On approximate solutions for convex semi-infinite programming with uncertainty

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Abstract

In this paper, we consider approximate solutions (also called ε-solutions) for semi-infinite optimization problems that objective function and constraint functions with uncertainty data are all convex, and establish robust counterpart of convex semi-infinite program and then consider approximate solutions for its. Moreover, the robust necessary condition and robust sufficient theorems are obtained. Then the duality results of the Lagrangian dual approximate solution is given by the robust optimization approach under a cone constraint qualification.

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1 Introduction

Focus on the convex semi-infinite programming (CSIP) as follows:

$$\begin{aligned} &(\text{CSIP}) \min w(x) \\ &\qquad \text{s. t. } h'_t(x) \leq 0, \forall t \in T, \end{aligned} \tag{1}$$

where $w : R^n \to R$, and $h'_t : R^n \to R, t \in T$, are convex and continuous functions, and T is an infinite set. When $w(x), h'_t(x)$ are all linear function, we call its as linear semi-infinite programming.

Dual theory plays an important role in the study of semi-infinite programming problems. Goberna [1] summarizes the 2000 publication on semi-infinite linear C (SILP), which aims to identify the most active research areas, major trends in applications. In each chapter of [2] contains detailed bibliographical introduction on (SILP) and extensions. Goberna [2] devoted to model-building by means of primal and dual (SILP) problems. The research on (SILP) duality is discussed the dual problems of convex semi-infinite case in [3, 4]. Semi-infinite programming traditionally assumes certain information. In real life, the information of optimization problems sometimes are uncertain, wrong or lacking, so it is important to discuss the dual problem under uncertain set.

Ben-Tal and Nemirovski et al. [5] propose a deterministic framework for the study of mathematical programming under uncertain data. The robust optimization methods for linear programming problems and convex optimization problems under data uncertainties are discussed successfully by El Ghaoui [6]. Consider as the data uncertainty, Goberna [7] using robust duality theory to work out convex programming problems. The research on the robust correspondence between dual problems and uncertain

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convex programming [8] shows that the value of the robust counterpart of primal problem is equal to the value of the optimistic counterpart of the dual primal (i. e. primal worst equals dual best).

Convex programming [9] which the constraint functions are finite with uncertain data can be summarized as follows:

$$\begin{aligned} &(\text{UCP}) \min w(x) \\ &\qquad \text{s. t. } h_i'(x, u_i) \leq 0, i = 1, \ldots, m, \end{aligned} \tag{2}$$

where $h_i' : R^n \times R^m \to R$, $h_i(\cdot, u_i)$ is convex and $u_i \in U_i \subseteq R^m$ is the uncertain parameter.

In recent years, many scholars have studied the robust convex optimization problem with data uncertainty. A selection problem of robust convex optimization is proposed in [10]. For (UCP) with data uncertainty considers robust counterpart and an optimistic counterpart. Jeyakumar and Li [11] prove that Lagrangian strong duality theorem, and then give a new robust characteristic cone, and give the necessary and sufficient conditions for the existence of strong duality. The optimistic correspondence is proposed by [12]. Sun et al. [13] studies the robust quasi-approximate optimal solution for a class of nonsmooth semi-infinite programming with uncertain data.

Lee and Lee [14] think about the approximate solution to robust convex optimization problem, and establish the duality theorem of Wolfe type dual problem with finite constraint function. Then Lee and Lee [15] give the ε- solution of the robust semi-infinite optimization problem. Under the closed convex cone, an approximate weak duality theorem and a strong duality theorem for the original problem and its Wolfe dual problem are established. Then, Zeng et al. [16] present some modified robust solutions for fractional (SIP) with uncertain information. Lagrangian dual with finite constraints is studied in existing literature [11], it shows strong dual holds (i. e. $\varepsilon = 0$) under the robust characteristic cone as following:

$$M = \bigcup_{u_i \in U_i, \lambda_i \geq 0} \text{epi}(\sum_{i=1}^{m} \lambda_i h_i(\cdot, u_i))^*,$$

where M is closed and convex, and $(\sum_{i=1}^{m} \lambda_i h_i(\cdot, u_i))^*$ denotes the conjugate function of $(\sum_{i=1}^{m} \lambda_i h_i(\cdot, u_i))$.

With uncertain constraint conditions (CSIP), can be summarized as follows:

$$\begin{aligned} &(\text{UCSIP}) \min w(x) \\ &\qquad \text{s. t. } h_t(x, u_t) \leq 0, \forall t \in T, \end{aligned} \tag{3}$$

where for any $t \in T, h_t : R^n \times R^m \to R$, are continuous convex functions, and $u_t \in R^m$ is an uncertain parameter, which belong to some convex compact set $U_t \subset R^m$.

Defined the uncertainty set-valued mapping U: $T \to R^m$ as $U(t) := U_t$ for all $t \in T$. And $u \in U$ implies that u is an element of U, i. e. , $u : T \to R^m$ and $u_t \in U_t$ for all $t \in T$.

The Lagrangian dual of (UCSIP) is given by

$$\begin{aligned} &(\text{LDUCSIP}) \max_{\lambda_t} \inf_{x \in R^n} \{w(x) + \sum_{t \in T} \lambda_t h_t(x, u_t)\} \\ &\qquad \text{s. t. } \lambda_t \geq 0, \end{aligned} \tag{4}$$

The robust counterpart of (UCSIP) can be summarized as follows:

$$\begin{aligned} &(\text{RCSIP}) \min w(x) \\ &\qquad \text{s. t. } h_t(x, u_t) \leq 0, \forall t \in T, u_t \in U_t. \end{aligned} \tag{5}$$

The best possible robust feasible solution is the one that solves the optimization problem (RCSIP) or which is same with (UCSIP). (RCSIP) is called the robust counterpart of the original uncertain problem (UCSIP).

Motivated by above, in this paper, we consider approximate solutions (i. e. $\varepsilon > 0$) for robust semi-infinite convex programming. By using the robust optimization method, the robust necessary condition and sufficient conclusions of (RCSIP) under closed convex cone constraints are established, denote the cone Γ as follows:

$$\Gamma := \bigcup_{u_t \in U_t, \lambda_t \geq 0, t \in T} \operatorname{epi}\Big(\sum_{t \in T} \lambda_t h_t(\cdot, u_t)\Big)^*. \tag{6}$$

where $(\sum_{t \in T} \lambda_t h_t(\cdot, u_t))^*$ is the conjugate function of $(\sum_{t \in T} \lambda_t h_t(\cdot, u_t))$. Moreover, under the closed convex cone Γ constraint qualification, between the primal problem and Lagrangian dual problem, we prove that an approximate weak duality result and strong duality theorem are held.

Denote the optimistic counterpart of (LDUCSIP) as follows,

$$\begin{aligned} &(\text{OLDCSIP}) \max_{\lambda_t} \inf_{x} \{w(x) + \sum_{t \in T} \lambda_t h_t(x, u_t)\} \\ &\qquad \text{s. t. } \lambda_t \geq 0, \forall t \in T, u_t \in U_t, x \in R^n. \end{aligned} \tag{7}$$

Denote the Lagrangian dual of the robust counterpart (RCSIP) as follows,

$$\begin{aligned} &(\text{LDRCSIP}) \max_{\lambda_t} \inf_{x} \sup_{u_t} \{w(x) + \sum_{t \in T} \lambda_t h_t(x, u_t)\} \\ &\qquad \text{s. t. } \lambda_t \geq 0, \forall t \in T, u_t \in U_t, x \in R^n. \end{aligned} \tag{8}$$

We kick the approximate weak dual theorem and strong dual theorem of (LDRCSIP) around in section 4. And get the feasible set of (UCSIP) is as follows:

$$F := \{x \in R^n | h_t(x, u_t) \leq 0, \forall t \in T, u_t \in U_t\}. \tag{9}$$

Let $\varepsilon \geq 0$. We define $\widehat{x}$ as an ε-solution of (RCSIP) if $\widehat{x}$ satisfied

$$w(x) \geq w(\widehat{x}) - \varepsilon,$$

for any $x \in F$.

The structure of the paper is as following. We introduce some preliminary knowledge and notations in section 2. Some conditions for the existence are discussed in Section 3. Approximate weak and strong theorem are given in Section 4. In section 5, we summarize the content of this article.

2 Notations and preliminaries

In order to show our conclusions are given in next section, recall some symbols and preliminary results. Representation R^n is the n-dimension Euclidean space, denote R^+ as the nonnegative quadrant of R, and denote the graph of set U as $\operatorname{gph}U := \{(t, u_t) | u_t \in U_t, t \in T\}$. Representation cl$A$, co$A$, and cone$A$ is the closure, the convex hull, and the conical hull severally. Let $w : R^n \to \widetilde{R}$ where $\widetilde{R}$ is a extended real set, denote as $\widetilde{R} = [-\infty, +\infty]$. Here, if for all $x \in R^n, w(x) > -\infty$ and exists $x' \in R^n$ such that $w(x') \in R$, then w is said to be proper .

Definition 2.1: δ_A be the indicator function, if $R^n \to R \cup +\infty$ where A is a closed and convex set, $A \in R^n$. $\delta_A(x) = 0$ when $x \in A$, and $\delta_A(x) = +\infty$, when $x \notin A$. i. e.

$$\delta_A(x) := \begin{cases} 0, & \text{if} x \in A, \\ +\infty, & \text{(otherwise)}. \end{cases} \tag{10}$$

Definition 2.2: The domain of w is dom$w$, dimension dom$w$ is as follows

$$\text{dom} w := \{x \in R^n | w(x) < +\infty\}. \tag{11}$$

Definition 2.3: Defined the epigraph of a function $w : R^n \to R \cup +\infty$, epi$w$ as follows

$$\text{epi} w := \{(x, r)^T \in R^{n+1} | w(x) \leq r\}. \tag{12}$$

Definition 2.4: Defined the conjugate function of w as $w^* : R^n \to R \cup \{+\infty\}$, for any proper convex function w on R^n, and for any $w^* \in R^n$,

$$w^*(x^*) = \sup\{\langle x^*, x\rangle - w(x) | x \in R^n\}. \tag{13}$$

Definition 2.5: If for any $\mu \in [0, 1]$, $x, y \in R^n$,satisfied

$$w((1 - \mu)x + \mu y) \leq (1 - \mu)w(x) + \mu w(y),$$

we call $w(x)$ is convex functions.

Then according to Definition 2. 5 epi$w$ is convex. When $-w$ is a convex function, the function w is a concave function.

The sub-differential of w at $x \in$ w is defined by

$$\partial_x w(x) := \{x^* \in R^n | \langle x^*, \widetilde{x} - x\rangle \leq w(\widetilde{x}) - w(x), \forall \widetilde{x} \in R^n\}. \tag{14}$$

If $x \notin$ dom$w, \partial_x w(x)$ is empty. More generally, for $\varepsilon \geq 0$, defined the ε-sub-differential of w at $x \in$ dom$w$ as follows:

$$\partial_\varepsilon w(x) := \{x^* \in R^n | \langle x^*, \widetilde{x} - x\rangle \leq w(\widetilde{x}) - w(x) + \varepsilon, \forall \widetilde{x} \in R^n\}. \tag{15}$$

For $x \notin$ dom$w, \partial_x w(x)$ is empty. We call w is a lower is a lower semi-continuous function if

$$\liminf_{\widetilde{x} \to x} w(\widetilde{x}) \geq w(x) \tag{16}$$

for all $x \in R^n$.

Definition 2.6: [14]For any $\varepsilon > 0$ there exists $\rho > 0$ such that for all $s \in T, U_s \subset U_t + \varepsilon B$ where B is a unit ball in R^m and $d(s, t) \leq \rho$ where d is the distance on U, then the set-valued mapping U :$T \to R^m$, where (T, d) is a metric space, called the upper semi-continuous at $t \in T$.

If for any $\varepsilon > 0$ there exists $\rho > 0$ such that for all $s, t \in T$, $U_s \subset U_t + \varepsilon B$ with $d(s, t) \leq \rho$, U is called uniformly upper semi-continuous on T.

In order to describe the relationship between the ε-sub-differential and the epigraph of a conjugate function we give following lemma, which plays a key role in the derivation of the main results.

Lemma 2.1: If w is said to be proper, $w = \{x \in R^n | w(x) < +\infty\} \neq \emptyset$. Let w be a proper lower semi-continuous convex function, $w : R^n \to R \cup \{+\infty\}$. Then

$$\text{epi}w^* = \bigcup_{\varepsilon \geq 0} \{(u, \langle u, \xi \rangle + \varepsilon - w(\xi)) : \xi \in \partial_\varepsilon w(\xi)\}. \tag{17}$$

where $\xi \in \text{dom}w$.

Lemma 2.2: If $\text{dom}w \cap \text{dom}h \neq \emptyset$, let $w, h : R^n \to R \cup \{+\infty\}$, and w, h be a be a proper lower semi-continuous convex function, then

$$\text{epi}(w + h)^* = \text{cl}(\text{epi}w^* + \text{epi}h^*). \tag{18}$$

Then

$$\text{epi}(w + h)^* = \text{epi}w^* + \text{epi}h^*, \tag{19}$$

where one of the functions w and h is continuous,

Lemma 2.3: Let I is an arbitrary index set, $h_i : R^n \to R \cup \{+\infty\}, i \in I$, and h_i be a proper lower semi-continuous convex function. If there exists $x' \in R^n$ such that $\sup_{i \in I} h_i(x') < +\infty$. Then,

$$\text{epi}\left(\sup_{i \in I} h_i\right)^* = \text{cl}\left(\text{co} \bigcup_{i \in I} \text{epi}h_i^*\right). \tag{20}$$

Lemma 2.4: Let $u_t \in R^m$, for any vector x, $h_t(x, u_t)$ is a convex function, $h_t : R^n \times R^m \to R, t \in T$, and h_t be continuous functions, then

$$\bigcup_{u_t \in U_t, \lambda_t \geq 0, t \in T} \text{epi}\left(\sum_{t \in T} \lambda_t h_t(\cdot, u_t)\right)^*. \tag{21}$$

is called the robust characteristic cone. This cone plays an important role in duality theory of next part. And following lemma prove cone is convex and closed under certain conditions.

Proof: Let $\lambda_t = 0, t \in T$, one observes that

$$(0, 0)^T \in \bigcup_{u_t \in U_t, \lambda_t \geq 0, t \in T} \text{epi}\left(\sum_{t \in T} \lambda_t h_t(\cdot, u_t)\right)^*.$$

Now let $\lambda > 0$ and $(a, b)^T \in \bigcup_{u_t \in U_t, \lambda_t \geq 0, t \in T} \text{epi}\left(\sum_{t \in T} \lambda_t h_t(\cdot, u_t)\right)^*$.

Then, there exist $u_t \in U_t, \alpha_t \geq 0, t \in T$, such that $(a, b)^T \in \text{epi}\left(\sum_{t \in T} \alpha_t h_t(\cdot, u_t)\right)^*$.

Let $\lambda_t = \alpha_t \lambda \geq 0$. Then,

$$\lambda(a, b)^T \in \lambda\text{epi}\left(\sum_{t \in T} \alpha_t h_t(\cdot, u_t)\right)^* = \text{epi}\left(\sum_{t \in T} \lambda_t h_t(\cdot, u_t)\right)^* \subseteq \bigcup_{u_t \in U_t, \lambda_t \geq 0, t \in T} \text{epi}\left(\sum_{t \in T} \lambda_t h_t(\cdot, u_t)\right)^*.$$

Hence, $\bigcup_{u_t \in U_t, \lambda_t \geq 0, t \in T} \text{epi}\left(\sum_{t \in T} \lambda_t h_t(\cdot, u_t)\right)^*$ is a cone.

Generally speaking, without the condition that $h_t(x,u_t)$ is a convex function most robust characteristic cone is not a convex cone .

We next illustrate that convexity of the robust characteristic cone under the concavity of $h(x,\cdot)$ and the convexity of U_t.

Lemma 2.5: Let $h_t : R^n \times R^m \to R, t \in T$, and h_t be continuous functions. Assume that for every $U_t \subseteq R^m$, is convex, for every $u_t \in R^m, t \in T, h_t(\cdot,u_t)$ be convex, $h_t(x,\cdot)$ is concave on U_t, for any $x \in R^n$. Then

$$\Gamma = \bigcup_{u_t \in U_t, \lambda_t \geq 0, t \in T} \operatorname{epi}\left(\sum_{t\in T} \lambda_t h_t(\cdot,u_t)\right)^*. \tag{22}$$

is convex.

Proof: In order to proof its convex, let $(a_i,b_i)^T \in \Gamma, i=1,2$, and $\mu \in [0,1]$. In order to illustrate Γ is convex, we now prove that $(\mu a^1 + (1-\mu)a^2, \mu b^1 + (1-\mu)b^2)^T \in \Gamma$.Obviously, Γ is a cone, $\mu(a^1,b^1)^T \in \Gamma$ and $(1-\mu)(a^2,b^2)^T \in \Gamma$. So for each $i=1,2$, there exist $u_t^1 \in U_t, t\in T$, for each $i=1,2$, there exist $\lambda_t^1, \lambda_t^2 \geq 0$ such that

$$\begin{aligned} \mu(a^1,b^1)^T &\in \operatorname{epi}\left(\sum_{t\in T}\lambda_t^1 h_t(\cdot,u_t^1)\right)^*, \\ (1-\mu)(a^2,b^2)^T &\in \operatorname{epi}\left(\sum_{t\in T}\lambda_t^2 h_t(\cdot,u_t^2)\right)^* \end{aligned} \tag{23}$$

According to the definition of epi$w$, that we have

$$\left(\sum_{t\in T}\lambda_t^1 h_t(\cdot,u_t^1)\right)^*(\mu a^1) + \left(\sum_{t\in T}\lambda_t^2 h_t(\cdot,u_t^2)\right)^*((1-\mu)a^2) \leq \mu b^1 + (1-\mu)b^2 \tag{24}$$

Next, as $t \in T$,

$$\lambda_t := \mu\lambda_t^1 + (1-\mu)\lambda_t^2 \tag{25}$$

$$u_t := \begin{cases} u_t^1, & \text{if}\lambda_t = 0, \\ \dfrac{\mu\lambda_t^1}{\lambda_t}u_t^1 + \dfrac{(1-\mu)\lambda_t^2}{\lambda_t}u_t^2, & \text{if}\lambda_t > 0. \end{cases} \tag{26}$$

If $\lambda_t = 0$, then $\lambda_t^1 = \lambda_t^2 = 0$, so

$$\mu\lambda_t^1 h_t(x,u_t^1) + (1-\mu)\lambda_t^2 h_t(x,u_t^2) = \lambda_t h_t(x,u_t) \tag{27}$$

If $\lambda_t \neq 0$, then suppose that $\lambda_t > 0$, then,

$$\begin{aligned} \mu\lambda_t^1 h_t(x,u_t^1) + (1-\mu)\lambda_t^2 h_t(x,u_t^2) &= \lambda_t\left(\frac{\mu\lambda_t^1}{\lambda_t}h_t(x,u_t^1) + \frac{(1-\mu)\lambda_t^2}{\lambda_t}h_t(x,u_t^2)\right) \\ &\leq \lambda_t h_t\left(x, \frac{\mu\lambda_t^1}{\lambda_t}u_t^1 + \frac{(1-\mu)\lambda_t^2}{\lambda_t}u_t^2\right) \\ &= \lambda_t h_t(x,u_t), \forall t \in T. \end{aligned} \tag{28}$$

According to the definition of concave function the second inequality holds. By (24), we have

$$
\begin{aligned}
\mu b^1+(1-\mu)b^2 &\geq \left(\sum_{t\in T}\lambda_t^1 h_t(\cdot,u_t^1)\right)^*(\mu a^1)+\left(\sum_{t\in T}\lambda_t^2 h_t(\cdot,u_t^2)\right)^*((1-\mu)a^2)\\
&= \sup_{x\in R^n}\left\{\langle \mu a^1,x\rangle-\sum_{t\in T}\lambda_t^1 h_t(x,u_t^1)\right\}+\sup_{x\in R^n}\left\{\langle (1-\mu)a^2,x\rangle-\sum_{t\in T}\lambda_t^2 h_t(x,u_t^2)\right\}\\
&\geq \sup_{x\in R^n}\left\{\langle \mu a^1+(1-\mu)a^2,x\rangle-\sum_{t\in T}(\lambda_t^1 h_t(x,u_t^1)+\lambda_t^2 h_t(x,u_t^2))\right\} \qquad (29)\\
&\geq \sup_{x\in R^n}\left\{\langle \mu a^1+(1-\mu)a^2,x\rangle-\sum_{t\in T}(\lambda_t^1 h_t(x,u_t)\right\}\\
&= \left(\sum_{t\in T}\lambda_t h_t(\cdot,u_t)\right)^*(\mu a^1+(1-\mu)a^2),
\end{aligned}
$$

where because of the definition of 2.3, the second equality holds, and the fourth inequality holds by (24). So $(\mu a^1+(1-\mu)a^2,\mu b^1+(1-\mu)b^2)^T\in\Gamma$.

Lemma 2.6: Let h_t: $R^n\times R^m\to R, t\in T$, be continuous convex functions, such that for each $u_t\in R^m, h_t(\cdot,u_t)$ is convex. Suppose that each U_t is convex and compact, there exists $\overline{x}\in R^n$ such that

$$h_t(\overline{x},u_t)<0,\forall u_t\in U_t,t\in T \tag{30}$$

And,

$$\Gamma=\bigcup_{u_t\in U_t,\lambda_t\geq 0,t\in T}\operatorname{epi}\left(\sum_{t\in T}\lambda_t h_t(\cdot,u_t)\right)^*. \tag{31}$$

is closed.

Proof: We defined that

$$\Gamma:=\bigcup_{u_t\in U_t,\lambda_t\geq 0,t\in T}\operatorname{epi}\left(\sum_{t\in T}\lambda_t h_t(\cdot,u_t)\right)^*. \tag{32}$$

According to the proposition of proper lower semi-continuous convex functions and functions $h_t(\cdot,u_t)$ is continuous, such that

$$\operatorname{epi}\left(\sum_{t\in T}\lambda_t h_t(\cdot,u_t)\right)^*:=\begin{cases}\sum_{t\in T}\lambda_t\operatorname{epi}(h_t(\cdot,u_t))^*, & \text{if}\quad \lambda_t\neq 0,\\ \{0\}\times R_+, & \text{if}\quad \lambda_t=0.\end{cases} \tag{33}$$

Contact with (32) can be obtained that

$$
\begin{aligned}
\Gamma &= \bigcup_{u_t\in U_t}\left\{\bigcup_{\lambda_t\geq 0}\sum_{t\in T}\lambda_t\quad \operatorname{epi}(h_t(\cdot,u_t))^*\cup\{0\}\times R_+\right\}\\
&= \bigcup_{u_t\in U_t}\text{co cone}\left\{\bigcup_{t\in T}\quad \operatorname{epi}(h_t(\cdot,u_t))^*\cup(0,1)\right\}.
\end{aligned} \tag{34}
$$

Then let $(s^\kappa,r^\kappa)^T\in\Gamma$ be a sequence with $(s^\kappa,r^\kappa)^T\to(x^*,r)\in R^n\times R$. In order to prove the result, we illustrate that $(x^*,r)^T\in\Gamma$. Because of $(s^\kappa,r^\kappa)^T\in\Gamma$, for each $\kappa\in N$, we have $u_t^\kappa\in U_t, t\in T$, such that

$$(s^\kappa,r^\kappa)^T\in\text{co cone}(\{\operatorname{epi}(h_t(\cdot,u_t^\kappa))^*:t\in T\}\cup(0,1)).$$

According to Caratheodory theorem that for all $\kappa \in N$, there exist $u_{t_i}^{\kappa} \in U_{t_i}^{\kappa}, t_i^{\kappa} \in T, \lambda_{t_i}^{\kappa} \geq 0, i = 1, \cdots, n+1$, and $\lambda_t^{\kappa} \geq 0$ such that

$$(s^{\kappa}, r^{\kappa})^T \in \sum_{i=1}^{n+1} \lambda_{t_i}^{\kappa} \text{epi}(h_{t_i}^{\kappa}(\cdot, u_{t_i}^{\kappa}))^* + \lambda_t^{\kappa}(0,1) \tag{35}$$

Because T is compact, we suppose that $t_i^{\kappa} \to t_i \in T$ as $\kappa \to \infty, i = 1, \ldots, n+1$.

Next, let $i = 1, \ldots, n+1$ and $\varepsilon > 0$ be fixed. Since U is uniformly upper semi-continuous, there exists $\rho > 0$ such that $U_t \subset U_{t_i} + \varepsilon B$ where B is a closed unit ball of R^n for any $t \in T$ with $d(t, t_i) \leq \rho$. Since $t_i^{\kappa} \to t_i$ as $\kappa \to \infty$, there exists $\kappa_i \in N$ such that for all $\kappa \geq \kappa_i, d(t_i^{\kappa}, t_i) \leq \rho$. So, for all $\kappa \geq \kappa_i, U_{t_i^{\kappa}} \subset U_{t_i} + \varepsilon B$. Since $u_{t_i^{\kappa}} \in U_{t_i^{\kappa}}$, there exists $\nu_{t_i} \in U_{t_i}$ such that $u_{t_i^{\kappa}} \in \nu_{t_i} + \varepsilon B$, i. e. , $\|u_{t_i^{\kappa}} - \nu_{t_i}\| < \varepsilon$.So, $\inf_{l_{t_i} \in U_{t_i}} \|u_{t_i^{\kappa}} - l_{t_i}\| < \varepsilon$. It according to that there exists $\kappa_i \in N$ such that for all $\kappa \geq \kappa_i, d(u_{t_i^{\kappa}}, U_{t_i}) \leq \varepsilon$. So, $d(u_{t_i^{\kappa}}, U_{t_i}) \to 0$ as $\kappa \to \infty$, i. e. , $u_{t_i^{\kappa}} \in U_{t_i}$. Hence, there exists $l_{t_i^{\kappa}}^* \in U_{t_i}, \kappa = 1, 2, \ldots$ such that $d(u_{t_i^{\kappa}}, U_{t_i}) = \|u_{t_i^{\kappa}} - l_{t_i^{\kappa}}^*\| \to 0$ as $\kappa \to \infty$. Since U_{t_i} is compact, we suppose that $z_{t_i^{\kappa}}^* \to u_{t_i} \in U_{t_i}$ as $\kappa \to \infty$. Then we get:

$$\begin{aligned}
\lim_{\kappa \to \infty} \|u_{t_i^{\kappa}} - u_{t_i}\| &= \lim_{\kappa \to \infty} \|(u_{t_i^{\kappa}} - l_{t_i^{\kappa}}^*) + (l_{t_i^{\kappa}}^* - u_{t_i})\| \\
&\leq \lim_{\kappa \to \infty} \|u_{t_i^{\kappa}} - l_{t_i^{\kappa}}^*\| + \lim_{\kappa \to \infty} \|z_{t_i^{\kappa}}^* - u_{t_i}\| \\
&= 0
\end{aligned} \tag{36}$$

So, if $\kappa \to \infty$ have $u_{t_i^{\kappa}} \to u_{t_i}$.

Next, we prove that $z^{\kappa} := \sum_{i=1}^{n+1} \lambda_{t_i}^{\kappa} + \lambda_t^{\kappa}$ is bounded. By the contract, we supposed that $z^{\kappa} \to \infty$. And we assume that $\frac{\lambda_{t_i}^{\kappa}}{z^{\kappa}} \to \sigma_{i \in R_+}, i = 1, \ldots, n+1$ and $\frac{\lambda_t^{\kappa}}{z^{\kappa}} \to \sigma_0 \in R_+$ with $\sum_{i=1}^{n+1} \sigma_i + \sigma_0 = 1$. Then according to (35) and the definition 2. 3, for any $x \in R^n$. we have

$$(s^{\kappa})^T x - \sum_{i=1}^{n+1} \lambda_{t_i}^{\kappa} h_{t_i}^{\kappa}(x, u_{t_i}^{\kappa}) \leq \left(\sum_{i=1}^{n+1} \lambda_{t_i}^{\kappa} h_{t_i}^{\kappa}(x, u_{t_i}^{\kappa}) \right)^* (s^{\kappa}) \leq r^{\kappa} - \lambda_t^{\kappa} \leq r^{\kappa} \tag{37}$$

Both sides of the last inequality divide by z^{κ} and taking the limit, we have that, $\sum_{i=1}^{n+1} \sigma_i h_{t_i}(x, u_{t_i}) \geq \sigma_0, \forall x \in R^n$. If $\sigma_i = 0$, $\forall i = 1, \ldots, n+1$,then we get that $0 = \sum_{i=1}^{n+1} \sigma_i h_{t_i}(x, u_{t_i}) \geq 1$. This is contrary to the assumption. Also, if $\sigma_i \neq 0$, for some i, then$\sum_{i=1}^{n+1} \sigma_i h_{t_i}(x, u_{t_i}) \geq 0$. The assumption is contrary to (30) as $0 < \sum_{i=1}^{n+1} \sigma_i \leq 1$. So, z_{κ} is bounded. Now, without loss of generality, because of $l = z^{\kappa}$ is bounded, such that $\lambda_{t_i}^{\kappa} \to \lambda_{t_i}$ and $\lambda_t^{\kappa} \to \lambda_t$. As, for each $x \in R^n$,

$$(s^{\kappa})^T x - \sum_{i=1}^{n+1} \lambda_{t_i}^{\kappa} h_{t_i}^{\kappa}(x, u_{t_i}^{\kappa}) \leq r^{\kappa} - \lambda_t^{\kappa} \tag{38}$$

it follows, by passing to the limit and notice that h_t is continuous, so, $(x^*)^T x - \sum_{i=1}^{n+1} \lambda_{t_i} h_{t_i}(x, u_{t_i}^{\kappa}) \leq$

$r-\lambda_t, \forall x \in R^n$,it follows that

$$
\begin{aligned}
(x^*, r-\lambda_t)^T &\in \operatorname{epi}\Big(\sum_{i=1}^{n+1} \lambda_{t_i} h_{t_i}(\cdot, u_{t_i}^{\kappa})\Big)^*, \\
(x^*, r)^T &\in \operatorname{epi}\Big(\sum_{i=1}^{n+1} \lambda_{t_i} h_{t_i}(\cdot, u_{t_i}^{\kappa})\Big)^* + \lambda_t(0,1) \\
&\subseteq \operatorname{epi}\Big\{\sum_{t\in T} \lambda_t (h_t(\cdot, u_t)\Big\}^* + \{0\}\times R_+ \\
&\subseteq \bigcup_{u_t\in U_t, \lambda_t\geq 0, t\in T} \operatorname{epi}\Big(\sum_{t\in T} \lambda_t h_t(\cdot, u_t)\Big)^* = \Gamma
\end{aligned}
\tag{39}
$$

So Γ is closed.

3 Necessary conditions for dual theorem

Some main necessary optimality conditions for a robust approximate optimal solution of (UCSIP) are discussed in this section. In order to show necessary conditions for dual theorem, we give the following Robust Farkas Lemma for convex function.

Lemma 3.1: Let $w : R^n \to R$ be a convex function and let $h_t : R^n \times R^m \to R, h_t(\cdot, u_t)$ is a convex function. Let $U_t \subseteq R^m, t \in T$, be compact and let $F := \{x \in R^n : h_t(x, u_t) \leq 0$, for all $u_t \in U_t, t \in T\}$ is not empty. Then following relationships are equivalent.

$$
\begin{aligned}
&\text{(i)}\{x \in R^n | h_t(x, u_t) \leq 0, \forall u_t \in U_t, t \in T\} \subseteq \{x \in R^n | w(x) \geq 0\}; \\
&\text{(ii)}(0,0)^T \in \operatorname{epi} w^* + \operatorname{cl}\operatorname{co}\Big\{\bigcup_{u_t\in U_t, \lambda_t\geq 0, t\in T} \operatorname{epi}\Big(\sum_{t\in T} \lambda_t h_t(\cdot, u_t)\Big)^*\Big\}.
\end{aligned}
\tag{40}
$$

Proof: By the definition of F. Then we will prove that

$$
\operatorname{epi}\sigma_F^* = \operatorname{cl}\operatorname{co}\Big\{\bigcup_{u_t\in U_t, \lambda_t\geq 0, t\in T} \operatorname{epi}(\sum_{t\in T} \lambda_t h_t(\cdot, u_t))^*\Big\}.
$$

For any $x \in R^n, \sigma_F(x) = \sup\limits_{u_t\in U_t, \lambda_t\geq 0, t\in T} \sum\limits_{t\in T} \lambda_t h_t(x, u_t)$. Thus we have

$$
\begin{aligned}
\operatorname{epi}\sigma_F^* &= \operatorname{cl}(\operatorname{epi}\sigma_F^*) \\
&= \operatorname{cl}\Big\{\bigcup_{u_t\in U_t, \lambda_t\geq 0, t\in T} \operatorname{epi}(\sum_{t\in T} \lambda_t h_t(\cdot, u_t))^*\Big\} \\
&= \operatorname{cl}\Big\{\operatorname{cl}\operatorname{co}\bigcup_{u_t\in U_t, \lambda_t\geq 0, t\in T} \operatorname{epi}(\sum_{t\in T} \lambda_t h_t(\cdot, u_t))^*\Big\} \\
&= \operatorname{cl}\operatorname{co}\Big\{\bigcup_{u_t\in U_t, \lambda_t\geq 0, t\in T} \operatorname{epi}(\sum_{t\in T} \lambda_t h_t(\cdot, u_t))^*\Big\},
\end{aligned}
\tag{41}
$$

where the third equality holds by Lemma 2.3.

So,besides (ii) holds, we get

$$
\begin{aligned}
(ii) &\Leftrightarrow (0,0)^T \in \text{epi}w^* + \text{epi}\sigma_F^* \qquad \text{according to (40)}\\
&\Leftrightarrow (0,0)^T \in \text{epi}(w+\sigma_F)^* \qquad \text{invokes Lemma 2.2}\\
&\Leftrightarrow (w+\sigma_F)^*(0) \le 0 \qquad \text{follows from the definition of 2.3}\\
&\Leftrightarrow (w+\sigma_F)^*(0) = \sup\left\{\langle x^*, 0\rangle - (w+\sigma_F)^*(x) : x \in R^n\right\} = \sup\left\{0 - (w+\sigma_F)^*(x)\right\} \qquad (42)\\
&\Leftrightarrow (w+\sigma_F)(x) \ge 0, \forall x \in R^n \qquad \text{by the definition of 2.4}\\
&\Leftrightarrow w(x) \ge 0, \forall x \in F\\
&\Leftrightarrow (i),
\end{aligned}
$$

According to Lemma 3.1, the following theorem is given:

Theorem 3.1: Let w be a convex function, where w is defined as $R^n \to R$ and define h_t as $R^n \times R^m \to R$, h_t be continuous for any $t \in T$, $h_t(\cdot, u_t)$ is convex on R^n for each $u_t \in R^m$. Let $\widehat{x} \in F$ and define $U_t \subseteq R^m, t \in T$ which is compact. Assume that Γ is closed and convex, then $\widehat{x}$ is an approximate solution ($\varepsilon-$solution) of (RCSIP) if and only if there exist $(\widehat{\lambda}_t) \ge 0$ and $\widehat{u}_t \in U_t, t \in T$, such that for any $x \in R^n$,

$$
w(x) + \sum_{t\in T} \widehat{\lambda}_t h_t(x, \widehat{u}_t) \ge w(\widehat{x}) - \varepsilon. \tag{43}
$$

Proof:(Sufficiency) Suppose that $\widehat{x}$ be an $\varepsilon-$solution of (RCSIP). So for any $x \in F$, $\widehat{x}$ satisfy $w(x) \ge w(\widehat{x}) - \varepsilon$. Then $F \subseteq \{x \in R^n : w(x) - w(\widehat{x}) + \varepsilon \ge 0\}$. Using Lemma 3.1,

$$
(0, \varepsilon - w(\widehat{x}))^T \in \text{epi}w^* + \text{cl co} \bigcup_{u_t\in U_t, \lambda_t\ge 0, t\in T} \text{epi}\left(\sum_{t\in T} \lambda_t h_t(\cdot, u_t)\right)^*. \tag{44}
$$

And cone $\bigcup_{u_t\in U_t, \lambda_t\ge 0, t\in T} \text{epi}\left(\sum_{t\in T} \lambda_t h_t(\cdot, u_t)\right)^*$ is closed and convex, so (44) is equivalent to

$$
(0, \varepsilon - w(\widehat{x}))^T \in \text{epi}w^* + \bigcup_{u_t\in U_t, \lambda_t\ge 0, t\in T} \text{epi}\left(\sum_{t\in T} \lambda_t h_t(\cdot, u_t)\right)^*. \tag{45}
$$

So let $\widehat{\lambda}_t \ge 0, u_t \in U_t$, then we have

$$
(0, \varepsilon - w(\widehat{x}))^T \in \text{epi}w^* + \text{epi}\left(\sum_{t\in T} \widehat{\lambda}_t h_t(\cdot, \widehat{u}_t)\right)^*. \tag{46}
$$

And because $h_t : R^n \times R^m \to R, \forall t \in T$, are continuous and $\widehat{\lambda}_t \ge 0$, and Lemma2.2, then we get,

$$
(0, \varepsilon - w(\widehat{x}))^T \in \text{epi}w^* + \left(\sum_{t\in T} \text{epi}(\widehat{\lambda}_t h_t(\cdot, \widehat{u}_t)\right)^*. \tag{47}
$$

And, $\exists \xi^* \in R^n, p \ge 0, s_t^* \in R^n$, and $q_t \ge 0, t \in T$ have:

$$
(0, \varepsilon - w(\widehat{x}))^T \in (\xi^*, w^*(\xi^*) + p) + \sum_{t\in T} (s_t^*, (\widehat{\lambda}_t h_t(\cdot, \widehat{u}_t))^*(s_t^*) + q_t). \tag{48}
$$

So,

$$
\begin{aligned}
&0 = \xi^* + \sum_{t\in T} s_t^*\\
&\varepsilon - w(\widehat{x}) = w^*(\xi^*) + p + \sum_{t\in T} ((\widehat{\lambda}_t h_t(\cdot, \widehat{u}_t))^*(s_t^*) + q_t).
\end{aligned} \tag{49}
$$

Hence, according to (49) for any $x \in R^n$,

$$-\left\langle \sum_{t\in T} s_t^*, x \right\rangle - w(x) = \langle s^*, x\rangle - w(x)$$
$$\le w^*(s)^* \tag{50}$$
$$= \varepsilon - w(\widehat{x}) - p - \sum_{t\in T}((\widehat{\lambda}_t h_t(\cdot, \widehat{u}_t))^*(s_t^*) + q_t),$$

where the second inequality holds by $w^*(s^*) = \sup\{\langle s^*, s\rangle - w(s), s \in R^n\}$. Thus, for any $x \in R^n$,

$$\begin{aligned} w(\widehat{x}) - \varepsilon &\le \left\langle \sum_{t\in T} s_t^*, x \right\rangle + w(x) - p - \sum_{t\in T}(\widehat{\lambda}_t h_t(\cdot, \widehat{u}_t))^*(s_t^*) - \sum_{t\in T} q_t \\ &\le \left\langle \sum_{t\in T} s_t^*, x \right\rangle + w(x) - \sum_{t\in T}(\widehat{\lambda}_t h_t(\cdot, \widehat{u}_t))^*(s_t^*) \\ &\le w(x) + \sum_{t\in T} \widehat{\lambda}_t h_t(x, \widehat{u}_t). \end{aligned} \tag{51}$$

According to the definition of conjugate function of $h_t(\cdot, \widehat{u}_t)$, the third inequality is true.

(Necessity) Supposed that $\exists(\widehat{\lambda}_t) \ge 0, \widehat{u}_t \in U_t, t \in T$, such that for any $x \in R^n$,

$$w(x) + \sum_{t\in T} \widehat{\lambda}_t h_t(x, u_t) \ge w(\widehat{x}) - \varepsilon.$$

For any $x \in F$ then we have (meet the constraint conditions),

$$w(x) \ge w(x) + \sum_{t\in T} \widehat{\lambda}_t h_t(x, \widehat{u}_t) \ge w(\widehat{x} - \varepsilon).$$

Hence we get the conclusion that $w(x) \ge w(\widehat{x}) - \varepsilon$. Then $\widehat{x}$ is an approximate solution of (RCSIP).

Theorem3. 2:(ε optimality theorem) Let w be a convex function, $w : R^n \to R$, and let h_t be continuous functions such that for each $u_t \in R^m$, $h_t(\cdot, u_t)$ is convex on R^n, for any $t \in T, h_t : R^n \times R^m \to R$. let $U_t \subseteq R^q, t \in T$ be compact. Assume that Γ is closed and convex. Let $\widehat{x} \in F$, then the following (i), (ii), (iii) are equivalent:

(i)$\widehat{x}$ is an $\varepsilon-$solution to (RCSIP);

(ii)$(0, \varepsilon - w(\widehat{x}))^T \in \mathrm{epi} w^* + \bigcup_{u_t \in U_t, \lambda_t \ge 0, t\in T} \mathrm{epi}(\sum_{t\in T} \lambda_t h_t(\cdot, u_t))^*$;

(iii)$\exists(\widehat{\lambda}_t) \ge 0, \widehat{u}_t \in U_t, \varepsilon_t \ge 0, t \in T$ and $\varepsilon_0 \ge 0$ such that $0 \in \partial_{\varepsilon_0} w(\widehat{x}) + \sum_{t\in T} \partial_{\varepsilon_t}(\widehat{\lambda}_t h_t(\cdot, \widehat{u}_t))(\widehat{x})$ and $\varepsilon_0 + \sum_{t\in T} \varepsilon_t - \varepsilon = \sum_{t\in T} \widehat{\lambda}_t h_t(\widehat{x}, \widehat{u}_t)$.

Proof: Assume that $\widehat{x}$ is an $\varepsilon-$solution of (RCSIP). Therefore, $w(x) \ge w(\widehat{x}) - \varepsilon$, for any $x \in F$, let $\varphi(x) := w(x) - w(\widehat{x}) + \varepsilon$, then we can get $\varphi(x) \ge 0$ for any $x \in F$, it follows

$$\begin{aligned} \varphi^*(u) &= \sup\{\langle u, x\rangle - \varphi(x) | x \in R^n\} \\ &= \sup\{\langle u, x\rangle - w(x) + w(\widehat{x}) - \varepsilon | x \in R^n\} \\ &= \sup\{\langle u, x\rangle - w(x) | x \in R^n\} + w(\widehat{x}) - \varepsilon \\ &= w^*(u) + w(\widehat{x}) - \varepsilon, \end{aligned} \tag{52}$$

and by Lemma 3. 1 we can get (i)$\Leftrightarrow$(ii)

Let $F := \{x \in R^n : h_t(x, u_t) \leq 0, \forall u_t \in U_t, t \in T\}$, then $F \neq \emptyset$ we have,
According to(ii), we get $(0, \varepsilon - w(\widehat{x}))^T \in \text{epi}w^* + \bigcup_{u_t \in U_t, \widehat{\lambda}_t \geq 0, t \in T} \text{epi}(\sum_{t \in T} \lambda_t h_t(\cdot, \widehat{u}_t))^*$.

That means there exists $\widehat{u}_t \in U_t, \widehat{\lambda}_t \geq 0$ such that

$$(0, \varepsilon - w(\widehat{x}))^T \in \text{epi}w^* + \text{epi}\left(\sum_{t \in T} \widehat{\lambda}_t h_t(\cdot, \widehat{u}_t)\right)^* \tag{53}$$

By Lemma 2.1, there exist $\varepsilon_0 \geq 0, \varepsilon_t \geq 0, t \in T$ then w and $u_t \in R^m, h_t(\cdot, u_t)$ are all convex, such that

$$\begin{aligned}(0, \varepsilon - w(\widehat{x}))^T \in & \bigcup_{\varepsilon_0 \geq 0} \{(\zeta_0, \langle \zeta_0, \widehat{x} \rangle + \varepsilon_0 - w(\widehat{x}))^T : \zeta_0 \in \partial_{\varepsilon_0} w(\widehat{x})\} \\ & + \sum_{t \in T} \bigcup_{\varepsilon_t \geq 0} \{(\zeta_t, \langle \zeta_t, \widehat{x} \rangle + \varepsilon_t - \widehat{\lambda}_t h_t(\widehat{x}, \widehat{u}_t))^T : \zeta_t \in \partial_{\varepsilon_t}(\widehat{\lambda}_t h_t(\cdot, \widehat{u}_t))(\widehat{x})\}.\end{aligned} \tag{54}$$

It equivalent to that there exist $\widehat{u}_t \in U_t, \widehat{\lambda}_t \geq 0, \widehat{\zeta}_0 \in \partial_{\varepsilon_0} w(\widehat{x})$ and $\widehat{\zeta}_t \in \partial_{\varepsilon_t}(\widehat{\lambda}_t h_t(\cdot, \widehat{u}_t))(\widehat{x}), t \in T$, such that

$$(0, \varepsilon - w(\widehat{x}))^T \in (\widehat{\zeta}_0, \langle \widehat{\zeta}_0, \widehat{x} \rangle + \varepsilon_0 - w(\widehat{x}))^T + \sum_{t \in T} (\widehat{\zeta}_t, \langle \widehat{\zeta}_t, \widehat{x} \rangle + \varepsilon_t - \widehat{\lambda}_t h_t(\widehat{x}, \widehat{u}_t))^T. \tag{55}$$

Then it can be written as there exist $u_t \in U_t, \lambda_t \geq 0, \widehat{\zeta}_0 \in \partial_{\varepsilon_0} w(\widehat{x}), \widehat{\zeta}_t \in \partial_{\varepsilon_t}$ and $\varepsilon_0 \geq 0, \varepsilon_t \geq 0, t \in T$ such that

$$\begin{aligned} & 0 = \widehat{\zeta}_0 + \sum_{t \in T} \widehat{\zeta}_t \\ & \varepsilon_0 + \sum_{t \in T} \varepsilon_t - \varepsilon = \sum_{t \in T} \widehat{\lambda}_t h_t(\widehat{x}, \widehat{u}_t), \end{aligned} \tag{56}$$

which is equivalent to (iii).

4 ε-duality theorem of Lagrangian dual

Next, using the approximate solution to (RCSIP), we consider a Lagrangian dual problem $(\text{LDRCSIP})_\varepsilon$, as follows

$$\begin{aligned} & (\text{LDRCSIP})_\varepsilon \max_{\lambda_t} \inf_{z} \sup_{u_t} \left\{ w(z) + \sum_{t \in T} \lambda_t h_t(z, u_t) \right\} \\ & \text{s. t. } \lambda_t \geq 0, \forall t \in T, u_t \in U_t, z \in R^n. \end{aligned} \tag{57}$$

If (z, u, λ) is a feasible solution of $(\text{LDRCSIP})_\varepsilon$, then z satisfies the following statement:$0 \in \partial_{\varepsilon_0} w(z) + \sum_{t \in T} \partial_{\varepsilon_t}(\lambda_t h_t(\cdot, u_t))(z)$ and $\varepsilon_0 \geq 0, \varepsilon_t \geq 0, \varepsilon_0 + \sum_{t \in T} \varepsilon_t \leq \varepsilon$ So the feasible set of $(\text{LDRCSIP})_\varepsilon$ is

$$\begin{aligned} F_L =: \Bigg\{ & (z, u, \lambda) | 0 \in \partial_{\varepsilon_0} w(z) + \sum_{t \in T} \partial_{\varepsilon_t}(\lambda_t h_t(\cdot, u_t))(z), \lambda_t \geq 0, \\ & \forall t \in T, u_t \in U_t, z \in R^n, \varepsilon_0 \geq 0, \varepsilon_t \geq 0, \varepsilon_0 + \sum_{t \in T} \varepsilon_t \leq \varepsilon \Bigg\}. \end{aligned} \tag{58}$$

Assume that $\varepsilon \geq 0$, and $(\widehat{x}, \widehat{u}, \widehat{\lambda})$ is the approximate solution of $(\text{LDRCSIP})_\varepsilon$, if for any $(z, u, \lambda) \in F_L$,

$$w(z) + \sum_{t\in T} \lambda_t h_t(z, u_t) \geq w(\widehat{x}) + \sum_{t\in T} \widehat{\lambda}_t h_t(\widehat{x}, \widehat{u}_t) - \varepsilon \tag{59}$$

And then we can obtain

$$\inf_{z\in R^n} \sup_{u_t\in U_t} \left\{ w(z) + \sum_{t\in T} \lambda_t h_t(z, u_t) \right\} \geq \inf_{\widehat{x}\in R^n} \sup_{\widehat{u}\in U_t} \left\{ w(\widehat{x}) + \sum_{t\in T} \widehat{\lambda}_t h_t(\widehat{x}, \widehat{u}_t) \right\} - \varepsilon \tag{60}$$

Theorem 4. 1: Suppose x and (z, u, λ) are feasible solutions of (RCSIP) and $(\text{LDRCSIP})_\varepsilon$,respectively. if

$$w(x) \geq \inf_{z\in R^n} \sup_{u_t\in U_t} w(z) + \sum_{t\in T} \lambda_t h_t(z, u_t) - \varepsilon \tag{61}$$

then we call that x satisfied approximate weak duality theorem.

Proof: Supposed that x is a feasible solution of (RSIP) and (z, u, λ) is a feasible solution of $(\text{LDRCSIP})_\varepsilon$. Then we have $\lambda_t \geq 0, t \in T, u_t \in U_t, z \in R^n, \varepsilon_0 \geq 0, \varepsilon_t \geq 0, \widehat{\zeta}_0 \in \partial_{\varepsilon_0} w(z)$ and $\widehat{\zeta}_t \in \partial_{\varepsilon_t}(\lambda_t h_t(\cdot, u_t))(z)$, such that $\varepsilon_0 + \sum_{t\in T} \varepsilon_t \leq \varepsilon$ and $\widehat{\zeta}_0 + \sum_{t\in T} \widehat{\zeta}_t$ Thus, we have

$$\begin{aligned}
& w(x) - w(z) - \sup_{u_t\in U_t} \sum_{t\in T} \lambda_t h_t(z, u_t) \\
& \geq \langle \widehat{\zeta}_0, x - z \rangle - \varepsilon_0 - \sup_{u_t\in U_t} \sum_{t\in T} \lambda_t h_t(z, u_t) \\
& = -\langle \sum_{t\in T} \widehat{\zeta}_t, x - z \rangle - \varepsilon_0 - \sup_{u_t\in U_t} \sum_{t\in T} \lambda_t h_t(z, u_t) \\
& \geq -\sum_{t\in T} \lambda_t (h_t(x, u_t) - h_t(z, u_t)) - \varepsilon_0 - \sum_{t\in T} \varepsilon_t - \sup_{u_t\in U_t} \sum_{t\in T} \lambda_t h_t(z, u_t) \\
& \geq -\sup_{u_t\in U_t} \sum_{t\in T} \lambda_t (h_t(x, u_t) - h_t(z, u_t)) - \varepsilon_0 - \sum_{t\in T} \varepsilon_t - \sup_{u_t\in U_t} \sum_{t\in T} \lambda_t h_t(z, u_t) \\
& = -\sup_{u_t\in U_t} \sum_{t\in T} \lambda_t h_t(x, u_t) - \varepsilon_0 - \sum_{t\in T} \varepsilon_t \\
& \geq -\varepsilon
\end{aligned} \tag{62}$$

And then

$$\begin{aligned}
w(x) &\geq w(z) + \sup_{u_t\in U_t} \left\{ \sum_{t\in T} \lambda_t h_t(z, u_t) \right\} - \varepsilon \\
w(x) &\geq \inf_{z\in R^n} \sup_{u_t\in U_t} \left\{ w(z) + \sum_{t\in T} \lambda_t h_t(z, u_t) \right\} - \varepsilon
\end{aligned} \tag{63}$$

Theorem 4. 2: Let w be a convex function for $w : R^n \to R$, for any $t \in T, h_t : R^n \times R^m \to R$, and let h_t be continuous such that $h_t(\cdot, u_t)$ is convex on R^n for each $u_t \in R^m$. Let $U_t \subseteq R^m, t \in T$, be compact. Assume that the closed convex cone constraint qualification holds. Let $\widehat{\lambda} := \widehat{\lambda}_t$, for any $\lambda_t \geq 0, t \in T$ and let $\widehat{u} = (u_t) \in U$. If $\widehat{x}$ is an ε-solution of (RCSIP), then $(\widehat{x}, \widehat{u}, \widehat{\lambda})$ is a 2ε-solution of $(\text{LDRCSIP})_\varepsilon$, then we call that x satisfied Approximate strong duality theorem.

Proof: Let $\widehat{x} \in F$, be an ε-solution of (RCSIP). Then, from Theorem 3. 2, there exist$\widehat{\lambda}_t \geq 0, \widehat{u}_t \in U_t, t \in T, \varepsilon_t \geq 0, t \in T$, and $\varepsilon_0 \geq 0$ such that

$$0 \in \partial_{\varepsilon_0} w(\widehat{x}) + \sum_{t\in T} \partial_{\varepsilon_t}(\widehat{\lambda}_t h_t(\cdot, \widehat{u}_t))(\widehat{x}) \text{ and } \varepsilon_0 + \sum_{t\in T} \varepsilon_t - \varepsilon = \sum_{t\in T} \widehat{\lambda}_t h_t(\widehat{x}, \widehat{u}_t). \tag{64}$$

Then according to Theorem4. 1,

$$\begin{aligned} w(\widehat{x}) + \sum_{t\in T}\widehat{\lambda}_t h_t(\widehat{x},\widehat{u}_t) - \inf_{z\in R^n}\sup_{u_t\in U_t}\left\{w(z)+\sum_{t\in T}\lambda_t h_t(z,u_t)\right\} &\geq -\varepsilon + \sum_{t\in T}\widehat{\lambda}_t h_t(\widehat{x},\widehat{u}_t) \\ &\geq -\varepsilon + \varepsilon_0 + \sum_{t\in T}\varepsilon_t - \varepsilon \\ &\geq -2\varepsilon \end{aligned} \tag{65}$$

Thus

$$-w(\widehat{x}) - \sum_{t\in T}\widehat{\lambda}_t h_t(\widehat{x},\widehat{u}_t) + \inf_{z\in R^n}\sup_{u_t\in U_t}\left\{w(z)+\sum_{t\in T}\lambda_t h_t(z,u_t)\right\} \leq 2\varepsilon$$

5 Conclusion

A convex semi-infinite optimization problem with uncertain information in the constraint function is established in this paper. Based on the robust optimization approach some approximate optimality qualifications and approximate dual theorem are all established under a closed and convex cone Γ. Then a Lagrangian dual problem is established, and the approximate weak dual and strong dual theorem with data uncertain are also given in this paper .

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