SYMMETRIC PERIODIC SOLUTIONS OF SYMMETRIC HAMILTONIANS IN $1:1$ RESONANCE

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Abstract. The aim of this work is to prove analytically the existence of symmetric periodic solutions of the family of Hamiltonian systems with Hamiltonian function $H(q_1, q_2, p_1, p_2) = \frac{1}{2}(q_1^2 + p_1^2) + \frac{1}{2}(q_2^2 + p_2^2) + a\ q_1^4 + b\ q_1^2 q_2^2 + c\ q_2^4$ with three real parameters a, b and c. Moreover, we characterize the stability of these periodic solutions as function of the parameters. Also, we find a first-order analytical approach of these symmetric periodic solutions.
We emphasize that these families of periodic solutions are different from those that exist in the literature.

1. Introduction

We consider the family of Hamiltonian systems in two degrees of freedom associated to the Hamiltonian function

$$H(q_1, q_2, p_1, p_2, \epsilon) = H_0(q_1, q_2, p_1, p_2) + \epsilon H_1(q_1, q_2), \tag{1}$$

where

$$H_0(q_1, q_2, p_1, p_2) = \frac{1}{2}(q_1^2 + p_1^2) + \frac{1}{2}(q_2^2 + p_2^2), \tag{2}$$

is the sum of two harmonic oscillators in 1:1 resonance and

$$H_1(q_1, q_2) = a\ q_1^4 + b\ q_1^2 q_2^2 + c\ q_2^4, \tag{3}$$

is a quartic perturbation where a, b, c are real parameters. Thus, the Hamiltonian system to study has the form

$$\begin{aligned} \dot{q}_1 &= p_1, \qquad \dot{p}_2 = -\,q_1 - \epsilon(4aq_1^3 + 2bq_1 q_2^2), \\ \dot{q}_1 &= p_2, \qquad \dot{p}_2 = -\,q_2 - \epsilon(2bq_1^2 q_2 + 4cq_2^3). \end{aligned} \tag{4}$$

According to the literature the Hamiltonian (1) has practical importance. In fact, it is related with galactic dynamics which is one of the most significant Astrophysics branches, see details in [5] and references therein. The dynamics of galaxies has been studied in several works under different points of view such as regular and chaotic behaviours. For example, Caranicolas [2] says that (1) may be considered to represent the potential field on the plane of symmetry of the central parts of a nonrotating triaxial galaxy. On the other hand, it has as physical application the modelling of the quasi-homogeneous core of a galaxy whose mass distribution presents two axes of symmetry (see [1],[2],[4] and [13]) or the classical counterpart of a symmetric molecule.

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As it is known, for $\epsilon = 0$ all the solutions of the integrable system (4) are 2π-periodic in t. If we now consider ϵ as a small positive parameter, we are interested in finding the conditions on the parameters a, b and c for which some solution of the perturbed system (4) can be continued using the symmetries of the problem. It is very well known see for example [10] and [11] that it is impossible to get periodic solutions as continuation (by Poincaré's method) of one periodic solution of the unperturbed problem, since the rank of the periodicity equation is far from maximal (fourth) in cartesian coordinates, in order to apply the Implicit Function Theorem.

The existence of periodic solutions of this model was studied, for example, in [6] where they considered the case $c = 0$. Moreover, in [7] the case $c \neq 0$ was studied. In both works the authors find at the most four families of periodic solutions for each positive energy level using according some restrictions on the parameters. They use the Averaging Method of First Order and convenient generalized polar coordinates. We point out that the families of periodic solutions in Theorem 4.1 and Theorem 4.2 (see Appendix) in [6] and [7], respectively, are parameterized by only one parameter, namely, ϵ. Furthermore, according to the Averaging Theory used by them, these solutions have period close to 2π. On the other hand, the authors do not give information about the type of stability.

In this work we find families of symmetric T-periodic solutions using symmetries of the problem and the Continuation Poincaré Method where the period of the solutions found is $T = 2\pi + O(\epsilon)$ and the reversibility symmetry is given by $S_1 : (q_1, q_2, p_1, p_2) \to (q_1, -q_2, -p_1, p_2)$. The results obtained for the reversibility symmetry S_1 are also valid for the reversibility symmetry $S_2 = (q_1, q_2, p_1, p_2) \to (-q_1, q_2, p_1, -p_2)$ applying a rotation at angle $\pi/2$.

Our result for S_1-symmetric periodic solutions is the following.

Theorem 1.1. *For the Hamiltonian system* (4) *associated to the Hamiltonian function* (1) *the following statements hold:*

(*a*) *If $a = 0$ and $b \neq 0$ then for ΔL and ϵ sufficiently small, there exists a family of S_1-symmetric periodic solutions $\varphi(t, z^{(1)}, \Delta L, \epsilon)$ parametrized by ΔL and ϵ with initial condition*

$$z^{(1)} = (\sqrt{2}\sqrt{\Delta L + h} + O(\epsilon^2), 0, 0, 0 + O(\epsilon^2)),$$

of the form

$$\begin{aligned} q_1(t, z^{(1)}, \Delta L, \epsilon) =& \sqrt{2}\sqrt{\Delta L + h}\cos t + O(\epsilon^2), \\ q_2(t, z^{(1)}, \Delta L, \epsilon) =& 0 + O(\epsilon^2), \\ p_1(t, z^{(1)}, \Delta L, \epsilon) =& \sqrt{2}\sqrt{\Delta L + h}\sin t + O(\epsilon^2), \\ p_2(t, z^{(1)}, \Delta L, \epsilon) =& 0 + O(\epsilon^2), \end{aligned} \tag{5}$$

and this family is linearly stable.

(*b*) *If $c = 0$ and $b \neq 0$ then for ΔL and ϵ sufficiently small, there exists a family of S_1-symmetric periodic solutions $\varphi(t, z^{(2)}, \Delta L, \epsilon)$ parametrized by ΔL and ϵ with initial condition*

$$z^{(2)} = (0 + O(\epsilon^2), 0, 0, \sqrt{2}\sqrt{\Delta L + h} + O(\epsilon^2)),$$

of the form

$$
(6)\qquad
\begin{aligned}
q_1(t, z^{(2)}, \Delta L, \epsilon) =& 0 + O(\epsilon^2),\\
q_2(t, z^{(2)}, \Delta L, \epsilon) =& \sqrt{2}\sqrt{\Delta L + h}\sin t + O(\epsilon^2),\\
p_1(t, z^{(2)}, \Delta L, \epsilon) =& 0 + O(\epsilon^2),\\
p_2(t, z^{(2)}, \Delta L, \epsilon) =& \sqrt{2}\sqrt{\Delta L + h}\cos t + O(\epsilon^2),
\end{aligned}
$$

and this family is linearly stable.

(c) If $b^2 = 36ac$, $b(b-6c) > 0$, and $c \neq 0$, then for ΔL and ϵ sufficiently small, there exists two families of S_1-symmetric periodic solutions $\varphi(t, z^{(1)}, \Delta L, \epsilon)$ parametrized by ΔL and ϵ with initial condition

$$
z^{(1)}_{\mp} = \left(2\sqrt{3}\sqrt{\tfrac{c(h+\Delta L)}{6c-b}} + O(\epsilon^2), 0, 0, \mp\tfrac{sgn(c)\sqrt{2}\sqrt{b(b-6c)^3(h+\Delta L)}}{(b-6c)^2} + O(\epsilon^2)\right),
$$

are of the form

$$
(7)\qquad
\begin{aligned}
q_1(t, z^{(1)}, \Delta L, \epsilon) =& 2\sqrt{3}\sqrt{\frac{c(h+\Delta L)}{6c-b}}\cos t - \epsilon\frac{4b^2}{\sqrt{3}c}\left(\frac{c(h+\Delta L)}{6c-b}\right)^{3/2}\sin^2 t\cos t + O(\epsilon^2),\\
q_2(t, z^{(1)}, \Delta L, \epsilon) =& \mp sgn(c)\sqrt{2}\sqrt{\frac{b(h+\Delta L)}{b-6c}}\sin t \mp \epsilon\frac{4\sqrt{2}b^2\,|c|\,(h+\Delta L)^2}{\sqrt{b(b-6c)^3(h+\Delta L)}}\sin^3 t + O(\epsilon^2),
\end{aligned}
$$

$$
\begin{aligned}
p_1(t, z^{(1)}, \Delta L, \epsilon) =& -2\sqrt{3}\sqrt{\frac{c(h+\Delta L)}{6c-b}}\sin t - \epsilon\frac{2b^2}{\sqrt{3}c}\left(\frac{c(h+\Delta L)}{6c-b}\right)^{3/2}(3\cos 2t + 1)\sin t + O(\epsilon^2),\\
p_2(t, z^{(1)}, \Delta L, \epsilon) =& \mp\sqrt{2}sgn(c)\sqrt{\frac{b(h+\Delta L)}{b-6c}}\cos t \mp \epsilon\frac{6\sqrt{2}b^2\,|c|\,(h+\Delta L)^2}{\sqrt{b(b-6c)^3(h+\Delta L)}}\sin t\sin 2t + O(\epsilon^2).
\end{aligned}
$$

These families of periodic solutions are unstable.

The proof of this theorem can be found in Section 3.

It is important to call the attention that if we consider the Hamiltonian system with Hamiltonian function as

$$
(8)\qquad H(q_1, q_2, p_1, p_2, \epsilon) = \frac{1}{2}(q_1^2 + p_1^2) + \frac{1}{2}(q_2^2 + p_2^2) + \epsilon(a\ q_1^4 + b\ q_1^2 q_2^2 + c\ q_2^4) + H^*(q, p, \epsilon),
$$

where $H^*(q, p, \epsilon) = O(\epsilon^2)$, our theorems are also valid imposing that H^* has the symmetry S_1 or S_2. If now we consider the Hamiltonian system with Hamiltonian function written as

$$
(9)\qquad H(q_1, q_2, p_1, p_2) = \frac{1}{2}(q_1^2 + p_1^2) + \frac{1}{2}(q_2^2 + p_2^2) + a\ q_1^4 + b\ q_1^2 q_2^2 + c\ q_2^4,
$$

we can introduce the small parameter ϵ by a ϵ^{-2}-symplectic change of variables given by $q_i = \epsilon q_i, p_i = \epsilon p_i$ such that the Hamiltonian (9) assumes the form

$$
(10)\qquad H(q_1, q_2, p_1, p_2, \epsilon) = \frac{1}{2}(q_1^2 + p_1^2) + \frac{1}{2}(q_2^2 + p_2^2) + \epsilon^2(a\ q_1^4 + b\ q_1^2 q_2^2 + c\ q_2^4),
$$

and our results are also valid.

This paper is organised as follows. In Section 2 we describe the symplectic variables which will be used to show in a better way the symmetries of the problem and we write the Hamiltonian (1) at these coordinates. We give a characterization and approximation of the symmetric periodic solutions and their initial conditions and we show how to study the stability of these solutions. We prove Theorem 1.1 in Section 3. We show the existence of four S_1-symmetric periodic solutions using variables (11) and (15) and the Poincaré Continuation Method. We study the stability of each solution found and we characterize these solutions in cartesian coordinates. In Section 4 we show results about periodic solution of the system (4) given by other authors and we compare their results with ours. We give a better approximation of the periodic solution given in [7] using Averaging Theory. In order to make the paper self-contained, we include the Appendix 5.1 where we have the main results about Averaging Theory and in Appendix 5.2 we have an approximation of the solutions via Averaging Theory.

2. Preliminary and statements of the main results

Before giving the demonstration of our main results we are going to describe the main ingredients in order to apply conveniently the Poincaré continuation method.

2.1. **Symplectic variables.** To facilitate the procedure, we use a type of symplectic variables $(L, Q,$
$l, P)$ as in [9], where the authors construct these variables such that the unperturbed Hamiltonian in (2) in the new coordinates depends only on L. They make a particularization for the $1:1:1$ resonance of the construction of local symplectic maps for resonant Hamiltonian systems with n degrees of freedom (see [3]; [8]). For our work, we use these symplectic variables in two degrees of freedom given by

$$\begin{aligned} q_1 &= \sqrt{2L - P^2 - Q^2}\cos l, & q_2 &= Q\cos l - P\sin l, \\ p_1 &= \sqrt{2L - P^2 - Q^2}\sin l, & p_2 &= P\cos l + Q\sin l, \end{aligned} \tag{11}$$

where $L > 0, 2L > Q^2 + P^2$ and $0 \leq l < 2\pi$. In variables (11) the Hamiltonian (1) assumes the form

$$\mathcal{H}(L, Q, l, P) = \mathcal{H}_0(L) + \epsilon\mathcal{H}_1(L, Q, l, P), \tag{12}$$

where $\mathcal{H}_0$ read as

$$\mathcal{H}_0(L) = L, \tag{13}$$

and

$$\begin{aligned} \mathcal{H}_1^{(1)}(L, Q, l, P) =& \mathcal{H}_1(L, Q, l, P) \\ =& a\left(2L - P^2 - Q^2\right)^2\cos^4 l + b\left(2L - P^2 - Q^2\right)(Q\cos l - P\sin l)^2\cos^2 l + \\ & c(Q\cos l - P\sin l)^4. \end{aligned} \tag{14}$$

It is important to call the attention that the variables (11) are not defined when $q_1^2 + p_1^2 = 0$ and so in order to study all phase spaces is that we introduce other symplectic variables through a rotation at an angle $\pi/2$ so, we define the coordinates (L, Q, l, P) of the form

$$\begin{aligned} q_1 &= Q\cos l - P\sin l, & q_2 &= \sqrt{2L - P^2 - Q^2}\cos l, \\ p_1 &= P\cos l + Q\sin l, & p_2 &= \sqrt{2L - P^2 - Q^2}\sin l, \end{aligned} \tag{15}$$

where $L > 0, 2L > Q^2 + P^2, 0 \leq l < 2\pi$. Note that they are not defined when $q_2^2 + p_2^2 = 0$ but they are in $q_1^2 + p_1^2 = 0$ which was what we wanted. Now the Hamiltonian (1) in coordinates (15) assumes the form (12), but in this case the perturbing function H_1 reads as

$$\begin{aligned}\mathcal{H}_1^{(2)}(L,Q,l,P) =&\mathcal{H}_1(L,Q,l,P)\\ =&a(Q\cos l - P\sin l)^4 + b\left(2L-P^2-Q^2\right)(Q\cos l - P\sin l)^2\cos^2 l - \\ &c\left(2L-P^2-Q^2\right)^2\cos^4 l.\end{aligned} \tag{16}$$

We will denote by $\varphi(t,z,\epsilon)=(q_1(t,z,\epsilon),q_2(t,z,\epsilon),p_1(t,z,\epsilon),p_2(t,z,\epsilon))$ the solution of the Hamiltonian system (4) and initial condition z and when $\epsilon=0$ we use the notation $\varphi_{osc}(t,z)=(q_1(t,z),$ $q_2(t,z),p_1(t,z),p_2(t,z))$. On the other hand, we will denote by $\psi(t,Z,\epsilon)=(L(t,Z,\epsilon),Q(t,Z,\epsilon),$ $l(t,Z,\epsilon),P(t,Z,\epsilon))$ the solution of the Hamiltonian system associated to the Hamiltonian (12) with initial condition Z and when $\epsilon=0$, we will write $\psi_{osc}(t,Z)=(L(t,Z),Q(t,Z),l(t,Z),P(t,Z))$.

2.2. **Characterization and approximation of the symmetric periodic solutions.** Initially we need to characterize the symmetric periodic solution as in cartesian coordinates as Lissajous variables. Next, it becomes necessary to characterize "good" initial conditions that allow us to generate periodic symmetric solutions of the complete Hamiltonian system. Then, we get a good approximation of the solutions of the Hamiltonian system (4) in the symplectic coordinates (11) or (15).

We start observing that the Hamiltonian function (1) is invariant under the reflections

$$S_1:(q_1,q_2,p_1,p_2)\to(q_1,-q_2,-p_1,p_2) \quad\text{and}\quad S_2:(q_1,q_2,p_1,p_2)\to(-q_1,q_2,p_1,-p_2). \tag{17}$$

There are two important consequences about Hamiltonian systems invariants under reflections:

(i) If $\varphi(t,q_1,q_2,p_1,p_2)=(q_1(t),q_2(t),p_1(t),p_2(t))$ is a solution of an Hamiltonian system invariant under S_1 and S_2, then $S_1\circ\varphi(-t,q_1,q_2,p_1,p_2)=(q_1(-t),-q_2(-t),-p_1(-t),p_2(-t))$ and $S_2\circ\varphi(-t,q_1,q_2,p_1,p_2)=(-q_1(-t),q_2(-t),p_1(-t),-p_2(-t))$ are also solutions.

(ii) Given the set of the fixed points of the symmetry S_1, namely, $\mathcal{L}_1=\{(q_1,0,0,p_2);q_1,p_2\in\mathbb{R}\}$ (resp. for the symmetry S_2, namely, $\mathcal{L}_2=\{(0,q_2,p_1,0);q_2,p_1\in\mathbb{R}\}$), if we consider an initial condition $(q_1,q_2,p_1,p_2)\in\mathcal{L}_1$ such that $\varphi(T/2,q_1,q_2,p_1,p_2)\in\mathcal{L}_1$ (resp. $\varphi(T/2,q_1,q_2,p_1,p_2)\in\mathcal{L}_2$), then the solution will be T-periodic and S_1-symmetric (resp. S_2-symmetric).

One important property of the variables (11) and (15) is that they allow us to characterize the symmetries more easily than the cartesian coordinates. In variables (11) as in (15) the Hamiltonian (1) assumes the form (12) where $\mathcal{H}_1(L,Q,l,P)$ was described in (14) and (16), respectively. For the following arguments we can assume that H_1 is an arbitrary function.

In our approach we are going to study only the case of the symmetry S_1 because the other case is similar. The first step is to consider a 2π-periodic solution of the unperturbed Hamiltonian system (i.e., associated to the Hamiltonian H_0) in cartesian coordinates denoted by $\varphi_{os}(t,z_0)$, such that its initial condition has the symmetry, that is,

$$z_0=(q_1^{(0)},q_2^{(0)},p_1^{(0)},p_2^{(0)})=(q_1,0,0,p_2)\in\mathcal{L}_1. \tag{18}$$

As during this approach we are working with the variables (11) and (15), we need to characterize the symmetric initial conditions (i.e., the points on $\mathcal{L}_1$) in these variables.

Lemma 2.1. *i) In symplectic variables* (11)*, an orbit hits $\mathcal{L}_1$ at time $t=T/2$ if $l(T/2)=0(mod\,\pi)$, and $Q(T/2)=0$. This set will be denoted by $\mathcal{L}_1^{(1)}$.*

ii) In symplectic variables (15)*, an orbit hits $\mathcal{L}_1$ at time $t=T/2$ if $l(T/2)=\frac{\pi}{2}(mod\ \pi)$ and $Q(T/2)=0$. This set will be denoted by $\mathcal{L}_1^{(2)}$.*

In variables (11) and (15) we will denote the periodic solution of the Hamiltonian system associated to $\mathcal{H}_0(L)$ by $\psi_{os}(t, Z_0^{(j)})$, with the convenient initial condition

$$Z_0^{(1)} = (L_0, Q_0, l_0, P_0) = (L_0, 0, 0, P_0) \in \mathcal{L}_1^{(1)}, \tag{19}$$

in the case $j = 1$, and in variables (15)

$$Z_0^{(2)} = (L_0, Q_0, l_0, P_0) = (L_0, 0, \pi/2, P_0) \in \mathcal{L}_1^{(2)}, \tag{20}$$

for $j = 2$. For instance L_0 and P_0 are arbitrary.

The second step is to perturb or modify the initial condition. In cartesian coordinates this means that we perturbed the initial conditions only in the directions of q_1 and p_2, such that the perturb initial condition remains at $\mathcal{L}_1$. For this purpose we will take a small perturbation of the initial condition $Z_0^{(j)}$ in a convenient way such that it still lies in the set of symmetry $\mathcal{L}_1^{(j)}$. We will do this by considering

$$Z^{(1)} = (L_0 + \Delta L, 0, 0, P_0 + \Delta P) \tag{21}$$

when we work with variables (11) and

$$Z^{(2)} = (L_0 + \Delta L, 0, \pi/2, P_0 + \Delta P), \tag{22}$$

when we consider the variables (15).

For any type of variables (11) or (15), we will take the initial condition $Z^{(j)}$ and we denote the solution of the complete Hamiltonian system (12) as

$$\psi(t, Z^{(j)}, \epsilon) = (L(t, Z^{(j)}, \epsilon), Q(t, Z^{(j)}, \epsilon), l(t, Z^{(j)}, \epsilon), P(t, Z^{(j)}, \epsilon)), \tag{23}$$

with

$$\begin{aligned} L(t, Z^{(j)}, \epsilon) =& L^{(0)}(t, Z^{(j)}) + \epsilon L^{(1)}(t, Z^{(j)}) + O(\epsilon^2), \\ Q(t, Z^{(j)}, \epsilon) =& Q^{(0)}(t, Z^{(j)}) + \epsilon Q^{(1)}(t, Z^{(j)}) + O(\epsilon^2), \\ l(t, Z^{(j)}, \epsilon) =& l^{(0)}(t, Z^{(j)}) + \epsilon l^{(1)}(t, Z^{(j)}) + O(\epsilon^2), \\ P(t, Z^{(j)}, \epsilon) =& P^{(0)}(t, Z^{(j)}) + \epsilon P^{(1)}(t, Z^{(j)}) + O(\epsilon^2). \end{aligned} \tag{24}$$

The third step is to get the approximation at first order in ϵ of the solution $\psi(t, Z_0^{(j)}, \epsilon)$. In any case,

$$\psi_{os}(t, Z_0) = (L^{(0)}(t, Z_0^{(j)}), Q^{(0)}(t, Z_0^{(j)}), l^{(0)}(t, Z_0^{(j)}), P^{(0)}(t, Z_0^{(j)})) = Z_0^{(j)} + (0, 0, -t, 0). \tag{25}$$

Using variational equations we obtain that the expressions for $L^{(1)}(t, Z^{(j)}), Q^{(1)}(t, Z^{(j)}), l^{(1)}(t, Z^{(j)})$ and $P^{(1)}(t, Z^{(j)})$ are given by

$$\begin{aligned} L^{(1)}(t, Z^{(j)}) &= \int_0^t \frac{\partial \mathcal{H}_1^{(j)}}{\partial l}(\psi_{os}(\tau, Z^{(j)}))d\tau, \\ Q^{(1)}(t, Z^{(j)}) &= \int_0^t \frac{\partial \mathcal{H}_1^{(j)}}{\partial P}(\psi_{os}(\tau, Z^{(j)}))d\tau, \\ l^{(1)}(t, Z^{(j)}) &= -\int_0^t \frac{\partial \mathcal{H}_1^{(j)}}{\partial L}(\psi_{os}(\tau, Z^{(j)}))d\tau, \\ P^{(1)}(t, Z^{(j)}) &= -\int_0^t \frac{\partial \mathcal{H}_1^{(j)}}{\partial Q}(\psi_{os}(\tau, Z^{(j)}))d\tau. \end{aligned} \tag{26}$$

2.3. **Study of the stability.** The fourth step is the study of stability of the periodic solutions. So, in order to calculate the characteristic multipliers associated one fixed S_1-symmetric T-periodic solution, we take the local cross

$$\Sigma = \{(L, Q, l, P) : l = 0, \mathcal{H} = L_0\},$$

let us $X = (X_1, X_2) = (Q, P)$ be the coordinates in Σ and we consider $\overline{Z} = (L_0, Q_0+X_1, l_0, P_0+X_2)$, where $\psi_{os}(t, (L_0, Q_0 + X_1, l_0, P_0 + X_2))$ is a solution of the Hamiltonian system associated to $\mathcal{H}_0$ with initial condition $(L_0, Q_0+X_1, l_0, P_0+X_2)$ at the level $\mathcal{H}_0 = L = L_0$. Let us define the Poincaré map $\mathcal{P} : \Sigma \to \Sigma$ as follows

$$\mathcal{P}(X, \epsilon) =(Q(\tau, \overline{Z}, \epsilon), P(\tau, \overline{Z}, \epsilon))$$

where $\tau = 2\pi + O(\epsilon)$ is the time of first return. So considering the approximation of the solutions given in (24), we have that

$$\begin{aligned}\mathcal{P}(X, \epsilon) &= (X_1, P_0 + X_2) + \epsilon \left(\int_0^\tau \frac{\partial \mathcal{H}}{\partial P}(\psi(t, \overline{Z}))dt, -\int_0^\tau \frac{\partial \mathcal{H}}{\partial Q}(\psi(t, \overline{Z}))dt \right) + O(\epsilon^2) \\ &= \left(X_1 + \epsilon \int_0^\tau \frac{\partial \mathcal{H}}{\partial P}(\psi(t, \overline{Z}))dt + O(\epsilon^2), P_0 + X_2 - \epsilon \int_0^\tau \frac{\partial \mathcal{H}}{\partial Q}(\psi(t, \overline{Z}))dt + O(\epsilon^2) \right).\end{aligned}$$

Expanding in Taylor series around $\epsilon = 0$, we obtain that the Poincaré map is given by

(27)
$$\mathcal{P}(X, \epsilon) = \left(X_1 + \epsilon \int_0^{2\pi} \frac{\partial \mathcal{H}}{\partial P}(\psi_{os}(t, \overline{Z}))dt + O(\epsilon^2,), P_0 + X_2 - \epsilon \int_0^{2\pi} \frac{\partial \mathcal{H}}{\partial Q}(\psi_{os}(t, \overline{Z}))dt + O(\epsilon^2) \right).$$

Differentiating $\mathcal{P}$ with respect to $X = (X_1, X_2)$, we arrive to

$$D_X\mathcal{P}(X, \epsilon) = \begin{pmatrix} 1 + \epsilon \frac{\partial}{\partial X_1} \left(\int_0^{2\pi} \frac{\partial \mathcal{H}}{\partial P}(\psi_{os}(t, \overline{Z}))dt \right) & \epsilon \frac{\partial}{\partial X_2} \left(\int_0^{2\pi} \frac{\partial \mathcal{H}}{\partial P}(\psi_{os}(t, \overline{Z}))dt \right) \\ -\epsilon \frac{\partial}{\partial X_1} \left(\int_0^{2\pi} \frac{\partial \mathcal{H}}{\partial Q}(\psi_{os}(t, \overline{Z}))dt \right) & 1 - \epsilon \frac{\partial}{d X_2} \left(\int_0^{2\pi} \frac{\partial \mathcal{H}}{\partial Q}(\psi_{os}(t, \overline{Z}))dt \right) \end{pmatrix} + O(\epsilon^2),$$

or equivalently,

(28) $$D_X\mathcal{P}(X, \epsilon) = I + 2\pi\epsilon A + O(\epsilon^2),$$

where

$$A = J \left(\frac{\partial^2 \overline{\mathcal{H}}_1}{\partial X_i \partial X_j} \right)_{X=0},$$

and

(29) $$\overline{\mathcal{H}}_1(X) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_1(\psi_{os}(\tau, (L_0, Q_0 + X_1, l_0, P_0 + X_2)))d\tau,$$

is the averaged Hamiltonian associated to $\mathcal{H}_1$. Therefore, the characteristic multipliers of the fixed periodic solution $\psi(t, Z)$ are given by

(30) $$1, 1, 1 + 2\pi\epsilon\lambda_1 + O(\epsilon^2), 1 + 2\pi\epsilon\lambda_2 + O(\epsilon^2),$$

where λ_1 and λ_2 are the eigenvalues of the matrix A.

3. Proof of Theorem 1.1

To prove our theorem, we consider the Hamiltonian system (4) with Hamiltonian function as in (1) and we make a symplectic change of variables (11), so, the Hamiltonian function assumes the form (12) with $\mathcal{H}_1$ as in (14) and the Hamiltonian system is given by

$$
\begin{aligned}
\dot{L} =&\epsilon \left[-2b(-2L+P^2+Q^2)(P\sin l - Q\cos l)(P\cos l + Q\sin l)\cos^2 l + \right.\\
&4c(P\sin l - Q\cos l)^3(P\cos l + Q\sin l) - 4a(-2L+P^2+Q^2)^2\sin l\cos^3 l + \\
&\left. b(-2L+P^2+Q^2)(Q\cos l - P\sin l)^2\sin 2l\right],\\
\dot{Q} =&\epsilon \left[4aP(-2L+P^2+Q^2)\cos^4 l + 2b(-2L+P^2+Q^2)(Q\cos l - P\sin l)\sin l\cos^2 l - \right.\\
&\left. 2bP(Q\cos l - P\sin l)^2\cos^2 l + 4c(P\sin l - Q\cos l)^3\right]\sin l,\\
\dot{l} =& -1 + \epsilon\left[4a(-2L+P^2+Q^2)\cos^4 l - 2b(Q\cos l - P\sin l)^2\cos^2 l\right],\\
\dot{P} =& -2\epsilon\left[2aQ(-2L+P^2+Q^2)\cos^3 l + b(-2L+P^2+Q^2)(P\sin l - Q\cos l)\cos^2 l - \right.\\
&\left. bQ(Q\cos l - P\sin l)^2\cos l + 2c(Q\cos l - P\sin l)^3\right]\cos l.
\end{aligned}
\tag{31}
$$

In order to get an approximation of the initial conditions and a approximations of S_1-symmetric periodic solutions of the Hamiltonian system (31), we consider a perturbed initial condition in two directions (two coordinates) of the form

$$Z^{(1)} = (L_0 + \Delta L, 0, 0, P_0 + \Delta P) \in \mathcal{L}_1^{(1)}.$$

Using the expression of $\mathcal{H}_1$ in (14) we get

$$
\begin{aligned}
\frac{\partial \mathcal{H}_1^{(1)}}{\partial L} =& 2b(Q\cos l - P\sin l)^2\cos^2 l - 4a(-2L+P^2+Q^2)\cos^4 l,\\
\frac{\partial \mathcal{H}_1^{(1)}}{\partial Q} =& 2\left[2aQ\cos^3 l\left(-2L+P^2+Q^2\right) + b\cos^2 l\left(-2L+P^2+Q^2\right)(P\sin l - Q\cos l) - \right.\\
&\left. bQ\cos l(Q\cos l - P\sin l)^2 + 2c(Q\cos l - P\sin l)^3\right]\cos l,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{H}_1^{(1)}}{\partial l} =& -4a(-2L+P^2+Q^2)^2\sin l\cos^3 l + b\sin 2l\left(-2L+P^2+Q^2\right)(Q\cos l - P\sin l)^2 - \\
&2b\cos^2 l\left(-2L+P^2+Q^2\right)(P\sin l - Q\cos l)(P\cos l + Q\sin l) - \\
&4c(Q\cos l - P\sin l)^3(P\cos l + Q\sin l),\\
\frac{\partial \mathcal{H}_1^{(1)}}{\partial P} =& 4aP(-2L+P^2+Q^2)\cos^4 l + 2b(-2L+P^2+Q^2)(Q\cos l - P\sin l)\sin l\cos^2 l - \\
&2bP(Q\cos l - P\sin l)^2\cos^2 l + 4c(P\sin l - Q\cos l)^3\sin l,
\end{aligned}
$$

and after some manipulations it follows that equations (26) assumes the form

$$
\begin{aligned}
L^{(1)}(t,Z^{(1)}) =& -\frac{1}{2}(P_0+\Delta P)^2\left[-2b(L_0+\Delta L)+(P_0+\Delta P)^2(b+2c)\right]\sin^4 t+\\
&\frac{1}{2}\left(2(L_0+\Delta L)-(P_0+\Delta P)^2\right)\left[a\left(4(L_0+\Delta L)-2(P_0+\Delta P)^2\right)-\right.\\
&\left. b(P_0+\Delta P)^2\right](1-\cos^4 t),\\
Q^{(1)}(t,Z^{(1)}) =&\frac{1}{8}(P_0+\Delta P)\left[-2a(L_0+\Delta L)(12t+8\sin 2t+\sin 4t)+\right.\\
&a(P_0+\Delta P)^2(12t+8\sin 2t+\sin 4t)-b(L_0+\Delta L)(\sin 4t-4t)+\\
&\left. b(P_0+\Delta P)^2(\sin 4t-4t)+c(P_0+\Delta P)^2(12t-8\sin 2t+\sin 4t)\right],\\
l^{(1)}(t,Z^{(1)}) =&\frac{a}{8}(12t+8\sin 2t+\sin 4t)\left((P_0+\Delta P)^2-2(L_0+\Delta L)\right)+\\
&\frac{b}{16}(P_0+\Delta P)^2(\sin 4t-4t),\\
P^{(1)}(t,Z^{(1)}) =&\frac{1}{2}b(P_0+\Delta P)\left((P_0+\Delta P)^2-2(L_0+\Delta L)\right)(1-\cos^4 t)-c(P_0+\Delta P)^3\sin^4 t.
\end{aligned}
$$

Considering (24), we obtain that the approximation of the solution with initial condition $Z_0^{(1)}$ in the first order is given by

$$
\begin{aligned}
L(t,Z^{(1)},\epsilon) =&L_0+\Delta L+\epsilon L^{(1)}(t,Z^{(1)})+O(\epsilon^2),\\
Q(t,Z^{(1)},\epsilon) =&\epsilon Q^{(1)}(t,Z^{(1)})+O(\epsilon^2),\\
l(t,Z^{(1)},\epsilon) =&-t+\epsilon l^{(1)}(t,Z^{(1)})+O(\epsilon^2),\\
P(t,Z^{(1)},\epsilon) =&P_p+\Delta P+\epsilon P^{(1)}(t,Z^{(1)})+O(\epsilon^2),
\end{aligned}
\tag{32}
$$

Next, since we are we looking for S_1-symmetric periodic solutions in coordinates (11), according to Lemma 2.1, the following two periodicity equations (32) must satisfy

$$
\begin{aligned}
\frac{1}{\epsilon}Q(T/2,\Delta L,\Delta P,\epsilon) =&Q^{(1)}(T/2,Z^{(1)})+O(\epsilon)=0,\\
l(T/2,\Delta L,\Delta P,\epsilon) =&-T/2+\frac{\epsilon}{8}\left\{a(6T+8\sin T+\sin 2T)\left((P_0+\Delta P)^2-2(L_0+\Delta L)\right)+\right.\\
&\left.\frac{b}{2}(P_0+\Delta P)^2(\sin 2T-2T)\right\}+O(\epsilon^2)=0.
\end{aligned}
\tag{33}
$$

Since, we want to avoid degeneration in the rank (this will be clear later), we introduce the "time" as a dependent variable, the system (33) must be modified, so the periodicity system (the system that characterizes the symmetric periodic solution but with period not necessarily of fixed period 2π) assumes the form

$$
\begin{aligned}
f_1(\tau,\Delta L,\Delta P,\epsilon) =&\frac{1}{8}(P_0+\Delta P)\left(-a(12\tau+8\sin 2\tau+\sin 4\tau)\left(2(L_0+\Delta L)-(P_0+\Delta P)^2\right)-\right.\\
&b(\sin 4\tau-4\tau)\left(\Delta L-(P_0+\Delta P)^2+L_0\right)+\\
&\left.c(P_0+\Delta P)^2(12\tau-8\sin 2\tau+\sin 4\tau)\right)+O(\epsilon),\\
f_2(\tau,\Delta L,\Delta P,\epsilon) =&-\tau+\frac{\epsilon}{16}\left(b(P_0+\Delta P)^2(\sin 4\tau-4\tau)-\right.\\
&\left.2a(12\tau+8\sin 2\tau+\sin 4\tau)\left(2(L_0+\Delta L)-(P_0+\Delta P)^2\right)\right)+O(\epsilon^2).
\end{aligned}
\tag{34}
$$

At this point we are going to consider as independent variables the pair $(\tau, \Delta P)$, that is, ΔL and ϵ are the other variables, and P_0 and L_0 are parameters. In order to apply the Implicit Function Theorem, we need to solve the system

$$
\begin{aligned}
f_1(\pi,0,0,0) =& \frac{1}{2}\pi P_0 \left(-6aL_0 + 3aP_0^2 + bL_0 - bP_0^2 + 3cP_0^2\right) = 0,\\
f_2(\pi,0,0,0) =& -\frac{1}{4}\pi \left(12aL_0 - 6aP_0^2 + bP_0^2\right) = 0.
\end{aligned}
\tag{35}
$$

The solutions of this system are

$$
\begin{aligned}
(i)\quad & P_0^{(1)} = 0, \quad \text{whenever} \quad a = 0,\\
(ii)\quad & P_0^{(2)} = -\sqrt{2L_0}\sqrt{\frac{b}{b-6c}}, \quad \text{whenever} \quad \frac{b}{b-6c} > 0,\\
(iii)\quad & P_0^{(3)} = \sqrt{2L_0}\sqrt{\frac{b}{b-6c}}, \quad \text{whenever} \quad \frac{b}{b-6c} > 0.
\end{aligned}
\tag{36}
$$

Moreover, differentiating the system (34) with respect to $(t, \Delta P)$ and evaluating in each P_0 given in (36) and $\tau = \pi, \Delta L = 0, \Delta P = 0, \epsilon = 0$, we have that

$$
\left.\frac{\partial(f_1, f_2)}{\partial(\tau, \Delta P)}\right|_{\tau=\pi,\Delta L=0,\Delta P=0,\epsilon=0,P=P_0^{(1)}} = \begin{pmatrix} 0 & \frac{bL_0\pi}{2} \\ -1 & 0 \end{pmatrix},
\tag{37}
$$

$$
\left.\frac{\partial(f_1, f_2)}{\partial(\tau, \Delta P)}\right|_{\tau=\pi,\Delta L=0,\Delta P=0,\epsilon=0,P=P_0^{(2)}} = \begin{pmatrix} -\frac{4\sqrt{2}b^2cL_0^{3/2}}{3(b-6c)^3}\sqrt{\frac{b(b-6c)^3}{c^2}} & \frac{\pi bL_0(b-6c)}{6c} \\ -1 & 0 \end{pmatrix},
\tag{38}
$$

$$
\left.\frac{\partial(f_1, f_2)}{\partial(\tau, \Delta P)}\right|_{\tau=\pi,\Delta L=0,\Delta P=0,\epsilon=0,P=P_0^{(3)}} = \begin{pmatrix} \frac{4\sqrt{2}b^2cL_0^{3/2}}{3(b-6c)^3}\sqrt{\frac{b(b-6c)^3}{c^2}} & \frac{\pi bL_0(b-6c)}{6c} \\ -1 & 0 \end{pmatrix},
\tag{39}
$$

and their respective determinants are $\frac{\pi}{2}L_0b$ for the matrix (37) and $\frac{\pi bL_0(b-6c)}{6c}$ for the matrix (38) and (39). Therefore, we are in position to apply the Implicit Function Theorem to the functions (f_1, f_2) . Thus, assuming the existence of the points P_0 in (36) for each fixed $L_0 > 0$, and that the previous determinants are non null, we arrive that for each of these points there is a unique differentiable function $\Delta P(\Delta L, \epsilon)$ and $\tau(\Delta L, \epsilon)$ for ΔL and ϵ sufficiently small, such that $\Delta P(0,0) = 0$ and $\tau(0,0) = \pi$ and the system $(f_1(\tau(\Delta L,\epsilon), \Delta L, \Delta P(\Delta L,\epsilon), \epsilon), f_2(\tau(\Delta L,\epsilon), \Delta L, \Delta P(\Delta L,\epsilon)\epsilon)) = (0,0)$ is satisfied. Therefore, we have found three $2\tau(\Delta L,\epsilon)$-periodic symmetric solutions with period close to 2π.

In order to obtain the approximation of the three families of S_1-symmetric periodic solutions in the variables (11), it is enough to substitute the values of $P_0^{(1)}$, $P_0^{(2)}$ and $P_0^{(3)}$ in (32).

The next step is to compute the characteristic multiplier of each solution found, we will follow the idea given in Section 2, so for $P = P_0^{(1)}$ in (36), we have that the matrix A is given by

$$
A = \begin{pmatrix} 0 & \frac{bL_0}{2} \\ -\frac{3}{2}bL_0 & 0 \end{pmatrix},
$$

whose eigenvalues are $\pm\frac{i}{2}\sqrt{3}bL_0$. Thus, the characteristic multipliers for this solution are

$$
1, 1, 1 + \epsilon i\sqrt{3}\pi bL_0 + O(\epsilon^2), 1 - \epsilon i\sqrt{3}\pi bL_0 + O(\epsilon^2) + O(\epsilon^2).
$$

For $P_0^{(2)}$ and $P_0^{(3)}$ given in (36), we have that the matrix A assumes the form

$$A = \begin{pmatrix} 0 & \frac{b(b-6c)L_0}{6c} \\ \frac{6bcL_0}{b-6c} & 0 \end{pmatrix},$$

whose eigenvalues are $\pm bL_0$. Thus, the characteristic multipliers for these solutions are

$$1, 1, 1 + \epsilon 2\pi bL_0 + O(\epsilon^2), 1 - \epsilon 2\pi bL_0 + O(\epsilon^2).$$

Finally to obtain the characterization of S_1-symmetric periodic solutions up to order ϵ in cartesian coordinates we recall that

$$\begin{aligned} q_1(t) =& \sqrt{2L(t) - P(t)^2 - Q(t)^2}\cos l(t), \quad q_2(t) = Q(t)\cos l(t) - P(t)\sin l(t), \\ p_1(t) =& \sqrt{2L(t) - P(t)^2 - Q(t)^2}\sin l(t), \quad p_2(t) = P(t)\cos l(t) + Q\sin l(t), \end{aligned}$$

and using (32) for each point $P_0^{(j)}$, we arrive to the description of the periodic solution as in (5) for $j = 1$ and as in (7) for $j = 2, 3$, respectively. Moreover, solution (5) es linearly stable and solutions (7) are unstable. Therefore we have proved item (a) and (c) of the theorem.

In order to prove item (b) of our theorem, we consider the Hamiltonian system associated to (12) with $\mathcal{H}_1$ like in (16) written by

$$\begin{aligned} \dot{L} =& \epsilon\left(-2b(-2L + P^2 + Q^2)(P\sin l - Q\cos l)(P\cos l + Q\sin l)\cos^2 l + \right. \\ & b\sin 2l(-2L + P^2 + Q^2)(Q\cos l - P\sin l)^2 - 4c\sin l\cos^3 l(-2L + P^2 + Q^2)^2 + \\ & \left. 4a(P\sin l - Q\cos l)^3(P\cos l + Q\sin l)\right), \\ \dot{Q} =& \epsilon\left(2b(-2L + P^2 + Q^2)(Q\cos l - P\sin l)\sin l\cos^2 l - 2bP(Q\cos l - P\sin l)^2\cos^2 l + \right. \\ & \left. 4cP(-2L + P^2 + Q^2)\cos^4 l + 4a(P\sin l - Q\cos l)^3\sin l\right), \\ \dot{l} =& -1 + \epsilon\left(4c(-2L + P^2 + Q^2)\cos^4 l - 2b(Q\cos l - P\sin l)^2\cos^2 l\right), \\ \dot{P} =& -2\epsilon\left(2a(Q\cos l - P\sin l)^3 + b(-2L + P^2 + Q^2)(P\sin l - Q\cos l)\cos^2 l - \right. \\ & \left. bQ(Q\cos l - P\sin l)^2\cos l + 2cQ(-2L + P^2 + Q^2)\cos^3 l\right)\cos l, \end{aligned} \tag{40}$$

Now, in order to get an approximation of the initial conditions and a approximations of periodic S_1-symmetric solutions of the perturbed Hamiltonian system (31), we consider a perturbed initial condition in two directions (two coordinates) of the form

$$Z^{(2)} = \left(L_0 + \Delta L, 0, \frac{\pi}{2}, P_0 + \Delta P\right) \in \mathcal{L}_1^{(2)}.$$

Using the expression of $\mathcal{H}_1$ in (16) we get

$$\begin{aligned} \frac{\partial \mathcal{H}_1^{(2)}}{\partial L} =& 2b(Q\cos l - P\sin l)^2\cos^2 l - 4c\left(-2L + P^2 + Q^2\right)\cos^4 l, \\ \frac{\partial \mathcal{H}_1^{(2)}}{\partial Q} =& 4a(Q\cos l - P\sin l)^3\cos l - 2b\left(-2L + P^2 + Q^2\right)(Q\cos l - P\sin l)\cos^3 l - \\ & 2bQ(Q\cos l - P\sin l)^2\cos^2 l + 4cQ\left(-2L + P^2 + Q^2\right)\cos^4 l, \end{aligned}$$

$$
\begin{aligned}
\frac{\partial \mathcal{H}_1^{(2)}}{\partial l} =& -2b\left(-2L+P^2+Q^2\right)(P\sin l - Q\cos l)(P\cos l + Q\sin l)\cos^2 l \\
& -4a(Q\cos l - P\sin l)^3(P\cos l + Q\sin l) + b\left(-2L+P^2+Q^2\right)(Q\cos l - P\sin l)^2\sin 2l - \\
& 4c\left(-2L+P^2+Q^2\right)^2\sin l\cos^3 l, \\
\frac{\partial \mathcal{H}_1^{(2)}}{\partial P} =& 4a(P\sin l - Q\cos l)^3\sin l + 2b\left(-2L+P^2+Q^2\right)(Q\cos l - P\sin l)\sin l\cos^2 l - \\
& 2bP(Q\cos l - P\sin l)^2\cos^2 l + 4cP\left(-2L+P^2+Q^2\right)\cos^4 l,
\end{aligned}
$$

and after some manipulations it follows that equations (26) assume the form

$$
\begin{aligned}
L^{(1)}(t, Z^{(2)}) =& \frac{1}{2}(P_0+\Delta P)^2\left[(2a+b)(P_0+\Delta P)^2 - 2b(L_0+\Delta L)\right](1-\cos^4 t) - \\
& \frac{1}{2}\left(2(L_0+\Delta L) - (P_0+\Delta P)^2\right)\left(4c(L_0+\Delta L) - (b+2c)(P_0+\Delta P)^2\right), \\
Q^{(1)}(t, Z^{(2)}) =& \frac{1}{8}(\Delta P + P_0)\left[a(\Delta P+P_0)^2(12t+8\sin 2t+\sin 4t) - b(\Delta L+L_0)(\sin 4t - 4t) + \right. \\
& b(\Delta P+P_0)^2(\sin 4t - 4t) - 2c(\Delta L+L_0)(12t - 8\sin 2t + \sin 4t) + \\
& \left. c(\Delta P+P_0)^2(12t-8\sin 2t+\sin 4t)\right], \\
l^{(1)}(t, Z^{(2)}) =& \frac{1}{16}\left[b(\Delta P+P_0)^2(\sin 4t - 4t) + 2c(12t - 8\sin 2t + \sin 4t)\left((\Delta P+P_0)^2 - 2(\Delta L+L_0)\right)\right], \\
P^{(1)}(t, Z^{(2)}) =& -a(\Delta P+P_0)^3\left(\cos^4 t - 1\right) - \frac{1}{2}b(\Delta P+P_0)\left((\Delta P+P_0)^2 - 2(\Delta L+L_0)\right)\sin^4 t.
\end{aligned}
$$

Considering (24), we obtain that the approximation of the solution with initial condition $Z_0^{(2)}$ in the first order is given by

$$
\begin{aligned}
L(t, Z^{(2)}, \epsilon) =& L_0 + \Delta L + \epsilon L^{(1)}(t, Z^{(2)}) + O(\epsilon^2), \\
Q(t, Z^{(2)}, \epsilon) =& \epsilon Q^{(1)}(t, Z^{(2)}) + O(\epsilon^2), \\
l(t, Z^{(2)}, \epsilon) =& -t + \epsilon l^{(1)}(t, Z^{(2)}) + O(\epsilon^2), \\
P(t, Z^{(2)}, \epsilon) =& P_p + \Delta P + \epsilon P^{(2)}(t, Z^{(2)}) + O(\epsilon^2).
\end{aligned} \tag{41}
$$

In order to have a S_1-symmetric periodic solution in coordinates (15), according to Lemma 2.1, the following two periodicity equations must satisfy

$$
\begin{aligned}
\frac{1}{\epsilon}Q(T/2, \Delta L, \Delta P, \epsilon) =& Q^{(1)}(T/2, Z^{(2)}) + O(\epsilon) = 0, \\
l(T/2, \Delta L, \Delta P, \epsilon) =& \frac{\pi}{2} - \frac{T}{2} + \frac{\epsilon}{16}\left[b(\Delta P+P_0)^2(\sin 2T - 2T) - \right. \\
& \left. 2c(6T - 8\sin T + \sin 2T)\left(2\Delta L - (\Delta P+P_0)^2 + 2L_0\right)\right] + O(\epsilon^2) = 0,
\end{aligned} \tag{42}
$$

Since, we want to avoid degeneration in the rank (this will be clear later), we introduce the "time" as a dependent variable, the system (42) must be modified, so the periodicity system (the system that characterizes the symmetric periodic solution but with period not necessarily of fixed period

2π) assumes the form

$$
\begin{aligned}
f_1(\tau,\Delta L,\Delta P,\epsilon)=&\frac{1}{8}(\Delta P+P_0)\left(a(\Delta P+P_0)^2(12\tau+8\sin 2\tau+\sin 4\tau)-\right.\\
&c(12\tau-8\sin 2\tau+\sin 4\tau)\left(2\Delta L-(\Delta P+P_0)^2+2L_0\right)-\\
&\left.b(\sin 4\tau-4\tau)\left(\Delta L-(\Delta P+P_0)^2+L_0\right)\right)+O(\epsilon),\\
f_2(\tau,\Delta L,\Delta P,\epsilon)=&\frac{\pi}{2}-\tau+\frac{\epsilon}{16}\left[b(\Delta P+P_0)^2(\sin 4\tau-4\tau)-\right.\\
&\left.2c(12\tau-8\sin 2\tau+\sin 4\tau)\left(2\Delta L-(\Delta P+P_0)^2+2L_0\right)\right]+O(\epsilon^2).
\end{aligned}
\tag{43}
$$

At this point we are going to consider as independent variables the pair $(\tau,\Delta P)$, that is, ΔL and ϵ are the other variables, and P_0 and L_0 are parameters. In order to apply the Implicit Function Theorem, we need to solve the system

$$
\begin{aligned}
f_1(\pi,0,0,0)=&\frac{1}{2}\pi P_0\left(3aP_0^2+bL_0-bP_0^2-6cL_0+3cP_0^2\right)=0,\\
f_2(\pi,0,0,0)=&\frac{1}{16}\left(-4\pi bP_0^2-24\pi c\left(2L_0-P_0^2\right)\right)=0.
\end{aligned}
\tag{44}
$$

The solutions of this system are

$$
\begin{aligned}
&(i)\;\; P_0^{(4)}=0,\quad \text{whenever}\quad c=0,\\
&(ii)\;\; P_0^{(5)}=-\sqrt{2}\sqrt{L_0}\sqrt{\frac{b}{b-6a}},\quad \text{whenever}\quad \frac{b}{b-6a}>0,\\
&(iii)\;\; P_0^{(6)}=\sqrt{2}\sqrt{L_0}\sqrt{\frac{b}{b-6a}},\quad \text{whenever}\quad \frac{b}{b-6a}.
\end{aligned}
\tag{45}
$$

Moreover, differentiating the system (43) with respect to $(\tau,\Delta P)$ and evaluating in each P_0 given in (45) and $\tau=\pi,\Delta L=0,\Delta P=0,\epsilon=0$, we have that

$$
a\;\frac{\partial(f_1,f_2)}{\partial(\tau,\Delta P)}\bigg|_{\tau=\pi,\Delta L=0,\Delta P=0,\epsilon=0,P=P_0^{(4)}}=\begin{pmatrix}0 & \frac{bL_0\pi}{2}\\ -1 & 0\end{pmatrix},
\tag{46}
$$

$$
\frac{\partial(f_1,f_2)}{\partial(\tau,\Delta P)}\bigg|_{\tau=\pi,\Delta L=0,\Delta P=0,\epsilon=0,P=P_0^{(5)}}=\begin{pmatrix}\frac{8\sqrt{2}abL_0}{6a-b}\sqrt{\frac{bL_0}{b-6a}} & \frac{b(b-6a)L_0\pi}{6a}\\ -1 & 0\end{pmatrix}
\tag{47}
$$

$$
\frac{\partial(f_1,f_2)}{\partial(\tau,\Delta P)}\bigg|_{\tau=\pi,\Delta L=0,\Delta P=0,\epsilon=0,P=P_0^{(6)}}=\begin{pmatrix}-\frac{8\sqrt{2}abL_0}{6a-b}\sqrt{\frac{bL_0}{b-6a}} & \frac{b(b-6a)L_0\pi}{6a}\\ -1 & 0\end{pmatrix},
\tag{48}
$$

and their respective determinants are $\frac{\pi}{2}L_0b$ for the matrix (46) and $\frac{\pi bL_0(b-6a)}{6a}$ for the matrix (47) and (48). Therefore, we are in position to apply the Implicit Function Theorem to the functions (f_1,f_2) . Thus, assuming the existence of the points P_0 in (45) for each fixed $L_0>0$, and that the previous determinants are non null, we arrive that for each of these points there is a unique differentiable function $\Delta P(\Delta L,\epsilon)$ and $\tau(\Delta L,\epsilon)$ for ΔL and ϵ sufficiently small, such that $\Delta P(0,0)=0$ and $\tau(0,0)=\pi$ and the system $(f_1(\tau(\Delta L,\epsilon),\Delta L,\Delta P(\Delta L,\epsilon),\epsilon),f_2(\tau(\Delta L,\epsilon),\Delta L,\Delta P(\Delta L,\epsilon)\epsilon))=(0,0)$ is satisfied. Therefore, we have found three $2\tau(\Delta L,\epsilon)$-periodic symmetric solutions with period close to 2π.

Finally to obtain the characterization of S_1-symmetric periodic solutions up to order ϵ in cartesian coordinates we recall that

$$\begin{aligned} q_1(t) &= Q(t)\cos l(t) - P(t)\sin l(t), & q_2(t) &= \sqrt{2L(t)-P(t)^2-Q(t)^2}\cos l(t),\\ p_1(t) &= P(t)\cos l(t) + Q(t)\sin l(t), & p_2(t) &= \sqrt{2L(t)-P(t)^2-Q(t)^2}\sin l(t), \end{aligned}$$

and using (41) for each point $P_0^{(j)}$, in (45) we arrive to the characterization of the periodic solution (6) for $j=4$. For $j=5,6$ we obtain the same solution than for $j=2,3$ for the relationship that exists between (11) and (15). Remembering the domain of both coordinates and that we can obtain (15) be applying a rotation at an angle of $\pi/2$ to (15). Therefore we just need to study the stability of the new found solution. In order to compute the characteristic multiplier of this solution, we will follow the same idea of Section 2.3, so for $P=P_0^{(4)}$ in (45), we have that the matrix A is given by

$$A = \begin{pmatrix} 0 & \frac{bL_0}{2} \\ -\frac{3}{2}bL_0 & 0 \end{pmatrix},$$

whose eigenvalues are $\pm\frac{i}{2}\sqrt{3}bL_0$. Thus, the characteristic multipliers for the solution are given by

$$1, 1, 1+\epsilon i\sqrt{3}\pi bL_0 + O(\epsilon^2), 1-\epsilon i\sqrt{3}\pi bL_0 + O(\epsilon^2) + O(\epsilon^2).$$

so the Theorem 1.1 is proved. $\square$

Remark 1. *We point out that the families of periodic solutions in Theorem 1.1 are parameterized by two parameters, namely, ΔL and ϵ. Also, note that the family of periodic solution given in items (a) and (b) are stable (linearly) and the two families given in item (c) are unstable.*

4. Concluding remarks

The existence of periodic solutions of this model was studied in [6] in the case $c=0$ and in [7] for $c\neq 0$. In these articles the authors use the Averaging Method of First Order and convenient generalized polar coordinates $(q_1,q_2,p_1,p_2)\to(r,\theta,\rho,\alpha)$ given by

$$q_1 = r\cos\theta,\quad p_1 = r\sin\theta,\quad q_2=\rho\cos(\alpha+\theta),\quad p_2=\rho\sin(\alpha+\theta). \tag{49}$$

Their main results about periodic orbits is summarized as follows. The first one is for $c=0$ and the main result is as follows an can be found in [6].

Theorem 4.1. *For ϵ sufficiently small in every level $H=h>0$ the perturbed Hamiltonian system has four periodic solutions bifurcating from the periodic orbits of the unperturbed Hamiltonian system, as follows:*
(a) It has at least four periodic orbits if $b(b-a)>0$, $(b-a)(b-2a)>0$, $b(b-3a)>0$ and $(b-6a)(b-3a)>0$. These come from $r^*=\sqrt{\frac{hb}{b-a}}$, $\rho^*=\sqrt{\frac{b(b-2a)}{b-a}}$ and $\alpha^*=0$; $r^*=\sqrt{\frac{hb}{b-a}}$, $\rho^*=\sqrt{\frac{b(b-2a)}{b-a}}$ and $\alpha^*=\pi$; $r^*=\sqrt{\frac{hb}{b-a}}$, $\rho^*=\sqrt{\frac{b(b-6a)}{b-3a}}$ and $\alpha^*=\frac{\pi}{2}$; $r^*=\sqrt{\frac{hb}{b-a}}$, $\rho^*=\sqrt{\frac{b(b-6a)}{b-3a}}$ and $\alpha^*=-\frac{\pi}{2}$;

(b) It has two periodic orbits if either $b(b-a)>0$ and $(b-a)(b-2a)>0$, or $hb(b-3a)>0$ and $h(b-6a)(b-3a)>0$. These come from $r^*=\sqrt{\frac{hb}{b-a}}$, $\rho^*=\sqrt{\frac{b(b-2a)}{b-a}}$ and $\alpha^*=0$; $r^*=\sqrt{\frac{hb}{b-a}}$, $\rho^*=\sqrt{\frac{b(b-2a)}{b-a}}$ and $\alpha^*=\pi$ or $r^*=\sqrt{\frac{hb}{b-a}}$, $\rho^*=\sqrt{\frac{b(b-6a)}{b-3a}}$ and $\alpha^*=\frac{\pi}{2}$; $r^*=\sqrt{\frac{hb}{b-a}}$, $\rho^*=\sqrt{\frac{b(b-6a)}{b-3a}}$ and $\alpha^*=-\frac{\pi}{2}$;

For $c\neq 0$ the main result is as follows an can be found in [7].

Theorem 4.2. *For ϵ sufficiently small in every level $H = h > 0$ the perturbed Hamiltonian system has four periodic solutions bifurcating from the periodic orbits of the unperturbed Hamiltonian system, as follows:*

(a) The first one comes from the periodic orbit $r^* = 0$ and $\rho^* = \sqrt{2h}$ with $\epsilon = 0$ if $|\frac{6c}{b} - 2| \leq 1$ and $(b - 6c)(b - 2c) \neq 0$.

(b) The second one comes from the periodic orbit $r^* = \sqrt{2h}$ and $\rho^* = 0$ with $\epsilon = 0$ if $|\frac{6a}{b} - 2| \leq 1$ and $(6a - b)(2a - b) \neq 0$.

(c) The third one comes from the periodic orbit $r^* = \sqrt{\frac{(6c-b)h}{3a-b+3c}}$ and $\rho^* = \sqrt{\frac{(6a-b)h}{3a-b+3c}}$ with $\epsilon = 0$ if $\frac{(6c-b)}{3a-b+3c} > 0$, $\frac{(6a-b)}{3a-b+3c} > 0$ and $b(6a - b)(2c - b)(3a - b + 3c) \neq 0$.

(d) The fourth one comes from the periodic orbit $r^* = \sqrt{\frac{(2c-b)h}{a-b+c}}$ and $\rho^* = \sqrt{\frac{(2a-b)h}{a-b+c}}$ with $\epsilon = 0$ if $\frac{(2c-b)}{a-b+c} > 0$, $\frac{(2a-b)}{a-b+c} > 0$ and $b(2a - b)(b - 2c)(a - b + c) \neq 0$.

We emphasize that the families of periodic solutions in Theorems 4.1 and 4.2 are parametrized by only one parameter, namely, ϵ. Furthermore, according to the Average Theory used by them, these solutions have period close to 2π.

In order to compare the periodic solutions given in [6] and [7] and that are stated in Theorems 4.1 and 4.2, respectively, with the periodic S_1-symmetric found solutions by us and stated in Theorem 1.1, is that we proceed to approximate in first order all solutions in a common coordinate system, namely, cartesian coordinates. It is also important to mention that we must first look at the conditions on the parameters for the existence of the solutions found, in each of the theorems mentioned.

According to the regions of existence of the periodic solutions on the parameters a, b, c we have the following: For the existence of the periodic solution given in item (a) of Theorem 4.2, they need that $c \neq 0$. Also, for $\epsilon = 0$ this is a periodic orbit of the harmonic oscillator contained in the plane (q_1, p_1). Taking into account the above conditions, this solution cannot coincide with the periodic S_1-symmetric solution given in item (b) of Theorem 1.1 . Analogously to the above, for the existence of the periodic solution given in item (b) of Theorem 4.2, they need that $a \neq 0$. Also, for $\epsilon = 0$ this is a periodic orbit of the harmonic oscillator contained in the plane (q_2, p_2). Taking into account the above conditions, this solution cannot coincide with the periodic S_1-symmetric solution given in item (a) of Theorem 1.1. For the third solution given in Theorem 4.2, they have that the region of existence of the solution intersects with that given in item (c) of Theorem 1.1, therefore we proceed to compare the analytical approaches of the solutions up to first order.

The periodic solution of Theorems 4.1 and 4.2 according to Averaging Theory are of the form

$$
\begin{aligned}
q_1(t) &= r^* \cos t + O(\epsilon), \quad q_2(t) = \rho^* \sin(t + \alpha^*) + O(\epsilon), \\
p_1(t) &= r^* \sin t + O(\epsilon), \quad p_2(t) = \rho^* \cos(t + \alpha^*) + O(\epsilon).
\end{aligned} \tag{50}
$$

Since we want to compare the solutions obtained in [7], it is necessary to obtain an approximation in epsilon order with greater precision. For this purpose, we will use the results of Appendix 5.2 that consist essentially in obtaining an approximation in order ϵ of the initial condition and the periodic solution obtain by Averaging Theory. For which we will consider that the system (4) in coordinates (49) assumes the form

$$
\begin{aligned}
\dot{r} &= -\,2r\epsilon \sin\theta \cos\theta \left(2ar^2 \cos^2\theta + bg^2 \cos^2(\alpha + \theta)\right), \\
\dot{\rho} &= -\,2\rho\epsilon \sin(\alpha + \theta) \cos(\alpha + \theta) \left(br^2 \cos^2\theta + 2c\rho^2 \cos^2(\alpha + \theta)\right), \\
\dot{\theta} &= -\,1 - 2\epsilon \cos^2\theta \left(2ar^2 \cos^2\theta + b\rho^2 \cos^2(\alpha + \theta)\right),
\end{aligned} \tag{51}
$$

$$\dot{\alpha} =2\epsilon\left(2ar^2\cos^4\theta + b\cos^2\theta\left(\rho^2 - r^2\right)\cos^2(\alpha+\theta) - 2c\rho^2\cos^4(\alpha+\theta)\right),$$

which verifies that

$$\mathcal{H} = \frac{1}{2}\left(\rho^2 + r^2\right) + \epsilon\left(ar^4\cos^4\theta + b\rho^2r^2\cos^2\theta\cos^2(\alpha+\theta) + c\rho^4\cos^4(\alpha+\theta)\right), \tag{52}$$

is a first integral. Now, considering θ as the new dependent variable and setting the energy level $h>0$, solving the equation $\mathcal{H}(r,\rho,\theta,\alpha)=h$ for ρ we obtain a new system in variables (r,α) and 2π-periodic in θ,

$$\begin{aligned} r' &=\epsilon F_0^{(1)}(\theta,r,\alpha) + \epsilon^2 F_1^{(2)}(\theta,r,\alpha) + O(\epsilon^3),\\ \alpha' &=\epsilon F_0^{(2)}(\theta,r,\alpha) + \epsilon^2 F_1^{(2)}(\theta,r,\alpha) + O(\epsilon^3), \end{aligned} \tag{53}$$

where

$$\begin{aligned} F_0^{(1)}(\theta,r,\alpha) &=2r\sin\theta\cos\theta\left(2ar^2\cos^2\theta + b\left(2h-r^2\right)\cos^2(\alpha+\theta)\right),\\ F_0^{(2)}(\theta,r,\alpha) &=-4\left(ar^2\cos^4\theta + b\left(h-r^2\right)\cos^2\theta\cos^2(\alpha+\theta) + c\left(r^2-2h\right)\cos^4(\alpha+\theta)\right),\\ F_1^{(1)}(\theta,r,\alpha) &=-r\sin\theta\cos^3\theta\left(r^2(b-2a) - 2ar^2\cos 2\theta + b\left(r^2-2h\right)\cos(2(\alpha+\theta)) - 2bh\right)^2,\\ F_1^{(2)}(\theta,r,\alpha) &=4\left(4ar^2\cos^4\theta - 2b\left(r^2-2h\right)\cos^2\theta\cos^2(\alpha+\theta)\right)\left(ar^2\cos^4\theta + \right.\\ &\qquad \left. b\left(h-r^2\right)\cos^2\theta\cos^2(\alpha+\theta) + c\left(r^2-2h\right)\cos^4(\alpha+\theta)\right). \end{aligned} \tag{54}$$

Note that they apply Averaging Theory to system (53) and according to the notation used in (63) we have that $F_0 = (F_0^{(1)}, F_0^{(2)})$ and $F_1 = (F_1^{(1)}, F_1^{(2)})$. Thus, the average system (64) assumes the form

$$\begin{aligned} r' &=\frac{\epsilon}{4} br\sin 2\alpha\left(r^2-2h\right),\\ \alpha' &=\frac{\epsilon}{2}\left(-3r^2(a+c) + b\cos 2\alpha\left(r^2-h\right) + 2b\left(r^2-h\right) + 6ch\right). \end{aligned} \tag{55}$$

Applying the Averaging Theorem 5.1, the authors in [7] obtain four families of periodic solutions given in Theorem 4.2. Following the notation given in the Appendix, we have that the functions $g(\theta,r,\alpha) = (g^{(1)}(\theta,r,\alpha), g^{(2)}(\theta,r,\alpha))$ are such that

$$\begin{aligned} g^{(1)}(\theta,r,\alpha) =&\frac{1}{4}r\left[-b\left(r^2-2h\right)\left((9ar^2 + 4b(h-r^2))\cos\theta + ar^2\cos 3\theta + 5bh\cos(2\alpha+\theta) + \right.\right.\\ & bh\cos(2\alpha+3\theta) - 5br^2\cos(2\alpha+\theta) - br^2\cos(2\alpha+3\theta) - ch\cos(2\alpha+\theta) -\\ & 2ch\cos(4\alpha+\theta) - 2ch\cos(4\alpha+3\theta) + 8cr^2\cos(2\alpha+\theta) + cr^2\cos(4\alpha+\theta) +\\ & \left. cr^2\cos(4\alpha+3\theta)\right)\sin 2\theta\sin(2\alpha+2\theta) - 4\left(r^2(b-2a) - 2ar^2\cos 2\theta + \right.\\ & \left. b(r^2-2h)\cos(2\alpha+2\theta) - 2bh\right)^2\cos^3\theta - 2\sin\theta\cos\theta\left(6ar^2\cos^2(\theta) + \right.\\ & \left. b\left(2h-3r^2\right)\cos^2(\alpha+\theta)\right)\left(\left(-5ar^2 - 4bh + 2br^2\right)\sin\theta - ar^2\sin 3\theta + \right.\\ & \left.\left. b\left(r^2-2h\right)\cos\theta\sin(2\alpha+2\theta)\right)\right]\sin\theta, \end{aligned}$$

$$
\begin{aligned}
g^{(2)}(\theta,r,\alpha) = {} & 4\left(4ar^2\cos^4\theta - 2b(r^2-2h)\cos^2\theta\cos^2(\alpha+\theta)\right)\left(ar^2\cos^4\theta + \right.\\
& \left. b\left(h-r^2\right)\cos^2\theta\cos^2(\alpha+\theta) + c\left(r^2-2h\right)\cos^4(\alpha+\theta)\right) - \\
& 2\sin\theta\sin(\alpha+\theta)\cos(\alpha+\theta)\left(b\left(h-r^2\right)\cos^2\theta + 2c\left(r^2-2h\right)\cos^2(\alpha+\theta)\right)\left(\cos\theta\left(9ar^2 + \right.\right.\\
& \left. 4b\left(h-r^2\right)\right) + ar^2\cos 3\theta + 5bh\cos(2\alpha+\theta) + bh\cos(2\alpha+3\theta) - 5br^2\cos(2\alpha+\theta) - \\
& br^2\cos(2\alpha+3\theta) - 16ch\cos(2\alpha+\theta) - 2ch\cos(4\alpha+\theta) - 2ch\cos(4\alpha+3\theta) + \\
& \left. 8cr^2\cos(2\alpha+\theta) + cr^2\cos(4\alpha+\theta) + cr^2\cos(4\alpha+3\theta)\right) + 2r^2\sin\theta\left(a\cos^4\theta - \right.\\
& \left. b\cos^2\theta\cos^2(\alpha+\theta) + c\cos^4(\alpha+\theta)\right)\left(\sin\theta\left(-5ar^2 - 4bh + 2br^2\right) - ar^2\sin 3\theta + \right.\\
& \left. b(r^2-2h)\cos\theta\sin(2(\alpha+\theta))\right).
\end{aligned}
$$

For $f_0(r,\alpha) = (f_0^{(1)}(r,\alpha), f_0^{(2)}(r,\alpha))$ and $f_1(r,\alpha) = (f_1^{(1)}(r,\alpha), f_1^{(2)}(r\alpha))$ we have that

$$
\begin{aligned}
f_0^{(1)}(r,\alpha) = {} & \frac{1}{4}br\left(r^2-2h\right)\sin 2\alpha, \\
f_0^{(2)}(r,\alpha) = {} & \frac{1}{2}\left[-3r^2(a+c) + b\left(r^2-h\right) + 2b\left(r^2-h\right)\cos 2\alpha + 6ch\right],
\end{aligned}
$$

$$
\begin{aligned}
f_1^{(1)}(r,\alpha) = {} & \frac{1}{64}br\left[68ahr^2 - 12ar^4 - \left(2h-r^2\right)\left(b\left(14h-19r^2\right)\cos 2\alpha + 32c\left(r^2-2h\right)\right) + \right.\\
& \left. 4b\left(12h^2 - 8hr^2 + r^4\right) + 24ch^2 + 4c\left(r^2-2h\right)^2\cos 4\alpha - 24chr^2 + 6cr^4\right]\sin 2\alpha, \\
f_1^{(2)}(r,\alpha) = {} & \frac{1}{32}\left[2\left(57a^2r^4 - 4b\left(-18ahr^2 + 11ar^4 + 8ch^2 + 4chr^2 - 4cr^4\right) - 72achr^2 + 6acr^4 + \right.\right.\\
& 2b^2\left(16h^2 - 19hr^2 + 5r^4\right) + bc\left(2h^2 - 3hr^2 + r^4\right)\cos 6\alpha - 136c^2h^2 + 136c^2hr^2 - \\
& \left. 34c^2r^4\right) + \left(b\left(76ahr^2 - 49ar^4 - 48ch^2 + 34chr^2 - 5cr^4\right) + 32acr^4 + \right.\\
& \left. 4b^2\left(12h^2 - 14hr^2 + 3r^4\right)\right)\cos 2\alpha + \left(2acr^4 + b^2\left(-6h^2 + 16hr^2 - 9r^4\right) + \right.\\
& \left.\left. 16bc\left(2h^2 - 3hr^2 + r^4\right)\right)\cos 4\alpha\right],
\end{aligned}
$$

and for $w_0(\theta,r,\alpha) = (w_0^{(1)}(\theta,r,\alpha), w_0^{(2)}(\theta,r,\alpha))$ and $w_1(\theta,r,\alpha) = (w_1^{(1)}(\theta,r,\alpha), w_1^{(2)}(\theta,r,\alpha))$ whose expressions can be found in Appendix 5.3.

Considering the general solution of the system (53) as in (76) and the general solution of (55) as in (77), with initial condition as in (79), where

$$
p = (r^*, \alpha^*) = \left(\frac{\sqrt{(6c-b)h}}{\sqrt{3a-b+3c}}, \frac{\pi}{2}\right),
$$

and η_1 is given in (80), therefore,

$$
\begin{aligned}
\eta = {} & \left(\frac{\sqrt{(6c-b)h}}{\sqrt{3a-b+3c}}, \frac{\pi}{2}\right) + \epsilon\left(\frac{h^2}{32(3a-b+3c)^{5/2}\sqrt{h(6c-b)}}\left(36a^2c(11b-26c) + \right.\right.\\
& \left.\left. a\left(-19b^3 + 54b^2c + 420bc^2 - 3240c^3\right) + b^2\left(5b^2 - 57bc + 202c^2\right)\right), 0\right) + O(\epsilon^2).
\end{aligned}
$$

Finally, according to the equation (81), we have that the 2π-periodic solution

$$
x(\theta,\eta,\epsilon) = (r(\theta,\eta,\epsilon), \alpha(\theta,\eta,\epsilon)
$$

where

(56)

$$
\begin{aligned}
r(\theta,\eta,\epsilon) =&\frac{\sqrt{h(6c-b)}}{\sqrt{3a-b+3c}}+\epsilon\frac{h^2}{32(3a-b+3c)^{5/2}\sqrt{h(6c-b)}}\left(12a^2\left(-4b^2+27bc+102c^2\right)+\right.\\
&2(b-6c)(-3a+b-3c)\left(4\cos 2\theta\left(b^2-4a(b+3c)\right)-\cos 4\theta\left(-8ab+12ac+b^2\right)\right)+\\
&\left.3a\left(5b^3-26b^2c-124bc^2-360c^3\right)-b^2\left(b^2+3bc-94c^2\right)\right),\\
\alpha(\theta,\eta,\epsilon) =&\frac{\pi}{2}+\epsilon\frac{1}{3a-b+3c}\left(h(b(a-c)\cos 2\theta+a(b-12c)+bc)\right)\sin 2\theta+O(\varepsilon^2).
\end{aligned}
$$

Since our objective is to compare the solutions, we will restrict the parameters a,b,c to $b^2=36ac$, $c<0$ and $b>0$, so (56) reads as

$$
\begin{aligned}
r(\theta,\eta,\epsilon) =&2\sqrt{3}\sqrt{\frac{ch}{6c-b}}+\epsilon\frac{b^2h^{3/2}}{(b-6c)^3}\sqrt{\frac{c}{3(6c-b)}}\left[b^2-2(b-6c)^2\cos 2\theta+(b-6c)^2\cos 4\theta-\right.\\
&\left.6bc-48c^2\right]+O(\epsilon^2),\\
\alpha(\theta,\eta,\epsilon) =&\frac{\pi}{2}+\frac{bh}{3(b-6c)}((b+6c)\cos 2\theta+b-6c)\sin 2\theta+O(\epsilon^2).
\end{aligned} \tag{57}
$$

As we already have the approximation of the 2π-periodic solution in time θ of system (53), we will proceed to recover the solution T-periodic in time t of the system (4), for this, we use the fact that we considered an energy level $h>0$ and solve the equation

$$\mathcal{H}(r(\theta,\eta,\epsilon),\rho(\theta,\eta,\epsilon),\alpha(\theta,\eta,\epsilon),\theta)=h,$$

with $\mathcal{H}$ as in (52) for $\rho(\theta,\eta,\epsilon)$ we have that

$$\rho(\theta,\eta,\epsilon)=\sqrt{2}\sqrt{\frac{bh}{b-6c}}-2\epsilon\sqrt{2}c\left(\frac{bh}{b-6c}\right)^{3/2}\cos 4\theta+O(\epsilon^2). \tag{58}$$

After that, we go back to time t using the equation $\dot\theta$ of system (51), replacing $r=r(\theta,\eta,\epsilon)$, $\alpha=\alpha(\theta,\eta,\epsilon)$, and $\rho=\rho(\theta,\eta,\epsilon)$ and expanding in Taylor series up to first order we get

$$
\begin{aligned}
F(\theta,\epsilon) &= \dot\theta(r(\theta,\eta,\epsilon),\alpha(\theta,\eta,\epsilon),\rho(\theta,\eta,\epsilon))\\
&=-1+\epsilon\frac{h}{b-6c}\left(48ac\cos^4\theta-b^2\sin^2 2\theta\right)+O(\epsilon^2).
\end{aligned} \tag{59}
$$

Integrating in the previous equality with respect to θ and using the Inverse Function Theorem we get that

$$\theta(t,\eta,\epsilon)=-t+\epsilon\frac{1}{3(b-6c)}4b^2h\sin t\cos^3 t+O(\epsilon^2). \tag{60}$$

Finally, replacing $\theta=\theta(t,\eta,\epsilon)$ in $r(\theta,\eta,\epsilon),\rho(\theta,\eta,\epsilon),\alpha(\theta,\eta,\epsilon)$, using the change of variables (49) and expanding in Taylor series around $\epsilon=0$ we arrive to

$$
\begin{aligned}
q_1(t,z,\epsilon) =& 2\sqrt{3}\sqrt{\frac{ch}{6c-b}}\cos t - \epsilon\frac{2b^2h}{\sqrt{3}(b-6c)^3}\sqrt{\frac{ch}{6c-b}}\left[(b-6c)^2\cos 2t - b^2 + \right.\\
& \left. 9bc + 6c^2\right]\cos t + O(\epsilon^2),\\
q_2(t,z,\epsilon) =& \sqrt{2}\sqrt{\frac{bh}{b-6c}}\sin t - 2\epsilon\left[\sqrt{2}c\left(\frac{bh}{b-6c}\right)^{3/2}\sin t\cos^2 2t\right] + O(\epsilon^2),\\
p_1(t,z,\epsilon) =& -2\sqrt{3}\sqrt{\frac{ch}{6c-b}}\sin t + \epsilon\frac{2b^2h}{\sqrt{3}(b-6c)^3}\sqrt{\frac{ch}{6c-b}}\left[3(b-6c)^2\cos 2t + b^2 - \right.\\
& \left. 15bc + 78c^2\right]\sin t + O(\epsilon^2),\\
p_2(t,z,\epsilon) =& \sqrt{2}\sqrt{\frac{bh}{b-6c}}\cos t - \epsilon\frac{bch}{b-6c}\sqrt{\frac{bh}{2b-12c}}(-2\cos t + 5\cos 3t + \cos 5t) + O(\epsilon^2),
\end{aligned}
\tag{61}
$$

where the initial condition is

$$
\begin{aligned}
z =& \left(2\sqrt{3}\sqrt{\frac{ch}{6c-b}}, 0, 0, \sqrt{2}\sqrt{\frac{bh}{b-6c}})\right) + \epsilon\left(-\frac{2\sqrt{3}b^2}{(b-6c)^2}\left(\frac{ch}{6c-b}\right)^{3/2}(b-14c), 0, 0,\right.\\
& \left. -\frac{4bch}{b-6c}\sqrt{\frac{bh}{2b-12c}}\right) + O(\epsilon^2).
\end{aligned}
\tag{62}
$$

Now, we will proceed to compare the initial conditions of the periodic solutions obtained in our Theorem 1.1, item (c) with those obtained in Theorem 4.2, item (c). The first thing to recall is that the initial condition of our theorem in cartesian coordinates is given by

$$
\begin{aligned}
z^{(1)} =& \left(2\sqrt{3}\sqrt{\tfrac{c(h+\Delta L)}{6c-b}} + O(\epsilon^2), 0, 0, \sqrt{2}c\sqrt{\tfrac{b(b-6c)^3((h+\Delta L)}{c^2(b-6c)}} + O(\epsilon^2)\right)\\
=& \left(2\sqrt{3}\sqrt{\tfrac{ch}{6c-b}} + \frac{\sqrt{3}\Delta L\sqrt{\frac{ch}{6c-b}}}{h} + O(\Delta L, \epsilon^2), 0, 0, \frac{\sqrt{2}\sqrt{bh}}{\sqrt{b-6c}} + \frac{b\Delta L}{\sqrt{2}\sqrt{b-6c}\sqrt{bh}} + O(\Delta L, \epsilon^2)\right).
\end{aligned}
$$

Then, since we have assumed that $z^{(1)} \in \mathcal{L}_1$ it follows that the second and third components are identically null, however, the initial condition in (62) does not necessarily happen. On the other hand, our initial condition $z^{(1)}$ depends on the parameters $(\epsilon, \Delta L)$ as opposed to (62) that depends only on ϵ.

On the other hand, the approximation of the family of periodic solutions obtained in (7) by us differs in order ϵ from those characterized in (61).

In addition we characterize the stability which was not performed by the authors in [7].

5. Appendix

5.1. **Averaging theory of first order.** We shall present the basic results from averaging theory that we need to prove the results of this paper.

The next theorem provides a first order approximation for the periodic solutions of a periodic differential system. Consider the differential equation

$$
\dot{x} = \epsilon F_0(t,x) + \epsilon^2 F_1(t,x,\epsilon), \quad x(0) = x_0 \tag{63}
$$

with $x \in D$, where D is an open subset of $\mathbb{R}^n$, $t \geq 0$. Moreover, we assume that both $F_0(t,x)$ and $F_1(t,x,\epsilon)$ are $T-$periodic in t. We also consider in D the averaged differential equation

$$\dot{y} = \epsilon f_0(y), \quad y(0) = x_0, \tag{64}$$

where

$$f_0(y) = \frac{1}{T}\int_0^T F_0(t,y)dt.$$

Under certain conditions, equilibrium solutions of the averaged equation turn out to correspond with $T-$periodic solutions of equation (63).

Theorem 5.1. *Consider the two initial value problems* (63) *and* (64). *Suppose:*

(i) F_0*, its Jacobian* $\partial F_0/\partial x$*, its Hessian* $\partial^2 F_0/\partial x^2$*,* F_1 *and its Jacobian* $\partial F_1/\partial x$ *are defined, continuous and bounded by a constant independent of* ϵ *in* $[0,\infty)\times D$ *and* $\epsilon \in (0,\epsilon_0]$.

(ii) F_0 *and* F_1 *are* $T-$*periodic in* t *(*T *independent of* ϵ*).*

Then the following statements hold.

(a) *If* p *is an equilibrium point of the averaged equation* (64) *and*

$$\det\left(\frac{\partial f_0}{\partial y}\right)\bigg|_{y=p} \neq 0, \tag{65}$$

then there exists a $T-$*periodic solution* $\varphi(t,\epsilon)$ *of equation* (63) *such that* $\varphi(0,\epsilon)\to p$ *as* $\epsilon \to 0$.

(b) *The stability or instability of the limit cycle* $\varphi(t,\epsilon)$ *is given by the stability or instability of the equilibrium point* p *of the averaged system* (64). *In fact the singular point* p *has the stability behavior of the Poincaré map associated to the limit cycle* $\varphi(t,\epsilon)$.

5.2. **Approximation of the solutions via averaging.** We consider the system,

$$\dot{x} = \epsilon F_0(t,x) + \epsilon^2 F_1(t,x) + O(\epsilon^3) \tag{66}$$

and let us use change of coordinates,

$$x = y + \epsilon w(t,y,\epsilon) = y + \epsilon w_0(t,y) + \epsilon^2 w_1(t,y) + O(\epsilon^3), \tag{67}$$

where w is a $T-$periodic function. Substituting (67) in F_0 and expanding in a Taylor series around $\epsilon = 0$,

$$\begin{aligned} F_0(t,x) = \;& F_0(t, y + \epsilon w_0(t,y)) \\ = \;& F_0(t,y) + \epsilon D_x F_0(t,y) w_0(t,y) + \tfrac{\epsilon^2}{2}\left[2 D_x F_0(t,y) w_1(t,y)\right. \\ & \left. + w_0(t,y)^T Hess F_0(t,y) w_0(t,y)\right] + O(\epsilon^3). \end{aligned} \tag{68}$$

Analogously,

$$\begin{aligned} F_1(t,x) = \;& F_1(t, y + \epsilon w_0(t,y) + \epsilon^2 w_1(t,y) + O(\epsilon^3)) \\ = \;& F_1(t,y) + \epsilon \tfrac{\partial F_1}{\partial x}(t,y) w_0(t,y) + O(\epsilon^2). \end{aligned} \tag{69}$$

Next, differentiating (67) with respect to t, we get

$$\dot{x} = \left(I + \epsilon \frac{\partial w_0}{\partial y}(t,y) + \epsilon^2 \frac{\partial w_1}{\partial y}(t,y)\right)\dot{y} + \epsilon \frac{\partial w_0}{\partial t}(t,y) + \epsilon^2 \frac{\partial w_1}{\partial t}(t,y) + O(\epsilon^3).$$

Using (68) and (69) we obtain

(70)
$$\begin{aligned}
\dot{y} = &\left(I + \epsilon \tfrac{\partial w_0}{\partial y}(t,y) + \epsilon^2 \tfrac{\partial w_1}{\partial y}(t,y))\right)^{-1} (\epsilon F_0(t,y) + \epsilon^2 F_1(t,y) \\
&-\epsilon \tfrac{\partial w_0}{\partial t}(t,y) - \epsilon^2 \tfrac{\partial w_1}{\partial t}(t,y) + O(\epsilon^3)\Big) \\
= &\left(I - \epsilon \tfrac{\partial w_0}{\partial y}(t,y) + O(\epsilon^2)\right) \Big[\epsilon \left(F_0(t,y) - \tfrac{\partial w_0}{\partial t}(t,y)\right) \\
&+\epsilon^2 \left(D_x F_0(t,y) w_0(t,y) + F_1(t,y) - \tfrac{\partial w_1}{\partial t}(t,y)\right) + O(\epsilon^3)\Big] \\
= &\ \epsilon \left(F_0(t,y) - \tfrac{\partial w_0}{\partial t}(t,y)\right) + \epsilon^2 \Big(D_x F_0(t,y) w_0(t,y) - \tfrac{\partial w_1}{\partial t}(t,y) \\
&+F_1(t,y) - \tfrac{\partial w_0}{\partial y}(t,y) \left(F_0(t,y) - \tfrac{\partial w_0}{\partial t}(t,y)\right)\Big) + O(\epsilon^3) \\
= &\ \epsilon f_0(y) + \epsilon^2 f_1(t,y) + O(\epsilon^3),
\end{aligned}$$

where,

(71)
$$\begin{aligned}
f_0(y) = &\ F_0(t,y) - \tfrac{\partial w_0}{\partial t}(t,y), \\
f_1(t,y) = &\ D_x F_0(t,y) w_0(t,y) + F_1(t,y) - \tfrac{\partial w_1}{\partial t}(t,y) - \tfrac{\partial w_0}{\partial y}(t,y) f_0(y).
\end{aligned}$$

Remark 2. *From the first equation in* (71)*, we have*

$$\frac{\partial w_0}{\partial t}(t,y) = F_0(t,y) - f_0(y),$$

thus

(72)
$$w_0(t,y) = \int_0^t F_0(s,y)ds - f_0(y)t = \int_0^t F_0(s,y)ds - \frac{t}{T}\int_0^T F_0(s,y)ds.$$

Remark 3. *From the second equation in* (71)*, we have,*

(73)
$$f_1(t,y) = D_x F_0(t,y) w_0(t,y) + F_1(t,y) - \frac{\partial w_1}{\partial t}(t,y) - \frac{\partial w_0}{\partial y}(t,y) f_0(y).$$

We define the following auxiliary function

(74)
$$g(t,y) = D_x F_0(t,y) w_0(t,y) + F_1(t,y) - \frac{\partial w_0}{\partial y}(t,y) f_0(y).$$

If we assume that $f_1(t,y) = f_1(y)$ is a function that depends only on y, we have that

$$f_1(y) = \frac{1}{T}\int_0^T g(t,y)dt$$

and so

$$\frac{\partial w_1}{\partial t}(t,y) = g(t,y) - f_1(y).$$

Therefore,

(75)
$$w_1(t,y) = \int_0^t g(s,y)ds - f_1(y)t.$$

Let

(76)
$$x(t,\xi,\epsilon) = x_0(t) + \epsilon x_1(t) + O(\epsilon^2),$$

be the general solution of (63) with initial condition ξ, and let

(77)
$$y(t,\eta,\epsilon) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + O(\epsilon^3),$$

be the general solution of (70) with initial condition η. We have that,

$$\eta = y(0,\eta,\epsilon) = y_0(0) + \epsilon y_1(0) + \epsilon^2 y_2(0) + O(\epsilon^3),$$

and by (70) and using a Taylor series around $\epsilon = 0$,

$$\begin{aligned}\dot{y}_0(t) + \epsilon\dot{y}_1(t) + \epsilon^2\dot{y}_2(t) + O(\epsilon^3) = &\ \dot{y} = \epsilon f_0(y) + \epsilon^2 f_1(y) + O(\epsilon^3)\\ = &\ \epsilon f_0(y_0(t) + \epsilon y_1(t) + O(\epsilon^2))\\ &+\epsilon^2 f_1(y_0(t) + \epsilon y_1(t) + O(\epsilon^2)) + O(\epsilon^3)\\ = &\ \epsilon f_0(y_0) + \epsilon^2\left(D_y f_0(y_0) y_1 + f_1(y_0)\right) + O(\epsilon^2).\end{aligned}$$

Thus,

$$\begin{aligned}\dot{y}_0(t) = &\ 0,\\ \dot{y}_1(t) = &\ f_0(y_0),\\ \dot{y}_2(t) = &\ D_y f_0(y_0) y_1 + f_1(y_0),\end{aligned}$$

and it follows that

$$\begin{aligned}y_0(t) = &\ z,\\ y_1(t) = &\ f_0(z)t,\\ y_2(t) = &\ \tfrac{t^2}{2} D_y f_0(z) f_0(z) + f_1(z)t.\end{aligned}$$

So, the general solution (77) is written as

$$y(t,\eta,\epsilon) = \eta + \epsilon f_0(\eta)t + \epsilon^2\left(\frac{t^2}{2} D_y f_0(\eta) f_0(\eta) + f_1(\eta)t\right) + O(\epsilon^3) \tag{78}$$

Now, we assume that $y(t,\eta,\epsilon)$ is a particular $T-$periodic solution of the system (70) where the initial condition

$$\eta = \eta(\epsilon) = p + \epsilon\eta_1 + \epsilon^2\eta_2 + O(\epsilon^3), \tag{79}$$

where p satisfies $f_0(p) = 0$ and $D_y f_0(p)$ is nondegenerate.
Now, we consider the $T-$periodic solution $y(t,\eta(\epsilon),\epsilon)$ obtained by Theorem 5.1 of the system (70). Our next aim is to describe the approximation of this family of initial conditions, that is, we are going to characterize η_1.

Because of the periodicity of the solution $y(t,\eta(\epsilon),\epsilon)$ and using (78), we must have

$$\begin{aligned}0 = &\ y(T,\eta(\epsilon),\epsilon) - \eta(\epsilon)\\ = &\ \epsilon T f_0(\eta(\epsilon)) + \epsilon^2\left[\tfrac{T^2}{2} D_y f_0(\eta(\epsilon)) f_0(\eta(\epsilon)) + f_1(\eta(\epsilon))T\right] + O(\epsilon^3).\end{aligned}$$

Or equivalently, developing in a Taylor series around $\epsilon = 0$, we get

$$\begin{aligned}0 = &\ f_0(\eta(\epsilon)) + \epsilon\left[\tfrac{T}{2} D_y f_0(\eta(\epsilon)) f_0(\eta(\epsilon)) + f_1(\eta(\epsilon))\right] + O(\epsilon^2)\\ = &\ f_0(p + \epsilon\eta_1 + \epsilon^2\eta_2 + O(\epsilon^3)) +\\ &+\epsilon\left[\tfrac{T}{2} D_y f_0(p + \epsilon\eta_1 + \epsilon^2\eta_2 + O(\epsilon^3)) f_0(p + \epsilon\eta_1 + \epsilon^2\eta_2 + O(\epsilon^3))\right.\\ &\left.+f_1(p + \epsilon\eta_1 + \epsilon^2\eta_2 + O(\epsilon^3))\right] + O(\epsilon^2)\\ = &\ f_0(p) + \epsilon\left[D_y f_0(p)\eta_1 + \tfrac{T}{2} D_y f_0(p) f_0(p) + f_1(p)\right] + O(\epsilon^2),\end{aligned}$$

from where,

$$\eta_1 = -[D_y f_0(p)]^{-1} f_1(p). \tag{80}$$

Coming back to the associated $T-$periodic solution $x(t,\eta(\epsilon),\epsilon)$ of system (63), we arrive to

$$
\begin{aligned}
x(t,\eta(\epsilon),\epsilon) = \;& y(t,\eta(\epsilon),\epsilon) + \epsilon w_0(t, y(t,\eta(\epsilon),\epsilon)) + O(\epsilon^2)\\
= \;& \eta(\epsilon) + \epsilon f_0(\eta(\epsilon))t + \epsilon^2 \left(\tfrac{t^2}{2} D_y f_0(\eta(\epsilon)) f_0(\eta(\epsilon)) + f_1(\eta(\epsilon))t\right)\\
& +\epsilon w_0 \Big(t, \eta(\epsilon) + \epsilon f_0(\eta(\epsilon))t + \epsilon^2 \Big(\tfrac{t^2}{2} D_y f_0(\eta(\epsilon)) f_0(\eta(\epsilon))\\
& +f_1(\eta(\epsilon))t) + O(\epsilon^3)\Big) + O(\epsilon^2)\\
= \;& p + \epsilon\eta_1 + O(\epsilon^2) + \epsilon f_0(\eta(\epsilon))t\\
& +\epsilon^2 \left(\tfrac{t^2}{2} D_y f_0(\eta(\epsilon)) f_0(\eta(\epsilon)) + f_1(\eta(\epsilon))t\right) + O(\epsilon^3)\\
& +\epsilon w_0 \big(t, p + \epsilon\eta_1 + O(\epsilon^2) + \epsilon f_0(\eta(\epsilon))t\\
& +\epsilon^2 \left(\tfrac{t^2}{2} D_y f_0(\eta(\epsilon)) f_0(\eta(\epsilon)) + f_1(\eta(\epsilon))t\right) + O(\epsilon^3)\big)\\
= \;& p + \epsilon\left[-[D_y f_0(p)]^{-1} f_1(p) + w_0(t,p)\right] + O(\epsilon^2).
\end{aligned} \tag{81}
$$

5.3. **Description of the functions w_0 and w_1.**

$$
w_0^{(1)}(\theta,r,\alpha) = -\frac{1}{4} r \left[\left(-5ar^2 - 4bh + 2br^2\right)\sin\theta - ar^2\sin 3\theta + b\left(r^2-2h\right)\sin(2\alpha+2\theta)\cos\theta\right]\sin\theta,
$$

$$
\begin{aligned}
w_0^{(2)}(\theta,r,\alpha) = -\frac{1}{4}\big[&\left(9ar^2 + 4b\left(h-r^2\right)\right)\cos\theta + ar^2\cos 3\theta + 5bh\cos(2\alpha+\theta) + bh\cos(2\alpha+3\theta)-\\
&5br^2\cos(2\alpha+\theta) - br^2\cos(2\alpha+3\theta) - 16ch\cos(2\alpha+\theta) - 2ch\cos(4\alpha+\theta)-\\
&2ch\cos(4\alpha+3\theta) + 8cr^2\cos(2\alpha+\theta) + cr^2\cos(4\alpha+\theta) + cr^2\cos(4\alpha+3\theta)\big]\sin\theta,
\end{aligned}
$$

$$
\begin{aligned}
w_1^{(1)}(\theta,r,\alpha) = -\frac{1}{1536} r \big[&2052a^2r^4\sin\theta - 800abr^4\sin\theta + 576bcr^4\sin\theta + 1956a^2r^4\sin 3\theta+\\
&384b^2r^4\sin 3\theta - 328abr^4\sin 3\theta - 192bcr^4\sin 3\theta + 756a^2r^4\sin 5\theta - 272abr^4\sin 5\theta+\\
&84a^2r^4\sin 7\theta - 280b^2r^4\sin(2\alpha+\theta) + 408abr^4\sin(2\alpha+\theta) + 24bcr^4\sin(2\alpha+\theta)+\\
&36bcr^4\sin(6\alpha+3\theta) - 339b^2r^4\sin(4\alpha+\theta) + 448bcr^4\sin(4\alpha+\theta)+\\
&36bcr^4\sin(6\alpha+\theta) + 248b^2r^4\sin(2\alpha+3\theta) - 648abr^4\sin(2\alpha+3\theta)+\\
&24bcr^4\sin(2\alpha+3\theta) - 51b^2r^4\sin(4\alpha+3\theta) + 64bcr^4\sin(4\alpha+3\theta)+\\
&200b^2r^4\sin(2\alpha+5\theta) - 624abr^4\sin(2\alpha+5\theta) + 129b^2r^4\sin(4\alpha+5\theta)-\\
&128bcr^4\sin(4\alpha+5\theta) - 12bcr^4\sin(6\alpha+5\theta) - 96abr^4\sin(2\alpha+7\theta)+\\
&33b^2r^4\sin(4\alpha+7\theta) - 12bcr^4\sin(6\alpha+7\theta) - 864b^2hr^2\sin\theta + 3264abhr^2\sin\theta-\\
&2304bchr^2\sin\theta - 1248b^2hr^2\sin 3\theta + 2208abhr^2\sin 3\theta + 768bchr^2\sin 3\theta+\\
&480abhr^2\sin 5\theta + 512b^2r^2h\sin 2\alpha+\theta) - 440abhr^2\sin(2\alpha+\theta) - 96bchr^2\sin(2\alpha+\theta)-\\
&144bchr^2\sin(6\alpha+3\theta) + 1060b^2hr^2\sin(4\alpha+\theta) - 1792bchr^2\sin(4\alpha+\theta)-\\
&144bchr^2\sin(6\alpha+\theta) - 1024b^2hr^2\sin(2\alpha+3\theta) + 1672abhr^2\sin(2\alpha+3\theta)-\\
&96bchr^2\sin(2\alpha+3\theta) + 100b^2hr^2\sin(4\alpha+3\theta) - 256bchr^2\sin(4\alpha+3\theta)-\\
&640b^2hr^2\sin(2\alpha+5\theta) + 1096abhr^2\sin(2\alpha+5\theta) - 428b^2hr^2\sin(4\alpha+5\theta)+\\
&512bchr^2\sin(4\alpha+5\theta) + 48bchr^2\sin(6\alpha+5\theta) + 168abhr^2\sin(2\alpha+7\theta)-\\
&108b^2hr^2\sin(4\alpha+7\theta) + 48bchr^2\sin(6\alpha+7\theta) - 24br^2\Big(a\left(32h-25r^2\right)-\\
&c\left(r^2-2h\right)^2 + b\left(3r^4 - 16hr^2 + 20h^2\right)\Big)\sin(2\alpha-\theta) + 1728b^2h^2\sin\theta+
\end{aligned}
$$

$$2304bch^2\sin\theta + 960b^2h^2\sin 3\theta - 768bch^2\sin 3\theta + 96b^2h^2\sin(2\alpha+\theta)+$$
$$96bch^2\sin(2\alpha+\theta) + 144bch^2\sin(6\alpha+3\theta) - 764b^2h^2\sin(4\alpha+\theta)+$$
$$1792bch^2\sin(4\alpha+\theta) + 144bch^2\sin(6\alpha+\theta) + 1056b^2h^2\sin(2\alpha+3\theta)+$$
$$96bch^2\sin(2\alpha+3\theta) + 4b^2h^2\sin(4\alpha+3\theta) + 256bch^2\sin(4\alpha+3\theta)+$$
$$480b^2h^2\sin(2\alpha+5\theta) + 340b^2h^2\sin(4\alpha+5\theta) - 512bch^2\sin(4\alpha+5\theta)-$$
$$48bch^2\sin(6\alpha+5\theta) + 84b^2h^2\sin(4\alpha+7\theta) - 48bch^2\sin(6\alpha+7\theta) + 24b\left(3ar^4-\right.$$
$$\left.4ahr^2 + c\left(r^2-2h\right)^2\right)\sin(2\alpha-3\theta)\Big]\sin\theta,$$

and

$$w_1^{(2)}(\theta,r,\alpha) = \frac{1}{384}\Big[3522a^2r^4\cos\theta + 744b^2r^4\cos\theta + 792c^2r^4\cos\theta - 2768abr^4\cos\theta+$$
$$1944acr^4\cos\theta - 1248bcr^4\cos\theta + 1026a^2r^4\cos 3\theta + 168b^2r^4\cos 3\theta + 24c^2r^4\cos 3\theta-$$
$$704abr^4\cos 3\theta + 216acr^4\cos 3\theta - 96bcr^4\cos 3\theta + 210a^2r^4\cos 5\theta - 80abr^4\cos 5\theta+$$
$$18a^2r^4\cos 7\theta + 880b^2r^4\cos(2\alpha+\theta) - 288c^2r^4\cos(2\alpha+\theta) - 2235abr^4\cos(2\alpha+\theta)+$$
$$1280acr^4\cos(2\alpha+\theta) - 702bcr^4\cos(2\alpha+\theta) - 32c^2r^4\cos(6\alpha+3\theta)-$$
$$9bcr^4\cos(6\alpha+3\theta) + 151b^2r^4\cos(4\alpha+\theta) - 384c^2r^4\cos(4\alpha+\theta)+$$
$$182acr^4\cos(4\alpha+\theta) + 96bcr^4\cos(4\alpha+\theta) - 128c^2r^4\cos(6\alpha+\theta) + 39bcr^4\cos(6\alpha+\theta)-$$
$$12c^2r^4\cos(8\alpha+\theta) + 400b^2r^4\cos(2\alpha+3\theta) + 192c^2r^4\cos(2\alpha+3\theta)-$$
$$1083abr^4\cos(2\alpha+3\theta)4 + 1088acr^4\cos(2\alpha+3\theta) - 606bcr^4\cos(2\alpha+3\theta)+$$
$$223b^2r^4\cos(4\alpha+3\theta) + 384c^2r^4\cos(4\alpha+3\theta) + 278acr^4\cos(4\alpha+3\theta)-$$
$$624bcr^4\cos(4\alpha+3\theta) - 12c^2r^4\cos(8\alpha+3\theta) + 88b^2r^4\cos(2\alpha+5\theta)-$$
$$321abr^4\cos(2\alpha+5\theta) + 128acr^4\cos(2\alpha+5\theta) + 103b^2r^4\cos(4\alpha+5\theta)+$$
$$254acr^4\cos(4\alpha+5\theta) - 240bcr^4\cos(4\alpha+5\theta) + 160c^2r^4\cos(6\alpha+5\theta)-$$
$$123bcr^4\cos(6\alpha+5\theta) + 12c^2r^4\cos(8\alpha+5\theta)4 - 33abr^4\cos(2\alpha+7\theta)+$$
$$15b^2r^4\cos(4\alpha+7\theta) + 30acr^4\cos(4\alpha+7\theta) - 27bcr^4\cos(6\alpha+7\theta)+$$
$$12c^2r^4\cos(8\alpha+7\theta) - 2136b^2hr^2\cos\theta - 3168c^2hr^2\cos\theta + 3552abhr^2\cos\theta-$$
$$2592achr^2\cos\theta + 3936bchr^2\cos\theta - 408b^2hr^2\cos 3\theta - 96c^2hr^2\cos 3\theta+$$
$$768abhr^2\cos 3\theta - 288achr^2\cos 3\theta + 288bchr^2\cos 3\theta + 96abhr^2\cos 5\theta-$$
$$2560b^2hr^2\cos(2\alpha+\theta) + 1152c^2hr^2\cos(2\alpha+\theta) + 3244abhr^2\cos(2\alpha+\theta)-$$
$$4032achr^2\cos(2\alpha+\theta) + 3264bchr^2\cos(2\alpha+\theta) + 128c^2hr^2\cos(6\alpha+3\theta)+$$
$$64bchr^2\cos(6\alpha+3\theta) - 508b^2hr^2\cos(4\alpha+\theta) + 1536c^2hr^2\cos(4\alpha+\theta)-$$
$$336achr^2\cos(4\alpha+\theta) + 64bchr^2\cos(4\alpha+\theta) + 512c^2hr^2\cos(6\alpha+\theta)-$$
$$80bchr^2\cos(6\alpha+\theta) + 48c^2hr^2\cos(8\alpha+\theta) - 1024b^2hr^2\cos(2\alpha+3\theta)-$$
$$768c^2hr^2\cos(2\alpha+3\theta) + 1324abhr^2\cos(2\alpha+3\theta) - 1728achr^2\cos(2\alpha+3\theta)+$$
$$1872bchr^2\cos(2\alpha+3\theta) - 556b^2hr^2\cos(4\alpha+3\theta) - 1536c^2hr^2\cos(4\alpha+3\theta)-$$

$$
\begin{aligned}
&720achr^2\cos(4\alpha+3\theta)+2080bchr^2\cos(4\alpha+3\theta)+48c^2hr^2\cos(8\alpha+3\theta- \\
&208b^2hr^2\cos(2\alpha+5\theta)+340abhr^2\cos(2\alpha+5\theta)-192achr^2\cos(2\alpha+5\theta)- \\
&244b^2hr^2\cos(4\alpha+5\theta)-432achr^2\cos(4\alpha+5\theta)+736bchr^2\cos(4\alpha+5\theta)- \\
&640c^2hr^2\cos(6\alpha+5\theta)+388bchr^2\cos(6\alpha+5\theta)-48c^2hr^2\cos(8\alpha+5\theta)+ \\
&36abhr^2\cos(2\alpha+7\theta)-36b^2hr^2\cos(4\alpha+7\theta)-48achr^2\cos(4\alpha+7\theta)+84bchr^2\cos(6\alpha+7\theta)- \\
&48c^2hr^2\cos(8\alpha+7\theta)+6\bigl(4\left(r^4-6hr^2+4h^2\right)b^2+\bigl(ar^2\left(44h-53r^2\right)-10c\left(r^4-3hr^2+2h^2\right)\bigr)b+ \\
&16c\Bigl(2a\left(r^2-h\right)r^2+c\left(r^2-2h\right)^2\Bigr)\Bigr)\cos(2\alpha-\theta)+1344b^2h^2\cos\theta+3168c^2h^2\cos\theta-2880bch^2\cos\theta+ \\
&192b^2h^2\cos 3\theta+96c^2h^2\cos 3\theta-192bch^2\cos(3\theta)+1728b^2h^2\cos(2\alpha+\theta)-1152c^2h^2\cos(2\alpha+\theta)- \\
&3720bch^2\cos(2\alpha+\theta)-128c^2h^2\cos(3(2\alpha+\theta))-92bch^2\cos(3(2\alpha+\theta))+370b^2h^2\cos(4\alpha+\theta)- \\
&1536c^2h^2\cos(4\alpha+\theta)-512bch^2\cos(4\alpha+\theta)-512c^2h^2\cos(6\alpha+\theta)+4bch^2\cos(6\alpha+\theta)- \\
&48c^2h^2\cos(8\alpha+\theta)+576b^2h^2\cos(2\alpha+3\theta)+768c^2h^2\cos(2\alpha+3\theta)-1320bch^2\cos(2\alpha+3\theta)+ \\
&322b^2h^2\cos(4\alpha+3\theta)+1536c^2h^2\cos(4\alpha+3\theta)-1664bch^2\cos(4\alpha+3\theta)-48c^2h^2\cos(8\alpha+3\theta)+ \\
&96b^2h^2\cos(2\alpha+5\theta)+130b^2h^2\cos(4\alpha+5\theta)-512bch^2\cos(4\alpha+5\theta)+640c^2h^2\cos(6\alpha+5\theta)- \\
&284bch^2\cos(6\alpha+5\theta)+48c^2h^2\cos(8\alpha+5\theta)+18b^2h^2\cos(4\alpha+7\theta)-60bch^2\cos(6\alpha+7\theta)+ \\
&48c^2h^2\cos(8\alpha+7\theta)-6b\left(5ar^4-4ahr^2+2c\left(r^4-3hr^2+2h^2\right)\right)\cos(2\alpha-3\theta)\bigr]\sin\theta.
\end{aligned}
$$

References

[1] N. Caranicolas, H. Varvoglis. Families of periodic orbits in a quartic potential, *Astron. Astrophys.* **141** (1984), 383-388.

[2] N. Caranicolas. Some properties of an intrinsically degenerate Hamiltonian system, *J Astrophys Astron.* **10** (1989), 197-202.

[3] D. Carrasco, J. Palacián, C. Vidal, J. Vidarte, P. Yanguas. Dynamics of axially symmetric perturbed Hamiltonians in 1:1:1 resonance, *J. Nonlinear Sci.* **28** (2018), no. 4, 1293-1359.

[4] G. Contopoulos, H. Papadaki, C. Polymilis. The structure of chaos in a potential without escapes, *Celestial Mech. Dynam. Astronom.* **60** (1994), no. 2 249-271.

[5] G. Contopoulos. Order and chaos in dynamical astronomy, Astronomy and Astrophysics Library. Springer-Verlag, Berlin, (2002).

[6] L. Jiménez-Lara, J. Llibre. Periodic orbits and nonintegrability of generalized classical Yang-Mills Hamiltonian systems, *J. Math. Phys.* **52** (2011), no. 3, 9 pp.

[7] E. Lacomba, J. Llibre. Dynamics of a galactic Hamiltonian system,*J. Math. Phys.* **53** (2012), no. 7 9pp.

[8] K. Meyer, J. Palacián, P. Yanguas. Singular reduction of resonant Hamiltonians, *Nonlinearity.* **31** (2018), no. 6, 2854-2894.

[9] J. Palacián, C. Vidal, J. Vidarte, P. Yanguas. Periodic Solutions and KAM Tori in triaxial potential, *SIAM J. Appl. Dyn. Syst.* **6** (2017), no. 1, 159-187.

[10] H. Poincaré. Les Méthodes Nouvelles de la Mecánique Céleste, (1987) Dover Publ.

[11] C. Siegel, J. Moser. Lectures on Celestial Mechanics, (1971). Springer-Verlag, New York.

[12] G. Torres -Espino and C. Vidal *Periodic solutions of a tumor-inmune system interacion under a periodic inmunotherapy*, Preprint.

[13] T. Zeeuw, D. Merrit, Stellar orbits in a triaxial galaxy. I. Orbits in the plane of rotations, *Astrophys. J..* **267** (1983), no. 2, 571-595.

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