lemma 2, system (1) is approximately null controllable in $\mathcal{V}$.

Conversely, thanks to Lemma 2, we only need to prove that the D-observability in $\mathcal{W}$ implies the D-observability in $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ for the adjoint problem (11).

When problem (11) is D-observable in $\mathcal{W}$, we can define a Hilbert norm in $\mathcal{W}$ by

$$||(\Phi_0, \Phi_1)||_{\mathcal{F}}^2 = \int_0^T \int_{\Gamma_1} |D^T \partial_\nu \Phi|^2 d\Gamma dt, \tag{18}$$

which induces a closure $\mathcal{F}$ of $\mathcal{W}$. From the definition of $\mathcal{F}$-norm, observation (12) implies $||(\Phi_0, \Phi_1)||_{\mathcal{F}}^2 = 0$, then, for $(\Phi_0, \Phi_1) \in \mathcal{F}$, observation (12) implies $(\Phi_0, \Phi_1) \equiv (0, 0)$, then $\Phi \equiv 0$. Thus, problem (11) is D-observable in $\mathcal{F}$.

Due to the hidden regularity[1] of problem (11): there exists a constant $c > 0$, such that

$$\int_0^T \int_{\Gamma_1} |\partial_\nu \Phi|^2 d\Gamma dt \leq c||(\Phi_0, \Phi_1)||_{(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N}^2, \tag{19}$$

namely, $||(\Phi_0, \Phi_1)||_{\mathcal{F}}^2 \leq c||(\Phi_0, \Phi_1)||_{(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N}^2$, then we have

$$(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N \subseteq \mathcal{F}, \tag{20}$$

hence, problem (11) is D-observable in $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$. The proof is complete. □

2.2 | Generalized approximate boundary synchronization

We now consider the generalized approximate boundary synchronization with respect to Θ_p for system (1) as $(\hat{U}_0, \hat{U}_1) \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$. Meanwhile we will explain the relationship between the generalized approximate boundary synchronization and the generalized exact boundary synchronization.

First, we consider the case that the coupling matrix A satisfies the following condition of Θ_p-compatibility:

$$A\,\mathrm{Ker}(\Theta_p) \subseteq \mathrm{Ker}(\Theta_p). \tag{21}$$

Then there exists $\bar{A}_p \in \mathbb{M}^{(N-p)\times(N-p)}(\mathbb{R})$, such that

$$\Theta_p A = \bar{A}_p \Theta_p. \tag{22}$$

Let $W = \Theta_p U$. We get the following self-closed reduced system:

$$\begin{cases} W'' - \Delta W + \bar{A}_p W = 0 & \text{in } (0,+\infty)\times\Omega, \\ W = 0 & \text{on } (0,+\infty)\times\Gamma_0, \\ W = \Theta_p DH & \text{on } (0,+\infty)\times\Gamma_1 \end{cases} \tag{23}$$

with the initial data $(W, W')(0) \in (\mathcal{H}_0)^{N-p} \times (\mathcal{H}_{-1})^{N-p}$. Correspondingly, the adjoint problem of reduced system (23) is

$$\begin{cases} \Psi'' - \Delta\Psi + \bar{A}_p^T \Psi = 0 & \text{in } (0,+\infty)\times\Omega, \\ \Psi = 0 & \text{on } (0,+\infty)\times\Gamma, \\ t = 0:\ (\Psi, \Psi') = (\Psi_0, \Psi_1) & \text{in } \Omega \end{cases} \tag{24}$$

with the initial data $(\Psi_0, \Psi_1) \in (\mathcal{H}_1)^{N-p} \times (\mathcal{H}_0)^{N-p}$. Thus, by definition and Lemma 2, we have

Theorem 3. Assume that A satisfies the condition of Θ_p-compatibility (21). Then system (1) is generalized approximately synchronizable with respect to Θ_p at time $T > 0$ if and only if the reduced system (23) is approximately null controllable, and equivalently if and only if the reduced adjoint problem (24) is $\Theta_p D$-observable, that is,

$$(\Theta_p D)^T \partial_\nu \Psi \equiv 0 \ \text{ on } [0,T]\times\Gamma_1 \tag{25}$$

implies $(\Psi_0, \Psi_1) \equiv (0, 0)$, then $\Psi \equiv 0$.

Now we consider the general case. For any given coupling matrix A, we define an extended matrix $\tilde{\Theta}_{\tilde{p}} (0 \le \tilde{p} \le p)$ of the generalized synchronization matrix Θ_p by

$$\operatorname{Im}(\tilde{\Theta}_{\tilde{p}}^T) = \operatorname{Span}(\Theta_p^T, A^T\Theta_p^T, \dots, (A^T)^{N-1}\Theta_p^T). \tag{26}$$

Let $\tilde{\Theta}_{\tilde{p}}$ be an $(N - \tilde{p}) \times N$ full row-rank matrix, denoted by

$$\tilde{\Theta}_{\tilde{p}} = \begin{pmatrix} \Theta_p \\ (x_{N-p+1}, \dots, x_{N-\tilde{p}})^T \end{pmatrix}. \tag{27}$$

Thus A always satisfies the condition of $\tilde{\Theta}_{\tilde{p}}$-compatibility:

$$A \operatorname{Ker}(\tilde{\Theta}_{\tilde{p}}) \subseteq \operatorname{Ker}(\tilde{\Theta}_{\tilde{p}}). \tag{28}$$

Obviously, A satisfies the condition of Θ_p-compatibility (21) if and only if $\tilde{p} = p$.

Then, $\tilde{W} = \tilde{\Theta}_{\tilde{p}} U$ satisfies the following extended reduced system:

$$\begin{cases} \tilde{W}'' - \Delta\tilde{W} + \bar{A}_{\tilde{p}} \tilde{W} = 0 & \text{in } (0,+\infty)\times\Omega, \\ \tilde{W} = 0 & \text{on } (0,+\infty)\times\Gamma_0, \\ \tilde{W} = \bar{D}_{\tilde{p}} H & \text{on } (0,+\infty)\times\Gamma_1 \end{cases} \tag{29}$$

with the corresponding initial data

$$t = 0: \quad (\tilde{W}, \tilde{W}') = \tilde{\Theta}_{\tilde{p}}(\hat{U}_0, \hat{U}_1) \ \text{ in } \Omega, \tag{30}$$

in which, $\bar{A}_{\tilde{p}}$ and $\bar{D}_{\tilde{p}}$ are given by

$$\tilde{\Theta}_{\tilde{p}} A = \bar{A}_{\tilde{p}} \tilde{\Theta}_{\tilde{p}}, \ \ \bar{D}_{\tilde{p}} = \tilde{\Theta}_{\tilde{p}} D, \tag{31}$$

respectively. Similarly to the related results of generalized exact boundary synchronization in [15], we have

Theorem 4. System (1) is generalized approximately synchronizable with respect to Θ_p if and only if it is generalized approximately synchronizable with respect to $\tilde{\Theta}_{\tilde{p}}$.

Proof. Noting (27) and (4), it is easy to get the sufficiency. We need only to verify the necessity part.

By definition, for any given initial data $(\hat{U}_0, \hat{U}_1) \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$, there exists a sequence $\{H_n\}$ of boundary controls, $H_n \in \mathcal{L}^M$ with compact support in $[0,T]$, such that (4) holds. Then, multiplying the coupled wave equations given in (1) by Θ_p from the left, we have

$$\Theta_p A U_n \to 0 \ (n \to +\infty) \text{ in } C^0_{\text{loc}}([T,+\infty); (\mathcal{H}_{-2})^{N-p}) \cap C^1_{\text{loc}}([T,+\infty); (\mathcal{H}_{-3})^{N-p}).$$

similarly, multiplying the coupled wave equations given in (1) by $\Theta_p A^{k-1}$ ($k = 2, \dots, N-1$) from the left successively, we have

$$\Theta_p A^k U_n \to 0\ (n \to +\infty) \text{ in } C^0_{\mathrm{loc}}([T,+\infty);(\mathcal{H}_{-2k})^{N-p}) \cap C^1_{\mathrm{loc}}([T,+\infty);(\mathcal{H}_{-1-2k})^{N-p}).$$

Then, by (26), for any given initial data $(\tilde{W}, \tilde{W}')(0) = \tilde{\Theta}_{\tilde{p}}(\hat{U}_0, \hat{U}_1) \in (\mathcal{H}_0)^{N-\tilde{p}} \times (\mathcal{H}_{-1})^{N-\tilde{p}}$, we have

$$\tilde{W}_n = \tilde{\Theta}_{\tilde{p}} U_n \to 0\ (n \to +\infty) \text{ in } C^0_{\mathrm{loc}}([T,+\infty);(\mathcal{H}_{-2(N-1)})^{N-\tilde{p}}) \cap C^1_{\mathrm{loc}}([T,+\infty);(\mathcal{H}_{-1-2(N-1)})^{N-\tilde{p}}).$$

By Remark 3, we have

$$(\tilde{W}_n, \tilde{W}_n')(T) \to (0,0)\ (n \to +\infty) \text{ in } (\mathcal{H}_{-2(N-1)})^{N-\tilde{p}} \times (\mathcal{H}_{-1-2(N-1)})^{N-\tilde{p}}.$$

Since $\mathcal{V}_0 = (\mathcal{H}_0)^{N-\tilde{p}} \times (\mathcal{H}_{-1})^{N-\tilde{p}}$ is dense in $\mathcal{V} = (\mathcal{H}_{-2(N-1)})^{N-\tilde{p}} \times (\mathcal{H}_{-1-2(N-1)})^{N-\tilde{p}}$, by Lemma 3, the extended reduced system (29) is approximately null controllable in $\mathcal{V} = (\mathcal{H}_{-2(N-1)})^{N-\tilde{p}} \times (\mathcal{H}_{-1-2(N-1)})^{N-\tilde{p}}$.

Thus, by Theorem 2, the extended reduced system (29) is approximately null controllable in $(\mathcal{H}_0)^{N-\tilde{p}} \times (\mathcal{H}_{-1})^{N-\tilde{p}}$. Noting that A satisfies the condition of $\tilde{\Theta}_{\tilde{p}}$-compatibility (28), by Theorem 3, system (1) is generalized approximately synchronizable with respect to $\tilde{\Theta}_{\tilde{p}}$. □

From the above theorem and Lemma 2, we have

Corollary 1. System (1) is generalized approximately synchronizable with respect to Θ_p if and only if the extended reduced system (29) is approximately null controllable, and equivalently if and only if the corresponding reduced adjoint problem

$$\begin{cases} \tilde{\Psi}'' - \Delta\tilde{\Psi} + \bar{A}^T_{\tilde{p}}\tilde{\Psi} = 0 & \text{in } (0,+\infty)\times\Omega, \\ \tilde{\Psi} = 0 & \text{on } (0,+\infty)\times\Gamma, \\ t = 0 : (\tilde{\Psi}, \tilde{\Psi}') = (\tilde{\Psi}_0, \tilde{\Psi}_1) & \text{in } \Omega \end{cases} \tag{32}$$

with the initial data $(\tilde{\Psi}_0, \tilde{\Psi}_1) \in (\mathcal{H}_1)^{N-\tilde{p}} \times (\mathcal{H}_0)^{N-\tilde{p}}$ is $\tilde{\Theta}_{\tilde{p}} D$-observable.

This gives the following

Corollary 2. If D satisfies

$$\mathrm{rank}(\tilde{\Theta}_{\tilde{p}} D) = N - \tilde{p}, \tag{33}$$

namely, A, D and Θ_p satisfy $\mathrm{Ker}(D^T) \cap \mathrm{Span}(\Theta_p^T, A^T\Theta_p^T, \dots, (A^T)^{N-1}\Theta_p^T) = \{0\}$, then system (1) is generalized approximately synchronizable with respect to Θ_p.

Proof. By Theorem 3 and the Holmgren uniqueness theorem[12], system (1) is generalized approximately synchronizable with respect to $\tilde{\Theta}_{\tilde{p}}$, then by Theorem 4 we get the conclusion. □

Furthermore, we give the following relationship between the approximate and exact generalized boundary synchronizations.

Theorem 5. System (1) is generalized approximately synchronizable with respect to Θ_p if and only if there exists a dense subspace $\mathcal{V}_0$ of $(\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$:

$$\bar{\mathcal{V}}_0 = (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N, \tag{34}$$

such that system (1) is generalized exactly synchronizable with respect to Θ_p in $\mathcal{V}_0$.

Proof. By Theorem 4, the generalized approximate boundary synchronization with respect to Θ_p of system (1) is equivalent to that with respect to $\tilde{\Theta}_{\tilde{p}}$, and then equivalent to the approximate boundary null controllability of the extended reduced system (29) in $(\mathcal{H}_0)^{N-\tilde{p}} \times (\mathcal{H}_{-1})^{N-\tilde{p}}$. By Theorem 1, this is equivalent to the exact boundary null controllability of the extended reduced system (29) in a dense subspace $\mathcal{D}$:

$$\bar{\mathcal{D}} = (\mathcal{H}_0)^{N-\tilde{p}} \times (\mathcal{H}_{-1})^{N-\tilde{p}}.$$

According to[16], this is equivalent to the generalized exact boundary synchronization with respect to $\tilde{\Theta}_{\tilde{p}}$ of system (1) in

$$\mathcal{V}_0 = \{U \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N : \tilde{\Theta}_{\tilde{p}} U \in \mathcal{D}\},$$

then equivalent to the generalized exact boundary synchronization with respect to Θ_p of system (1) in $\mathcal{V}_0$. Since $\bar{\mathcal{V}}_0 = \{U \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N : \tilde{\Theta}_{\tilde{p}} U \in \bar{\mathcal{D}}\} = (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$, (34) holds. The proof is complete. □

Corollary 3. If system (1) is generalized approximately synchronizable with respect to Θ_p, then either A satisfies the condition of Θ_p-compatibility (21), or there exists a full row-rank extended matrix of Θ_p:

$$\tilde{\Theta}_{p-1} = \begin{pmatrix} \Theta_p \\ x^T \end{pmatrix} \tag{35}$$

such that system (1) is generalized approximately synchronizable with respect to $\tilde{\Theta}_{p-1}$, in which x^T is an N-dimensional row vector.

Proof. By Theorem 5, system (1) possesses the generalized exact boundary synchronization with respect to Θ_p in a dense subspace $\mathcal{V}_0$ of $(\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$, hence for any given initial data $(\hat{U}_0, \hat{U}_1) \in \mathcal{V}_0$, there exists a boundary control $H \in \mathcal{L}^M$ with compact support in $[0, T]$, such that the solution U to problem (1)–(2) satisfies

$$t \geq T: \quad \Theta_p U \equiv 0. \tag{36}$$

If A does not satisfy the condition of Θ_p-compatibility (21), then, similarly to [16], there exists an extended matrix $\tilde{\Theta}_{p-1}$ as (35), such that

$$t \geq T: \quad \tilde{\Theta}_{p-1} U \equiv 0, \tag{37}$$

namely system (1) possesses the generalized exact boundary synchronization with respect to $\tilde{\Theta}_{p-1}$ in $\mathcal{V}_0$. Then by Theorem 5, we get the conclusion. □

Remark 4. Similarly to the generalized exact boundary synchronization, if system (1) is not generalized approximately synchronizable by $(p-1)$ groups, then A must satisfy the condition of Θ_p-compatibility (21), provided that system (1) is generalized approximately synchronizable with respect to Θ_p. □

2.3 | Number of total controls

For system (1), the boundary controls H act not only directly on some state variables on the boundary through the boundary control matrix D, but also indirectly on the rest state variables in the domain through the coupling matrix A. According to the following Lemma 4, $\text{rank}(D, AD, \ldots, A^{N-1}D)$ can be regarded as the number of total controls acting on system (1), which stands for the total number of state variables influenced by boundary controls H both directly through the boundary control matrix D and indirectly through the coupling matrix A.

Proposition 1.

$$\text{rank}(D, AD, \ldots, A^{N-1}D) = N - p \tag{38}$$

is equivalent to that the maximum dimension of A^T-invariant subspaces included in $\text{Ker}(D^T)$ is equal to p, and also equivalent to that there exists an invertible matrix X such that

$$A^* = XAX^{-1} = \begin{pmatrix} \bar{A}_p^* & l^* \\ b^* & \tilde{A}_p^* \end{pmatrix}, \quad D^* = XD = \begin{pmatrix} \bar{D}^* \\ d^* \end{pmatrix}, \tag{39}$$

where $\bar{A}_p^* \in \mathbb{M}^{(N-p)\times(N-p)}(\mathbb{R})$, $\tilde{A}_p^* \in \mathbb{M}^{p\times p}(\mathbb{R})$, $l^*, b^{*T} \in \mathbb{M}^{(N-p)\times p}(\mathbb{R})$, $\bar{D}^* \in \mathbb{M}^{(N-p)\times M}(\mathbb{R})$, $d^* \in \mathbb{M}^{p\times M}(\mathbb{R})$, and

$$b^* = 0, \ d^* = 0, \tag{40}$$

satisfies

$$\text{rank}(\bar{D}^*, \bar{A}_p^* \bar{D}^*, \ldots, \bar{A}_p^{*N-p-1} \bar{D}^*) = N - p. \tag{41}$$

Here, condition (40) means that the final p rows $(\eta_1, \ldots, \eta_p)^T$ of X satisfy

$$(\eta_1, \ldots, \eta_p)^T A = \tilde{A}_p^* (\eta_1, \ldots, \eta_p)^T \quad \text{and} \quad (\eta_1, \ldots, \eta_p)^T D = 0, \tag{42}$$

namely, $\text{Span}\{\eta_1, \ldots, \eta_p\}$ is an A^T-invariant subspace included in $\text{Ker}(D^T)$.

Proof. If (38) holds, then there exists an invertible matrix X such that the last p rows of $X(D, AD, \ldots, A^{N-1}D)$ are all zeros. Hence A^* and D^* given by (39) satisfy

$$(D^*, A^*D^*, \ldots, A^{*N-1}D^*) = X(D, AD, \ldots, A^{N-1}D), \tag{43}$$

the last p rows of which are also all zeros, therefore $d^* = 0$ and the last p rows of

$$A^* D^* = \begin{pmatrix} \bar{A}_p^* \bar{D}^* \\ b^* \bar{D}^* \end{pmatrix}, A^{*2} D^* = \begin{pmatrix} \bar{A}_p^{*2} \bar{D}^* \\ b^* \bar{A}_p^* \bar{D}^* \end{pmatrix}, \dots, A^{*N-1} D^* = \begin{pmatrix} \bar{A}_p^{*N-1} \bar{D}^* \\ b^* \bar{A}_p^{*N-2} \bar{D}^* \end{pmatrix} \tag{44}$$

are all zeros, namely,

$$b^*(\bar{D}^*, \bar{A}_p^* \bar{D}^*, \dots, \bar{A}_p^{*N-2} \bar{D}^*) = 0. \tag{45}$$

Since the last p rows of (43) are all zeros:

$$X(D, AD, \dots, A^{N-1} D) = \begin{pmatrix} \bar{D}^* & \bar{A}_p^* \bar{D}^* & \dots & \bar{A}_p^{*N-1} \bar{D}^* \\ 0 & 0 & \dots & 0 \end{pmatrix}, \tag{46}$$

noting (38), we have

$$\begin{aligned} \operatorname{rank}(\bar{D}^*, \bar{A}_p^* \bar{D}^*, \dots, \bar{A}_p^{*N-p-1} \bar{D}^*) &= \operatorname{rank}(X(D, AD, \dots, A^{N-1} D)) \\ &= \operatorname{rank}(D, AD, \dots, A^{N-1} D) = N - p, \end{aligned} \tag{47}$$

then by (45) we get $b^* = 0$.

Conversely, if there exists an invertible matrix X such that A^* and D^* given by (39) satisfy (40) and (41), then we have (43) and (46), therefore by (47) we get (38). The proof is complete. □

Now we explain that the number of total controls $\operatorname{rank}(D, AD, \dots, A^{N-1} D)$ is indeed the total number of state variables influenced by boundary controls.

Lemma 4. Under assumption (38), there exists an invertible linear transformation such that system (1) under this transformation has only $(N - p)$ state variables depending on boundary controls, while the other p state variables are independent of boundary controls.

Proof. By Proposition 1, there exists an invertible matrix X such that (39)–(41) hold. Let

$$U^* = XU = (u^{*(1)}, \dots, u^{*(N)})^T. \tag{48}$$

Multiplying problem (1)–(2) by X from the left, we get

$$\begin{cases} U^{*\prime\prime} - \Delta U^* + A^* U^* = 0 & \text{in } (0, +\infty) \times \Omega, \\ U^* = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ U^* = D^* H & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \tag{49}$$

with the corresponding initial data

$$t = 0 : \ (U^*, U^{*\prime}) = X(\hat{U}_0, \hat{U}_1) \ \text{ in } \Omega. \tag{50}$$

By (40), the last p components $\hat{u}_p^* = (u^{*(N-p+1)}, \dots, u^{*(N)})^T$ of U^* satisfy the following self-closed system

$$\begin{cases} \hat{u}_p^{*\prime\prime} - \Delta \hat{u}_p^* + \tilde{A}_p^* \hat{u}_p^* = 0 & \text{in } (0, +\infty) \times \Omega, \\ \hat{u}_p^* = 0 & \text{on } (0, +\infty) \times \Gamma, \end{cases} \tag{51}$$

which is purely determined by the corresponding initial data, and independent of boundary controls H; while, the former $(N-p)$ components $\hat{u}_{N-p}^* = (u^{*(1)}, \dots, u^{*(N-p)})^T$ satisfy

$$\begin{cases} \hat{u}_{N-p}^{*\prime\prime} - \Delta \hat{u}_{N-p}^* + \bar{A}_p^* \hat{u}_{N-p}^* + l^* \hat{u}_p^* = 0 & \text{in } (0, +\infty) \times \Omega, \\ \hat{u}_{N-p}^* = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \hat{u}_{N-p}^* = \bar{D}^* H & \text{on } (0, +\infty) \times \Gamma_1. \end{cases} \tag{52}$$

For any given initial data (50), noting that $\hat{u}_p^*$ is independent of boundary controls, the difference $v = \hat{u}_{(N-p)1}^* - \hat{u}_{(N-p)2}^* = (v^{(1)}, \dots, v^{(N-p)})^T$ of $\hat{u}_{(N-p)1}^*$ and $\hat{u}_{(N-p)2}^*$, corresponding to different boundary controls H_1 and H_2, respectively, satisfies

$$\begin{cases} v'' - \Delta v + \bar{A}_p^* v = 0 & \text{in } (0, +\infty) \times \Omega, \\ v = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ v = \bar{D}^* h & \text{on } (0, +\infty) \times \Gamma_1, \\ t = 0 : (v, v') = (0, 0) & \text{in } \Omega, \end{cases} \tag{53}$$

in which $h = H_1 - H_2$ is an M dimensional non-zero vector function.

Then we need only to verify that for each component $v^{(i)}$ $(i = 1, \dots, N-p)$ of v, there exists a non-zero vector function h such that

$$v^{(i)} \not\equiv 0.$$

Let h be smooth enough and satisfy the compatibility conditions of all orders with the zero initial data when $(t, x) \in \{0\} \times \Gamma_1$ for problem (53). By well-posedness, the corresponding solution v to problem (53) is smooth enough up to the boundary. Denote $\Box v = v'' - \Delta v$. Noting the coupled wave equations given in (53) and the boundary condition

$$v = \bar{D}^* h$$

on the boundary Γ_1, we have

$$-\Box v = \bar{A}_p^* \bar{D}^* h$$

on the boundary Γ_1. Applying the operator $\Box$ on the coupled wave equations given in (53), we get

$$\Box^2 v = (\bar{A}_p^*)^2 \bar{D}^* h$$

on the boundary Γ_1. Similarly, we have

$$(-1)^k \Box^k v = (\bar{A}_p^*)^k \bar{D}^* h, \ k = 3, 4, \dots$$

on the boundary Γ_1. Hence, for the i-th component $v^{(i)}$ $(i = 1, \dots, N-p)$ of v, we get

$$(-1)^k \Box^k v^{(i)} = ((\bar{A}_p^*)^k \bar{D}^*)_i h, \ k = 0, 1, \dots \tag{54}$$

on the boundary Γ_1, in which $((\bar{A}_p^*)^k \bar{D}^*)_i$ stands for the i-th row of the matrix $(\bar{A}_p^*)^k \bar{D}^*$.

Noting (41), each row of the matrix $(\bar{D}^*, \bar{A}_p^* \bar{D}^*, \dots, (\bar{A}_p^*)^{N-p-1} \bar{D}^*)$ is a non-zero vector, thus, for $i = 1, \dots, N-p$, we may assume that in its i-th row $((\bar{D}^*)_i, (\bar{A}_p^* \bar{D}^*)_i, \dots, ((\bar{A}_p^*)^{N-p-1} \bar{D}^*)_i)$, we have

$$((\bar{A}_p^*)^{\tau_i} \bar{D}^*)_i \neq (0, \dots, 0),$$

where $0 \leq \tau_i \leq N - p - 1$. Hence, by (54), for each component $v^{(i)}$ $(i = 1, \dots, N-p)$ of v, there exists a non-zero vector function h such that

$$(-1)^{\tau_i} \Box^{\tau_i} v^{(i)} = ((\bar{A}_p^*)^{\tau_i} \bar{D}^*)_i h \not\equiv 0,$$

then $v^{(i)} \not\equiv 0$. Therefore, all the components of $\hat{u}_{N-p}^*$ depend on boundary controls H. The proof is complete. □

Theorem 6. If system (1) is generalized approximately synchronizable with respect to Θ_p, then we have the following lower-bound estimate for the number of total controls:

$$\text{rank}(D, AD, \dots, A^{N-1}D) \geq N - p. \tag{55}$$

Proof. Taking an invertible square matrix

$$\tilde{X} = \begin{pmatrix} \Theta_p \\ \hat{x}_p^T \end{pmatrix},$$

let $U^* = \tilde{X} U$. The former $(N-p)$ components $\hat{u}_{N-p}^* = \Theta_p U$ satisfy (4). Clearly, under any invertible linear transformation, U^* has at least $(N-p)$ components depending on applied boundary controls H, thus the same is true for U: under any invertible linear transformation, U has at least $(N-p)$ components depending on applied boundary controls H. Therefore, by Lemma 4, we have (55). In fact, if (55) fails, we may write $\text{rank}(D, AD, \dots, A^{N-1}D) = N - r < N - p$. By Lemma 4, there exists an invertible linear transformation X such that $U^* = XU$ has only $(N-r)$ components depending on H. Noting $N - r < N - p$, this leads to a contradiction. □

We have still a stronger conclusion as follows.

Theorem 7. If system (1) is generalized approximately synchronizable with respect to Θ_p, then

$$\text{rank}(\tilde{\Theta}_{\tilde{p}}(D, AD, \dots, A^{N-1}D)) = N - \tilde{p}, \tag{56}$$

in which, $\tilde{\Theta}_{\tilde{p}}$ is an extended matrix of Θ_p, given by (26), and $\tilde{p} = N - \dim \text{Im}(\tilde{\Theta}_{\tilde{p}}^T)$. In particular, we have

$$\text{rank}(\Theta_p(D, AD, \dots, A^{N-1}D)) = N - p. \tag{57}$$

Proof. Since system (1) is generalized approximately synchronizable with respect to Θ_p, by Corollary 1, the extended reduced system (29) is approximately null controllable, namely, generalized approximately synchronizable with respect to the corresponding Θ_0. Then by Lemma 4, we have

$$\mathrm{rank}(\bar{D}_{\tilde{p}}, \bar{A}_{\tilde{p}}\bar{D}_{\tilde{p}}, \dots, \bar{A}_{\tilde{p}}^{N-\tilde{p}+1}\bar{D}_{\tilde{p}}) = N - \tilde{p}, \tag{58}$$

where $\bar{A}_{\tilde{p}}$ and $\bar{D}_{\tilde{p}}$ satisfy (31), hence $\tilde{\Theta}_{\tilde{p}}(D, AD, \dots, A^{N-1}D) = (\bar{D}_{\tilde{p}}, \bar{A}_{\tilde{p}}\bar{D}_{\tilde{p}}, \dots, \bar{A}_{\tilde{p}}^{N-1}\bar{D}_{\tilde{p}})$. Therefore we have (56), then by (27), we have also (57). □

Remark 5. To be generalized approximately synchronizable with respect to Θ_p for system (1), Theorem 7 shows that (56) is a necessary condition. However, it can be also a sufficient condition in certain special cases where (58) is sufficient for the extended reduced system (29) to be approximately null controllable (see examples in[8,10]). □

Remark 6. From (56), the generalized approximate boundary synchronization with respect to Θ_p of system (1) implies that the number of total controls $\mathrm{rank}(D, AD, \dots, A^{N-1}D) \geq N - \tilde{p} \geq N - p$ might be greater than $(N - p)$. Therefore, under the minimal number $(N - p)$ of total controls, we have $\tilde{p} = p$, namely, A satisfies the condition of Θ_p-compatibility (21) (A stronger result can be seen in Lemma 6 below). □

3 | GENERALIZED APPROXIMATELY SYNCHRONIZABLE STATES

The aim of this section is to define and study the generalized approximately synchronizable states when system (1) possesses the generalized approximate boundary synchronization with respect to Θ_p.

3.1 | Definition of generalized approximately synchronizable states

Let $\{\epsilon_1, \dots, \epsilon_p\}$ be a basis of $\mathrm{Ker}(\Theta_p)$:

$$\mathrm{Ker}(\Theta_p) = \mathrm{Span}\{\epsilon_1, \dots, \epsilon_p\}. \tag{59}$$

$\{\epsilon_1, \dots, \epsilon_p\}$ is called the generalized synchronization basis, and p the grouping number. Now we define the generalized approximate boundary synchronization with respect to $\{\epsilon_1, \dots, \epsilon_p\}$ for system (1), then we will show that it is equivalent to the generalized approximate boundary synchronization with respect to Θ_p.

Definition 3. System (1) is generalized approximately synchronizable with respect to $\{\epsilon_1, \dots, \epsilon_p\}$ at time $T > 0$, if for any given initial data $(\hat{U}_0, \hat{U}_1) \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$, there exist a sequence $\{H_n\}$ of boundary controls, $H_n \in \mathcal{L}^M$ with compact support in $[0, T]$, and a sequence $\{\tilde{u}_n\}$ of p-dimensional vector functions $\tilde{u}_n = (\tilde{u}_n^{(1)}, \dots, \tilde{u}_n^{(p)})^T \in C^0_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_0)^p) \cap C^1_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_{-1})^p)$, such that the corresponding sequence $\{U_n\}$ of solutions to problem (1)–(2) satisfies

$$U_n - (\epsilon_1, \dots, \epsilon_p)\tilde{u}_n \to 0 \ (n \to +\infty) \ \text{ in } C^0_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_0)^N) \cap C^1_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_{-1})^N), \tag{60}$$

where $\{\tilde{u}_n\}$ is called the sequence of generalized approximately synchronizable states.

Remark 7. For any given initial data and the sequence $\{H_n\}$ of applied boundary controls, the sequence $\{\tilde{u}_n\}$ of generalized approximately synchronizable states defined by (60) is unique in the sense of neglecting a vanishing sequence. In fact, if there is another sequence $\{\tilde{\tilde{u}}_n\}$ satisfying (60):

$$U_n - (\epsilon_1, \dots, \epsilon_p)\tilde{\tilde{u}}_n \to 0 \ (n \to +\infty) \ \text{ in } C^0_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_0)^N) \cap C^1_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_{-1})^N), \tag{61}$$

then by (60) and (61), we have

$$(\epsilon_1, \dots, \epsilon_p)(\tilde{\tilde{u}}_n - \tilde{u}_n) \to 0 \ (n \to +\infty) \ \text{ in } C^0_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_0)^N) \cap C^1_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_{-1})^N).$$

Noting that $\epsilon_1, \dots, \epsilon_p$ are linearly independent, we have

$$\tilde{\tilde{u}}_n - \tilde{u}_n \to 0 \ (n \to +\infty) \ \text{ in } C^0_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_0)^p) \cap C^1_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_{-1})^p). \tag{62}$$

Conversely, if the sequence $\{\tilde{\tilde{u}}_n\}$ satisfies (62), then by (60) we have that $\{\tilde{\tilde{u}}_n\}$ satisfies (61). Thus, any one of the sequences of generalized approximately synchronizable states can be taken as a representative in discussion. □

Definition 4. Assume that system (1) possesses the generalized approximate boundary synchronization (60) with respect to $\{\epsilon_1, \dots, \epsilon_p\}$. Assume furthermore that there exist a p-dimensional vector function $u_* = (u_*^{(1)}, \dots, u_*^{(p)})^T \in C^0_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_0)^p) \cap C^1_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_{-1})^p)$, and a sequence $\{H_n\}$ of boundary controls, $H_n \in \mathcal{L}^M$ with compact support in $[0, T]$, such that the corresponding sequence $\{U_n\}$ of solutions to problem (1)–(2) satisfies

$$U_n \to (\epsilon_1, \dots, \epsilon_p)u_* \ (n \to +\infty) \text{ in } C^0_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_0)^N) \cap C^1_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_{-1})^N), \tag{63}$$

then system (1) is called to be generalized approximately synchronizable with respect to $\{\epsilon_1, \dots, \epsilon_p\}$ at time $T > 0$ in the pinning sense, and u_* is called the generalized approximately synchronizable state.

3.2 | Properties of generalized approximately synchronizable states

In this subsection, we will show that the generalized approximate boundary synchronizations (4) defined by Θ_p and (60) defined by $\{\epsilon_1, \dots, \epsilon_p\}$ are equivalent, and describe the relationship among the generalized approximate boundary synchronization, the approximate boundary null controllability and the sequence of generalized approximately synchronizable states. Moreover, we will give a governing system for the sequence of generalized approximately synchronizable states, whose corresponding solutions are just the sequence $\{\tilde{u}_n\}$ of generalized approximately synchronizable states given in (60). Therefore we can utilize this governing system to study the sequence $\{\tilde{u}_n\}$ of generalized approximately synchronizable states as well as the generalized approximately synchronizable state u_* in the pinning sense.

According to [15], by means of Θ_p we can take an invertible transformation

$$X = \begin{pmatrix} \Theta_p \\ (y_1, \dots, y_p)^T \end{pmatrix}, \tag{64}$$

in which $\{y_1, \dots, y_p\}$ and $\{\epsilon_1, \dots, \epsilon_p\}$ are bi-orthonormal:

$$(y_1, \dots, y_p)^T(\epsilon_1, \dots, \epsilon_p) = I_p, \tag{65}$$

where I_p is an identity matrix of order p. Under this transformation (called the generalized synchronization transformation), the state variable U of system (1) turns into

$$\tilde{U} = XU = \begin{pmatrix} W \\ V \end{pmatrix}, \tag{66}$$

where

$$W = \Theta_p U, \ V = (y_1, \dots, y_p)^T U, \tag{67}$$

in which W is called the null controllable part, while V is called the synchronizable state part.

Lemma 5. The generalized approximate boundary synchronization (4) is equivalent to

$$W_n \to 0 \ (n \to +\infty) \text{ in } C^0_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_0)^{N-p}) \cap C^1_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_{-1})^{N-p}), \tag{68}$$

and also equivalent to

$$U_n - (\epsilon_1, \dots, \epsilon_p)V_n \to 0 \ (n \to +\infty) \text{ in } C^0_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_0)^N) \cap C^1_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_{-1})^N), \tag{69}$$

where $W_n = \Theta_p U_n$ and $V_n = (y_1, \dots, y_p)^T U_n$.

Proof. Obviously, under the generalized synchronization transformation, the generalized approximate boundary synchronization (4) is just (68). Then by (66), we have

$$\tilde{U}_n - \begin{pmatrix} 0 \\ V_n \end{pmatrix} \to 0 \ (n \to +\infty) \text{ in } C^0_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_0)^N) \cap C^1_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_{-1})^N). \tag{70}$$

By (66), $U = X^{-1}\tilde{U}$. It follows from the bi-orthonormal relation (65) that the last p columns of X^{-1} must be $(\epsilon_1, \dots, \epsilon_p)$, namely, X^{-1} can be written as $X^{-1} = (x_1, \dots, x_{N-p}, \epsilon_1, \dots, \epsilon_p)$, hence

$$X^{-1}\begin{pmatrix} 0 \\ V_n \end{pmatrix} = (\epsilon_1, \dots, \epsilon_p)V_n.$$

Multiplying (70) by X^{-1} from the left, we get (69). On the other hand, multiplying (69) by Θ_p from the left and noting (59), we have (4). □

Noting that (69) is just the generalized approximate boundary synchronization with respect to $\{\epsilon_1, \dots, \epsilon_p\}$ given by (60), we have

Theorem 8. The generalized approximate boundary synchronizations (4) and (60) are equivalent, in which Θ_p and $\{\epsilon_1, \dots, \epsilon_p\}$ satisfy (59).

From (69), the sequence $\{\tilde{u}_n\}$ of generalized approximately synchronizable states given in (60) can be chosen as $\{V_n\}$, then we have

Theorem 9. Under the generalized synchronization transformation (66), the generalized approximate boundary synchronization with respect to Θ_p for system (1) is equivalent to the approximate boundary null controllability for the former $(N-p)$ variables W of the transformed $\tilde{U}$, while the left p variables V correspond to the sequence of generalized approximately synchronizable states.

Under the generalized synchronization transformation (66), the state variables $\tilde{U}$ satisfy

$$\begin{cases} \tilde{U}'' - \Delta\tilde{U} + \tilde{A}\tilde{U} = 0 & \text{in } (0,+\infty)\times\Omega, \\ \tilde{U} = 0 & \text{on } (0,+\infty)\times\Gamma_0, \\ \tilde{U} = XDH & \text{on } (0,+\infty)\times\Gamma_1, \\ t=0:\ (\tilde{U},\tilde{U}') = X(\hat{U}_0,\hat{U}_1) & \text{in } \Omega, \end{cases} \tag{71}$$

in which

$$\tilde{A} = XAX^{-1} = \begin{pmatrix} \bar{A}_p & Z_1 \\ Z_2^T & \tilde{A}_p \end{pmatrix}, \tag{72}$$

where $\bar{A}_p$ and $\tilde{A}_p$ are square matrices of order $(N-p)$ and p, respectively.

Splitting problem (71) into sub-problems corresponding to the null controllable part W and the synchronizable state part V, the original problem (1)–(2) can be decomposed to the following two sub-problems:

$$\begin{cases} W'' - \Delta W + \bar{A}_p W = -Z_1 V & \text{in } (0,+\infty)\times\Omega, \\ W = 0 & \text{on } (0,+\infty)\times\Gamma_0, \\ W = \Theta_p DH & \text{on } (0,+\infty)\times\Gamma_1, \\ t=0:\ (W,W') = \Theta_p(\hat{U}_0,\hat{U}_1) & \text{in } \Omega \end{cases} \tag{73}$$

and

$$\begin{cases} V'' - \Delta V + \tilde{A}_p V = -Z_2^T W & \text{in } (0,+\infty)\times\Omega, \\ V = 0 & \text{on } (0,+\infty)\times\Gamma_0, \\ V = (y_1,\dots,y_p)^T DH & \text{on } (0,+\infty)\times\Gamma_1, \\ t=0:\ (V,V') = (y_1,\dots,y_p)^T(\hat{U}_0,\hat{U}_1) & \text{in } \Omega \end{cases} \tag{74}$$

under the generalized synchronization transformation.

If A satisfies the condition of Θ_p-compatibility (21), then in problem (73) for the null controllable part W, $Z_1 = 0$ and $\bar{A}_p$ satisfies (22), hence problem (73) turns into the aforementioned reduced problem (23). As to problem (74) for the synchronizable state part V, $\tilde{A}_p$ satisfies

$$A(\epsilon_1,\dots,\epsilon_p) = (\epsilon_1,\dots,\epsilon_p)\tilde{A}_p. \tag{75}$$

Thus we have

Theorem 10. Assume that A satisfies the condition of Θ_p-compatibility (21), and that system (1) is generalized approximately synchronizable with respect to Θ_p. Then there exists a sequence $\{\tilde{u}_n\}$ of generalized approximately synchronizable states satisfying (60), in which $\tilde{u}_n$ satisfies

$$\begin{cases} \tilde{u}_n'' - \Delta\tilde{u}_n + \tilde{A}_p\tilde{u}_n = 0 & \text{in } (T,+\infty)\times\Omega, \\ \tilde{u}_n = 0 & \text{on } (T,+\infty)\times\Gamma, \end{cases} \tag{76}$$

and $\tilde{A}_p$ is given by (75). Moreover, for each $\tilde{u}_n$, the attainable set of its values $(\tilde{u}_n,\tilde{u}_n')(T)$ at $t=T$ is the whole space $(\mathcal{H}_0)^p\times(\mathcal{H}_{-1})^p$ as the initial data $(\hat{U}_0,\hat{U}_1)$ varies in $(\mathcal{H}_0)^N\times(\mathcal{H}_{-1})^N$.

Proof. First we prove that there exists $(\tilde{u}_n, \tilde{u}'_n)(T) = (\tilde{u}_{nT}, \tilde{u}'_{nT})$, such that the sequence $\{\tilde{u}_n\}$ of solutions to the corresponding problem (76) is actually the sequence of generalized approximately synchronizable states satisfying (60). By Remark 7 and Lemma 5, we need only to show that $\tilde{u}_n$ obtained by system (76) with the value $(\tilde{u}_{nT}, \tilde{u}'_{nT})$ given at $t = T$ satisfies

$$\tilde{u}_n - V_n \to 0 \ (n \to +\infty) \ \text{ in } C^0_{\rm loc}([T, +\infty); (\mathcal{H}_0)^p) \cap C^1_{\rm loc}([T, +\infty); (\mathcal{H}_{-1})^p) \tag{77}$$

with V_n given in (69). Since $V_n = (y_1, \dots, y_p)^T U_n$ satisfies

$$\begin{cases} V_n'' - \Delta V_n + \tilde{A}_p V_n = -Z_2^T W_n & \text{in } (T, +\infty) \times \Omega, \\ V_n = 0 & \text{on } (T, +\infty) \times \Gamma, \\ t = T \,:\, (V_n, V_n') = (V_{nT}, V'_{nT}) & \text{in } \Omega, \end{cases} \tag{78}$$

taking $(\tilde{u}_n, \tilde{u}'_n)(T) = (V_{nT}, V'_{nT})$, the corresponding problem (76) has the same initial and boundary conditions as problem (78). By well-posedness, for any given $T_1 > T$, there exists a constant $c_1 > 0$, such that

$$\|(\tilde{u}_n, \tilde{u}'_n) - (V_n, V'_n)\|(t)_{(\mathcal{H}_1)^p \times (\mathcal{H}_0)^p} \le c_1 \|W_n\|_{L^2([T,T_1];(\mathcal{H}_0)^{N-p})}, \tag{79}$$

then by (68) we get (77).

Now we show that, for each $\tilde{u}_n$, the attainable set of $(\tilde{u}_n, \tilde{u}'_n)(T)$ is the whole space $(\mathcal{H}_0)^p \times (\mathcal{H}_{-1})^p$. For any given $(\hat{u}_T, \hat{u}'_T) \in (\mathcal{H}_0)^p \times (\mathcal{H}_{-1})^p$, taking $(\tilde{u}_n, \tilde{u}'_n)(T) = (\hat{u}_T, \hat{u}'_T)$, and solving problem (76) backward on the time interval $[0, T]$ with homogeneous Dirichlet boundary condition, we get $(\tilde{u}_n, \tilde{u}'_n)(0) = (\hat{u}_0, \hat{u}_1)$. Let the initial data of U be $(\hat{U}_0, \hat{U}_1) = (\epsilon_1, \dots, \epsilon_p)(\hat{u}_0, \hat{u}_1)$, and let the boundary control be $H \equiv 0$. Since A satisfies the condition of Θ_p-compatibility (21), (75) holds. Multiplying problem (76) by $(\epsilon_1, \dots, \epsilon_p)$ from the left, we get that $U = (\epsilon_1, \dots, \epsilon_p)\tilde{u}_n$ is the solution to system (1) with the initial data $(\hat{U}_0, \hat{U}_1) = (\epsilon_1, \dots, \epsilon_p)(\hat{u}_0, \hat{u}_1)$, and it actually realizes the generalized exact boundary synchronization with respect to Θ_p for system (1) and then the generalized approximate boundary synchronization, while, $\tilde{u}_n$ is actually the corresponding generalized exactly synchronizable state and then the generalized approximately synchronizable state, satisfying $(\tilde{u}_n, \tilde{u}'_n)(T) = (\hat{u}_T, \hat{u}'_T)$. □

When A does not satisfy the condition of Θ_p-compatibility (21), similarly to [16], applying Theorem 4 and noting that A satisfies the condition of $\tilde{\Theta}_{\tilde{p}}$-compatibility (28), we can obtain corresponding results through the extended matrix $\tilde{\Theta}_{\tilde{p}}$ defined by (26).

3.3 | Determination of generalized approximately synchronizable state

In general, the generalized approximately synchronizable states depend on applied boundary controls, even if the generalized approximate boundary synchronization of system (1) has the pinning sense. In what follows, under the assumption that system (1) possesses the generalized approximate boundary synchronization, we consider the situation that the synchronizable state part V is independent of applied boundary controls. Then by Theorem 9, there exists a sequence $\{\tilde{u}_n\}$ of generalized approximately synchronizable states with $\tilde{u}_n \equiv V$, hence there exists a generalized approximately synchronizable state $u_* = V$. Thus, in this case, the generalized approximate boundary synchronization is in the pinning sense and the generalized approximately synchronizable state u_* does not depend on applied boundary controls.

Lemma 6. Assume that system (1) is generalized approximately synchronizable with respect to Θ_p under the minimal number (38) of total controls, then

(i) A must satisfy the following condition of Θ_p-strong compatibility:

$$\begin{cases} \mathrm{Ker}(\Theta_p) = \mathrm{Span}\{\epsilon_1, \dots, \epsilon_p\} \text{ is } A\text{-invariant}, \\ \text{there exists } \mathrm{Span}\{y_1, \dots, y_p\} \text{ which is } A^T\text{-invariant and bio-orthonomal to } \mathrm{Span}\{\epsilon_1, \dots, \epsilon_p\}\ , \end{cases} \tag{80}$$

and

$$\mathrm{Ker}(D^T) \supseteq \mathrm{Span}\{y_1, \dots, y_p\}. \tag{81}$$

(ii) Problem (74) of the synchronizable state part $V = (y_1, \dots, y_p)^T U$ becomes the following self-closed problem with homogeneous boundary condition:

$$\begin{cases} V'' - \Delta V + \tilde{A}_p V = 0 & \text{in } (0, +\infty) \times \Omega, \\ V = 0 & \text{on } (0, +\infty) \times \Gamma, \\ t = 0 \,:\, (V, V') = (y_1, \dots, y_p)^T (\hat{U}_0, \hat{U}_1) & \text{in } \Omega, \end{cases} \tag{82}$$

whose solution is completely determined by the initial data and independent of the sequence $\{H_n\}$ of applied boundary controls. Hence, system (1) is generalized approximately synchronizable with respect to Θ_p in the pinning sense, and the generalized approximately synchronizable state

$$u_* = V \ (t \geq T) \tag{83}$$

does not depend on the sequence $\{H_n\}$ of applied boundary controls.

Proof. By Proposition 1, (42) holds under the minimal number (38) of total controls, then there exists an A^T-invariant subspace $\mathrm{Span}\{y_1, \dots, y_p\} \subseteq \mathrm{Ker}(D^T)$. Thus, problem (74) of the corresponding synchronizable state part $V = (y_1, \dots, y_p)^T U$ turns into problem (82) with homogeneous boundary condition, whose solution is independent of the sequence $\{H_n\}$ of applied boundary controls.

Since system (1) possesses the generalized approximate boundary synchronization (4), i.e. (60), multiplying (60) by $(y_1, \dots, y_p)^T$ from the left, we get

$$V - (y_1, \dots, y_p)^T(\epsilon_1, \dots, \epsilon_p)u_n \to 0 \ \text{ in } C^0_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_0)^p) \cap C^1_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_{-1})^p)$$

as $n \to +\infty$, then for any given initial data $(\hat{U}_0, \hat{U}_1)$ we have

$$(y_1, \dots, y_p)^T(\epsilon_1, \dots, \epsilon_p)u_n \to V \ \text{ in } C^0_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_0)^p) \cap C^1_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_{-1})^p)$$

as $n \to +\infty$. Noting that $y_1, \dots, y_p$ are linearly independent, the attainable set of the solution V to problem (82) is the whole space $C^0_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_0)^p) \cap C^1_{\mathrm{loc}}([T, +\infty); (\mathcal{H}_{-1})^p)$ as $(\hat{U}_0, \hat{U}_1)$ varies in $(\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$. Therefore, $(y_1, \dots, y_p)^T(\epsilon_1, \dots, \epsilon_p)$ must be invertible. Without loss of generality, assume

$$(y_1, \dots, y_p)^T(\epsilon_1, \dots, \epsilon_p) = I_p,$$

otherwise we can take $(y_1, \dots, y_p)^T$ as $((y_1, \dots, y_p)^T(\epsilon_1, \dots, \epsilon_p))^{-1}(y_1, \dots, y_p)^T$. Thus, there exists a generalized approximately synchronizable state $u_* = V$ satisfying the generalized approximate boundary synchronization (63) in the pinning sense.

It remains to prove the Θ_p-compatibility (21). In fact, taking $n \to +\infty$ in the sequence $\{U_n\}$ of solutions to system (1) at $t = T$, and noting the coupled wave equations given in (82), we have

$$(A(\epsilon_1, \dots, \epsilon_p) - (\epsilon_1, \dots, \epsilon_p)\tilde{A}_p)V(T) = 0.$$

Since $V(T)$ varies in the whole space $(\mathcal{H}_0)^p$, we have $A(\epsilon_1, \dots, \epsilon_p) = (\epsilon_1, \dots, \epsilon_p)\tilde{A}_p$, namely, A satisfies the condition of Θ_p-compatibility (21). □

Remark 8. The generalized approximate boundary synchronization with respect to Θ_p can be actually realized under the minimal number (38) of total controls. For this purpose, we need only that the coupling matrix A satisfies the condition of Θ_p-strong compatibility (80), namely, $Z_1 = Z_2 = 0$ in (72), and the boundary control matrix $D = X^{-1}\begin{pmatrix}\Theta_p D \\ 0\end{pmatrix}$ satisfies $\mathrm{rank}(\Theta_p D) = N - p$, in which the invertible matrix X is given by (64). □

Theorem 11. Assume that system (1) is generalized approximately synchronizable with respect to Θ_p.

(i) If the synchronizable state part (74) does not depend on applied boundary controls, then A should satisfy the condition of Θ_p-strong compatibility (80), and D satisfies (81).

(ii) If A satisfies the condition of Θ_p-strong compatibility (80), and D satisfies (81), then the generalized approximate boundary synchronization with respect to Θ_p is in the pinning sense, and the generalized approximately synchronizable state u_* satisfies (83).

Proof. If the synchronizable state part (74) is independent of applied boundary controls, then by Lemma 4, the number of total controls $\mathrm{rank}(D, AD, \dots, A^{N-1}D) \leq N - p$. By Theorem 6, we have (55), hence by Lemma 6 we get the conclusion (i).

The converse result (ii) holds obviously. □

Corollary 4. Assume that A satisfies the condition of Θ_p-strong compatibility (80). If system (1) is generalized approximately synchronizable with respect to Θ_p for boundary control matrix D, then for the following boundary control matrix

$$\hat{D} = D - (\epsilon_1, \dots, \epsilon_p)(y_1, \dots, y_p)^T D, \tag{84}$$

system (1) is generalized approximately synchronizable with respect to Θ_p in the pinning sense, and the generalized approximately synchronizable state u_* satisfies (83), then is independent of the sequence of applied boundary controls.

Proof. Noting that $\hat{D}$ satisfies

$$(y_1, \dots, y_p)^T \hat{D} = 0, \quad \Theta_p \hat{D} = \Theta_p D, \tag{85}$$

for this adjusted boundary control matrix $\hat{D}$, $W = \Theta_p U$ still satisfies the original reduced system (23). By Theorem 3, if system (1) is generalized approximately synchronizable for boundary control matrix D, the reduced system (23) is approximately null controllable, hence system (1) is still generalized approximately synchronizable for the adjusted boundary control matrix $\hat{D}$. Thus by Theorem 11 we get the conclusion. □

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Conflict of interest

This work does not have any conflicts of interest.

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