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Identical Approximation Operator and Regularization Method for the Cauchy problem of 2D Heat Conduction Equation

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ABSTRACT
In this paper, an identical approximate regularization method is extended to the Cauchy problem of two-dimensional heat conduction equation, this kind of problem is severely ill-posed. The convergence rates are obtained under a priori regularization parameter choice rule. Numerical results are presented for two examples with smooth and continuous but not smooth boundaries, and compared the identical approximate regularization solutions which are displayed in paper. The numerical results show that our method is effective, accurate and stable to solve the ill-posed Cauchy problems.

KEYWORDS
Cauchy problem; 2D Heat conduction equation; Identical approximation operator; Regularization method; Convergence estimate

1. Introduction

In many industrial heat transfer situations, one wishes to determine the temperature from transient temperature measurement at one or more interior locations. Especially, during the past several decades many researchers were interested in the special case of estimating a surface condition from interior measurements which has be known as the inverse heat conduction problem (IHCP). Mathematically, the IHCP is ill-posed or improperly-posed in the concept of Hadamard [1]. The Cauchy problem of one-dimensional heat equation has been widely researched over the last decades [2, 3]. Relatively, the results on ill-posed heat conduction equation in the 2D case are few. Ref.[4–6] used Fourier method, simplified Tikhonov regularization method, modified kernel method respectively, to solve problem (1). Liu [7] utilized a revised Tikhonov regularization method for a Cauchy problem of 2D heat conduction equation.

In this paper, we propose an identical approximation regularization method to treat the following Cauchy problem of the two-dimensional heat conduction equation in semi

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infinite plate:

$$\begin{cases} u_t = u_{xx} + u_{yy}, & 0 < x < 1, y > 0, t > 0, \\ u(0,y,t) = g(y,t), & y \geq 0, t \geq 0, \\ u_x(0,y,t) = 0, & y \geq 0, t \geq 0, \\ u(x,y,0) = 0, & 0 \leq x \leq 1, y \geq 0, \\ u(x,0,t) = 0, & 0 \leq x \leq 1, t \geq 0, \\ u(x,y,t)|_{y\to\infty} \; bounded, & 0 \leq x \leq 1, t \geq 0. \end{cases} \tag{1}$$

Assume that $u(x,y,t) \equiv 0$, for $\forall\ 0 \leq x \leq 1, (y,t) \in \{(y,t) : y < 0, t < 0\}$, and we suppose throughout the paper that all the functions involving x belong to Sobelov space $H^p(\mathbb{R}^2)$ for some $p \in \mathbb{R}$ and that the Cauchy data g is given inexactly by g^δ satisfying

$$\|g - g^\delta\|_p \leq \delta. \tag{2}$$

and there holds the following *a priori* bound,

$$\|u(1,\cdot,\cdot)\|_p \leq E, \tag{3}$$

where E is a finite positive constant, and $\|\cdot\|_p$ always denotes the Sobolev-H^p norm, i.e.

$$\|f\|_p := \Big(\int_{\mathbb{R}^2} |\hat{f}(\omega,\eta)|^2 (1+\omega^2+\eta^2)^p d\omega d\eta\Big)^{1/2}, \tag{4}$$

where, $f \in L^2(\mathbb{R}^2)$, and $\hat{f}(\omega,\eta)$ is the Fourier transform of function $f(x,y)$ respect to the variable $(x,y) \in \mathbb{R}^2$,

$$\hat{f}(\omega,\eta) = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(x,y)e^{-i(\omega x+\eta y)}dxdy,\ \ (\omega,\eta) \in \mathbb{R}^2.$$

We will see in the following text that the Cauchy problem (1) is severely ill-posed. So a suitable regularization method should be used. The character of the regularization method depends largely on the selection of regularization parameters, Therefore, the appropriate regularization parameter selection plays an important role in the regularization methods.

Definition 1.1. (see [8]) A regularization parameter $\mu = \mu(\delta)$ is called admissible if $\mu(\delta) \to 0$ and

$$\sup\{\|T_{\mu(\delta)}y^\delta - x\| : y^\delta \in Y, \|Tx - y^\delta\| \leq \delta\} \to 0, \delta \to 0,$$

for every $x \in X$. Where T is a linear compact operator between Hilbert spaces X and Y over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $T_{\mu(\delta)} : Y \to X$ is a regularization strategy of a family of linear and bounded operators, such that $\lim\limits_{\mu\to 0} T_{\mu(\delta)}Tx = x$, for all $x \in X$.

Manselli, Miller [2] and Murio [3, 9] used the mollification method with the Weierstrass kernel to solve inverse heat conduction problem(IHCP). Hào [10, 11] generalized their works with de la Vallée Poussin kernel and Dirichlet kernel. However, the regularization parameters μ which established on the Dirichlet kernel and de la Vallée Poussin kernel in Ref. [10, 11] are increasing with the decrease of noise data δ, which means that the selection of regular parameters does not consistent with definition 1.1. We modify the Dirichlet kernel and de la Vallée Poussin kernel of Ref.[10, 11], and find that the modified Dirichlet operator and modified de la Vallée Poussin operator belongs to the identical approximation operator. As it turns out that the regularization parameter which we propose, meets the need of definition 1.1.

The convolution method which was proposed firstly by Wei in Ref.[12], is related to the mollification method, but there is an essential difference. The convolution method aims at mollifying the equation but the mollification method aims at mollifying the improper data. We note that the kernels which were used in the mollification method and convolution method are consistent with identical approximation kernel. Therefore, we present the following definition.

Definition 1.2. (see [13, 14]) Suppose that T is the measurement function in $\mathbb{R}^n$, parameter $\mu > 0$ and operator $T_\mu(x) = \mu^{-n}T(x/\mu)$. Let $X \in \mathbb{R}^n$ is the subset of the all measurement functions. In a sense of convergence, for any $f \in X$, there hold

$$f * T_\mu \to f, \mu \to 0. \tag{5}$$

Then T is called the identical approximation kernel in X, and the operator $T_\mu : f| \to f * T_\mu$ is called the identical approximation operator in X.

If a identical approximate operator is applied to mollify the equation or mollify the improper data. We call the regular method that corresponding to this operator is **identical approximate regularization method.**

Some choices for the identical approximation operators:

(1) The Gaussian identical approximation operator [15]:

$$G_\mu(x) = \frac{1}{(\mu\sqrt{\pi})^n} \prod_{j=1}^{n} e^{-(\frac{x_j}{\mu})^2}. \tag{6}$$

The Fourier transform $\hat{G}_\mu(\xi)$ of $G_\mu(x)$ is

$$(2\pi)^{n/2}\hat{G}_\mu(\xi) = \prod_{j=1}^{n} e^{-\frac{\mu^2\xi^2}{4}}.$$

(2) The Dirichlet identical approximation operator:

$$D_\mu(x) = \frac{1}{\pi^n} \prod_{j=1}^{n} \frac{\sin(x_j/\mu)}{x_j}. \tag{7}$$

It's Fourier transform is [16]:

$$(2\pi)^{n/2}\hat{D}_\mu(\xi) = \begin{cases} 1, & \triangle_\mu = \{\xi_j \ |\xi_j| < 1/\mu\}, \\ 0, & |\xi| \geq 1/\mu, \end{cases}$$

(3) The de la Vallée Poussin identical approximation operator:

$$V_\mu(x) = (\frac{\mu}{\pi})^n \prod_{j=1}^{n} \frac{\cos(x_j/\mu) - \cos(2x_j/\mu)}{x_j^2}. \tag{8}$$

It's Fourier transform is [16]

$$(2\pi)^{\frac{n}{2}} \hat{V}_\mu = \prod_{j=1}^{n} \lambda(\xi_j),$$

where,

$$\lambda(\xi_j) = \begin{cases} 1, & |\xi_j| < 1/\mu, \\ 2 - \mu\xi_j, & 1/\mu < |\xi_j| \le 2/\mu, \\ 0, & |\xi_j| > 2/\mu. \end{cases}$$

Here, $x = (x_1, x_2, \cdots, x_n)$, $\mu > 0$ is regularization parameter.

Remark 1. Let $T(x)$ denotes the identical approximate kernel, and $T_\mu(x)$ denotes the identical approximate operator, then $\int_{\mathbb{R}^n} T(x)dx = 1$ and $\int_{\mathbb{R}^n} T_\mu(x)dx = 1$ holds.

Adapting the Fourier transform respect to the variables (y, t) to problem (1), we obtain

$$\hat{u}(x, \omega, \eta) = \hat{g}(\omega, \eta)\cosh(\beta x), \tag{9}$$

where,

$$\beta := \sqrt{\omega^2 + i\eta} = \sqrt{\frac{\sqrt{\omega^4 + \eta^2} + \omega^2}{2}} + isign(\eta)\sqrt{\frac{\sqrt{\omega^4 + \eta^2} - \omega^2}{2}}.$$

Noting equality (9), the function $\cosh(\mu x)$ is unbounded when $|\omega| \to \infty, |\eta| \to \infty$. So problem (1) is severely ill-posed.

In the present paper, we are to develop an efficient and stable identical approximate regularization method for the solution of the Cauchy problem associated with problem (1).

The remainder of the paper is organized as follows: In section 2, we apply the identical approximate regularization method to solve problem (1), and is devoted to some stable estimates in $0 < x \le 1$ under the assumption of *a priori* bound. The efficiency of the method is demonstrated in section 3, where, together with a detailed description of the algorithm, we present the results of two numerical experiments. A summary and some conclusions are presented in section 4.

2. Identical approximate regularization method

Instead of solving the problem (1) with the data $g(x, y)$, we will solve the following problem with the $(V_\mu * g^\delta)(x, y)$, where, $g^\delta(x, y)$ is the noise data. We denote the solution of the following modified problem by $u^{\mu,\delta}$, which satisfies the system:

$$\begin{cases} u_t^{\mu,\delta} = u_{xx}^{\mu,\delta} + u_{yy}^{\mu,\delta},\ 0 < x < 1, y > 0, t > 0, \\ u^{\mu,\delta}(0,y,t) = (T_\mu * g^\delta)(x,y),\ y \geq 0, t \geq 0, \\ u_x^{\mu,\delta}(0,y,t) = 0,\ y \geq 0, t \geq 0, \\ u^{\mu,\delta}(x,y,0) = 0,\ 0 \leq x \leq 1, y \geq 0, \\ u^{\mu,\delta}(x,0,t) = 0,\ \ 0 \leq x \leq 1, t \geq 0, \\ u^{\mu,\delta}(x,y,t)|_{y\to\infty}\ bounded,\ 0 \leq x \leq 1, t \geq 0. \end{cases} \tag{10}$$

By similar method to solving problem (1), the solution of the problem (10) is:

$$\hat{u}^{\mu,\delta}(x,\omega,\eta) = (T_\mu \hat{*} g^\delta)(\omega,\eta)\cosh(\beta x).$$

We need following lemma.

Lemma 2.1. *([17]) $a \geq b \geq 0, x \geq 0, \sigma = sign(\eta), \eta \in \mathbb{R}$, then we have*
(1) $|\cosh(a + i\sigma b)| \geq \frac{\sqrt{1-2e^{-\frac{\pi}{2}}}}{2} e^a$;
(2) $|\cosh(x(a + i\sigma b))| \leq e^{xa}$.

We will consider the following two identical approximate operators and their regularization methods.

2.1. Dirichlet identical approximate operator and error estimates

If we take the identical approximate operator as the Dirichlet operator, we get the error estimates as follows:

Theorem 2.2. *Suppose that $u(x,y,t)$ is the exact solution of problem (1), and $u^{\mu,\delta}(x,y,t)$ is the regular solution of problem (1) with modifying data by Dirichlet identical approximate operator. Assumptions (2) and (3) are satisfied, then for $0 < x < d$, we have the following error estimate:*

$$\|u - u^{\mu,\delta}\|_p \leq 8Ece^{-\frac{(1-x)}{\sqrt{2}\mu}} + \delta e^{2x/\mu}. \tag{11}$$

If the regularization parameter is selected by

$$\mu = \frac{2}{\ln(E/\delta)}, \tag{12}$$

then for the sufficiently small δ, we have a convergence estimate

$$\|u - u^{\mu,\delta}\|_p \leq 8cE^{1-\frac{1-x}{2\sqrt{2}}}\delta^{\frac{1-x}{2\sqrt{2}}} + E^x\delta^{1-x}, \tag{13}$$

where, $c = \frac{2}{\sqrt{1-2e^{-\frac{\pi}{2}}}}$.

Proof. According to the property of the integral, there is

$$\|u-u^{\mu,\delta}\|_p^2=\sum_{k=1}^{9}\int\int_{D_k}|\hat{u}-\hat{u}^{\mu,\delta}|^2(1+\omega^2+\eta^2)^s d\omega d\eta$$

$$=\sum_{k=1}^{8}\int\int_{D_k}|\hat{u}|^2(1+\omega^2+\eta^2)^s d\omega d\eta+\int\int_{D_9}|\hat{u}-\hat{u}^{\mu,\delta}|^2\cosh^2(\beta x)(1+\omega^2+\eta^2)^s d\omega d\eta$$

$$=\sum_{k=1}^{8}\int\int_{D_k}|P(x,\omega,\eta)\hat{u}|^2(1+\omega^2+\eta^2)^s d\omega d\eta+$$

$$\int\int_{D_9}|\cos(\beta x)(\hat{\varphi}-\hat{\varphi}^{\delta})|^2(1+\omega^2+\eta^2)^s d\omega d\eta.$$

Here, for every $D_k,\ k=1,2,\cdots,9$ (see Figure 1,where $\alpha=1/\mu$)

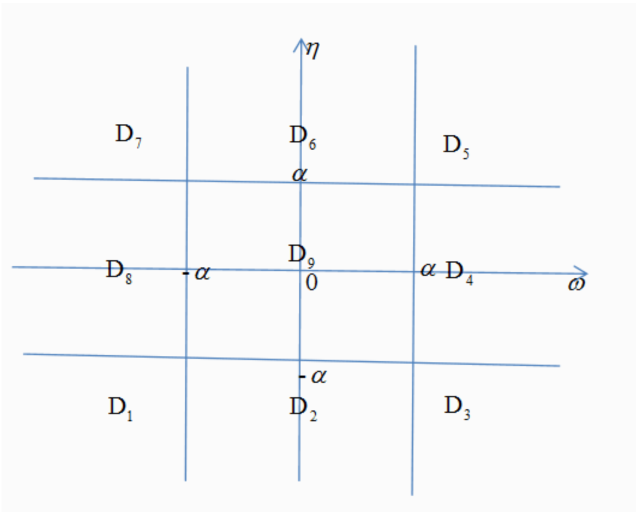


Fig. 1. For every D_k $(k=1,2,\cdots,9)$

$$D_1=(-\infty,-1/\mu)\times(-\infty,-1/\mu),\ D_2=(-1/\mu,1/\mu)\times(-\infty,-1/\mu),$$

$$D_3=(1/\mu,+\infty)\times(-\infty,-1/\mu),\ D_4=(1/\mu,+\infty)\times(-1/\mu,1/\mu),$$

$$D_9=(-1/\mu,1/\mu)\times(-1/\mu,1/\mu),\ D_8=(-\infty,-1/\mu)\times(-1/\mu,1/\mu),$$

$$D_7=(-\infty,-1/\mu)\times(1/\mu,+\infty),\ D_6=(-1/\mu,1/\mu)\times(1/\mu,+\infty),$$

$$D_5=(1/\mu,+\infty)\times(1/\mu,+\infty).$$

and equality

$$P(x,\omega,\eta)=\frac{\cos(\beta x)}{\cosh(\beta)}\quad(\omega,\eta\in\mathbb{R}),$$

where,

$$\beta=\sqrt{\omega^2+i\eta}=\sqrt{\frac{\sqrt{\omega^4+\eta^2}+\omega^2}{2}}+isign(\eta)\sqrt{\frac{\sqrt{\omega^4+\eta^2}-\omega^2}{2}}.$$

We have

$$\|u-u^{\mu,\delta}\|_p^2\quad\leq\sum_{k=1}^{8}(\sup_{D_k}|P(x,\omega,\eta)|)^2\|\hat{u}(1,\cdot,\cdot)\|_p^2+\delta^2(\sup_{D_9}|\cos(\beta x)|)^2.$$

In $D_1=(-\infty,-1/\mu)\times(-\infty,-1/\mu)$, by using Lemma 2.1, we have

$$\sqrt{\frac{\sqrt{\omega^4+\eta^2}+\omega^2}{2}}\geq\sqrt{\frac{\sqrt{1/\mu^4+1/\mu^2}+1/\mu^2}{2}}\geq\sqrt{\frac{\sqrt{1/\mu^2}+1/\mu^2}{2}}=1/\mu.$$

Then, in D_1 there is,

$$P(x,\omega,\eta)\leq\frac{e^{xa}}{\frac{\sqrt{1-2e^{-\frac{\pi}{2}}}}{2}e^a}=ce^{-(1-x)/\mu},$$

where,

$$c=\frac{2}{\sqrt{1-2e^{-\frac{\pi}{2}}}},\ a=\sqrt{\frac{\sqrt{\omega^4+\eta^2}+\omega^2}{2}}.$$

Similarly, in the domains D_3,D_5,D_7, we can also obtain $P(x,\omega,\eta)\leq ce^{-(1-x)/\mu}$. And

$$a=\sqrt{\frac{\sqrt{\omega^4+\eta^2}+\omega^2}{2}}\geq\sqrt{\frac{\sqrt{0+1/\mu^2}+0}{2}}=\frac{1}{\sqrt{2}\mu},(\omega,\eta)\in D_2,$$

consequently

$$P(x,\omega,\eta)\leq ce^{-(1-x)/\sqrt{2}\mu},(\omega,\eta)\in D_2.$$

$$a=\sqrt{\frac{\sqrt{\omega^4+\eta^2}+\omega^2}{2}}\geq\sqrt{\frac{\sqrt{0+1/\mu^4}+1/\mu^2}{2}}=1/\mu,(\omega,\eta)\in D_4,$$

$$P(x,\omega,\eta)\leq ce^{-(1-x)/\mu},(\omega,\eta)\in D_4;$$

$$a \leq \sqrt{\frac{\sqrt{1/\mu^4 + 1/\mu^2} + 1/\mu^2}{2}} \leq \sqrt{\frac{1/\mu + 1/\mu^2 + 1/\mu^2}{2}} \leq \sqrt{1/\mu + \sqrt{1/\mu}}, (\omega, \eta) \in D_9,$$

$$v(x,\omega,\eta) = \cosh(\mu x) \leq e^{xa} \leq e^{x(1/\mu + \sqrt{1/\mu})} \leq e^{x(1/\mu+\sqrt{1/\mu})}, (\omega,\eta) \in D_9.$$

By inequality $\sqrt{A+B} < \sqrt{A} + \sqrt{B} \quad (A > 0, B > 0)$ and the property of $v(x,\omega,\eta)$, we have

$$\begin{aligned}
\|u - u^{\mu,\delta}\|_p &= \sum_{k=1}^{8} \sup_{D_k} |P(x,\omega,\eta)| \|\hat{u}(1,\cdot,\cdot)\|_p + \delta \sup_{D_9} |v(x,\omega,\eta)| \\
&\leq 4Ece^{-\frac{(1-x)}{\mu}} + 4Ece^{-\frac{(1-x)}{\sqrt{2}\mu}} + \delta e^{x(1/\mu+\sqrt{1/\mu})} \\
&\leq 8Ece^{-\frac{(1-x)}{\sqrt{2}\mu}} + \delta e^{x(1/\mu+\sqrt{1/\mu})}.
\end{aligned}$$

For $\sqrt{1/\mu} < 1/\mu$, $(0 < \mu < 1)$, then we have

$$\|u - u^{\mu,\delta}\|_p \leq 8Ece^{-\frac{(1-x)}{\sqrt{2}\mu}} + \delta e^{2x/\mu}.$$

If we take the regularization parameter as $\mu = \frac{2}{\ln(E/\delta)}$, the following estimate holds

$$\|u - u^{\mu,\delta}\|_p \leq 8cE^{1-\frac{1-x}{2\sqrt{2}}}\delta^{\frac{1-x}{2\sqrt{2}}} + E^x\delta^{1-x}.$$

$\square$

Theorem 2.3. *Suppose that $u(1,y,t)$ is the exact solution of problem (1), and $u^{\mu,\delta}(1,y,t)$ is the regular solution of problem (1) with modifying data by Dirichlet identical approximate operator. Assumptions (2) and (3) are satisfied, then with $\mu = \frac{4}{\ln(E/\delta)}$, we have the following error estimate:*

$$\|u(1,\cdot,\cdot) - u^{\mu,\delta}(1,\cdot,\cdot)\|_p \leq 8E(\frac{4}{\ln(E/\delta)})^{r-s} + \delta^{1/2}E^{1/2} \to 0, \delta \to 0^+. \quad (14)$$

Proof. Similar proof with the Theorem 2.2, we have

$$\begin{aligned}
&\|u(1,\cdot,\cdot) - u^{\mu,\delta}(1,\cdot,\cdot)\|_p^2 \\
&= \sum_{k=1}^{9} \int\int_{D_k} |\hat{u}(1,\omega,\eta) - \hat{u}^{\mu,\delta}(1,\omega,\eta)|^2 (1+\omega^2+\eta^2)^p d\omega d\eta \\
&= \sum_{k=1}^{8} \int\int_{D_k} |\hat{u}(1,\omega,\eta)|^2 (1+\omega^2+\eta^2)^p d\omega d\eta + \\
&\quad \int\int_{D_9} |\hat{u}(1,\omega,\eta) - \hat{u}^{\mu,\delta}(1,\omega,\eta)|^2 (1+\omega^2+\eta^2)^p d\omega d\eta.
\end{aligned}$$

Let

$$q(\omega,\eta)=\frac{1}{(1+\omega^2+\eta^2)^{\frac{r-p}{2}}}\quad(\omega,\eta\in\mathbb{R}).$$

By the property of the integral, there is

$$\begin{aligned}&\|u(1,\cdot,\cdot)-u^{\mu,\delta}(1,\cdot,\cdot)\|_p^2\\ &\le\sum_{k=1}^{8}(\sup_{D_k}|q(\omega,\eta)|)^2\|\hat{u}(1,\cdot,\cdot)\|_p^2+\delta^2(\sup_{D_9}|v(x,\omega,\eta)|)^2.\end{aligned}$$

Here, D_k $(k=1,2,\cdots,9)$ are same as Theorem 2.2.
By using the inequality $\sqrt{A+B+C}<\sqrt{A}+\sqrt{B}+\sqrt{C}\quad(A,B,C>0)$ and the monotonicity of function $q(\omega,\eta)$, we have

$$\begin{aligned}&\|u(1,\cdot,\cdot)-u^{\mu,\delta}(1,\cdot,\cdot)\|_p\\ &\le\sum_{k=1}^{8}\sup_{D_k}|q(\omega,\eta)|\|\hat{u}(1,\cdot,\cdot)\|_p+\delta\sup_{D_9}|v(x,\omega,\eta))|\\ &\le\frac{8E}{(1+1/\mu^2)^{\frac{r-p}{2}}}+\delta e^{2/\mu}\ (\ r>p\ge 0).\end{aligned}$$

If we choose the parameter $\mu=\frac{4}{\ln(E/\delta)}$, and use $1+1/\mu^2>1/\mu^2$, then the convergence estimate (14) can be arrive at. $\square$

Remark 2. When $p=0$, the above results in the Sobolev space return to the conclusions of $L^2(\mathbb{R}^2)=H^0(\mathbb{R}^2)$ space.

2.2. De la Vallée Poussin identical approximate operator and error estimates

Theorem 2.4. *Let $u(x,y,t)$, $u^{\mu,\delta}(x,y,t)$ be the exact solution and the de la Vallée Poussin regularization solution of 2D heat conduction problem (1), respectively. Suppose that g is given approximately by g^δ with (2) and bound $\|u(d,\cdot,\cdot)\|_p\le E$ hold, then with regularization parameter $\mu=\frac{2}{\ln(E/\delta)}$ the following stability estimate holds*

$$\|u-u^{\mu,\delta}\|_p\le 59cE^{1-\frac{1-x}{2\sqrt{2}}}\delta^{\frac{1-x}{2\sqrt{2}}}+E^x\delta^{1-x}.\tag{15}$$

Here, $C=59E\frac{C_1}{C_2}+36C_1$.

Proof. According to the property of integral, there is

$$\|u-u^{\mu,\delta}\|_p^2=\sum_{k=1}^{25}\int_{D_k}|\hat{u}-\hat{u}^{\mu,\delta}|^2(1+\omega^2+\eta^2)^s d\omega d\eta$$
$$=\sum_{k=1}^{16}\int_{D_k}|\hat{u}|^2(1+\omega^2+\eta^2)^s d\omega d\eta+\sum_{k=17}^{25}\int_{D_k}|\hat{u}-\hat{u}^{\mu,\delta}|^2(1+\omega^2+\eta^2)^s d\omega d\eta$$
$$\leq E^2\sum_{k=1}^{16}(\sup_{D_k}|A(\omega,\eta)|)^2+\sum_{k=17}^{25}\int_{D_k}|(\hat{g}-(T_\mu \hat{*} g^\delta))B(\omega,\eta)|^2(1+\omega^2+\eta^2)^s d\omega d\eta.$$

Here, $\mathbb{R}^2=\bigcup_{k=1}^{25} D_k$ (see Fig.2 , $\alpha=1/\mu$).

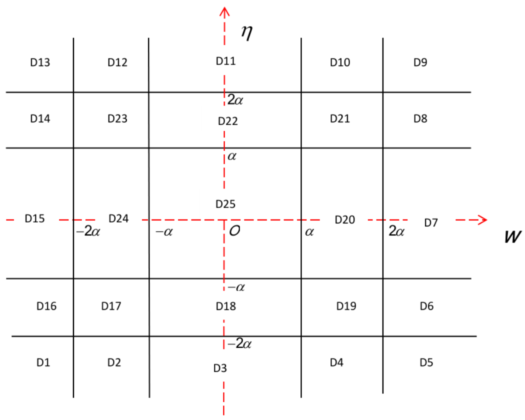


Fig. 2. For every D_k $(k=1,2,\cdots,25)$

$$A(\omega,\eta)=\frac{\cosh(\beta x)}{\cosh(\beta)},\ B(\omega,\eta)=\cosh(\beta x),\quad (\omega,\eta\in\mathbb{R}).$$

and

$$
\begin{aligned}
&\sum_{k=17}^{25}\int_{D_k}|(\hat{g}-(T_\mu \hat{*} g^\delta))B(\omega,\eta)|^2(1+\omega^2+\eta^2)^s d\omega d\eta = \\
&\int_{D_{17}}|(\hat{g}-(2-\mu\omega)(2-\mu\eta)\hat{g}^\delta)B(\omega,\eta)|^2(1+\omega^2+\eta^2)^s d\omega d\eta + \\
&\int_{D_{18}}|(\hat{g}-(2-\mu\eta)\hat{g}^\delta)B(\omega,\eta)|^2(1+\omega^2+\eta^2)^s d\omega d\eta + \\
&\int_{D_{19}}|(\hat{g}-(2-\mu\omega)(2-\mu\eta)\hat{g}^\delta)B(\omega,\eta)|^2(1+\omega^2+\eta^2)^s d\omega d\eta + \\
&\int_{D_{20}}|(\hat{g}-(2-\mu\omega)\hat{g}^\delta)B(\omega,\eta)|^2(1+\omega^2+\eta^2)^s d\omega d\eta + \\
&\int_{D_{21}}|(\hat{g}-(2-\mu\omega)(2-\mu\eta)\hat{g}^\delta)B(\omega,\eta)|^2(1+\omega^2+\eta^2)^s d\omega d\eta + \\
&\int_{D_{22}}|(\hat{g}-(2-\mu\eta)\hat{g}^\delta)B(\omega,\eta)|^2(1+\omega^2+\eta^2)^s d\omega d\eta + \\
&\int_{D_{23}}|(\hat{g}-(2-\mu\omega)(2-\mu\eta)\hat{g}^\delta)B(\omega,\eta)|^2(1+\omega^2+\eta^2)^s d\omega d\eta + \\
&\int_{D_{24}}|(\hat{g}-(2-\mu\omega)\hat{g}^\delta)B(\omega,\eta)|^2(1+\omega^2+\eta^2)^s d\omega d\eta + \\
&\int_{D_{25}}|(\hat{g}-\hat{g}^\delta)B(\omega,\eta)|^2(1+\omega^2+\eta^2)^s d\omega d\eta.
\end{aligned}
$$

It is easy to get $9 < (2-\mu\omega)(2-\mu\eta) < 16$ in

$$D_{17} = (-2/\mu, -1/\mu) \times (-2/\mu, -1/\mu).$$

Taking into account the Minkowski inequality, we have

$$
\begin{aligned}
&(\int_{D_{17}}|(\hat{g}-(2-\mu\omega)(2-\mu\eta)\hat{g}^\delta)B(\omega,\eta)|^2(1+\omega^2+\eta^2)^s d\omega d\eta)^{1/2} \\
&= \|(\hat{g}-(2-\mu\omega)(2-\mu\eta)\hat{g}^\delta)B(\omega,\eta)(1+\omega^2+\eta^2)^{s/2}\|_{L^2(D_{17})} \\
&\le \|\hat{g}B(\omega,\eta)(1+\omega^2+\eta^2)^{s/2}\|_{L^2(D_{17})} + 16\|(\hat{g}^\delta-\hat{g}+\hat{g})B(\omega,\eta)(1+\omega^2+\eta^2)^{s/2}\|_{L^2(D_{17})} \\
&\le 17\|\hat{g}B(\omega,\eta)(1+\omega^2+\eta^2)^{s/2}\|_{L^2(D_{17})} + 16\|(\hat{g}^\delta-\hat{g})B(\omega,\eta)(1+\omega^2+\eta^2)^{s/2}\|_{L^2(D_{17})} \\
&\le 17E\sup_{D_{17}}|A(\omega,\eta)| + 16\delta\sup_{D_{17}}|B(\omega,\eta)|
\end{aligned}
$$

Similar with above method, we can obtain the other inequality in $D_k(k=18,\cdots,24)$. Furthermore, if we take regular parameter $\mu = \frac{2}{\ln(E/\delta)}$, then we have

$$\|u-u^{\mu,\delta}\|_p \le 59cE^{1-\frac{1-x}{2\sqrt{2}}}\delta^{\frac{1-x}{2\sqrt{2}}} + E^x\delta^{1-x}. \tag{16}$$

. $\square$

By similar methods, we have the following error estimate at boundary $x = 1$.

Theorem 2.5. *Let $u(1, y, t)$ be the exact solution of Cauchy problem (1), and $u^{\mu,\delta}(1, y, t)$ be the de la Vallée Poussin regular solution. Assumptions (2) and $\|u(1, \cdot, \cdot)\|_r \le E, (r \ge p > 0)$ hold, then with $\mu = \frac{4}{\ln(E/\delta)}$, the following stability estimate holds*

$$\|u(1, \cdot, \cdot) - u^{\mu,\delta}(1, \cdot, \cdot)\|_p \le 59E(\frac{4}{\ln(E/\delta)})^{r-p} + \delta^{1/2}E^{1/2} \to 0, \delta \to 0^+. \tag{17}$$

3. Numerical experiments

In this section, we present two numerical examples to illustrate the effectiveness of the identical approximate regularization method.

In the numerical examples, we select the discrete interval as $[0, 10] \times [0, 10]$, and take $N = 101$, $E = \|u(1, \cdot, \cdot)\|_2$. the measurement data $g^\delta(x, y)$ is obtained as follows

$$g^\delta(x, y) = g + \epsilon(2randn(size(g)) - 1),$$

where

$$g = (g(x_i, y_j))^T_{N\times N},\ x_i = \frac{20(i-1)}{N-1},\ y_j = \frac{10(j-1)}{N-1}, (i, j = 1, 2, \cdots, N).$$

The error level δ is given by

$$\delta = \|g - g^\delta\|_2 = \sqrt{\frac{1}{N \times N}\sum_{i=1}^{N}\sum_{j=1}^{N}(g(x_i, y_j) - g^\delta(x_i, y_j))^2}.$$

Let u and $u^{\mu,\delta}$ present the exact solution and identical approximation regularization solution, respectively, and

$$rel(u) = \frac{\|u - u^{\mu,\delta}\|_{l^2}}{\|u\|_{l^2}}$$

denotes the relative error of the exact solution and identical approximation regularization solution. $rel(u)_G$ presents the relative error of exact solution and Gaussian regular solution, $rel(u)_D rel(u)_P$ denote the relative error of the exact solution and Dirichlet, de la Vallée Poussin identical approximate solutions, respectively. The regularization parameter is determined by $\mu = 4/\ln(E/\delta)$.

Example 3.1. Let $g(y, t) = \frac{yt}{25}e^{-(y-5)^2-(t-5)^2)}$ be the exact data of problem (1).

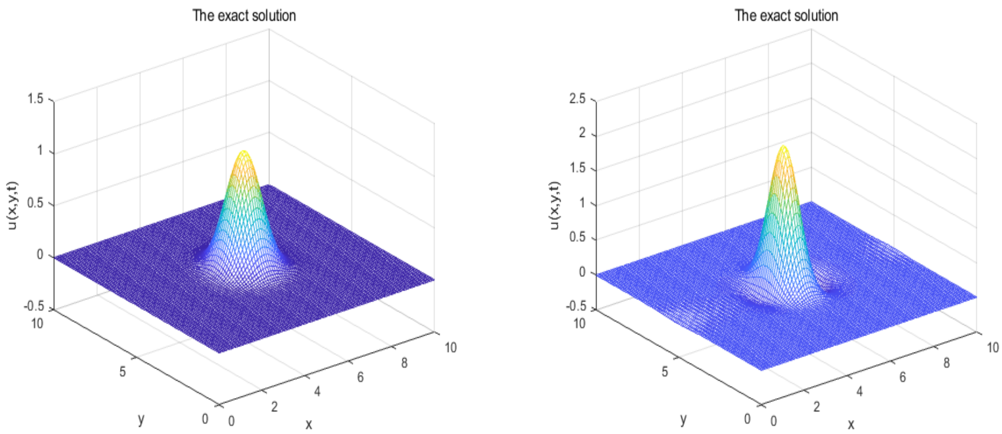
To study the numerical stability of our identical approximation algorithm, we use different noisy levels with $\delta = 1 \times 10^{-2}, 1 \times 10^{-3}, 1 \times 10^{-4}, 1 \times 10^{-5}$ at $x = 0.3$.

Table 1. displays the relative error of the exact solution and three kinds identical approximate solutions, respectively with different error levels δ.

We note that as the amount of noise in the data increases, the numerical solution in examples 3.1 is tend to stable. Fig.3 shows the exact solution for example 3.1 at $x = 0.3$ and $x = 0.8$. Fig. 4-5 displays the two kinds reconstructed solutions with $\delta = 10^{-3}$, $\delta = 10^{-5}$, respectively. Fig.6-7 show the error between the exact solution and identical approximation solutions under condition $\delta = 10^{-3}$ with $x = 0.3$, $x = 0.8$, respectively.

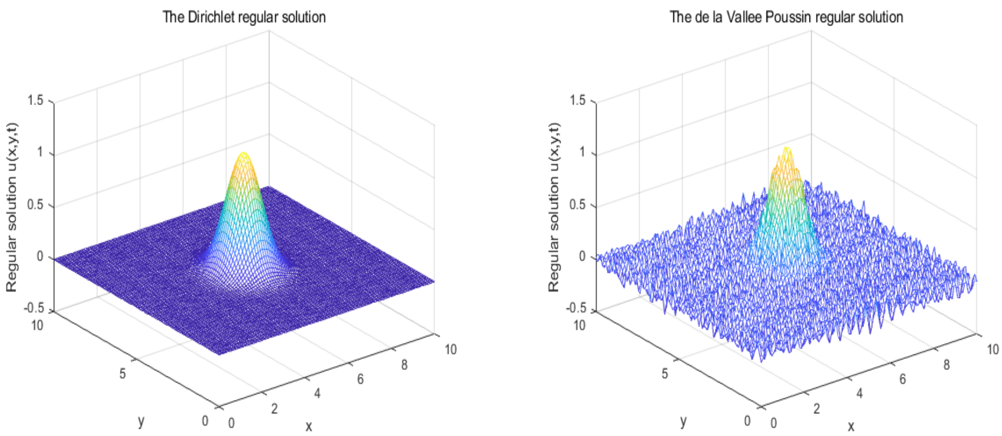
Table 1. The relative error at $x = 0.3$ for Example 3.1.

error level δ	$rel(u)_G$	$rel(u)_D$	$rel(u)_P$
1×10^{-2}	0.3566	0.3456	1.7985
1×10^{-3}	0.0660	0.0352	0.2639
1×10^{-4}	0.0479	0.0044	0.0383
1×10^{-5}	0.0384	$8.9160E-04$	0.0064



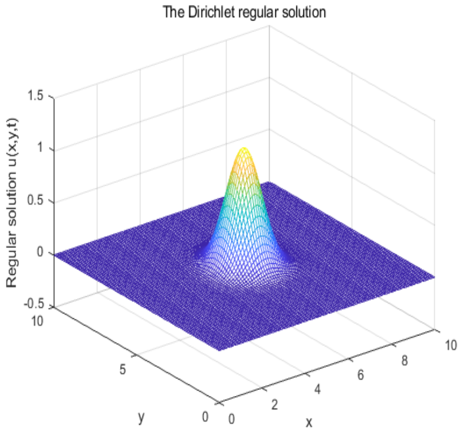
(a) The exact solution at $x = 0.3$ (b) The exact solution at $x = 0.8$

Figure 3. Example 3.1: The exact solution (a) at $x = 0.3$ (b) at $x = 0.8$.

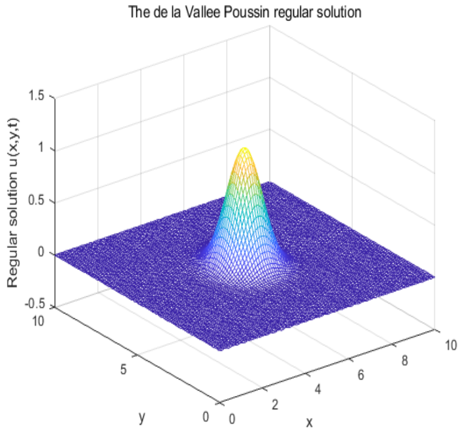


(a) Dirichlet regularization solution. (b) de la Vallée Poussin regularization solution.

Figure 4. Example 3.1: The regularization solution at $x = 0.3$, $\delta = 10^{-3}$ (a) Dirichlet (b) de la Vallée Poussin.

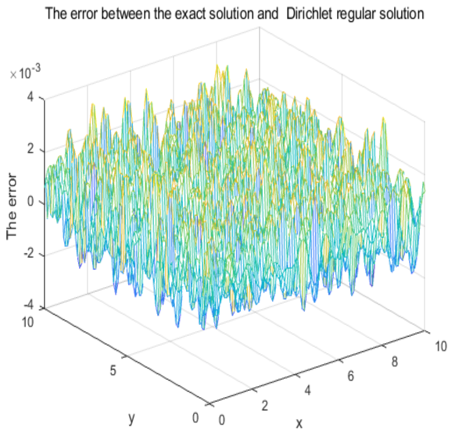


(a) Dirichlet regularization solution.

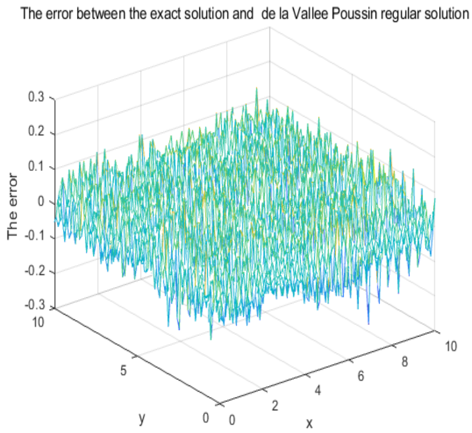


(b) De la Vallée Poussin regularization solution.

Figure 5. Example 3.1: The regularization solution at $x = 0.3$, $\delta = 10^{-5}$, (a) Dirichlet (b) de la Vallée Poussin.

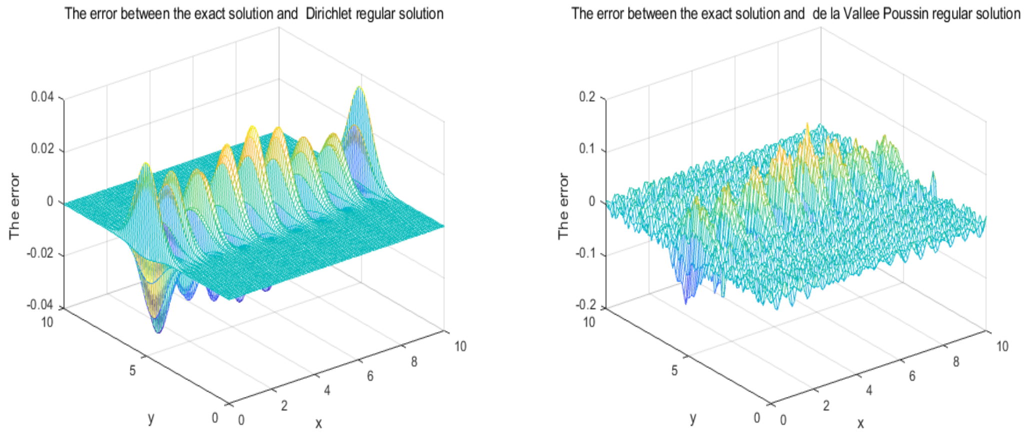


(a) The error between exact solution and Dirichlet regular solution.



(b) The error between exact solution and De la Vallée Poussin regular solution.

Figure 6. Example 3.1: The error between exact solution and regularization solution at $x = 0.3$, $\delta = 10^{-3}$.



(a) The error between exact solution and Dirichlet regular solution.

(b) The error between exact solution and De la Vallée Poussin regular solution.

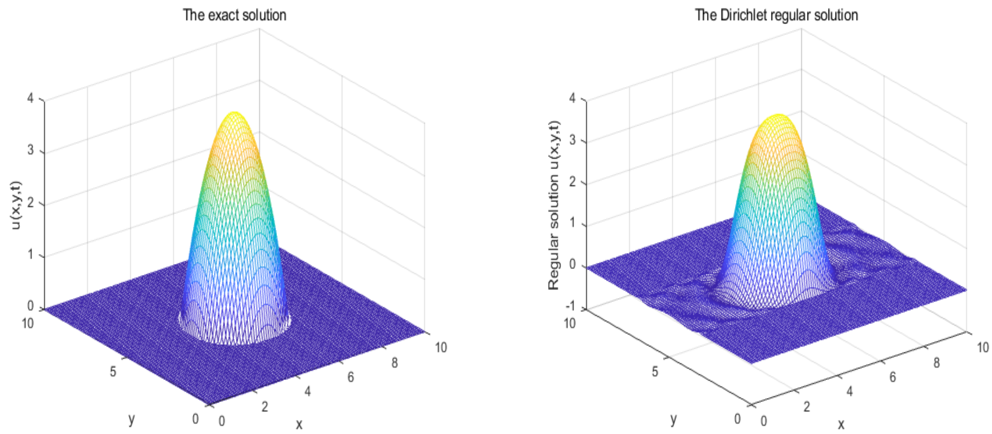
Figure 7. Example 3.1: The error between exact solution and regularization solution at $x = 0.8$, $\delta = 10^{-5}$.

Example 3.2. Taking continuous but not smooth function

$$g(y,t) = \begin{cases} 4-(y-5)^2-(t-5)^2, & (y-5)^2+(t-5)^2 \le 4, \\ 0, & (y-5)^2+(t-5)^2 > 4. \end{cases}$$

as the input data of problem (1).

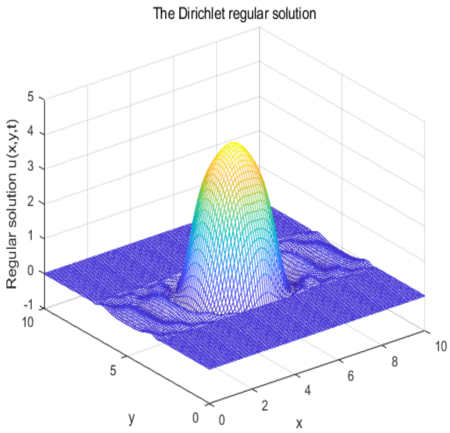
Figure 8-9 display the numerical results of the example 3.2. Figures 8 shows the exact data and Dirichlet regular solution at $x = 0$ with $\delta = 10^{-3}$, Figures 9 shows Dirichlet regular solution and de la Vallee Poussin regular solution at $x = 0.3$ with $\delta = 10^{-3}$. We can easily see that for not smooth function the proposed algorithm is feasible and effective. In two examples we see that the methods which we adopt are



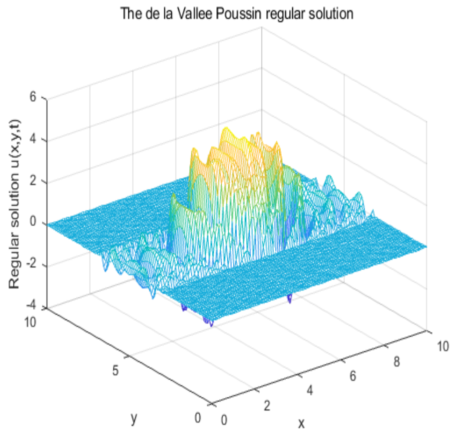
(a) The exact data.

(b) The Dirichlet regularization solution at $\delta = 10^{-3}$.

Figure 8. Example 3.2: The exact data and regular solution at $x = 0$.



(a) Dirichlet regularization solution.



(b) De la Vallée Poussin regularization solution.

Figure 9. Example 3.2: The comparison of the regular solution at $x = 0.3$, $\delta = 10^{-3}$.

stable and effect. Thereby, the stability of our proposed method is verified.

4. Conclusions

In this article, a new regularization method is proposed to solve the Cauchy problem of two-dimensional heat conduction equation, and the stable approximate estimates are obtained. Two numerical examples are investigated, the numerical examples do verify the numerical stability of the presented method. Furthermore, the accuracy of the procedure is quite acceptable.

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Disclosure statement

The authors declare that they have no competing interests.

Notes on contributor(s)

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

References

[1] Hadamard J., Lectures on Cauchy problem's in Linear Partial Differential Equations, *Dover Publications,* New York (1953).

[2] Manselli P., Miller K., Calculation of the surface temperature and heat flux on one side of a wall from measurements on the opposite side, *Ann. Mat. Pura Appl.* **123**(4) (1980) 161–183.

[3] Murio D. A., Numerical method for inverse transient heat conduction problems, *Revista de la Unión Matemátic Argentina.* **30** (1981) 25–46.

[4] Qian Z., Fu C.L., Regularization strategies for a two-dimensional inverse heat conduction problem, *Inverse problems.* **23** (3) (2007) 1053–1068.

[5] Li J., Wang F., Simplified Tikhonov regularization for a two kinds of parabolic equations, *Journal of the Korean Mathematical Society.* **48** (2) (2011) 311–327.

[6] Zhao J., Liu S., Liu T., A modifeid kernel method for solving Cauchy problem of two-dimensional heat conduction equation, *Advances in Applied Mathematics and Mechanics,* **7** (2) (2015) 31–42.

[7] Liu S., Feng L., A revised Tikhonov regularization method for a Cauchy problem of two-dimensional heat conduction equation, *Mathematical Problems in Engineering,* **7** (2) (2015) 31–42.

[8] Kirsch A., An introduction to the mathematical Theory of Inverse Problems (Second Edition), *Applied mathematical Sciences,* Springer (2011).

[9] Murio D. A., The mollification method and the numerical solution of ill-posed problems, *John Wiley and Sons Inc,* Wiley (2011).

[10] Hào D N, A mollification method for ill-posed problems, *Numer. Math.68* (1994) 469–506.

[11] Hào D N. A Mollification Method for a Noncharacteristic Cauchy Problem for a Parabolic Equation, *Journal of Mathematical Analysis and Applications*, **99** (3) (1996) 873-909.

[12] Zheng G H, Wei T. Two regularization methods for solving a Riesz-Feller space-fractional backward diffusion problem, *Inverse Problems*, **26** (11) (2010) 1-21.

[13] Lu S Z, Wang K Y. Real analysis (Second Edition), *Beijing Normal University Press,* Beijing (2005).

[14] Zhou M Q. Harmonic analysis (Real analysis method), *Peking University Press,* Beijing (1999)

[15] Hao C, Li F X, Fu C L, A mollification regularization method for the Cauchy problem of an elliptic equation in a multi-dimensional case, *Inverse Problems in Science and Engineering.* **18** (2010) 971–982.

[16] Nikol'skii S M, Approximation of functions of several variables and imbedding theorems, *Berlin-Heidelberg-New York,* Springer-Verlag (1975).

[17] Liu S, Regularization methods for ill-posed problems of several partial differetial equations, *Harbin Institute of Technology,* Harbin, China, 2014