FINITE TIME BLOW-UP AND GLOBAL SOLUTIONS FOR A CLASS OF FINITELY DEGENERATE PSEUDO-PARABOLIC EQUATION

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Abstract

In this paper, a class of finitely degenerate pseudo-parabolic equation, are studied. By a potential well method, we obtain a threshold result for the solutions to exist globally or to blow up in finite time for sub critical and critical initial energy. The asymptotic behavior of the global solutions, blow-up rate, a necessary and sufficient condition for blow-up solution, a upper bound and a lower bound for blow-up time of local solution are also given. When the initial energy is super critical, an abstract criterion is given for the solutions to exist globally or to blow up in finite time, in terms of two variational numbers. These generalize some recent results obtained in [7] and correct the proof of some results obtained by R. Xu in [25] and [26]

Keywords: finitely degenerate parabolic equation; global existence; blow-up; decay estimate

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1 Introduction

In this paper, we consider the the following finitely degenerate semilinear pseudo-parabolic equation

$$u_t - \Delta_X u - \Delta_X u_t = |u|^{p-2}u,\ (x,t) \in \Omega \times (0,\infty)\,, \tag{1.1}$$

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with the boundary conditions

$$u(x,t)=0,\ (x,t)\in\partial\Omega\times(0,\infty), \tag{1.2}$$

and the initial condition

$$u(x,0)=u_0(x),\ x\in\Omega, \tag{1.3}$$

where Ω is a bounded open domain such that $\Omega\subset\subset\Omega'$ is an open domain of $\mathbb{R}^n$ with $n\geqslant 2$, $X=(X_1,\ldots,X_n)$ is a system of C^∞ smooth vector fields defined on Ω' with $X_j=-X_j^*$. We define the operator $\Delta_X=\sum_{j=1}^{m}X_j^2$ is a self-adjoint operator. We assume that $p\in\left(2,\dfrac{2\nu}{\nu-2}\right)$ where $\nu>2$ is the generalized Metivier index of X on Ω. Moreover, we suppose the following hypotheses:

- $[A_1]$ $\partial\Omega$ is C^∞ smooth and non-characteristic for the system of vector fields X;
- $[A_2]$ X satisfies the Hörmander's condition on Ω' (see [13]), i.e., X together with their commutators

$$X_j=\left[X_{j_1},\left[X_{j_2},\ldots,\left[X_{j_{k-1}},X_{j_k}\right]\cdots\right]\right],\ 1\leqslant j_i\leqslant m,$$

  up to a certain fixed length $k\geqslant Q$, span the tangent space at each point of Ω. Here $Q>1$ is called the Hörmander index of X on Ω, which is defined as the smallest positive integer for the Hörmander's condition being satisfied.

If X satisfies the Hörmander's condition above, we call that X is finite degenerate vector fields and the operator Δ_X is finitely degenerate elliptic operator. Such kinds of degenerate operators arise from both physical applications and mathematical problems, for example, see [14, 18].

When $X=(\partial_{x_1},\ldots,\partial_{x_n})$ (i.e., Hörmander index $Q=1$), Δ_X is the standard Laplacian Δ, then we have the pseudo-parabolic equation

$$(\mathrm{Id}-\Delta)u_t-\Delta u=f(u),\ (x,t)\in\Omega\times(0,\infty). \tag{1.4}$$

The pseudo-parabolic equation has many important physical backgrounds such as the seepage of homogeneous fluids through a fissured rock [2], the aggregation of populations [21] (where u is the population density) and the unidirectional propagation of nonlinear dispersive long waves [3, 24]. Equation (1.1) is employed in the analysis of nonstationary processes in the area of semiconductors [15, 16], where the term $(\mathrm{Id}-\Delta)u_t$ regarded as the free electron density rate, term Δu is regarded as the linear dissipation of the free charge current and up is a source of free electron current. Equation (1.4) is also named a Sobolev type model or a Sobolev-Galpern type model [23]. In the past decades, a great deal of mathematical effort has been devoted to the study of existence and uniqueness of solutions, regularity, asymptotic behavior and blow-up of the solutions for such kinds of linear and nonlinear pseudo-parabolic equations

An important special case of this model is the Benjamin-Bona-Mahony-Burgers (BBMB) equation

$$u_t + u_x + uu_x - \nu u_{xx} - \alpha^2 u_{xxt} = 0. \tag{1.5}$$

Equation (1.5) was studied by Amick et al. in [1] with $\nu > 0$, $\alpha = 1$ and $(x,t) \in \mathbb{R} \times (0,\infty)$, in which the solution of (1.5) with initial data in $L^1 \cap H^2$ decays to zero in L^2 norm as $t \to \infty$. With $\nu > 0$, $\alpha = 1$ and $(x,t) \in \Omega \times (0,\infty)$ the model has the form (1.5) was also investigated earlier by Bona and Dougalis [4], where uniqueness, global existence and continuous dependence of solutions on initial and boundary data were established and the solutions were shown to depend continuously on $\nu > 0$ and on $\alpha > 0$. The results obtained in [1] were developed by many authors, such as by Zhang for equations of the form

$$u_t - \nu u_{xx} - u_{xxt} - u_x + u^m u_x = 0,$$

where $m \geqslant 0$, see [28]. In [25], Runzhang Xu and Jia Su considered the following pseudo-parabolic equation with a power source term

$$u_t - \Delta u - \Delta u_t = u^p, \ (x,t) \in \Omega \times (0,T),$$

where where $1 < p < \infty$ if $n = 1, 2$; $1 < p \leqslant 2^*$ if $n \geqslant 3$. By using the modified potential well method and the comparison principle, they obtained some results about the global existence and finite time blow-up of the solutions for the above problem with initial data at high energy level.

There were also many profound works on the initial value problems of high order nonlinear pseudo-parabolic equations, for example, we refer to two typical papers [8] and [29]. In [8], Y. Cao et al. established the global existence of classical solutions and the blow-up in a finite time for the viscous diffusion equation of higher order

$$u_t + k_1 u_{xxxx} - k_2 u_{txx} - (\Phi(u_x))_x + A(u) = 0, \ (x,t) \in (0,1) \times (0,\infty),$$

with Navier boundary conditions, where $k_1 > 0$, $k_2 > 0$ and Φ, A are appropriately smooth and $u(0) = u_0 \in C^{1+\beta}$ with $\beta \in (0,1)$. In [29], Zhao and Xuan studied the following pseudo-parabolic equation of fourth order

$$u_t - \alpha u_{xx} - \gamma u_{xxt} + \beta u_{xxxx} + f(u)_x = 0, \ (x,t) \in \mathbb{R} \times (0,\infty). \tag{1.6}$$

They obtained the existence of the global smooth solutions for the initial value problem of and discussed the convergence of solutions as $\beta \to 0$.

On the other hand, a numerous of nonlocal pseudo-parabolic (or parabolic) equations with nonlocal terms or nonlocal boundary conditions have been widely studied in the last few decades, we refer to [5, 6] and [9]. In [5], Bouziani studied the solvability of solutions for the nonlinear pseudo-parabolic equation

$$u_t - \frac{\partial}{\partial x}(a(x,t)u_x) - \frac{\partial^2}{\partial t \partial x}(a(x,t)u_x) = f(x,t,u,u_x), \ (x,t) \in (\alpha,\beta) \times (0,T),$$

subject to the nonlocal boundary condition

$$u(\alpha,t)=\int_\alpha^\beta u(x,t)\,\mathrm{d}x,$$

with $u_0(\alpha)=\displaystyle\int_\alpha^\beta u_0(x)\,\mathrm{d}x=0$. In [9], Dai and Huang considered the well-posedness and solvability of solutions for the nonlinear pseudo-parabolic equation

$$u_t+\frac{\partial}{\partial x}\left(a(x,t)\,u_{xt}\right)=F\left(x,t,u,u_x,u_{xx}\right),\ (x,t)\in(\alpha,\beta)\times(0,T),$$

and the nonlocal moment boundary conditions

$$\int_\alpha^\beta u(x,t)\,\mathrm{d}x=\int_\alpha^\beta xu(x,t)\,\mathrm{d}x=0.$$

Our aim in this paper is to extend the potential well method due to Payne and Sattinger in [22] to study some threshold results for the existence and nonexistence of global solutions to Eq. (1.1). Roughly speaking, our main results can be described as follows: When the initial energy $J(u_0)$ is less than or equal the mountain pass level d, then the solution u to (1.1) exists globally if it starts from the stable sets $\mathscr{W}$ and fails to exist globally if it begins in the unstable sets $\mathscr{U}$. We also give decay estimate for global solution, blow-up rate, blow-up rate, upper bound, lower bound of blow-up time. Finally, we give some characterizations of high energy initial data that lead to the non-global existence.

The plan of the paper is as follows: in Section 2, we present preliminaries. In Section 3, we introduce the definition of weak solution and state the existence of local weak solutions results for problem (1.1)-(1.3). Section 4 will be devoted to construct the stable and unstable sets that are invariant under the flows of (1.1)-(1.3). Section 5 and 6 presents some global existence and nonexistence when $J(u_0)<d$ and $J(u_0)=d$. In case global solutions exist, we also the decay properties of the solution. In case the solution blows up in finite time, we also the blow-up rate, upper bound, lower bound of blow-up time. Finally in Section 7, we prove the boundedness, convergence to equilibria of global solution and the characterizations of non-global solution at high energy initial data

2 Preliminary results and notations

Associated with the system of vector fields $X=(X_1,\ldots,X_m)$, we can introduce the following weighted Sobolev space:

$$H_X^1(\Omega')=\left\{u\in L^2(\Omega'):X_ju\in L^2(\Omega'),\ \forall j\in\overline{1,m}\right\},\tag{2.1}$$

which is a Hilbert space with norm

$$\|u\|^2_{H^1_X(\Omega')}=\|u\|^2_{L^2(\Omega')}+\|Xu\|^2_{L^2(\Omega')}=\|u\|^2_{L^2(\Omega')}+\sum_{j=1}^m\|X_ju\|^2_{L^2(\Omega')}.\tag{2.2}$$

The space $H_{X,0}^{1}(\Omega)$ is defined by the closure of $C_c^{\infty}(\Omega)$ in $H_X^1(\Omega')$, which is also a Hilbert space.

We introduce two definitions and some known properties of $H_{X,0}^{1}(\Omega)$.

Definition 2.1 (Metivier condition, [20])**.** Assume that the system of vector fields X satisfies the Hörmander's condition in Ω' with Hörmander index Q. Let $V_j(x)$ $(1 \leqslant j \leqslant Q)$, spanned by all commutators of $X_1, \ldots, X_m$ of length $\leqslant j$, be the subspaces of the tangent space at each $x \in \Omega'$. If $\mu_j = \dim V_j(x)$ is constant in a neighborhood of each $x \in \overline{\Omega} \subset \Omega'$, then we say that X satisfies Metivier condition on Ω. The Metivier index

$$\mu = \sum_{j=1}^{Q} j\left(\mu_j - \mu_{j-1}\right),\ \mu_0 = 0,$$

is also called the Hausdorff dimension or homogeneous dimension of Ω related to the subelliptic metric induced by X.

The Metivier's condition is an important condition on the study of finitely degenerate elliptic operator. However, there exist a lot of vector fields which do not satisfy the Metivier's condition, for example, Grushin type vector fields. Thus, we need to introduce the following

Definition 2.2 (Generalized Metivier index, [11])**.** In Definition 2.1, set

$$\mu(x) = \sum_{j=1}^{Q} j\left(\mu_j(x) - \mu_{j-1}(x)\right),\ \mu_0 = 0, \tag{2.3}$$

where $\mu_j(x)$ is the dimension of $V_j(x)$ for $x \in \Omega'$. Then, for $\Omega \subset\subset \Omega'$, we define

$$\nu = \max_{x \in \overline{\Omega}} \mu(x). \tag{2.4}$$

as the generalized Metivier index of Ω, which is also called non-isotropic dimension of Ω related to X. Here $\mu(x)$ is also called pointwise homogeneous dimension or non-isotropic dimension at x. Observe that $\nu = \mu$ if the Metivier's condition is satisfied.

Lemma 2.1 (Weighted Poincaré inequality, [10])**.** *Assume that the system of vector fields X satisfies Hörmander's condition on Ω, $\partial\Omega$ is C^{∞} smooth and non-characteristic for X. Then the first eigenvalue λ_1 of the operator $-\Delta_X$ is strictly positive and one has*

$$\lambda_1 \|v\|_{L^2(\Omega)}^2 \leqslant \|Xv\|_{L^2(\Omega)}^2,\ \forall v \in H_{X,0}^1(\Omega). \tag{2.5}$$

By Lemma 2.1, we can use $\|Xv\|_{L^2(\Omega)} = \sqrt{\sum_{j=1}^{m} \|X_j v\|_{L^2(\Omega)}^2}$ as a equivalent norm of the space $H_{X,0}^1(\Omega)$.

Lemma 2.2. *Assume that the system of vector fields X satisfies Hörmander's condition on on Ω, $\partial\Omega$ is C^{∞} smooth and non-characteristic for X. Then the Dirichlet eigenvalue problem*

$$\begin{cases} -\Delta_X u = \lambda u,\ x \in \Omega, \\ u = 0,\ x \in \partial\Omega \end{cases} \tag{2.6}$$

has a sequence of discrete eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots$, *and* $\lim\limits_{n\to\infty} \lambda_n = \infty$. *Moreover, the corresponding eigenfunctions* $\{w_j\}$ *constitute an orthogonal basis of the Sobolev space* $H^1_{X,0}(\Omega)$ *or an orthonormal basis of* $L^2(\Omega)$.

Lemma 2.3 (Weighted Sobolev embedding theorem)**.** *Assume that the system of vector fields* X *satisfies Hörmander's condition on on* Ω *with Hörmander index* $Q > 1$, $\partial\Omega$ *is* C^∞ *smooth and non-characteristic for* X. *Then for any* $u \in C^\infty\left(\overline{\Omega}\right)$, *we have*

$$\|u\|_{L^{p^*}(\Omega)} \leqslant C\left(\|Xu\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right),$$

where C *is a positive constant,* $p^* = \dfrac{p\nu}{\nu - p}$ *for* $p \in [1, \nu)$, $\nu \geqslant n + Q - 1 > 2$ *is generalized Metivier index of* X *on* Ω.

Remark 2.1. For $1 < p < \nu$, $1 < q < p^*$, similar to the classical Sobolev compactly embedding, we can deduce that the embedding $W^{1,p}_X(\Omega) \hookrightarrow L^q(\Omega)$ is compact.

3 Local well-posedness of weak solutions

Definition 3.1 (Weak solution)**.** A function u is called a weak solution of problem (1.1)-(1.3) on $(0, T)$ if and only if the function u belongs to the following functional space

$$W_T = \left\{u \in L^\infty\left(0, T; H^1_{X,0}(\Omega)\right) : u_t \in L^2\left(0, T; H^1_{X,0}(\Omega)\right)\right\}, \tag{3.1}$$

satisfies (1.1) in the distribution sense, i.e.,

$$\langle u'(t), v\rangle_{H^1_{X,0}(\Omega)} + \langle Xu(t), Xv\rangle_{L^2(\Omega)} = \left\langle |u(t)|^{p-2}u(t), v\right\rangle_{L^2(\Omega)}, \forall v \in H^1_{X,0}(\Omega), \tag{3.2}$$

and the initial condition

$$u(0) = u_0, \tag{3.3}$$

where $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$, $\langle\cdot,\cdot\rangle_{H^1_{X,0}(\Omega)}$ mean the inner product in $L^2(\Omega)$ and $H^1_{X,0}(\Omega)$.

Remark 3.1. Since $u \in W_T$, we have $u \in C\left([0, T]; H^1_{X,0}(\Omega)\right)$ and therefore (3.3) make sense.

By Faedo–Galerkin method and contraction mapping principle, we obtain the following result.

Theorem 3.1. *Let* $[A_1]$, $[A_2]$ *and* $p \in \left(2, \dfrac{2\nu}{\nu - 2}\right)$ *hold. Then for each* $H^1_{X,0}(\Omega)$, *there exists* $T_0 > 0$ *such that the Problem* (1.1)-(1.3) *possess a unique solution* $u \in W_{T_0}$.

Definition 3.2 (Maximal existence time)**.** Let u be a weak solution of (1.1)-(1.3). We define the maximal existence time T_∞ of u as follows:

1. If u exists for all $t \in [0, \infty)$ then $T_\infty = \infty$.
2. If there exists $t_0 \in (0, \infty)$ such that u exists for $t \in [0, t_0)$, but doesn't exist at $t = t_0$, then $T_\infty = t_0$.

Definition 3.3 (Finite time blow-up)**.** Let u be a weak solution of (1.1)-(1.3). We call that u blows up in finite time if the maximal existence time T_∞ is finite and $\lim_{t\to T_\infty^{-1}} \|Xu(t)\|_{L^2(\Omega)} = \infty$ and that T_∞ is the blow-up time.

4 Stationary problem and potential wells

Stationary solutions of problem (1.1)-(1.3) solve the nonlinear elliptic problem

$$\begin{cases} -\Delta_X u = |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Problem (4.1) may be tackled with critical-point theory. We consider the potential energy functional $J : H^1_{X,0}(\Omega) \longrightarrow \mathbb{R}$ and Nehari functional $I : H^1_{X,0}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\|Xu\|^2_{L^2(\Omega)} - \frac{1}{p}\|u\|^p_{L^p(\Omega)}, \tag{4.2}$$

and

$$I(u) = \|Xu\|^2_{L^2(\Omega)} - \|u\|^p_{L^p(\Omega)}. \tag{4.3}$$

It is clear that the functional J and I are belong to $C^1\left(H^1_{X,0}(\Omega), \mathbb{R}\right)$ and

$$J(u) = \frac{p-2}{2p}\|Xu\|^2_{L^2(\Omega)} - \frac{I(u)}{p}. \tag{4.4}$$

We put

$$S_p = \sup_{u\in H^1_{X,0}(\Omega)\setminus\{0\}} \frac{\|u\|_{L^p(\Omega)}}{\|Xu\|_{L^2(\Omega)}}. \tag{4.5}$$

In addition, since $(\nu-2)p < 2\nu$, the embedding $H^1_{X,0}(\Omega) \hookrightarrow L^p(\Omega)$ is compact and the supremum in (4.5) is attained. In such case, the functional J satisfies the Palais–Smale condition and therefore (4.1) admits at least a nonnegative solution whose energy can be characterized by

$$d = \inf_{u\in H^1_{X,0}(\Omega)\setminus\{0\}} \sup_{\lambda\in(0,\infty)} J(\lambda u). \tag{4.6}$$

Notice that the critical points of J are (weak) solutions of (4.1) and by the Moser iteration scheme and elliptic regularity, any weak solution of (4.1) is in fact a smooth classical solution. Clearly, (4.1) also admits a negative mountain pass solution.

Let $u \in H^1_{X,0}(\Omega)\setminus\{0\}$ and consider the fibrering map $j : (0,\infty) \longrightarrow \mathbb{R}$, defined by

$$j(\lambda) = J(\lambda u) = \frac{\lambda^2}{2}\|Xu\|^2_{L^2(\Omega)} - \frac{\lambda^p}{p}\|u\|^p_{L^p(\Omega)}. \tag{4.7}$$

Then we have the following lemma.

Lemma 4.1. *Let $u \in H^1_{X,0}(\Omega) \setminus \{0\}$. Then we possess*

1. $\lim\limits_{\lambda \to 0^+} j(\lambda) = 0$ *and* $\lim\limits_{\lambda \to \infty} j(\lambda) = -\infty$*;*
2. *There is a unique* $\lambda_* = \lambda_*(u) > 0$ *such that* $j'(\lambda_*) = 0$*;*
3. j *is increasing on* $(0, \lambda_*)$*, decreasing on* (λ_*, ∞) *and attains its maximum at* λ_**;*
4. $I(\lambda u) > 0$ *for all* $\lambda \in (0, \lambda_*)$*,* $I(\lambda u) > 0$ *for all* $\lambda \in (\lambda_*, \infty)$ *and* $I(\lambda_* u) = 0$*.*

Proof of Lemma 4.2. For $u \in H^1_{X,0}(\Omega) \setminus \{0\}$, by the definition of j, we have

$$j(\lambda) = J(\lambda u) = \frac{\lambda^2}{2} \|Xu\|^2_{L^2(\Omega)} - \frac{\lambda^p}{p} \|u\|^p_{L^p(\Omega)}. \tag{4.8}$$

It is clear that $\lim\limits_{\lambda \to 0^+} j(\lambda) = 0$ and $\lim\limits_{\lambda \to \infty} j(\lambda) = -\infty$ hold due to $\|u\|_{L^p(\Omega)} \neq 0$. Now, by straightforward calculation, we obtain

$$j'(\lambda) = \lambda \|Xu\|^2_{L^2(\Omega)} - \lambda^{p-1} \|u\|^p_{L^p(\Omega)} = 0 \Longleftrightarrow \lambda = \lambda_* = \|Xu\|^{\frac{2}{p-2}}_{L^2(\Omega)} \|u\|^{-\frac{p}{p-2}}_{L^p(\Omega)} > 0. \tag{4.9}$$

So there is a unique $\lambda_* = \lambda_*(u) > 0$ such that $j'(\lambda_*) = 0$. Furthermore, j is increasing on $(0, \lambda_*)$, decreasing on (λ_*, ∞) and attains its maximum at λ_*. On the other hand, we have

$$I(\lambda u) = \lambda^2 \|Xu\|^2_{L^2(\Omega)} - \lambda^p \|u\|^p_{L^p(\Omega)} = \lambda j'(\lambda). \tag{4.10}$$

So $I(\lambda u) > 0$ for all $\lambda \in (0, \lambda_*)$, $I(\lambda u) > 0$ for all $\lambda \in (\lambda_*, \infty)$ and $I(\lambda_* u) = 0$. Lemma 4.1 is proved. □

Lemma 4.2. *Let* $S_p = \sup\limits_{u \in H^1_{X,0}(\Omega) \setminus \{0\}} \dfrac{\|u\|_{L^p(\Omega)}}{\|Xu\|_{L^2(\Omega)}}$*. The following statements hold:*

1. *If* $0 < \|Xu\|_{L^2(\Omega)} < S_p^{-\frac{p}{p-2}}$ *then* $I(u) > 0$*;*
2. *If* $I(u) < 0$ *then* $\|Xu\|_{L^2(\Omega)} > S_p^{-\frac{p}{p-2}}$*;*
3. *If* $I(u) = 0$ *and* $u \neq 0$ *then* $\|Xu\|_{L^2(\Omega)} \geqslant S_p^{-\frac{p}{p-2}}$*.*

Proof of Lemma 4.2. For every $H^1_{X,0}(\Omega)$, we have

$$I(u) = \|Xu\|^2_{L^2(\Omega)} - \|u\|^p_{L^p(\Omega)} \geqslant \|Xu\|^2_{L^2(\Omega)} - S_p^p \|Xu\|^p_{L^2(\Omega)}. \tag{4.11}$$

This follows

1. If $0 < \|Xu\|_{L^2(\Omega)} < S_p^{-\frac{p}{p-2}}$ then $I(u) > 0$;

2. Assume that $I(u) < 0$. From (4.11), we get

$$1 - S_p^p \|Xu\|_{L^2(\Omega)}^{p-2} < 0 \Longleftrightarrow \|Xu\|_{L^2(\Omega)} > S_p^{-\frac{p-2}{p}}.$$

3. If $I(u) = 0$ and $u \neq 0$ then from (4.11), we obtain

$$1 - S_p^p \|Xu\|_{L^2(\Omega)}^{p-2} \leqslant 0 \Longleftrightarrow \|Xu\|_{L^2(\Omega)} \geqslant S_p^{-\frac{p-2}{p}}.$$

Lemma 4.2 is proved. □

All nontrivial stationary solutions belong to the so-called Nehari manifold defined by

$$\mathscr{N} = \left\{u \in H^1_{X,0}(\Omega) \setminus \{0\} : I(u) = 0\right\}. \tag{4.12}$$

By virtue of Lemma 4.1, it is clear that $\mathscr{N}$ is not empty and it is easy to show that each half line starting from the origin $H^1_{X,0}(\Omega)$ intersects exactly once the manifold $\mathscr{N}$ and that $\mathscr{N}$ separates the two unbounded sets

$$\mathscr{N}_+ = \left\{u \in H^1_{X,0}(\Omega) : I(u) > 0\right\} \cup \{0\},\ \mathscr{N}_- = \left\{u \in H^1_{X,0}(\Omega) : I(u) < 0\right\}. \tag{4.13}$$

The next lemma gives the variational characterization of d.

Lemma 4.3. *The following statement holds:*

$$d = \frac{p-2}{2p} S_p^{-\frac{2p}{p-2}} = \inf_{u \in \mathscr{N}} J(u) > 0. \tag{4.14}$$

Proof of Lemma 4.3. From the definition of d, we have

$$\begin{aligned} d &= \inf_{u \in H^1_{X,0}(\Omega)\setminus\{0\}} \sup_{\lambda \in (0,\infty)} J(\lambda u) = \inf_{u \in H^1_{X,0}(\Omega)\setminus\{0\}} J(\lambda_* u) \\ &= \inf_{u \in H^1_{X,0}(\Omega)\setminus\{0\}} \frac{p-2}{2p} \|Xu\|_{L^2(\Omega)}^{\frac{2p}{p-2}} \|u\|_{L^p(\Omega)}^{-\frac{2p}{p-2}} \\ &= \frac{p-2}{2p} S_p^{-\frac{2p}{p-2}}. \end{aligned}$$

Next, we prove that $d = \inf_{u \in \mathscr{N}} J(u)$. From the definition of $\mathscr{N}$, it follows from Lemma 4.1 that for any $u \in H^1_{X,0}(\Omega)$ we have $\lambda_* u \in \mathscr{N}$. As a result, we have

$$\sup_{\lambda \in (0,\infty)} J(\lambda u) = J(\lambda_* u) \geqslant \inf_{u \in \mathscr{N}} J(u),$$

or

$$d = \inf_{u \in H^1_{X,0}(\Omega)\setminus\{0\}} \sup_{\lambda \in (0,\infty)} J(\lambda u) \geqslant \inf_{u \in \mathscr{N}} J(u). \tag{4.15}$$

On the other hand, if $u \in \mathscr{N}$ then we find that (using Lemma 4.1) the only critical point in $(0, \infty)$ of the mapping j is $\lambda_* = 1$. Therefore,

$$J(u) = J(\lambda_* u) = \sup_{\lambda \in (0,\infty)} J(\lambda u) \geqslant \inf_{u \in H^1_{X,0}(\Omega) \setminus \{0\}} \sup_{\lambda \in (0,\infty)} J(\lambda u) = d.$$

or

$$\inf_{u \in \mathscr{N}} J(u) \geqslant d. \tag{4.16}$$

Combining (4.15) and (4.16), the result of Lemma 4.3 is obtained. □

5 Low energy initial data

This section is devoted to the behaviors of the solution of problem (1.1)-(1.3) under the condition that $J(u_0) < d$. We will give a threshold result for the solutions to exist globally or to blow up in finite time.

We consider the (open) sublevels of J

$$J^\alpha = \left\{ u \in H^1_{X,0}(\Omega) : J(u) < \alpha \right\}, \ (\alpha \in \mathbb{R}), \tag{5.1}$$

and we introduce the stable set $\mathscr{W}$ and the unstable set $\mathscr{U}$ defined by

$$\mathscr{W} = J^d \cap \mathscr{N}_+, \ \mathscr{U} = J^d \cap \mathscr{N}_-. \tag{5.2}$$

Note that by Definition 3.1 the weak solution u satisfies the following energy equality

$$\int_0^t \|u(s)\|^2_{H^1_{X,0}} \mathrm{d}s + J(u(t)) = J(u_0), \ \forall t \in [0, T_\infty). \tag{5.3}$$

Next, by using the potential wells above we can obtain the following invariance for some sets under the flow of (1.1)-(1.3).

Lemma 5.1. *The sets $\mathscr{W}$ and $\mathscr{U}$ are invariant under the semiflow of* (1.1)-(1.3).

Proof of Lemma 5.1. The proof can be shown by using contradiction method in a similar manner to Lions [19]. So we here omit them. □

Theorem 5.1. *Let $[A_1]$, $[A_2]$ and $p \in \left(2, \dfrac{2\nu}{\nu-2}\right)$ hold. Then weak solution u to* (1.1)-(1.3) *exists globally provided $u_0 \in \mathscr{W}$. Furthermore, we have the following estimates*

$$\|u(t)\|_{H^1_{X,0}(\Omega)} \leqslant \|u_0\|_{H^1_{X,0}(\Omega)} \exp(-\kappa t), \ \forall t \in [0, \infty), \tag{5.4}$$

where

$$\kappa = 1 - S_p^p \left(\frac{2pJ(u_0)}{p-2} \right)^{\frac{p-2}{2}} > 0.$$

Proof of Theorem 5.1. By Lemma 5.1, we have $I(u(t)) \geqslant 0$ for all $t \in [0, T_\infty)$. From definition of J, we obtain

$$d > J(u_0) \geqslant J(u(t)) \geqslant \frac{p-2}{2p} \|Xu(t)\|_{L^2(\Omega)}^2, \ \forall t \in [0, \infty). \tag{5.5}$$

Therefore, by virtue of (5.5), the Continuation Principle yields $T_\infty = \infty$.

From (4.11) and (5.5), we have

$$I(u(t)) \geqslant \left(1 - S_p^p \|Xu(t)\|_{L^2(\Omega)}^{p-2}\right) \|Xu(t)\|_{L^2(\Omega)}^2 \geqslant \kappa \|u(t)\|_{H_{X,0}^1(\Omega)}^2, \ \forall t \in [0, \infty).$$

By using u as a test function to equation (1.1), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|_{H_{X,0}^1(\Omega)}^2 = -I(u(t)) \leqslant -\kappa \|u(t)\|_{H_{X,0}^1(\Omega)}^2, \ \forall t \in [0, \infty).$$

Then, solving the ordinary differential inequality yields

$$\|u(t)\|_{H_{X,0}^1(\Omega)} \leqslant \|u_0\|_{H_{X,0}^1(\Omega)} \exp(-\kappa t), \ \forall t \in [0, \infty).$$

This completes the proof of Theorem 5.1. □

Next remark shows that global solutions of (1.1)-(1.3) have small time oscillations while blow-up solutions have large time oscillations. For blow-up case, see Theorem 5.5.

Remark 5.1. From (5.3), we have $t \longmapsto J(u(t))$ is decreasing; since it is also bounded from below by Lemma 4.2, we know that $\lim\limits_{t\to\infty} J(u(t)) = L$ for some $L \in (-\infty, J(u_0))$. Exactly, we have $L \in [0, J(u_0))$. If $L < 0$ then exists $t_* > 0$ such that $J(u(t_*)) < 0$. It leads to the solution blows up in finite time, see Theorem 5.4. Again by (5.3), we obtain

$$\int_0^\infty \|u'(s)\|_{H_{X,0}^1(\Omega)}^2 \mathrm{d}s = J(u_0) - L < \infty.$$

This fact implies $u_t \in L^2\left(0, \infty; H_{X,0}^1(\Omega)\right)$.

Furthermore, for any $k > 0$, by the Fubini theorem, Hölder inequality, we obtain

$$\begin{aligned}
\int_\Omega |u(x, t+k) - u(x, t)| \mathrm{d}x &= \int_\Omega \left| \int_t^{t+k} u'(x, s) \mathrm{d}s \right| \mathrm{d}x \leqslant \int_t^{t+k} \mathrm{d}s \int_\Omega |u'(x, s)| \mathrm{d}x \\
&\leqslant \sqrt{k|\Omega|} \sqrt{\int_t^{t+k} \|u'(s)\|_{L^2(\Omega)}^2 \mathrm{d}s} \\
&\leqslant \sqrt{k|\Omega|} \sqrt{\int_t^{t+k} \|u'(s)\|_{H_{X,0}^1(\Omega)}^2 \mathrm{d}s} \\
&= \sqrt{k|\Omega|} \sqrt{J(u(t)) - J(u(t+k))}.
\end{aligned}$$

This fact implies for any $k > 0$, we have $\lim\limits_{t\to\infty} \|u(t+k) - u(t)\|_{L^1(\Omega)} = 0$.

We next prove the instability of u starting from the unstable sets $\mathscr{U}$ which consists of both non-positive and positive initial energy. More precisely, one has the following theorem.

Theorem 5.2. *Let $[A_1]$, $[A_2]$ and $p \in \left(2, \dfrac{2\nu}{\nu-2}\right)$ hold. If $u_0 \in \mathscr{U}$ then the weak solution of the problem (1.1)-(1.3) blows up at finite time and the lifespan T_∞ of the solution u satisfies the following estimates*

$$\frac{1}{(p-2)\,S_p^p\,\|u_0\|^p_{H^1_{X,0}(\Omega)}} \leqslant T_\infty \leqslant \frac{4\,(p-1)\,\|u_0\|^2_{H^1_{X,0}(\Omega)}}{p(p-2)^2\,(d-J\,(u_0))}. \tag{5.6}$$

Proof of Theorem 5.2. By contradiction, we assume that $T_\infty = \infty$. The main tool in proving the blow-up result is the concavity method (introduced by Levine [17]) where the basis idea of the method is to construct a positive defined functional M of the solution by the energy inequality and show that $M^{-\alpha}$ is concave function of time variable. For this purpose, with $T_0 > 0$, $\beta > 0$ and $\tau > 0$ specified later, we define the auxiliary functional $M : [0, T] \longrightarrow \mathbb{R}$ by

$$M\,(t) = \int_0^t \|u\,(s)\|^2_{H^1_{X,0}}\,\mathrm{d}s + (T_0 - t)\,\|u_0\|^2_{H^1_{X,0}} + \beta(t+\tau)^2. \tag{5.7}$$

By direct computation, we achieve that

$$M'\,(t) = 2\int_0^t \langle u'\,(s)\,, u\,(s)\rangle_{H^1_{X,0}(\Omega)}\mathrm{d}s + 2\beta\,(t+\tau)\,, \tag{5.8}$$

and

$$M''\,(t) = 2\langle u'\,(t)\,, u\,(t)\rangle_{H^1_{X,0}(\Omega)} + 2\beta = 2\beta - 2I\,(u\,(t))\,. \tag{5.9}$$

From (5.7) and (5.8), we have $M\,(t) \geqslant \beta\tau^2$ for all $t \in [0, T_0]$ and $M'\,(0) = 2\beta\tau > 0$.

From (5.8), thanks to Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
\frac{1}{4}[M'\,(t)]^2 &\leqslant \left[\int_0^t \langle u'\,(s)\,, u\,(s)\rangle_{H^1_{X,0}(\Omega)}\mathrm{d}s + \beta\,(t+\tau)\right]^2 \\
&\leqslant \left[\int_0^t \|u\,(s)\|^2_{H^1_{X,0}(\Omega)}\,\mathrm{d}s + \beta(t+\tau)^2\right]\left[\int_0^t \|u'\,(s)\|^2_{H^1_{X,0}(\Omega)}\,\mathrm{d}s + \beta\right] \\
&\leqslant M\,(t)\left[\int_0^t \|u'\,(s)\|^2_{H^1_{X,0}(\Omega)}\,\mathrm{d}s + \beta\right].
\end{aligned} \tag{5.10}$$

Combine (5.7)-(5.10), we get

$$M''\,(t)\,M\,(t) - \frac{p}{2}[M'\,(t)]^2 \geqslant 2M\,(t)\left[(1-p)\,\beta - I\,(u\,(t)) - p\int_0^t \|u'\,(s)\|^2_{H^1_{X,0}(\Omega)}\,\mathrm{d}s\right]. \tag{5.11}$$

By Lemma 4.2, 5.1 and (5.3), we achieve

$$-I\,(u\,(t)) - p\int_0^t \|u'\,(s)\|^2_{H^1_{X,0}}\,\mathrm{d}s > p\,(d - J\,(u_0))\,. \tag{5.12}$$

Choose $\beta \in \left(0, \dfrac{p\left(d-J\left(u_0\right)\right)}{p-1}\right]$, (5.11) and (5.12) lead to

$$M''\left(t\right)M\left(t\right)-\frac{p}{2}[M'\left(t\right)]^2 \geqslant 0,\ \forall t\in[0,T_0]\,. \tag{5.13}$$

By direct computation, from (5.13), we achieve that

$$M\left(t\right) \geqslant \left[\left(1-\frac{p}{2}\right)\frac{M'\left(0\right)}{M^{\frac{p}{2}}\left(0\right)}t+M^{1-\frac{p}{2}}\left(0\right)\right]^{-\frac{2}{p-2}},\ \forall t\in[0,T_0]\,. \tag{5.14}$$

If we choose $\tau \in \left(\dfrac{\|u_0\|^2_{H^1_{X,0}(\Omega)}}{(p-2)\,\beta},\infty\right)$ and $T_0 \in \left[\dfrac{\beta\tau^2}{(p-2)\,\beta\tau-\|u_0\|^2_{H^2_{X,0}(\Omega)}},\infty\right)$, we will have

$$T_* = \frac{M^{1-\frac{p}{2}}\left(0\right)}{\left(\frac{p}{2}-1\right)\frac{M'(0)}{M^{\frac{p}{2}}(0)}} = \frac{2M\left(0\right)}{(p-2)\,M'\left(0\right)} = \frac{T_0\,\|u_0\|^2_{H^1_{X,0}(\Omega)}+\beta\tau^2}{(p-2)\,\beta\tau} \in (0,T_0]\,.$$

From (5.14), we get $\lim\limits_{t\to T_\infty^-} M\left(t\right)=\infty$. This is a contradiction with the fact that the solution is global and it shows that the solution blows up at finite time.

To derive the upper bound for T_∞, we know that

$$T_\infty \leqslant \frac{T_\infty\,\|u_0\|^2_{H^1_{X,0}(\Omega)}+\beta\tau^2}{(p-2)\,\beta\tau} \Longleftrightarrow T_\infty \leqslant \frac{\beta\tau^2}{(p-2)\,\beta\tau-\|u_0\|^2_{H^1_{X,0}(\Omega)}},\ \forall\,(\beta,\tau)\in\mathcal{O}, \tag{5.15}$$

where

$$\mathcal{O}=\left\{(\beta,\tau)\in\mathbb{R}^2 : 0<\beta\leqslant\frac{p\left(d-J\left(u_0\right)\right)}{p-1},\ \frac{\|u_0\|^2_{H^1_{X,0}(\Omega)}}{(p-2)\,\beta}<\tau<\infty\right\}.$$

Furthermore, we have

$$\begin{aligned}\frac{\beta\tau^2}{(p-2)\,\beta\tau-\|u_0\|^2_{H^1_{X,0}(\Omega)}} &\geqslant \frac{\beta\left[\frac{2\|u_0\|^2_{H^1_{X,0}(\Omega)}}{(p-2)\beta}\right]^2}{\frac{2(p-2)\beta\|u_0\|^2_{H^1_{X,0}(\Omega)}}{(p-2)\beta}-\|u_0\|^2_{H^1_{X,0}(\Omega)}} = \frac{4\,\|u_0\|^2_{H^1_{X,0}(\Omega)}}{(p-2)^2\beta}\\ &\geqslant \frac{4\,(p-1)\,\|u_0\|^2_{H^1_{X,0}(\Omega)}}{p(p-2)^2\left(d-J\left(u_0\right)\right)}.\end{aligned} \tag{5.16}$$

Combine (5.15) and (5.16), we obtain

$$T_\infty \leqslant \frac{4\,(p-1)\,\|u_0\|^2_{H^1_{X,0}(\Omega)}}{p(p-2)^2\left(d-J\left(u_0\right)\right)} = T_\infty^{\max}. \tag{5.17}$$

Finally, we seek a lower bound for the blow-up time T_∞ for the solution u. We put

$$\Psi(t)=\frac{1}{2}\|u(t)\|^2_{H^1_{X,0}(\Omega)},\ \forall t\in[0,T_\infty), \tag{5.18}$$

and note that $\lim\limits_{t\to T_\infty^-}\Psi(t)=\infty$. We have

$$\begin{aligned}\Psi'(t)=\langle u'(t),u(t)\rangle_{H^1_{X,0}(\Omega)}&=\|u(t)\|^p_{L^p(\Omega)}-\|Xu(t)\|^2_{L^2(\Omega)}\\&\leqslant S_p^p\|Xu(t)\|^p_{L^2(\Omega)}\leqslant S_p^p\|u(t)\|^p_{H^1_{X,0}(\Omega)}\leqslant S_p^p(2\Psi(t))^{\frac{p}{2}}.\end{aligned} \tag{5.19}$$

By direct calculation, from (5.19), we obtain

$$t\geqslant\frac{1}{S_p^p}\int_0^t\frac{\Psi'(s)}{(2\Psi(s))^{\frac{p}{2}}}\mathrm{d}s=\frac{1}{2S_p^p}\int_{\|u_0\|^2_{H^1_{X,0}(\Omega)}}^{\|u(t)\|^2_{H^1_{X,0}(\Omega)}}s^{-\frac{p}{2}}\mathrm{d}s. \tag{5.20}$$

Let $t\to T_\infty^-$, we have the lower bound for T_∞ as follow

$$T_\infty\geqslant\frac{1}{2S_p^p}\int_{\|u_0\|^2_{H^1_{X,0}(\Omega)}}^{\infty}s^{-\frac{p}{2}}\mathrm{d}s=\frac{1}{(p-2)\,S_p^p\,\|u_0\|^p_{H^1_{X,0}(\Omega)}}=T_\infty^{\min}. \tag{5.21}$$

This completes the proof of Theorem 5.2. □

Theorem 5.2 give us one sufficient condition for the existence of blow-up solution. Now, we give one necessary condition for the existence of blow-up for solution of problem (1.1)-(1.3).

Theorem 5.3. *Let $[A_1]$, $[A_2]$ and $p\in\left(2,\dfrac{2\nu}{\nu-2}\right)$ hold. If the solution of problem* (1.1)-(1.3) *blows up in finite time then $\lim\limits_{t\to T_\infty^-}J(u(t))=-\infty$.*

Proof of Theorem 5.3. Notice first that, for every $t>0$, there holds

$$\begin{aligned}\int_0^t\|u'(s)\|^2_{H^1_{X,0}(\Omega)}\mathrm{d}s&\geqslant\frac{1}{t}\left(\int_0^t\|u'(s)\|_{H^1_{X,0}(\Omega)}\mathrm{d}s\right)^2\\&\geqslant\frac{1}{t}\Big(\|u(t)\|_{H^1_{X,0}(\Omega)}-\|u_0\|_{H^1_{X,0}(\Omega)}\Big)^2.\end{aligned} \tag{5.22}$$

Hence, by (5.3), we obtain

$$J(u(t))\leqslant J(u_0)-\frac{1}{t}\Big(\|u(t)\|_{H^1_{X,0}(\Omega)}-\|u_0\|_{H^1_{X,0}(\Omega)}\Big)^2.$$

Since $\lim\limits_{t\to T_\infty^-}\|u(t)\|_{H^1_{X,0}(\Omega)}=\infty$, we conclude that $\lim\limits_{t\to T_\infty^-}J(u(t))=-\infty$. Theorem 5.3 is proved. □

Remark 5.2. As a byproduct of our proof it is clear that $T_\infty < \infty$ if and only if $\lim_{t\to T_\infty^-} J(u(t)) = -\infty$. In particular, the blow up has a full characterization in terms of (negative) energy blow up.

The next results gives the blow-up rate of solution.

Theorem 5.4. *Let* $[A_1]$, $[A_2]$ *and* $p \in \left(2, \frac{2\nu}{\nu-2}\right)$ *hold. If* $J(u_0) < 0$ *then the weak solution of the problem* (1.1)-(1.3) *blows up at finite time and the lifespan* T_∞ *of the solution* u *satisfies the following estimates*

$$\frac{1}{(p-2)\, S_p^p \left\|u_0\right\|_{H^1_{X,0}(\Omega)}^p} \leqslant T_\infty \leqslant -\frac{\left\|u_0\right\|_{H^1_{X,0}(\Omega)}^2}{p(p-2)\, J(u_0)}, \tag{5.23}$$

and

$$\left\|u(t)\right\|_{H^1_{X,0}(\Omega)}^2 \geqslant 2\Big[\left\|u_0\right\|_{H^1_{X,0}(\Omega)}^{2-p} + p(p-2)\, J(u_0) \left\|u_0\right\|_{H^1_{X,0}(\Omega)}^{-p} t\Big]^{\frac{2}{2-p}}, \ \forall t \in [0, T_\infty). \tag{5.24}$$

Proof of Theorem 5.4. By contradiction, we assume that $T_\infty = \infty$. To prove the blowup result for this case we borrow some techniques from [10]. Set

$$H(t) = -J(u(t)), \ \forall t \in [0, \infty). \tag{5.25}$$

It is clear that $H(0) > 0$. Moreover, it follows from (5.3) that

$$H'(t) = \left\|u'(t)\right\|_{H^1_{X,0}(\Omega)}^2 \geqslant 0, \ \forall t \in [0, \infty).$$

which implies that $H(t) \geqslant H(0) > 0$ for all $t \in [0, \infty)$. Recalling (5.18) and (5.19), we obtain, for any $t \in [0, \infty)$, that

$$\Psi'(t) = -I(u(t)) = \frac{p-2}{p} \left\|Xu(t)\right\|_{L^2(\Omega)}^2 - pJ(u(t)) \geqslant pH(t), \tag{5.26}$$

which, combined with Cauchy-Schwarz inequality, yields

$$\begin{aligned} \Psi(t)\, H'(t) &= \frac{1}{2} \left\|u'(t)\right\|_{H^1_{X,0}(\Omega)}^2 \left\|u(t)\right\|_{H^1_{X,0}(\Omega)}^2 \\ &\geqslant \frac{1}{2} \left\langle u'(t), u(t)\right\rangle_{H^1_{X,0}(\Omega)}^2 = \frac{1}{2}[\Psi'(t)]^2 \geqslant \frac{p}{2}\Psi'(t)\, H(t). \end{aligned} \tag{5.27}$$

From (5.27), it follows that

$$\left(H(t)\, \Psi^{-\frac{p}{2}}(t)\right)' = \Psi^{-\frac{p}{2}-1}(t) \left[H'(t)\, \Psi'(t) - \frac{p}{2} H(t)\, \Psi'(t)\right] \geqslant 0.$$

Therefore,

$$0 < H(0)\, \Psi^{-\frac{p}{2}}(0) \leqslant H(t)\, \Psi^{-\frac{p}{2}}(t) \leqslant \frac{1}{p}\Psi'(t)\, \Psi^{-\frac{p}{2}}(t) = \frac{2}{p(2-p)} \frac{\mathrm{d}}{\mathrm{d}t}\left(\Psi^{-\frac{p}{2}+1}(t)\right). \tag{5.28}$$

Integrating (5.28) over $[0,t]$, we arrive at

$$H(0)\,\Psi^{-\frac{p}{2}}(0)\,t \leqslant \frac{2}{p(2-p)}\left[\Psi^{-\frac{p}{2}+1}(t)-\Psi^{-\frac{p}{2}+1}(0)\right],$$

or equivalently

$$0 \leqslant \Psi^{-\frac{p}{2}+1}(t) \leqslant \Psi^{-\frac{p}{2}+1}(0) - \frac{p(p-2)}{2}H(0)\,\Psi^{-\frac{p}{2}}(0)\,t,\ \forall t\in[0,\infty). \tag{5.29}$$

Recalling that $p>2$, (5.29) can not hold for all $t>0$. Therefore, there must exist a finite time $T_\infty<\infty$ such that $\lim\limits_{t\to T_\infty^-}\Psi(t)=\infty$, i.e. u blows up in finite time. Moreover, it can be inferred from (5.29) that

$$T_\infty \leqslant \frac{\Psi^{-\frac{p}{2}+1}(0)}{\frac{p(p-2)}{2}H(0)\,\Psi^{-\frac{p}{2}}(0)} = \frac{2\Psi(0)}{p(p-2)H(0)} = -\frac{\|u_0\|^2_{H^1_{X,0}(\Omega)}}{p(p-2)J(u_0)},$$

and

$$\|u(t)\|^2_{H^1_{X,0}(\Omega)} \geqslant 2\left[\|u_0\|^{2-p}_{H^1_{X,0}(\Omega)} + p(p-2)J(u_0)\|u_0\|^{-p}_{H^1_{X,0}(\Omega)}\,t\right]^{\frac{2}{2-p}},\ \forall t\in[0,T_\infty).$$

This completes the proof of Theorem 5.4. □

Finally, we show that (in case of blow-up) the $H^1_{X,0}(\Omega)$ norm of u_t diverges at a higher rate when compared with the $H^1_{X,0}(\Omega)$ norms of the solution u.

Theorem 5.5. *Let* $[A_1]$, $[A_2]$ *and* $p\in\left(2,\dfrac{2\nu}{\nu-2}\right)$ *hold. If the solution of problem* (1.1)-(1.3) *blows up in finite time then*

$$\liminf_{t\to T_\infty}\frac{\|u'(t)\|_{H^1_{X,0}(\Omega)}}{\|u(t)\|_{H^1_{X,0}(\Omega)}}>0.$$

Proof of Theorem 5.5. From (5.19), thanks to Cauchy-Schwartz inequality, for $\epsilon>0$, we obtain

$$\Psi'(t)=\langle u'(t),u(t)\rangle_{H^1_{X,0}(\Omega)} \leqslant \epsilon\|u(t)\|^2_{H^1_{X,0}(\Omega)} + \frac{1}{4\epsilon}\|u'(t)\|^2_{H^1_{X,0}(\Omega)}. \tag{5.30}$$

Combine (5.3) and (5.30), we get

$$\begin{aligned} J(u_0) \geqslant J(u(t)) &= \frac{p-2}{2p}\|Xu(t)\|^2_{L^2(\Omega)} - \frac{\epsilon}{p}\|u(t)\|^2_{H^1_{X,0}(\Omega)} - \frac{1}{4p\epsilon}\|u'(t)\|^2_{H^1_{X,0}(\Omega)} \\ &\geqslant \frac{p-2-2\epsilon(1+S_2^2)}{2p}\|Xu(t)\|^2_{L^2(\Omega)} - \frac{1}{4p\epsilon}\|u'(t)\|^2_{H^1_{X,0}(\Omega)}, \end{aligned}$$

or equivalently

$$\frac{\|u'(t)\|^2_{H^1_{X,0}(\Omega)}}{\|Xu(t)\|^2_{L^2(\Omega)}} \geqslant 4p\epsilon\left[\frac{p-2-2\epsilon(1+S_2^2)}{2p} - \frac{J(u_0)}{\|Xu(t)\|^2_{L^2(\Omega)}}\right],\ \forall t\in[0,T_\infty). \tag{5.31}$$

Choose $\epsilon > 0$ such that $p - 2 - 2\epsilon\left(1 + S_2^2\right) > 0$, we obtain

$$\liminf_{t \to T_\infty} \frac{\|u'(t)\|^2_{H^1_{X,0}(\Omega)}}{\|Xu(t)\|^2_{L^2(\Omega)}} \geqslant 2\epsilon\left[p - 2 - 2\left(1 + S_2^2\right)\right] > 0.$$

This completes the proof of Theorem 5.5. $\square$

Remark 5.3 (Sharp Condition for $J(u_0) < d$)**.** Let $[A_1]$, $[A_2]$ and $p \in \left(2, \dfrac{2\nu}{\nu - 2}\right)$ hold. Assume that $u_0 \in H^1_{X,0}(\Omega)$ and $J(u_0) < d$. If $I(u_0) > 0$, problem (1.1)-(1.3) admits a global weak solution; if $I(u_0) < 0$, all solutions to problem (1.1)-(1.3) blow up in finite time.

6 Critical energy initial data

For the critical case $J(u_0) = d$, the invariance of $\mathscr{W}$ cannot be proved in general. By using the method of approximation, we can still prove the global existence of weak solutions.

Theorem 6.1. *Let $[A_1]$, $[A_2]$ and $p \in \left(2, \dfrac{2\nu}{\nu - 2}\right)$ hold. If $J(u_0) = d$ and $I(u_0) \geqslant 0$, then problem (1.1)-(1.3) admits a global weak solution $u \in L^\infty\left(0, T; H^1_{X,0}(\Omega)\right)$ with $u_t \in L^2\left(0, T; H^1_{X,0}(\Omega)\right)$ and $u(t) \in \overline{\mathscr{W}} = \mathscr{W} \cup \partial\mathscr{W}$ for all $t \in [0, \infty)$. Moreover, if $I(u(t)) > 0$. Moreover,*

1. *If $I(u(t)) > 0$ for all $[0, \infty)$ then u does not vanish and there exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$\|u(t)\|_{H^1_{X,0}(\Omega)} \leqslant C_1 \exp(-C_2 t), \ \forall t \in [0, \infty).$$

2. *If $I(u_0) > 0$ and there exists $t_* \in (0, \infty)$ such that $I(u(t_*)) = 0$ then $u(t) = 0$ for all $t \in [t_*, \infty)$.*

3. *If $I(u_0) = 0$ then $u(t) = u_0$ for all $t \in [0, \infty)$.*

Proof of Theorem 6.1. First $J(u_0) = d$ implies that $u_0 \neq 0$. Let $u_{0m} = \lambda_m u_0$ where $\lambda_m = 1 - \dfrac{1}{2m}$ for all $m \in \mathbb{N}$. Consider the initial conditions

$$u(x, 0) = u_{0m}, \ x \in \Omega, \tag{6.1}$$

and the corresponding problem (1.1)-(1.2) and (6.1). Noticing that $I(u_0) \geqslant 0$, by Lemma 4.1, we can deduce that there exists a unique $\lambda_* = \lambda_*(u_0) \geqslant 1$ such that $I(\lambda_* u_0) = 0$. Then from $\lambda_m < 1 \leqslant \lambda_*$, we get $I(u_{0m}) > 0$ and $J(u_{0m}) < d$ for all $m \in \mathbb{N}$. From Theorem 5.1, it follows that for each m problem (1.1), (1.2), (6.1) admits a global weak solution

$u_m \in L^\infty\left(0,\infty; H^1_{X,0}(\Omega)\right)$ with $u'_m \in L^2\left(0,\infty; H^1_{X,0}(\Omega)\right)$ and $u_m(t) \in \mathscr{W}$ for $t \in [0,\infty)$ satisfying

$$\langle u'_m(t), v\rangle_{H^1_{X,0}(\Omega)} + \langle Xu_m(t), Xv\rangle = \left\langle |u_m(t)|^{p-2} u_m(t), v\right\rangle, \forall v \in H^1_{X,0}(\Omega), \tag{6.2}$$

and

$$\int_0^t \|u'_m(s)\|^2_{H^1_{X,0}(\Omega)} \,\mathrm{d}s + J(u_m(t)) = J(u_{0m}) < d. \tag{6.3}$$

From (6.3) and

$$J(u_m(t)) = \frac{p-2}{2p}\|Xu_m(t)\|^2_{L^2(\Omega)} + \frac{I(u_m(t))}{p} \geqslant \frac{p-2}{2p}\|Xu_m(t)\|^2_{L^2(\Omega)}, \forall t \in [0,\infty),$$

we can get

$$\begin{aligned}
&\|u_m\|_{L^\infty\left(0,\infty;H^1_{X,0}\right)} \leqslant \sqrt{\frac{2pd}{p-2}}, \forall m \in \mathbb{N},\\
&\|u'_m\|_{L^2\left(0,\infty;H^1_{X,0}(\Omega)\right)} \leqslant \sqrt{d}, \forall m \in \mathbb{N},\\
&\|u_m\|_{L^\infty\left(0,T;L^{p'}(\Omega)\right)} \leqslant C_*\left(\frac{2pd}{p-2}\right)^{\frac{p-1}{2}}, \forall m \in \mathbb{N}.
\end{aligned}$$

Therefore, up to a subsequence, we may pass to the limit and obtain a weak solution u with the above regularity.

Next, we consider the following three cases.

1. Assume that $I(u(t)) > 0$ for all $t \in [0,\infty)$ then u does not vanish in finite time. Furthermore, from

   $$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{H^1_{X,0}(\Omega)} = \langle u'(t), u(t)\rangle_{H^1_{X,0}(\Omega)} = -I(u(t)) < 0, \forall t \in [0,\infty),$$

   which implies that $u_t \neq 0$. Therefore, by (5.3) for any $t_0 > 0$ (suitably small) we have

   $$0 < J(u(t_0)) = d - \int_0^t \|u'(s)\|^2_{H^1_{X,0}(\Omega)}\,\mathrm{d}s = d_1 < d.$$

   Taking $t = t_0$ as the initial time and by Theorem 5.1, there exist constants $C_1 > 0$ and $C_2 > 0$ such that

   $$\|u(t)\|_{H^1_{X,0}(\Omega)} \leqslant C_1 \exp(-C_2 t), \forall t \in [0,\infty).$$

2. If $I(u_0) > 0$ and there exists $t_* \in (0,\infty)$ such that $I(u(t_*)) = 0$ then $u'(t) \neq 0$ for all $t \in (0,t_0)$. It leads to $\int_0^{t_*} \|u'(s)\|^2_{H^1_{X,0}(\Omega)}\,\mathrm{d}s > 0$. Applying (5.3) again, we have

   $$J(u(t_*)) = d - \int_0^{t_*} \|u'(s)\|^2_{H^1_{X,0}(\Omega)}\,\mathrm{d}s = d_1 < d.$$

   By the definition of d, we know $u(t_*) = 0$. Define $u(t) = 0$ for all $t \geqslant t_*$. Then it is seen that such a u is a weak solution of (1.1)-(1.3) that vanishes in finite time.

3. Suppose $J(u_0) = d$ and $I(u_0) = 0$. From definition of d, we have $J(u_0) = \min_{u \in \mathcal{N}} J(u)$. Obviously,

$$ {}_{H_X^{-1}(\Omega)}\langle I'(u_0), u_0\rangle_{H_{X,0}^1(\Omega)} = 2\|Xu_0\|_{L^2(\Omega)}^2 - p\|u_0\|_{L^p(\Omega)}^p = (2-p)\|u_0\|_{L^p(\Omega)}^p > 0. $$

Thanks to Lagrange multiplier theorem, there exists $\lambda \in \mathbb{R}$ such that

$$ {}_{H_X^{-1}(\Omega)}\langle J'(u_0) - \lambda I'(u_0), v\rangle_{H_{X,0}^1(\Omega)} = 0, \forall v \in H_{X,0}^1(\Omega). \tag{6.4} $$

With $v = u_0$ in (6.4), we obtain

$$ \lambda(p-2)\|u_0\|_{L^p(\Omega)}^2 = 0 \Longleftrightarrow \lambda = 0. \tag{6.5} $$

Combine (6.4) and (6.5), we get

$$ {}_{H_X^{-1}(\Omega)}\langle J'(u_0), v\rangle_{H_{X,0}^1(\Omega)} = 0, \forall v \in H_{X,0}^1(\Omega), $$

or equivalently

$$ \langle Xu_0, Xv\rangle = \left\langle |u_0|^{p-2}u_0, v\right\rangle, \forall v \in H_{X,0}^1(\Omega). $$

So u_0 is stationary solution. Then $u(t) = u_0$ for all $t \geqslant 0$.

The proof is complete. □

Remark 6.1. In the proof of Theorem 0.1 in [26], in order to prove $I(u(t)) > 0$ for any $t > 0$, they wrote, "Let us suppose by contradiction that $t_0 > 0$ is the first time that $I(u(t_0)) = 0$. By the definition of d, we see $J(u(t_0)) \geqslant d$". This is not right. Since $J(u(t_0)) \geqslant d$ cannot be obtained only by $I(u(t_0)) = 0$, unless they had proved $u(t_0) \neq 0$, but which cannot be obtained by any one result in [26]. So, the result in the original paper [25] still holds.

Theorem 6.2. *Let $[A_1]$, $[A_2]$ and $p \in \left(2, \frac{2\nu}{\nu-2}\right)$ hold. If $J(u_0) = d$ and $I(u_0) < 0$, then the weak solution u to problem* (1.1)-(1.3) *blows up in finite time.*

Proof of Theorem 6.2. Since $J(u_0) = d$, $I(u_0) < 0$, by the continuity of $J(u(\cdot))$ and $I(u(\cdot))$ with respect to t, there exists a $t_0 > 0$ such that $J(u(t)) > 0$ and $I(u(t)) < 0$ for all $t \in [0, t_0]$. From $\langle u'(t), u(t)\rangle_{H_{X,0}^1(\Omega)} = -I(u(t))$, we have $u'(t) \neq 0$ for $t \in (0, t_0)$. Furthermore, we have

$$ J(u(t_0)) = J(u_0) - \int_0^{t_0} \|u'(s)\|_{H_{X,0}^1(\Omega)}^2 \,\mathrm{d}s < J(u_0) = d. $$

Taking $t = t_0$ as the initial time and reminder of the proof is almost the same as that of Theorem 5.2 and hence is omitted. □

Remark 6.2 (Sharp Condition for $J(u_0) < d$)**.** Let $[A_1]$, $[A_2]$ and $p \in \left(2, \frac{2\nu}{\nu-2}\right)$ hold. Assume that $u_0 \in H_{X,0}^1(\Omega)$ and $J(u_0) = d$. If $I(u_0) \geqslant 0$, problem (1.1)-(1.3) admits a global weak solution; if $I(u_0) < 0$, all solutions to problem (1.1)-(1.3) blow up in finite time.

7 High energy initial data

In this section, we investigate the conditions to ensure the existence of global solutions or blow-up solutions to problem (1.1)-(1.3) with $J(u_0) > d$. Inspired by some ideas from [12, 25], where a class of semilinear parabolic and pseudo-parabolic equations were studied, respectively, we give some sufficient conditions for the solutions to exist globally or not with arbitrarily high initial energy. Before stating the main results in this section, for $\alpha > d$, we put

$$\mathscr{N}_\alpha = J^\alpha \cap \mathscr{N} = \{u \in \mathscr{N} : J(u) < \alpha\} \neq \emptyset. \tag{7.1}$$

The above alternative characterization of d also shows that

$$\operatorname{dist}_{H^1_{X,0}(\Omega)}(0, \mathscr{N}) = \inf_{u \in \mathscr{N}} \|Xu\|_{L^2(\Omega)} = \sqrt{\frac{2pd}{p-2}} > 0. \tag{7.2}$$

We also define two variational numbers

$$\lambda_s = \inf\left\{\|u\|_{H^1_{X,0}(\Omega)} : u \in \mathscr{N}_s\right\},\ \Lambda_s = \sup\left\{\|u\|_{H^1_{X,0}(\Omega)} : u \in \mathscr{N}_s\right\}. \tag{7.3}$$

It is clear that λ_s is nonincreasing in s and Λ_s is nondecreasing in s.

The next lemma shows that λ_s and Λ_s are finite numbers.

Lemma 7.1. *Let* $[A_1]$, $[A_2]$ *and* $p \in \left(2, \frac{2\nu}{\nu-2}\right)$ *hold. Then for any* $s > d$, λ_s *and* Λ_s *defined in* (7.3) *satisfy*

$$0 < \lambda_s \leqslant \Lambda_s < \infty.$$

Proof of Lemma 7.1. We first prove that $\Lambda_s < \infty$. Let $s > d$, by definition of $\mathscr{N}_s$, for any $u \in \mathscr{N}_s$, we have that $J(u) < s$ and $I(u) = 0$. On the other hand, by the definition of functionals I and J one has

$$s > J(u) = \frac{p-2}{2p}\|Xu\|^2_{L^2(\Omega)} \Longleftrightarrow \|Xu\|_{L^2(\Omega)} < \frac{2ps}{p-2},\ \forall u \in \mathscr{N}_s.$$

Hence, we obtain

$$\Lambda_s = \sup\left\{\|u\|_{H^1_{X,0}(\Omega)} : u \in \mathscr{N}_s\right\} < \infty.$$

We finally prove $\lambda_s > 0$. Let $u \in \mathscr{N}_s$ then $u \in H^1_{X,0}(\Omega) \setminus \{1\}$, $J(u) < s$ and $I(u) = 0$. And therefore, we get

$$\|Xu\|^2_{L^2(\Omega)} = \|u\|^p_{L^p(\Omega)} \lesssim \|Xu\|^p_{L^2(\Omega)}.$$

This fact implies $\|Xu\|^{p-2}_{L^2(\Omega)} \gtrsim 1$ and therefore $\lambda_s > 0$. The proof is complete. □

Remark 7.1. We have the set $J^\alpha \cap \mathscr{N}_+$ is bounded in $H^1_{X,0}(\Omega)$ for any $\alpha \in \mathbb{R}$.

To state the next theorem, we denote by

$$\mathscr{S} = \left\{ u \in H^1_{X,0} : u \text{ is stationary solution of } (1.1)\text{-}(1.3) \right\}. \tag{7.4}$$

In the following, we let $T_\infty(u_0)$ denote the maximal existence time of the solution with initial condition $u_0 \in H^1_{X,0}(\Omega)$. If $T_\infty(u_0) = \infty$, we also define the ω-limit set $\omega(u_0)$ of the initial data $u_0 \in H^1_{X,0}(\Omega)$ by

$$\omega(u_0) = \bigcap_{t \geqslant 0} \overline{\{u(s) : s \geqslant t\}}^{H^1_{X,0}(\Omega)}. \tag{7.5}$$

Let u be a solution to (1.1)-(1.3) associated with $u_0 \in H^1_{X,0}(\Omega)$ on the maximal existence time interval $[0, T_\infty)$. We then introduce the sets

$$\mathcal{G} = \left\{ u_0 \in H^1_{X,0}(\Omega) : T_\infty(u_0) = \infty \right\}, \tag{7.6}$$

$$\mathcal{G}_0 = \left\{ u_0 \in H^1_{X,0}(\Omega) : T_\infty(u_0) = \infty, \lim_{t\to\infty} \|Xu(t)\|_{L^2(\Omega)} = 0 \right\}, \tag{7.7}$$

and

$$\mathcal{B} = \left\{ u_0 \in H^1_{X,0}(\Omega) : T_\infty(u_0) < \infty \right\}. \tag{7.8}$$

It is obvious that

$$\mathcal{G}_0 \subset \mathcal{G},\ \mathcal{G} \cap \mathcal{B} = \emptyset,\ \mathcal{G} \cup \mathcal{B} = H^1_{X,0}(\Omega).$$

The next lemma shows the characterization of the ω-limit set $\omega(u_0)$.

Lemma 7.2. *Let $[A_1]$, $[A_2]$ and $p \in \left(2, \dfrac{2\nu}{\nu-2}\right)$ hold. Then we have $\varphi \in \omega(u_0)$ if and only if there exist a sequence $\{t_n\} \subset (0,\infty)$ such that $\lim\limits_{n\to\infty} t_n = \infty$ and*

$$\lim_{n\to\infty} \|u(t_n) - \varphi\|_{H^1_{X,0}(\Omega)} = 0.$$

Furthermore, $\omega(u_0) \neq \emptyset$ and $\omega(u_0) \subset \mathscr{S}$.

Proof of Lemma 7.2. First, we assume that if there exist a sequence $\{t_n\} \subset (0,\infty)$ such that $\lim\limits_{n\to\infty} t_n = \infty$ and $\lim\limits_{n\to\infty} \|u(t_n) - \varphi\|_{H^1_{X,0}(\Omega)} = 0$. Then, for any $t \geqslant 0$, there exist $N \in \mathbb{N}$ such that

$$u(t_n) \in \{u(s) : s \geqslant t\},\ \forall n > N.$$

Let $n \to \infty$, we get $\varphi \in \overline{\{u(s) : s \geqslant t\}}^{H^1_{X,0}(\Omega)}$. And therefore, we have $\varphi \in \omega(u_0)$.

Next, suppose that $\varphi \in \omega(u_0) = \bigcap\limits_{t\geqslant 0} \overline{\{u(s) : s \geqslant t\}}^{H^1_{X,0}(\Omega)}$. Then $\varphi \in \overline{\{u(s) : s \geqslant n\}}^{H^1_{X,0}(\Omega)}$ for all $n \in \mathbb{N}$. So for each $n \in \mathbb{N}$, there exist $t_n \in [n, \infty)$ such that $\|u(t_n) - \varphi\|_{H^1_{X,0}(\Omega)} < \dfrac{1}{n}$ for all $n \in \mathbb{N}$. This fact implies $\lim\limits_{n\to\infty} t_n = \infty$ and $\lim\limits_{n\to\infty} \|u(t_n) - \varphi\|_{H^1_{X,0}(\Omega)} = 0$.

Next, we prove that $\omega(u_0) \neq \emptyset$. Thanks to Theorem 5.3, we may assume that $J(u(t)) \in [0, J(u_0)]$ and by (5.3), we have $\lim_{t\to\infty} J(u(t)) = L$ for some L. Again by (5.3), we obtain

$$\int_0^\infty \|u'(s)\|^2_{H^1_{X,0}(\Omega)}\,\mathrm{d}s = J(u_0) - L < \infty.$$

This fact implies $u_t \in L^2\left(0,\infty; H^1_{X,0}(\Omega)\right)$. By direct computation, we achieve that

$$\left|{}_{H_X^{-1}(\Omega)}\langle J'(u(t)), v\rangle_{H^1_{X,0}(\Omega)}\right| = \left|-\langle u'(t), v\rangle_{H^1_{X,0}(\Omega)}\right| \leqslant \|u'(t)\|_{H^1_{X,0}(\Omega)}\|v\|_{H^1_{X,0}(\Omega)},\ \forall v \in H^1_{X,0}(\Omega).$$

Therefore, we obtain

$$\|J'(u(t))\|_{H_X^{-1}(\Omega)} \leqslant \|u'(t)\|_{H^1_{X,0}(\Omega)},\ \forall t \in [0,\infty).$$

And this fact implies

$$\int_0^\infty \|J'(u(s))\|^2_{H_X^{-1}(\Omega)}\,\mathrm{d}s \leqslant \int_0^\infty \|u'(s)\|^2_{H^1_{X,0}(\Omega)}\,\mathrm{d}s < \infty.$$

From $u \in C\left([0,\infty); H^1_{X,0}(\Omega)\right)$ and $J \in C^1\left(H^1_{X,0}(\Omega); \mathbb{R}\right)$, it implies that there exists $\{t_n\} \subset (0,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \|J'(u(t_n))\|_{H_X^{-1}(\Omega)} = 0$. We know that J satisfies the Palais–Smale condition then without loss of generality, we may assume that $\lim_{n\to\infty} u(t_n) = \varphi$. Therefore, we have $\omega(u_0) \neq \emptyset$. Finally, we prove that $\omega(u_0) \subset \mathscr{S}$. Let $\varphi \in \omega(u_0)$ then there exists a sequence $\{t_n\} \subset (0,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} u(t_n) = \varphi$. We denote v is the weak solution of problem (1.1)-(1.3) with $u(0) = \varphi$. Then for all $t \geqslant 0$, we have

$$J(v(t)) = J(S(t)\varphi) = J\left(S(t)\left(\lim_{n\to\infty} u(t_n)\right)\right) = \lim_{n\to\infty} J(u(t_n + t)) = \lim_{t\to\infty} J(u(t)) = L,$$

where $S(t): H^1_{X,0}(\Omega) \longrightarrow H^1_{X,0}(\Omega)$ is the nonlinear semigroup associated to problem (1.1)-(1.3). Combine with (5.3), we have $v(t) = \varphi$ for all $t \geqslant 0$ or equivalently $\varphi \in \mathscr{S}$. Lemma 7.2 is proved. $\square$

We now give an abstract criterion for global existence and blow-up in finite time of solutions to (1.1)-(1.3) in terms of the variational values λ_s and Λ_s.

Theorem 7.1. *Let $[A_1]$, $[A_2]$ and $p \in \left(2, \dfrac{2\nu}{\nu-2}\right)$ hold. If $J(u_0) > d$, then the following statements hold*

1. *If $u_0 \in \mathscr{N}_+$ and $\|u_0\|_{H^1_{X,0}(\Omega)} \leqslant \lambda_{J(u_0)}$ then $u_0 \in \mathcal{G}_0$.*

2. *If $u_0 \in \mathscr{N}_-$ and $\|u_0\|_{H^1_{X,0}(\Omega)} \geqslant \Lambda_{J(u_0)}$ then $u_0 \in \mathcal{B}$.*

Proof of Theorem 7.1. We first assume that $u_0 \in \mathcal{N}_+$ and $\|u_0\|_{H^1_{X,0}(\Omega)} \leqslant \lambda_{J(u_0)}$. Then we possess $u(t) \in \mathcal{N}_+$ for all $t \in [0, T_\infty(u_0))$. Indeed, by contradiction we assume that there $t_0 > 0$ such that $u(t) \in \mathcal{N}_+$ for $t \in [0, t_0)$ and $u(t_0) \in \mathcal{N}$. Then by (5.3) and (5.19), we get

$$\|u(t_0)\|_{H^1_{X,0}(\Omega)} < \|u_0\|_{H^1_{X,0}(\Omega)} \leqslant \lambda_{J(u_0)},\ J(u(t_0)) < J(u_0).$$

This contradicts to the definition of $\lambda_{J(u_0)}$ and therefore one has $u(t) \in \mathcal{N}_+$ for all $t \in [0, T_\infty(u_0))$. Again by (5.3), we have $u(t) \in J^{J(u_0)} \cap \mathcal{N}_+$ for all $t \in [0, T_\infty(u_0))$ and then $\{u(t), t \in [0, T_\infty(u_0))\}$ remains bounded in $H^1_{X,0}(\Omega)$. So that $u_0 \in \mathcal{G}$. We next prove $u_0 \in \mathcal{G}_0$. Let $\omega \in \omega(u_0)$, we deduce from (5.3) and (5.19) that

$$\|\omega\|_{H^1_{X,0}(\Omega)} < \lambda_{J(u_0)},\ J(\omega) < J(u_0).$$

Then by definition of $\lambda_{J(u_0)}$ we get $\omega(u_0) \cap \mathcal{N} = \emptyset$, and hence $\omega(u_0) = \{0\}$ due to Lemma 7.2. And that leads to $u_0 \in \mathcal{G}_0$.

We finally prove the second claim of theorem. Let $u_0 \in \mathcal{N}_-$ satisfy $\|u_0\|_{H^1_{X,0}(\Omega)} \geqslant \Lambda_{J(u_0)}$. By using similar arguments above, we get $u(t) \in \mathcal{N}_-$ for $t \in [0, T_\infty(u_0))$. By contradiction, we assume that $T_\infty(u_0) = \infty$, then for every $\omega \in \omega(u_0)$, from (5.3) and (5.19) we have

$$\|\omega\|_{H^1_{X,0}(\Omega)} > \Lambda_{J(u_0)},\ J(\omega) < J(u_0).$$

By definition of $\Lambda_{J(u_0)}$, we get $\omega(u_0) \cap \mathcal{N} = \emptyset$. However, since $\mathrm{dist}_{H^1_{X,0}(\Omega)}(0, \mathcal{N}_-) > 0$, we also have $0 \notin \omega(u_0)$. And therefore $\omega(u_0) = \emptyset$ which contradicts to $T_\infty(u_0) = \infty$. Thus $u_0 \in \mathcal{B}$. This completes the proof of Theorem 7.1. $\square$

We now give an other criterion for blow-up in finite time of solutions to (1.1)-(1.3).

Theorem 7.2. *At first, we claim that*

$$\frac{p-2}{2p(1+S_2^2)} \|u_0\|^2_{H^2_{X,0}(\Omega)} > J(u_0), \tag{7.9}$$

$$T_\infty \leqslant \frac{8(p-1)\|u_0\|^2_{H^1_{X,0}(\Omega)}}{(p-2)^2 \left[\frac{(p-2)}{1+S_2^2}\|u_0\|^2_{H^1_{X,0}(\Omega)} - pJ(u_0)\right]}. \tag{7.10}$$

Proof of Theorem 7.2. At first, we claim that $u_0 \in \mathcal{N}_-$. Indeed, we have

$$J(u_0) < \frac{p-2}{2p(1+S_2^2)} \|u_0\|^2_{H^2_{X,0}(\Omega)} \leqslant \frac{p-2}{2p} \|Xu_0\|^2_{L^2(\Omega)}. \tag{7.11}$$

This fact implies $I(u_0) < 0$. We assume that $T_\infty = \infty$ and prove that $I(u(t)) < 0$ for all $t \in [0, \infty)$. If not, there would exist a $t_0 \in (0, \infty)$ such that $I(u(t)) < 0$ for $t \in [0, t_0)$ and $I(u(t_0)) = 0$. By (5.19), we have $\|u(t_0)\|^2_{H^1_{X,0}(\Omega)} > \|u_0\|^2_{H^1_{X,0}(\Omega)}$. Furthermore, we get

$$J(u_0) < \frac{p-2}{2p(1+S_2^2)} \|u_0\|^2_{H^2_{X,0}(\Omega)} \leqslant \frac{p-2}{2p(1+S_2^2)} \|u(t_0)\|^2_{H^1_{X,0}(\Omega)} \leqslant \frac{p-2}{2p} \|Xu(t_0)\|^2_{L^2(\Omega)}.$$

On other hand, we have

$$J\left(u_0\right) \geqslant J\left(u\left(t_0\right)\right) = \frac{p-2}{2p}\left\|Xu\left(t_0\right)\right\|_{L^2(\Omega)}^2 .$$

That is contraction. So $I\left(u\left(t\right)\right) < 0$ for all $t \in [0, \infty)$.

With the same notation and calculation in proof of Theorem 5.2, we have an estimate

$$\begin{aligned} -I\left(u\left(t\right)\right) - p\int_0^t \left\|u'\left(s\right)\right\|_{H^1_{X,0}(\Omega)}^2 \mathrm{d}s &= \frac{p-2}{2}\left\|Xu\left(t\right)\right\|_{L^2(\Omega)}^2 - pJ\left(u_0\right) \\ &\geqslant p\left[\frac{p-2}{2p\left(1+S_2^2\right)}\left\|u_0\right\|_{H^1_{X,0}(\Omega)}^2 - J\left(u_0\right)\right] > 0. \end{aligned} \tag{7.12}$$

We choose $\beta \in \left(0, \frac{1}{p-1}\left[\frac{p-2}{2\left(1+S_2^2\right)}\left\|u_0\right\|_{H^1_{X,0}(\Omega)}^2 - pJ\left(u_0\right)\right]\right)$. Then, we have the estimate (5.13). The remainder of the proof is almost the same as that of Theorem 5.2 and hence is omitted. □

Remark 7.2. In this remark, we point out that the technique used in [25] is not true. From the last line on page 2761 of [25], the author has the estimate

$$|\Omega|^{\frac{1-p}{2}}\left\|u\right\|_2^{p-1} < \left\|u\right\|_{p+1}^{p+1} = \left\|u\right\|_{H_0^1}^2 \leqslant \frac{2\left(p+1\right)}{p-1}J\left(u_0\right).$$

After that, the author take supremum both side and implied that

$$\Lambda_{J(u_0)}^{p+1} \leqslant \frac{2\left(p+1\right)}{p-1}|\Omega|^{\frac{1-p}{2}}J\left(u_0\right) \leqslant \left\|u_0\right\|_{H_0^1}^{p+1}.$$

This not right by definition of Λ_α. Because $\Lambda_\alpha = \sup\left\{\left\|u\right\|_{H_0^1} : u \in \mathscr{N}_\alpha\right\}$.

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