Received 26 January 2019; | Revised ; | Accepted 10 November 2019
DOI: xxx/xxxx

RESEARCH ARTICLE

Algebraic Techniques for Least Squares Problems in Elliptic Complex Matrix Theory and Their Applications

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Summary

In this study, we introduce concepts of norms of elliptic complex matrices and derive the least squares solution, the pure imaginary least squares solution, and the pure real least squares solution with the least norm for the elliptic complex matrix equation $AX = B$ by using the real representation of elliptic complex matrices. To prove the authenticity of our results and to distinguish them from existing ones, some illustrative examples are also given. Elliptic numbers are generalized form of complex and so real numbers. Thus, the obtained results extend, generalize and complement some known least squares solutions results from the literature.

KEYWORDS:
Elliptic complex numbers, Elliptic complex matrices, Least squares solution with the minimum norm, Real representation of elliptic complex matrices

MSC CLASSIFICATION:
30G35; 15A09

1 | INTRODUCTION

The least squares solution with the least norm in matrix theory has important applications in modeling human faces, gene analysis, information retrieval and extraction, size reduction and data compression, signal and image processing-restoration, computational mathematics, some fields of pure and applied mathematics and so on [1,2,3,4,5,6,7,8,9,10,11]. With the rapid development of these fields, more and more researchers are interested in least squares problems and have obtained many valuable results. For the least squares solution with the least norm, they mainly consider real matrix equations, complex matrix equations and quaternion matrix equations.

On the other hand, there is an approach that allows one to generalize the complex numbers with three different classes of generalized complex numbers [12]. In this approach, generalized complex numbers are two-component number of the form

$$z = x + iy \ \ (x, y \in \mathbb{R})$$

where $i^2 = p \ \ (p \in \mathbb{R})$. Depending on the sign of p, the number systems are classified as follows:

1. for $p < 0$; Elliptic complex numbers (In particular, $p = -1$ corresponds to complex numbers),
2. for $p = 0$; Parabolic (dual) numbers,
3. for $p > 0$; Hyperbolic numbers.

Each of these number systems has many applications in science and technology[13,14,15,16,17,18,19,20]. On the other hand, since many physical systems have elliptical behaviours, it is getting more and more necessary for us to further study the theoretical properties and numerical computations of elliptic complex numbers and their matrices.
In this paper, we introduce concepts of norms of elliptic complex matrices and investigate three kinds elliptic complex least squares problem such

$$\|AX - B\| = min.$$

by using the real representation of elliptic complex matrices.
Problem 1: Let $A \in \mathbb{C}_p^{m\times n}$, $B \in \mathbb{C}_p^{m\times q}$,

$$C_L = \{X \,:\, X \in \mathbb{C}_p^{n\times q}, \|AX - B\| = min.\}$$

Find out $X_E \in \mathbb{C}_p^{n\times q}$ such that $\|X_E\| = \min\limits_{X\in C_L}\{\|X\|\}$ where X_E is called elliptic least square solution with the least norm of the elliptic complex matrix equation $AX = B$.
Problem 2: Let $A \in \mathbb{C}_p^{m\times n}$, $B \in \mathbb{C}_p^{m\times q}$,

$$I_L = \{X \,:\, X \in \mathbb{IC}_p^{n\times q}, \|AX - B\| = min.\}$$

Find out $X_I \in \mathbb{IC}_p^{n\times q}$ such that $\|X_I\| = \min\limits_{X\in I_L}\{\|X\|\}$ where X_I is called pure imaginary least square solution with the least norm of the elliptic complex matrix equation $AX = B$.
Problem 3: Let $A \in \mathbb{C}_p^{m\times n}$, $B \in \mathbb{C}_p^{m\times q}$,

$$R_L = \{X \,:\, X \in \mathbb{R}^{n\times q}, \|AX - B\| = min.\}$$

Find out $X_R \in \mathbb{R}^{n\times q}$ such that $\|X_R\| = \min\limits_{X\in R_L}\{\|X\|\}$ where X_R is called pure real least square solution with the least norm of the elliptic complex matrix equation $AX = B$.
Throughout this paper, the following notations are used. Let $\mathbb{R}$, $\mathbb{C}_p$ and $\mathbb{IC}_p$ denote the sets of real, elliptic complex and pure elliptic complex numbers, respectively. $\mathbb{R}^{m\times n}$ and $\mathbb{C}_p^{m\times n}$ denote the set of all matrices on $\mathbb{R}$ and $\mathbb{C}_p$, respectively. Moreover, all computations in this study are performed on an Intel i7-3630QM@2.40 Ghz/16GB computer using MATLAB R2016a software.

2 | ALGEBRAIC PROPERTIES OF ELLIPTIC COMPLEX NUMBERS

The set of elliptic complex numbers is denoted by

$$\mathbb{C}_p = \left\{z = x + iy : \ x, y \in \mathbb{R},\ i^2 = p < 0,\ p \in \mathbb{R}\right\}. \tag{1}$$

The conjugate and norm of elliptic complex number $z = x + iy$ are defined as

$$\overline{z} = x - iy \quad \text{and} \quad \|z\| = \sqrt{z\overline{z}} = \sqrt{x^2 - py^2}, \tag{2}$$

respectively. $p-$*multiplication* of the elliptic complex numbers $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2 \in \mathbb{C}_p$ is defined

$$z_1 z_2 = \left(x_1 + iy_1\right)\left(x_2 + iy_2\right) = \left(x_1x_2 + py_1y_2\right) + i\left(x_1y_2 + x_2y_1\right).$$

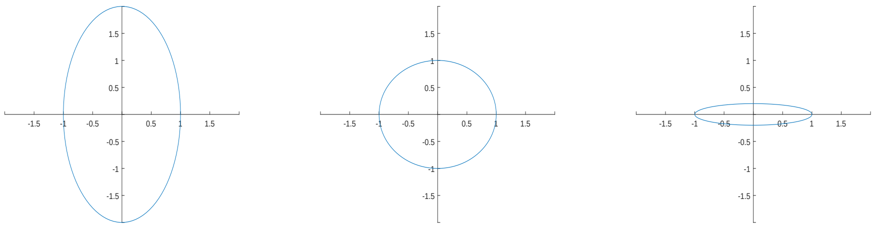
We note that $\mathbb{C}_p$ is 2D vector space over a field $\mathbb{R}$ according to addition and scalar multiplication. Also, each elliptic complex number can be represented in a single form in a plane which is called elliptic complex plane. In elliptic complex plane, the distance between the elliptic complex numbers z_1 and z_2 is defined as

$$\left\|z_1 - z_2\right\| = \sqrt{\left(x_1 - x_2\right)^2 - p\left(y_1 - y_2\right)^2}. \tag{3}$$

In elliptic complex plane, unit circle are defined by requiring

$$\|z\| = \sqrt{x^2 - py^2} = 1$$

as in Figure 1. In special case $p = -1$, the elliptic complex plane corresponds to the Euclidean plane[12,21].



(a) (b) (c)

FIGURE 1 Unit circles in $\mathbb{C}_{-0.5}$, $\mathbb{C}_{-1}$, $\mathbb{C}_{-5}$

Theorem 1. [22] Let $z = x + iy \in \mathbb{C}_p$ be given. Then z and its conjugate $\overline{z}$ satisfy the following universal similarity factorization equality (or for short USFE):

$$P^{-1}\begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} P = \begin{pmatrix} x & py \\ y & x \end{pmatrix} = \phi_p(z) \in \mathbb{R}^{2\times 2} \tag{4}$$

where P is

$$P = P^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ \frac{i}{p} & -1 \end{pmatrix}. \tag{5}$$

In here, $\phi_p(z) = \begin{pmatrix} x & py \\ y & x \end{pmatrix}$ is called fundamental matrix of z. USFE over the elliptic complex numbers clearly reveals three basic facts:

1. $\mathbb{C}_p$ is algebraically isomorphic to the matrix algebra

$$\mathbb{C}'_p = \left\{ \begin{pmatrix} x & py \\ y & x \end{pmatrix} : \ x,\ y \in \mathbb{R} \right\} \subseteq \mathbb{R}^{2\times 2} \tag{6}$$

   through the bijective map

$$\begin{aligned} \phi_p : \quad & \mathbb{C}_p \quad \to \quad \mathbb{C}'_p \\ & z = x + iy \to \phi_p(z) = \begin{pmatrix} x & py \\ y & x \end{pmatrix} \end{aligned} \tag{7}$$

2. Every elliptic complex number $z = x + iy \in \mathbb{C}_p$ has a real matrix representation

$$\phi_p(z) = \begin{pmatrix} x & py \\ y & x \end{pmatrix} \tag{8}$$

   over the real number field.

3. All real matrices in $\mathbb{C}'_p$ can uniformly be diagonalized over the elliptic complex numbers.

Meanwhile, we denote any elliptic complex number $z = x + iy := \begin{pmatrix} x \\ y \end{pmatrix}$. Then the *p−multiplication* of z and $z_1 = x_1 + iy_1$ can be written by the aid of the ordinary matrix multiplication as

$$zz_1 = \begin{pmatrix} x & py \\ y & x \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \phi_p(z)\, z_1.$$

Theorem 2. [22] Let $z_1 = x_2 + iy_2 \ and \ z_2 = x_2 + iy_2 \in \mathbb{C}_p$, the followings are satisfied:

1. $\phi_p\left(\phi_p\left(z_1\right) z_2\right) = \phi_p\left(z_1\right)\phi_p\left(z_2\right),$
2. $z_1 = z_2 \ \Leftrightarrow \ \phi_p\left(z_1\right) = \phi_p\left(z_2\right),$

3. $\phi_p\left(z_1+z_2\right)=\phi_p\left(z_1\right)+\phi_p\left(z_2\right)$,
4. $\phi_p\left(z_1 z_2\right)=\phi_p\left(z_1\right)\phi_p\left(z_2\right)$,
5. $\phi_p\left(\lambda z_1\right)=\lambda\phi_p\left(z_1\right)$,
6. $trace\left(\phi_p\left(z_1\right)\right)=z_1+\overline{z_1}$,
7. $\det\left(\phi_p\left(z_1\right)\right)=\left\|z_1\right\|^2$.

3 | ELLIPTIC COMPLEX MATRICES

The set of $m\times n$ matrices with elliptic complex number entries, which is denoted by $\mathbb{C}_p^{m\times n}$, is a ring under usual matrix addition and multiplication. Conjugate of $A=A_1+iA_2\in\mathbb{C}_p^{m\times n}$ ($A_1, A_2\in\mathbb{R}^{m\times n}$) is $\overline{A}=A_1-iA_2$. A matrix $A^T=A_1^T+iA_2^T\in\mathbb{C}_p^{n\times m}$ is transpose of $A=A_1+iA_2\in\mathbb{C}_p^{m\times n}$. Also $A^*=\left(\overline{A}\right)^T=A_1^T-iA_2^T\in\mathbb{C}_p^{n\times m}$ is called conjugate transpose[23].

Theorem 3. [23] A and B be elliptic complex matrices of appropriate sizes. Then the followings are satisfied:

1. $\left(A^{-1}\right)^{-1}=A$,
2. $(AB)^{-1}=B^{-1}A^{-1}$,
3. $\left(A^k\right)^{-1}=\left(A^{-1}\right)^k$ where k is any positive integer number,
4. $(\lambda A)^T=\lambda A^T$,
5. $(AB)^T=B^TA^T$,
6. $\left(A^k\right)^T=\left(A^T\right)^k$ where k is any positive integer number,
7. $\overline{\left(\overline{A}\right)}=A,\ \ (A^*)^*=A$,
8. $\overline{(A+B)}=\overline{A}+\overline{B},\ \ (A+B)^*=A^*+B^*$,
9. $\overline{(AB)}=\overline{A}\,\overline{B},\ \ (AB)^*=B^*A^*$.

Theorem 4. [22] Let $A=A_1+iA_2\in\mathbb{C}_p^{m\times n}$ be given. Then A and its conjugate $\overline{A}$ satisfy the following equality:

$$P_{2m}\begin{pmatrix} A & 0\\ 0 & \overline{A}\end{pmatrix}Q_{2n}=\begin{pmatrix} A_1 & pA_2\\ A_2 & A_1\end{pmatrix}=\Phi_p(A)\in\mathbb{R}^{2m\times 2n} \tag{9}$$

where

$$P_{2m}=P_{2m}^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix} I_m & iI_m\\ \frac{i}{p}I_m & -I_m\end{pmatrix},\ Q_{2m}=Q_{2m}^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix} I_n & iI_n\\ \frac{i}{p}I_n & -I_n\end{pmatrix} \tag{10}$$

In here, $\Phi_p(A)=\begin{pmatrix} A_1 & pA_2\\ A_2 & A_1\end{pmatrix}$ is called fundamental matrix of A.

Theorem 5. [22] Let $A,\ B\in\mathbb{C}_p^{m\times n},\ C\in\mathbb{C}_p^{n\times q}$ and $\lambda\in\mathbb{C}_p$ then the followings are satisfied:

1. $\Phi_p\left(I_n\right)=I_{2n}$,
2. $\Phi_p(A+B)=\Phi_p(A)+\Phi_p(B)$,
3. $\Phi_p(AC)=\Phi_p(A)\Phi_p(C)$,
4. $\Phi_p(\lambda A)=\lambda\Phi_p(A)$,

5. If $m = n$ and A is regular then $\Phi_p(A)$ is regular and $\left(\Phi_p(A)\right)^{-1} = \Phi_p\left(A^{-1}\right)$.
6. $\left(\Phi_p(A)\right)^{-} = \Phi_p(A^{-})$, where A^{-} and $\left(\Phi_p(A)\right)^{-}$ are Moore-Penrose generalized inverse of A and $\Phi_p(A)$, respectively.
7. For $A = \frac{1}{2}\left(I_m \; iI_m\right)\Phi_p(A)\begin{pmatrix} I_n \\ \frac{i}{p}I_n \end{pmatrix}$.

4 | NORMS OF ELLIPTIC COMPLEX MATRICES

This section gives the definition and properties of Frobenius norm of elliptic complex matrices.

Definition 1. Frobenius norm of an elliptic complex matrix $A = A_1 + iA_2 \in \mathbb{C}_p^{m\times n}$ is defined by

$$\|A\| = \sqrt{trace(AA^*)} = \sqrt{\|A_1\|^2 - p\|A_2\|^2}.$$

For all $\lambda \in \mathbb{C}_p$ and for all elliptic complex matrix $A,\ B \in \mathbb{C}_p^{m\times n}$, we have

- $\|\lambda A\| = \|\lambda\|\,\|A\|$,
- $\|A + B\| \leq \|A\| + \|B\|$,
- $\|A\| \geq 0$,
- $\|A\| = 0 \Leftrightarrow A = 0_{m\times n}$, where $0_{m\times n}$ is zero matrix with $m \times n$ dimension.

Theorem 6. Let $A = A_1 + iA_2 \in \mathbb{C}_p^{m\times n}$. Then we get $\|A\| \leq \frac{1}{\sqrt{2}}\left\|\Phi_p(A)\right\|$.

Proof. For $A = A_1 + iA_2 \in \mathbb{C}_p^{m\times n}$, we have

$$\left\|\Phi_p(A)\right\| = \sqrt{\|A_1\|^2 + \|pA_2\|^2 + \|A_2\|^2 + \|A_1\|^2}$$

$$= \sqrt{2\|A_1\|^2 + \left(p^2+1\right)\|A_2\|^2}.$$

On the other hand, we get

$$\left\|\Phi_p(A)\right\|^2 = 2\|A_1\|^2 + \left(p^2+1\right)\|A_2\|^2$$

$$= 2\|A_1\|^2 + \left(p^2+2p+1-2p\right)\|A_2\|^2$$

$$= 2\|A_1\|^2 + (p+1)^2\|A_2\|^2 - 2p\|A_2\|^2.$$

Thus

$$\frac{\left\|\Phi_p(A)\right\|^2}{2} = \|A_1\|^2 - p\|A_2\|^2 + \frac{(p+1)^2}{2}\|A_2\|^2.$$

Also, since

$$\|A\|^2 = \|A_1\|^2 - p\|A_2\|^2 \leq \|A_1\|^2 - p\|A_2\|^2 + \frac{(p+1)^2}{2}\|A_2\|^2$$

we have

$$\|A\| \leq \frac{1}{\sqrt{2}}\left\|\Phi_p(A)\right\|.$$

It is clear that for complex numbers, $\|A\| = \frac{1}{\sqrt{2}}\left\|\Phi_{-1}(A)\right\|$ equality is obtained.

□

5 | ALGEBRAIC METHOD FOR LEAST SQUARES PROBLEM FOR ELLIPTIC COMPLEX MATRICES

Lemma 1. [24] The real matrix equation $AX = B$, with $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times q}$, has a solution $X \in \mathbb{R}^{n\times q}$ if and only if $AA^{-}B = B$. In this case it has the general solution $X = A^{-}B + \left(I_n - A^{-}A\right)Z$, where $Z \in \mathbb{R}^{n\times q}$ is an arbitrary real matrix, and it has the unique solution $X = A^{-}B$ for the case when $rank\,(A) = n$.

Lemma 2. [24] The least squares solutions of the matrix equation $AX = B$, with $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times q}$, can be represented as $X = A^{-}B + \left(I_n - A^{-}A\right)Z$, where $Z \in \mathbb{R}^{n\times q}$ is an arbitrary real matrix, and the least squares solution with the least norm of the matrix equation $AX = B$ is $X = A^{-}B$.

Now, we consider the elliptic complex least squares problem

$$\|AX - B\| = \min. \tag{11}$$

where $A \in \mathbb{C}_p^{m\times n}$, $B \in \mathbb{C}_p^{m\times q}$ and $X \in \mathbb{C}_p^{n\times q}$. We define the real representation of the elliptic complex least squares problem of (11) by

$$\left\|\Phi_p(A)\,Y - \Phi_p(B)\right\| = \min. \tag{12}$$

Theorem 7. The elliptic complex least squares problem of (11) has a solution X if and only if the real least squares problem in (12) has a solution $Y = \Phi_p(X)$.

Proof. By Theorems 5 and 6, we have

$$\|AX - B\| \leq \frac{1}{\sqrt{2}}\left\|\Phi_p(AX) - \Phi_p(B)\right\| = \frac{1}{\sqrt{2}}\left\|\Phi_p(A)\,\Phi_p(X) - \Phi_p(B)\right\|.$$

This means that the elliptic complex least squares problem (11) has a solution X if and only if the real least squares problem (12) has a solution $Y = \Phi_p(X)$.

□

Theorem 8. Let $A \in \mathbb{C}_p^{m\times n}$, $B \in \mathbb{C}_p^{m\times q}$. The least squares solution with the least norm of the matrix equation $AX = B$ can be represented as

$$X = \frac{1}{2}\left(I_n\; iI_n\right)\left(\Phi_p(A)\right)^{-}\Phi_p(B)\begin{pmatrix} I_q \\ \frac{i}{p}I_q \end{pmatrix}. \tag{13}$$

Proof. By Lemma 2, the least squares solution with the least norm of (12) is $\Phi_p(X) = \left(\Phi_p(A)\right)^{-}\Phi_p(B)$. Then by the Theorem 7, the least squares solution with the least norm of (11) becomes

$$X = \frac{1}{2}\left(I_n\; iI_n\right)\left(\Phi_p(A)\right)^{-}\Phi_p(B)\begin{pmatrix} I_q \\ \frac{i}{p}I_q \end{pmatrix}.$$

□

Theorem 9. Let $A = A_1 + iA_2 \in \mathbb{C}_p^{m\times n}$, $B = B_1 + iB_2 \in \mathbb{C}_p^{m\times q}$. The pure imaginary least solution with the least norm of the matrix equation $AX = B$ can be represented as

$$X = i\left(\overleftarrow{A}\right)^{-}\left(\overrightarrow{B}\right) \tag{14}$$

where

$$\overleftarrow{A} = \begin{pmatrix} pA_2 \\ pA_1 \\ A_1 \\ pA_2 \end{pmatrix} \quad and \quad \overrightarrow{B} = \begin{pmatrix} B_1 \\ pB_2 \\ B_2 \\ B_1 \end{pmatrix}.$$

Proof. Suppose that $X = iX_2$ is a pure imaginary elliptic complex matrix. Then $\Phi_p(X) = \begin{pmatrix} 0 & pX_2 \\ X_2 & 0 \end{pmatrix}$. Also,

$$\begin{aligned}
\|AX - B\| &\leq \frac{1}{\sqrt{2}} \left\|\Phi_p(A)\Phi_p(X) - \Phi_p(B)\right\| \\
&= \frac{1}{\sqrt{2}} \left\| \begin{pmatrix} A_1 & pA_2 \\ A_2 & A_1 \end{pmatrix} \begin{pmatrix} 0 & pX_2 \\ X_2 & 0 \end{pmatrix} - \begin{pmatrix} B_1 & pB_2 \\ B_2 & B_1 \end{pmatrix} \right\| \\
&= \frac{1}{\sqrt{2}} \left\| \begin{pmatrix} pA_2X_2 & pA_1X_2 \\ A_1X_2 & pA_2X_2 \end{pmatrix} - \begin{pmatrix} B_1 & pB_2 \\ B_2 & B_1 \end{pmatrix} \right\| \\
&= \frac{1}{\sqrt{2}} \left\| \begin{pmatrix} pA_2 \\ pA_1 \\ A_1 \\ pA_2 \end{pmatrix} X_2 - \begin{pmatrix} B_1 \\ pB_2 \\ B_2 \\ B_1 \end{pmatrix} \right\|.
\end{aligned}$$

Thus, we have

$$X = i\begin{pmatrix} pA_2 \\ pA_1 \\ A_1 \\ pA_2 \end{pmatrix}^{-} \begin{pmatrix} B_1 \\ pB_2 \\ B_2 \\ B_1 \end{pmatrix}.$$

□

Theorem 10. Let $A = A_1 + iA_2 \in \mathbb{C}_p^{m\times n}$, $B = B_1 + iB_2 \in \mathbb{C}_p^{m\times q}$. The pure real least solution with the least norm of the matrix equation $AX = B$ can be represented as

$$X = \left(\overrightarrow{A}\right)^{-} \left(\overrightarrow{B}\right). \tag{15}$$

where

$$\overrightarrow{A} = \begin{pmatrix} A_1 \\ pA_2 \\ A_2 \\ A_1 \end{pmatrix} \ \ and \ \ \overrightarrow{B} = \begin{pmatrix} B_1 \\ pB_2 \\ B_2 \\ B_1 \end{pmatrix}.$$

Proof. Suppose that $X = X_1$ is a pure real elliptic complex matrix. Then $\Phi_p(X) = \begin{pmatrix} X_1 & 0 \\ 0 & X_1 \end{pmatrix}$. Also,

$$\begin{aligned}
\|AX - B\| &\leq \frac{1}{\sqrt{2}} \left\|\Phi_p(A)\Phi_p(X) - \Phi_p(B)\right\| \\
&= \frac{1}{\sqrt{2}} \left\| \begin{pmatrix} A_1 & pA_2 \\ A_2 & A_1 \end{pmatrix} \begin{pmatrix} X_1 & 0 \\ 0 & X_1 \end{pmatrix} - \begin{pmatrix} B_1 & pB_2 \\ B_2 & B_1 \end{pmatrix} \right\| \\
&= \frac{1}{\sqrt{2}} \left\| \begin{pmatrix} A_1X_1 & pA_2X_1 \\ A_2X_1 & A_1X_1 \end{pmatrix} - \begin{pmatrix} B_1 & pB_2 \\ B_2 & B_1 \end{pmatrix} \right\| \\
&= \frac{1}{\sqrt{2}} \left\| \begin{pmatrix} A_1 \\ pA_2 \\ A_2 \\ A_1 \end{pmatrix} X_1 - \begin{pmatrix} B_1 \\ pB_2 \\ B_2 \\ B_1 \end{pmatrix} \right\|.
\end{aligned}$$

Thus, we have

$$X = \begin{pmatrix} A_1 \\ pA_2 \\ A_2 \\ A_1 \end{pmatrix}^{-} \begin{pmatrix} B_1 \\ pB_2 \\ B_2 \\ B_1 \end{pmatrix}.$$

□

6 | NUMERICAL ALGORITHMS

We now provide numerical algorithms for problems 1,2 and 3 based on the Theorems 8, 9 and 10.

Algorithm 1 Pseudocode for Problem 1

---

1. Begin.
2. **Input** $A,\ B$ and p.
3. Form $\Phi_p(A)$ and $\Phi_p(B)$ according to equation (9).
4. Compute $X = \frac{1}{2}\left(I_n\, iI_n\right)\left(\Phi_p(A)\right)^{-}\Phi_p(B)\begin{pmatrix} I_q \\ \frac{i}{p}I_q \end{pmatrix}$.
5. **Output** X.
6. Stop.

Algorithm 2 Pseudocode for Problem 2

---

1. Begin.
2. **Input** $A,\ B$ and p.
3. Form $\overleftarrow{A}$ and $\overrightarrow{B}$ according to Theorem 9.
4. Calculate $X = i\left(\overleftarrow{A}\right)^{-}(\overrightarrow{B})$.
5. **Output** X.
6. Stop.

Algorithm 3 Pseudocode for Problem 3

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1. Begin.
2. **Input** $A,\ B$ and p.
3. Form $\overrightarrow{A}$ and $\overrightarrow{B}$ according to Theorem 10.
4. Compute $X = \left(\overrightarrow{A}\right)^{-}(\overrightarrow{B})$.

5. **Output** X.

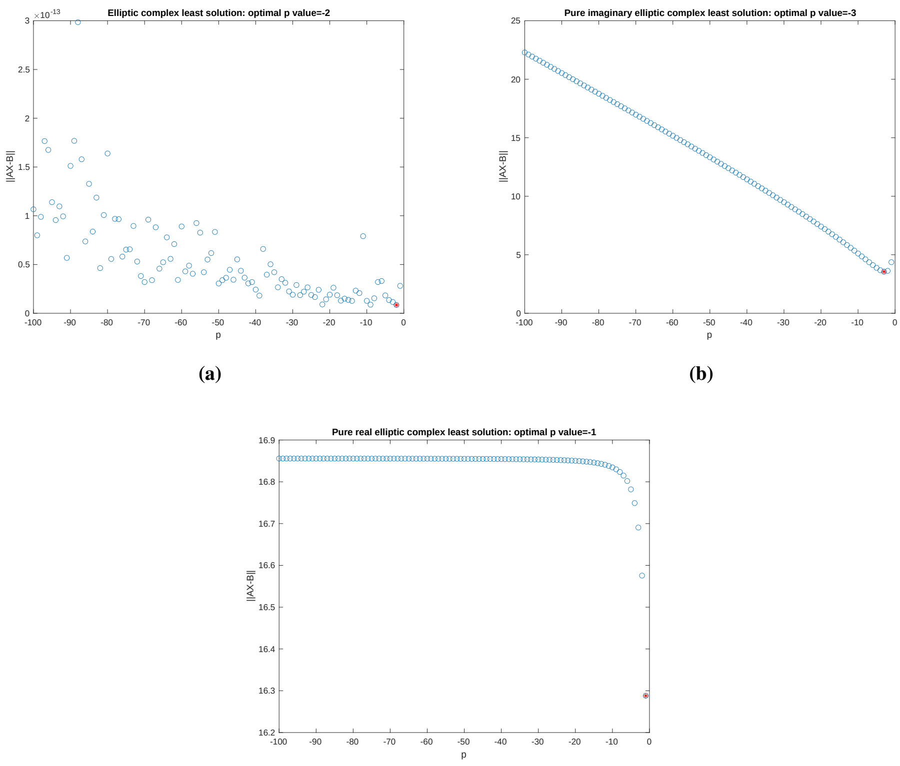
6. Stop.

7 | NUMERICAL EXAMPLES

Let

$$A = \begin{pmatrix} 2+8i & 1+i & 1+6i \\ 1+3i & 2+5i & 1+7i \\ 1+4i & 1+9i & 2+2i \end{pmatrix}, \quad B = \begin{pmatrix} 8+0.5i & 1-0.5i & 6-0.5i \\ 3-0.5i & 5+0.5i & 7-0.5i \\ 4-0.5i & 9-0.5i & 2+0.5i \end{pmatrix}.$$

Figure 2 shows the least norms for elliptic, pure imaginary and pure real solutions of elliptic complex least square problem according to the values of $p \in [-100, -1]$.



(a) (b) (c)

FIGURE 2 Optimal p values for the Problem 1, 2, 3, respectively

According to this graph, the optimal p value of the elliptic least squares solution with the least norm is -2, the optimal p value of the pure imaginary elliptic least squares solution with the least norm is -3 and the optimal p value of the pure real elliptic least squares solution with the least norm is -1.

Now, find elliptic, pure imaginary and pure real solutions of elliptic complex least square problems according to the optimal p values.

With Algorithm 1, we can get easily

$$\Phi_{-2}(A) = \begin{pmatrix} 2 & 1 & 1 & -16 & -2 & -12 \\ 1 & 2 & 1 & -6 & -10 & -14 \\ 1 & 1 & 2 & -8 & -18 & -4 \\ 8 & 1 & 6 & 2 & 1 & 1 \\ 3 & 5 & 7 & 1 & 2 & 1 \\ 4 & 9 & 2 & 1 & 1 & 2 \end{pmatrix}$$

and

$$\Phi_{-2}(B) = \begin{pmatrix} 8 & 1 & 6 & -1 & 1 & 1 \\ 3 & 5 & 7 & 1 & -1 & 1 \\ 4 & 9 & 2 & 1 & 1 & -1 \\ 0.5 & -0.5 & -0.5 & 8 & 1 & 6 \\ -0.5 & 0.5 & -0.5 & 3 & 5 & 7 \\ -0.5 & -0.5 & 0.5 & 4 & 9 & 2 \end{pmatrix}.$$

From $\Phi_{-2}(X) = \left(\Phi_{-2}(A)\right)^{-}\Phi_{-2}(B)$, we have

$$\Phi_{-2}(X) = \begin{pmatrix} 0.2159 & -0.2127 & 0.0934 & 0.9506 & 0.0118 & 0.0118 \\ -0.0903 & 0.0322 & 0.1546 & 0.0118 & 0.9506 & 0.0118 \\ -0.0290 & 0.2771 & -0.1515 & 0.0118 & 0.0118 & 0.9506 \\ -0.4753 & -0.0059 & -0.0059 & 0.2159 & -0.2127 & 0.0934 \\ -0.0059 & -0.4753 & -0.0059 & -0.0903 & 0.0322 & 0.1546 \\ -0.0059 & -0.0059 & -0.4753 & -0.0290 & 0.2771 & -0.1515 \end{pmatrix}.$$

Thus, we get

$$X = \begin{pmatrix} 0.2159 - 0.4753i & -0.2127 - 0.0059i & 0.0934 - 0.0059i \\ -0.0903 - 0.0059i & 0.0322 - 0.4753i & 0.1546 - 0.0059i \\ -0.0290 - 0.0059i & 0.2771 - 0.0059i & -0.1515 - 0.4753i \end{pmatrix}.$$

With Algorithm 2, we can get easily

$$\overleftarrow{A} = \begin{pmatrix} -24 & -3 & -18 \\ -9 & -15 & -21 \\ -12 & -27 & -6 \\ -6 & -3 & -3 \\ -3 & -6 & -3 \\ -3 & -3 & -6 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \\ -24 & -3 & -18 \\ -9 & -15 & -21 \\ -12 & -27 & -6 \end{pmatrix}, \quad \vec{B} = \begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \\ -1.5 & 1.5 & 1.5 \\ 1.5 & -1.5 & 1.5 \\ 1.5 & 1.5 & -1.5 \\ 0.5 & -0.5 & -0.5 \\ -0.5 & 0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 \\ 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix}.$$

From $X = i\left(\overleftarrow{A}\right)^{-}\vec{B}$, we have

$$X = \begin{pmatrix} -0.1560i & -0.0004i & -0.0256i \\ -0.0004i & -0.1812i & -0.0004i \\ -0.0256i & -0.0004i & -0.1560i \end{pmatrix}.$$

With Algorithm 3, we can get easily

$$\overrightarrow{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \\ -8 & -1 & -6 \\ -3 & -5 & -7 \\ -4 & -9 & -2 \\ 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \ \overrightarrow{B} = \begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \\ -0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 \\ -0.5 & 0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 \\ 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix}.$$

From $X = \left(\overrightarrow{A}\right)^{-}\overrightarrow{B}$ we have

$$X = \begin{pmatrix} 0.3646 & -0.3889 & 0.2422 \\ -0.0498 & 0.0726 & 0.1951 \\ -0.0969 & 0.5342 & -0.2194 \end{pmatrix}.$$

8 | CONCLUSION

In this study, we derive the least squares solution, the pure imaginary least squares solution, and the pure real least squares solution with the least norm for the elliptic complex matrix equation $AX = B$ by using the real representation of elliptic complex matrices. The least squares method has important applications in modeling human faces, gene analysis, information retrieval and extraction, size reduction and data compression, signal and image processing-enhancement processes. The use of elliptic matrices in these application areas will enable the previously known definitions and theorems to be interpreted with a wider perspective, and by selecting the ideal space for the problems, great flexibility and efficiency will be brought to existing techniques.

CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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