The Stabilization Of The Problem Of Transmission Of The Wave Equation with dynamical control *

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Abstract: In this paper, we address exponential stablization of transmission problem of the wave equation with dynamical boundary conditions. Using the classical energy method and multiplier technique, we prove that the energy of system exponentially decays.

Keywords: Wave equation; Problem of transmission; Dynamical boundary; Stabilization.

AMS subject classification: 35B35, 35L05.

1. Introduction and Main Results

Let Ω be a bounded domain (open, nonempty, and connected) in $\mathbb{R}^n (n \geq 1)$ with a boundary $\Gamma = \partial\Omega$ of class C^2 which consists of two parts, S_1 and S_2, such that $S_1 \cap S_2 = \varnothing$. S_1 can be an empty set or non-empty and $S_2 \neq \varnothing$. Let S_0 be a regular hypersurface of class C^2 which separates Ω into two domains, Ω_1 and Ω_2. In addition, S_0 satisfies $S_1 \cap S_0 = S_2 \cap S_0 = \varnothing$. Obviously, $S_1 \subset \Gamma_1 = \partial\Omega_1$ and $S_2 \subset \Gamma_2 = \partial\Omega_2$ (see Figure 1).

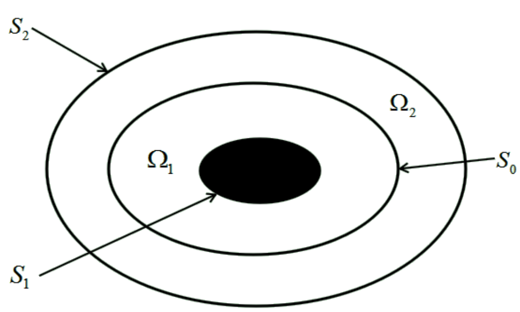

Figure 1: transmission domain

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We conside stabilization for the problem of transmission of the wave equation with dynamical boundary conditions

$$\begin{cases} u_i'' - a_i \Delta u_i = 0 & \text{in } \ \Omega_i \times (0,\infty), i=1,2, \\ u_i(x,0) = u_i^0(x), u_i'(x,0) = u_i^1(x) & \text{in } \ \Omega_i, i=1,2, \\ u(x,t) = 0 & \text{on } \ S_1 \times (0,\infty), \\ m u_2''(x,t) + a_2 \dfrac{\partial u_2(x,t)}{\partial \nu} = F(t) & \text{on } \ S_2 \times (0,\infty), \\ u_1 = u_2, a_1 \dfrac{\partial u_1}{\partial \nu} = a_2 \dfrac{\partial u_2}{\partial \nu} & \text{on } \ S_0 \times (0,\infty), \end{cases} \tag{1.1}$$

where Δ denotes the Laplace operator in the space variables, the prime $'$ denotes the derivate with respect to the time variable. ν denotes the unit normal on Γ and S_0 directing towards the exterior of Ω and Ω_1, a_1 and a_2 are positive constants. $F(t)$ is the boundary feedback control. Moreover

$$m \in L^\infty(S_2); \quad m(x) > 0, \quad \forall x \in S_2. \tag{1.2}$$

Transmission problems describe the phenomenon of waves propagating from one substance to another, therefore it is of practical significance. The problem of transmission for the wave equation had received many mathematicians attention for many years (see [11], [12], [14], [18], [19], [20], [24] and the references therein). In [18], Lions considered the problem of exact controllability with Dirichlet boundary conditions of transmission for the wave equation in the domain as described in Figure 1 and established the results of exact controllability (see [18, p.379, Th.5.1]) when $a_1 \geq a_2$. Later, Liu and Williams in [12] considered the following problem of stabilization with a domain as shown in Figure 1.

$$\begin{cases} u_i'' - a_i \Delta u_i + q u_i = 0 & \text{in } \ \Omega_i \times (0,\infty), i=1,2, \\ u_i(x,0) = u_i^0(x), u_i'(x,0) = u_i^1(x) & \text{in } \ \Omega_i, i=1,2, \\ u_1 = 0 & \text{on } \ S_1 \times (0,\infty), \\ \dfrac{\partial u_2}{\partial \nu} + \alpha(x) u_2 + \sigma(x) u_2' = 0 & \text{on } \ S_2 \times (0,\infty), \\ u_1 = u_2, a_1 \dfrac{\partial u_1}{\partial \nu} = a_2 \dfrac{\partial u_2}{\partial \nu} & \text{on } \ S_0 \times (0,\infty), \end{cases}$$

where the function $q : \Omega \to \mathbb{R}, \quad \alpha, \sigma : S_2 \to \mathbb{R}$ are nonnegative and satisfy $q \in L^\infty(\Omega), \quad \alpha, \sigma \in C^1(S_2)$. And they obtained the exponential stabilization under $a_1 \geq a_2$. Then in 2002, Liu [11] addressed the problem of control of the transmission wave equation and showed that such a system can be controlled by introducing both boundary control along the exterior boundary and distributed control near the transmission boundary. In [24], Ramos and Souza prove the equivalence between the exponential stability previously proven by Liu and Williams in [12] and the inequality observability on the boundary in the one-dimensional case.

We say that equation (1.1) is with dynamical boundary conditions when $m(x) \neq 0$. Dynamical boundary conditions play an important role in various fields, such as physics, biomedicine, and noise suppression and control of elastic structures (see [1], [7], [20], [15],[25] and the references therein). The dynamical terms on the boundary may change the stability property of the system. For example:

$$\begin{cases} u'' - \Delta u = 0 & \text{in } \ \Omega \times (0,\infty), \\ u(x,0) = u^0(x), u'(x,0) = u^1(x) & \text{in } \ \Omega, \\ u(x,t) = 0 & \text{on } \ \Gamma_0 \times (0,\infty), \\ m u'' + \dfrac{\partial u}{\partial \nu} = -\dfrac{\partial u'}{\partial \nu} & \text{on } \ \Gamma_1 \times (0,\infty). \end{cases}$$

It is shown that the system is exponentially stable under suitable conditions on Ω when $m = 0$, while the case when $m > 0$ the above system is not uniformly stable (see [7] and [15]). There has been extensive work for the wave equation with dynamical boundary conditions. Then, let us recall some works related to the problem we address. In the one-dimensional case, Conrad and Mifdal in [6] consider following problem of stabilization with $F(t) = -\alpha u + f(u'_x - \alpha u')$,

$$
\begin{cases}
u'' - a(x)\Delta u = 0 & \text{in } \ \Omega \times (0, \infty), \\
u(x,0) = u^0(x), u'(x,0) = u^1(x) & \text{in } \ \Omega, \\
M u_1''(x,t) + a_1 \dfrac{\partial u_1(x,t)}{\partial \nu} = 0 & \text{on } \ \Gamma_1 \times (0, \infty), \\
m u_2''(x,t) + a_2 \dfrac{\partial u_2(x,t)}{\partial \nu} = F(t) & \text{on } \ \Gamma_2 \times (0, \infty),
\end{cases}
\tag{1.3}
$$

where $a(x)$ is a positive function, and they obtained the exponential stabilization under some conditions. After that, In the N-dimensional($N \geq 2$) case, [4] discussed the asymptotic stabilization of the solutions with $F(t) = -a(x)u'$. [26] consider the wave equation with variable coefficients and a dynamical Neumann boundary control when $u(x,t) = 0$ on Γ_1 and by applying a boundary feedback control obtain the exponential decay for the solutions.

We note that in the previous work the problem of transmission for wave equation with dynamical boundary conditions has not been considered yet. Therefore, the purpose of this paper is to study how the dynamic boundary conditions on $\partial\Omega$ affect the stabilization of the system. In this article, the feedback law is of the form

$$
F(t) = -\beta u_2' - \gamma a_2 \frac{\partial u_2'}{\partial \nu}, \tag{1.4}
$$

where the constants β and γ are positive numbers such that $\beta\gamma < m$. Let u be a regular solution of system (1.1). We set

$$
\eta = m u_2'(x,t) + \gamma a_2 \frac{\partial u_2}{\partial \nu} \quad x \in S_2. \tag{1.5}
$$

Then we associate to system (1.1) the energy functional $E(t)$ as

$$
E(t) = \frac{1}{2}\Big(\int_{\Omega_1} a_1 |\nabla u_1|^2 + (u_1')^2 dx + \int_{\Omega_2} a_2 |\nabla u_2|^2 + (u_2')^2 dx + \int_{S_2} \frac{1}{m - \beta\gamma} \eta^2 d\sigma\Big). \tag{1.6}
$$

We can readily verify that

$$
\dot{E}(t) = -\beta \int_{S_2} (u')^2 d\sigma - \frac{\gamma}{m - \beta\gamma} \int_{S_2} (\eta')^2 d\sigma. \tag{1.7}
$$

Hence the energy decreases with time.

The main result of this paper reads as follows.

Theorem 1.1. *Let ν denote the unit normal on Γ and S_0 directing towards the exterior of Ω and Ω_1. Assume there is a vector field $h(x) = x - x_0$ such that*

(i) $h \cdot \nu \leq 0$ a.e. on S_1 with respect to the (n-1)-dimensional surface measure;

(ii)$(a_1 - a_2) h \cdot \nu \geq 0$ a.e. on S_0 with respect to the (n-1)-dimensional surface measure.

Then there are positive constants C, ω such that

$$
E(t) \leq C e^{-\omega t}, \quad \forall t \geq 0, \tag{1.8}
$$

for all solutions u of (1.1) with $(u^0, u^1, \eta^0) \in H^1_{S_1}(\Omega) \times L^2(\Omega) \times L^2(S_2)$.

The plan for the rest of this paper is as follows. In Section 2, we discuss the well-posedness of problem (1.1) through semigroup theory. In Section 3, we prove Theorem 1.1.

2. Well-posedness of the problem

In this part, we study the well-posedness of the problem (1.1). Set

$$u = \begin{cases} u_1, & x \in \Omega_1, \\ u_2, & x \in \Omega_2, \end{cases} \quad u^0 = \begin{cases} u_1^0, & x \in \Omega_1, \\ u_2^0, & x \in \Omega_2, \end{cases} \quad u^1 = \begin{cases} u_1^1, & x \in \Omega_1, \\ u_2^1, & x \in \Omega_2, \end{cases} \quad a(x) = \begin{cases} a_1, & x \in \Omega_1, \\ a_2, & x \in \Omega_2. \end{cases} \tag{2.1}$$

In the sequel, u, u^0, u^1 always means (2.1); an integral of u on a domain Ω means the sum of two integrals of u_1 and u_2 on the subdomains Ω_1 and Ω_2; an equation related to u holds on a domain Ω means that the equation holds on the subdomains Ω_1 and Ω_2. Set

$$H^1_{S_1}(\Omega) = \{u \in H^1(\Omega), u = 0 \ \text{ on } \ S_1\},$$

then let us consider the Hilbert space

$$\Upsilon = H^1_{S_1}(\Omega) \times L^2(\Omega) \times L^2(S_2),$$

equipped with the inner product

$$\langle (u, v, \eta), (\tilde{u}, \tilde{v}, \tilde{\eta}) \rangle_\Upsilon = \int_\Omega a(x) \nabla u \cdot \nabla \tilde{u} + v\tilde{v} dx + \int_{S_2} \frac{1}{m - \beta\gamma} \eta\tilde{\eta} d\sigma. \tag{2.2}$$

We define a operator $\mathbb{T}$ on Υ by

$$\mathbb{T}(u, v, \eta) = (v, a(x)\Delta u, -\frac{1}{\gamma}\eta + (\frac{m}{\gamma} - \beta)v). \tag{2.3}$$

with domain

$$D(\mathbb{T}) = \{(u, v, \eta) \in H^2(\Omega_1, \Omega_2) \times H^1_{S_1}(\Omega) \times L^2(S_2); \Delta u_i \in L^2(\Omega_i), \eta = mv|_{S_2} + \gamma a_2 \frac{\partial u_2}{\partial \nu}\}, \tag{2.5}$$

where

$$H^2(\Omega_1, \Omega_2) = \{u \in H^1(\Omega) : u_i = u|_{\Omega_i} \in H^2(\Omega_i), \quad i = 1, 2; \quad u = 0 \ \text{ on } S_1$$
$$\text{and } \ a_1 \frac{\partial u_1}{\partial \nu} = a_2 \frac{\partial u_2}{\partial \nu} \ \text{ on } S_0\}.$$

Setting $v = u'$, $\eta = mu_2' + \gamma a_2 \frac{\partial u_2}{\partial \nu}$ and $\Phi(t) = (u(t), v(t), \eta(t))$, problem (1.1) can be formulated as an abstract Cauchy problem:

$$\begin{cases} \Phi'(t) = \mathbb{T}\Phi(t), \\ \Phi(0) = \Phi_0 = (u^0, u^1, \eta^0), \end{cases} \tag{2.4}$$

on the Hilbert space Υ for an initial condition $\Phi(0) = (u^0, u^1, \eta^0)$.

We will show that $\mathbb{T}$ generates a C_0 semigroup on Υ. Now we are able to state a well-posedness result for the closed-loop system (2.4):

Theorem 2.1. *For any initial datum $\Phi_0 \in \Upsilon$, there exists a unique weak solution $\Phi \in C([0, \infty), \Upsilon)$ of system (2.4). Moreover, if $\Phi_0 \in D(\mathbb{T})$, then there exists a unique strong solution $\Phi \in C([0, \infty), D(\mathbb{T})) \cap C^1([0, \infty), \Upsilon)$.*

Proof. We prove that $\mathbb{T}$ is dissipative. Let $\Phi = (u, v, \eta) \in D(\mathbb{T})$. Using Green's formula, one can obtain

$$\langle \mathbb{T}\Phi, \Phi \rangle_\Upsilon = -\beta \int_{S_2} v^2 d\sigma - \frac{\gamma}{m - \beta\gamma} \int_{S_2} (\eta')^2 d\sigma \leq 0. \tag{2.6}$$

Now, given $(\bar{a},\bar{b},\bar{c})\in\Upsilon$, we seek $(u,v,\eta)\in D(\mathbb{T})$ solution of the equation $(I-\mathbb{T})(u,v,\eta)=(\bar{a},\bar{b},\bar{c})$, that is,

$$\begin{cases} u_i - v_i = \bar{a} & \text{in } \Omega_i, i=1,2,\\ v_i - a(x)\Delta u_i = \bar{b} & \text{in } \Omega_i, i=1,2,\\ u=0 & \text{on } \ S_1,\\ \eta + \dfrac{1}{\gamma}\eta - (\dfrac{m}{\gamma}-\beta)v = \bar{c} & \text{on } \ S_2,\\ u_1=u_2, a_1\dfrac{\partial u_1}{\partial\nu} = a_2\dfrac{\partial u_2}{\partial\nu} & \text{on } \ S_0. \end{cases} \tag{2.7}$$

Then eliminating v and using $\eta = mv+\gamma a_2\frac{\partial u_2}{\partial\nu}$, we find that u satisfies the system

$$\begin{cases} u_i - a(x)\Delta u_i = \bar{a}+\bar{b} & \text{in } \Omega_i, i=1,2,\\ u=0 & \text{on } \ S_1,\\ (m+\beta)u_2 + (\gamma+1)a_2\dfrac{\partial u_2}{\partial\nu} = (m+\beta)\bar{a}+\bar{c} & \text{on } \ S_2,\\ u_1=u_2, a_1\dfrac{\partial u_1}{\partial\nu} = a_2\dfrac{\partial u_2}{\partial\nu} & \text{on } \ S_0. \end{cases} \tag{2.8}$$

Using Green's formula, one can prove that the system (2.8) is equivalent to the following variational equation:

$$\begin{aligned} &\int_\Omega u\psi + a(x)\nabla u\cdot\nabla\psi dx + \int_{S_2}\frac{m+\beta}{\gamma+1}u_2\psi d\sigma\\ =&\int_\Omega \psi(\bar{a}+\bar{b})dx + \int_{S_2}\frac{m+\beta}{\gamma+1}\psi\bar{a} + \frac{1}{\gamma+1}\psi\bar{c}d\sigma, \end{aligned} \tag{2.9}$$

for any $\psi\in H^1_{S_1}(\Omega)=\{\psi\in H^1(\Omega),\psi|_{S_1}=0\}$. Using Lax-Milgram Theorem, one can prove that (2.9) admits a unique solution $u\in H^2(\Omega_1,\Omega_2)$. Then we can get range$(I-\mathbb{T})=\Upsilon$. Thus, Lummer-Phillips Theorem [22] leads us to claim that $\mathbb{T}$ generates a C_0 semigroup of contractions $S(t)$ on Υ. Finally, the well-posedness result follow from semigroup theory [22].

3. Exponential Stabilization

In this section, we prove Theorem 1.1. The idea of the proof is simple. It suffices to show that there exist positive constants $T>0$ and $0<\rho<1$ such that (see [9], [11] and [16])

$$E(t)\le\rho E(0), \qquad \forall t\ge T.$$

However, it is not easy to verify this inequality. For this, we construct two auxiliary functions to estimate the energy and finally obtain the main result.

Here we define two functions

$$V_1(t) = \int_\Omega h(x)\cdot\nabla u u' dx,$$

$$V_2(t) = \frac{1}{2}\int_\Omega (n-1)uu'dx.$$

Lemma 3.1. *Let h be a vector field on $\overline{\Omega}$. Suppose that $u(x,t)$ is a solution of the system (1.1). Then*

$$\dot{V}_1(t) = B_1 + I_1,$$

where we denote the boundary term

$$
\begin{aligned}
B_1 =& \int_{S_1} \frac{1}{2}(h\cdot\nu)((u_1')^2 - a_1|\nabla u_1|^2) + a_1\frac{\partial u_1}{\partial \nu} h\cdot\nabla u_1 d\sigma \\
&+ \int_{S_2} \frac{1}{2}(h\cdot\nu)((u_2')^2 - a_2|\nabla u_2|^2) + a_2\frac{\partial u_2}{\partial \nu} h\cdot\nabla u_2 d\sigma \\
&+ \int_{S_0} a_1\frac{\partial u_1}{\partial \nu} h\cdot\nabla u_1 - a_2\frac{\partial u_2}{\partial \nu} h\cdot\nabla u_2 + \frac{1}{2}(h\cdot\nu)a_2|\nabla u_2|^2 - \frac{1}{2}(h\cdot\nu)a_1|\nabla u_1|^2 d\sigma \\
=& B_1(S_1) + B_1(S_2) + B_1(S_0).
\end{aligned}
$$

And the internal term

$$
I_1 = \int_\Omega -\frac{1}{2}\mathrm{div}h((u')^2 - a(x)|\nabla u|^2) - \sum_{i,j=1}^{n} a(x)(\partial_i u)(\partial_i h_j)(\partial_j u)dx.
$$

Proof. Multipling (1.1)-1 by $h\cdot\nabla u$ and integrating on Ω, we can get the above result after a straightforward calculation.

Lemma 3.2. *Suppose that $u(x,t)$ is a solution of the system (1.1). Then*

$$
\dot{V}_2(t) = B_2 + I_2,
$$

where we denote the boundary term

$$
B_2 = \frac{n-1}{2}\int_{S_2} a_2\frac{\partial u_2}{\partial \nu} u_2 d\sigma.
$$

And the internal term

$$
I_2 = \frac{n-1}{2}\int_\Omega ((u')^2 - a(x)|\nabla u|^2)dx.
$$

Proof. Using Green's formula and system (1.1), we have

$$
\begin{aligned}
\dot{V}_2(t) =& \frac{n-1}{2}\int_\Omega (u')^2 + uu'' dx \\
=& \frac{n-1}{2}\int_\Omega (u')^2 + a(x)u\Delta u dx \\
=& \frac{n-1}{2}\int_\Omega ((u')^2 - a(x)|\nabla u|^2)dx + \frac{n-1}{2}\int_{S_2} a_2\frac{\partial u_2}{\partial \nu} u_2 d\sigma \\
=& I_2 + B_2.
\end{aligned}
$$

Lemma 3.3. *Suppose that the assumptions (i) and (ii) in Theorem 1.1 hold and $h = x - x_0$. Let u solve (1.1). Then there exist constants $C_1, C_2, C_3 > 0$ such that*

$$
E(t) + \dot{V}_1(t) + \dot{V}_2(t) \le \frac{1}{2}\int_{S_2} \frac{1}{m-\beta\gamma}\eta^2 d\sigma + C_1\int_{S_2} (\frac{\partial u}{\partial \nu})^2 + (u')^2 d\sigma, \tag{3.1}
$$

$$
|V_1(t)| \le C_2 E(t), \qquad |V_2(t)| \le C_3 E(t). \tag{3.2}
$$

Proof. Obviously the estimate (3.2) is true. Now, we prove (3.1). First we estimate the boundary term B_1 given in Lemma 3.1. Since $u_1 = 0$ on S_1, then $\nabla u_1 = \frac{\partial u_1}{\partial \nu}\nu$. From condition (i) of Theorem 1.1, it follows that

$$\begin{aligned} B_1(S_1) =& \int_{S_1} \frac{1}{2}(h\cdot\nu)((u_1')^2 - a_1|\nabla u_1|^2) + a_1 \frac{\partial u_1}{\partial \nu} h\cdot \nabla u_1 d\sigma \\ =& \frac{1}{2}\int_{S_1} (h\cdot\nu)((u_1')^2 - a_1|\nabla u_1|^2 + 2a_1|\nabla u_1|^2) d\sigma \\ \le& 0. \end{aligned} \tag{3.3}$$

Set $R = \sup\limits_{x\in\Omega}|h(x)|$. Using the fact $\frac{\partial u_2}{\partial \nu} = \nabla u_2 \cdot \nu$ on S_2, we obtain

$$\begin{aligned} B_1(S_2) =& \int_{S_2} \frac{1}{2}(h\cdot\nu)((u_2')^2 - a_2|\nabla u_2|^2) + a_2 \frac{\partial u_2}{\partial \nu} h\cdot \nabla u_2 d\sigma \\ \le& \int_{S_2} \frac{1}{2}(h\cdot\nu)((u_2')^2 + a_2|\nabla u_2|^2) d\sigma + \int_{S_2} R a_2 |\nabla u_2|^2 d\sigma \\ \le& \frac{R}{2}\int_{S_2} (u_2')^2 d\sigma + \frac{3R}{2}\int_{S_2} a_2|\nabla u_2|^2 d\sigma. \end{aligned} \tag{3.4}$$

Since $u_1 = u_2$ on S_0, we have

$$\nabla(u_2 - u_1) = \frac{\partial (u_2 - u_1)}{\partial \nu}\nu, \quad \text{on} \quad S_0,$$

then

$$\begin{aligned} |\nabla u_2|^2 =& |\nabla u_1|^2 + 2(\frac{\partial u_2}{\partial \nu} - \frac{\partial u_1}{\partial \nu})\frac{\partial u_1}{\partial \nu} + (\frac{\partial u_2}{\partial \nu} - \frac{\partial u_1}{\partial \nu})^2 \\ =& |\nabla u_1|^2 + (\frac{\partial u_2}{\partial \nu})^2 - (\frac{\partial u_1}{\partial \nu})^2. \end{aligned}$$

Therefore,

$$\begin{aligned} B_1(S_0) =& \int_{S_0} a_1\frac{\partial u_1}{\partial \nu} h\cdot\nabla u_1 - a_2\frac{\partial u_2}{\partial \nu} h\cdot\nabla u_2 + \frac{1}{2}(h\cdot\nu)a_2|\nabla u_2|^2 - \frac{1}{2}(h\cdot\nu)a_1|\nabla u_1|^2 d\sigma \\ =& \int_{S_0} a_1\frac{\partial u_1}{\partial \nu} h\cdot\nabla u_1 - a_2\frac{\partial u_2}{\partial \nu}[h\cdot\nabla u_1 + (\frac{\partial u_2}{\partial \nu} - \frac{\partial u_1}{\partial \nu})h\cdot\nu] \\ &+ \frac{1}{2}[a_2(|\nabla u_1|^2 + (\frac{\partial u_2}{\partial \nu})^2 - (\frac{\partial u_1}{\partial \nu})^2) - a_1|\nabla u_1|^2] h\cdot\nu d\sigma \\ =& \int_{S_0} a_1\frac{\partial u_1}{\partial \nu} h\cdot\nabla u_1 - a_1\frac{\partial u_1}{\partial \nu}[h\cdot\nabla u_1 + (\frac{a_1}{a_2}\frac{\partial u_1}{\partial \nu} - \frac{\partial u_1}{\partial \nu})h\cdot\nu] \\ &+ \frac{1}{2}[a_2(|\nabla u_1|^2 + \frac{a_1^2}{a_2^2}(\frac{\partial u_1}{\partial \nu})^2 - (\frac{\partial u_1}{\partial \nu})^2) - a_1|\nabla u_1|^2] h\cdot\nu d\sigma \\ =& \int_{S_0} \frac{1}{2}(a_2 - a_1)|\nabla u_1|^2(h\cdot\nu) - \frac{(a_2-a_1)^2}{2a_2}(\frac{\partial u_1}{\partial \nu})^2(h\cdot\nu) d\sigma. \end{aligned}$$

This shows that $B_1(S_0) \le 0$ because of condition (ii) from Theorem 1.1.

Then, we estimate the internal term I_1. Since $h = x - x_0$, we have

$$\begin{aligned} I_1 =& \int_\Omega -\frac{1}{2}\mathrm{div}h((u')^2 - a(x)|\nabla u|^2) - \sum_{i,j=1}^n a(x)(\partial_i u)(\partial_i h_j)(\partial_j u) dx \\ =& \int_\Omega -\frac{1}{2}\mathrm{div}h((u')^2 - a(x)|\nabla u|^2) - a(x)|\nabla u|^2 dx. \end{aligned} \tag{3.5}$$

Finally, we estimate the boundary term B_2. Applying Young's inequality, trace theorem and Poincaré inequality, we have

$$
\begin{aligned}
B_2(S) =& \frac{n-1}{2}\int_{S_2} a_2 \frac{\partial u_2}{\partial \nu} u_2 d\sigma \\
\leq & \frac{a_2(n-1)}{2}\Big(\int_{S_2} \varepsilon (u_2)^2 d\sigma + \int_{S_2} C_\varepsilon (\frac{\partial u_2}{\partial \nu})^2 d\sigma\Big) \\
\leq & \frac{a_2(n-1)}{2}\Big(C\varepsilon \int_\Omega |\nabla u|^2 dx + C_\varepsilon \int_{S_2} (\frac{\partial u_2}{\partial \nu})^2 d\sigma\Big),
\end{aligned}
\tag{3.6}
$$

where C, ε and C_ε are positive constants. Combining the above inequalities (3.3)-(3.6), if ε is small enough, we get

$$
\begin{aligned}
& E(t) + \dot{V}_1(t) + \dot{V}_2(t) \\
\leq & \frac{1}{2}\int_\Omega a(x)|\nabla u|^2 + (u')^2 dx + \frac{1}{2}\int_{S_2} \frac{1}{m-\beta\gamma}\eta^2 d\sigma + \int_\Omega -\frac{1}{2}\mathrm{div}h((u')^2 - a(x)|\nabla u|^2) \\
& - a(x)|\nabla u|^2 dx + \frac{R}{2}\int_{S_2} (u_2')^2 d\sigma + \frac{3R}{2}\int_{S_2} a_2 |\nabla u_2|^2 d\sigma + \frac{n-1}{2}\int_\Omega (u')^2 - a(x)|\nabla u|^2 dx \\
& + \frac{a_2(n-1)}{2}\Big(C\varepsilon \int_\Omega |\nabla u|^2 dx + C_\varepsilon \int_{S_2} (\frac{\partial u_2}{\partial \nu})^2 d\sigma\Big) \\
\leq & \frac{1}{2}\int_{S_2} \frac{1}{m-\beta\gamma}\eta^2 d\sigma + + \frac{R}{2}\int_{S_2} (u_2')^2 d\sigma + \frac{3R}{2}\int_{S_2} a_2 |\nabla u_2|^2 d\sigma + \frac{C_\varepsilon a_2(n-1)}{2}\int_{S_2} (\frac{\partial u_2}{\partial \nu})^2 d\sigma \\
\leq & \int_{S_2} \frac{1}{2(m-\beta\gamma)}\eta^2 d\sigma + C_1 \int_{S_2} (\frac{\partial u_2}{\partial \nu})^2 + (u')^2 d\sigma,
\end{aligned}
$$

where $C_1 = \max\{\frac{R}{2}, \frac{3R + C_\varepsilon a_2(n-1)}{2}\}$.

Lemma 3.4. *Suppose that the assumptions (i) and (ii) in Theorem 1.1 hold. Let u solve problem (1.1). Then there exists a time $T_0 > 0$ and a positive constant C_T such that*

$$
E(0) \leq C_T \Big\{ \int_0^T \int_{S_2} ((u_2')^2 + (\frac{\partial u_2}{\partial \nu})^2 + \eta^2) d\sigma dt \Big\},
\tag{3.7}
$$

for all $T > T_0$.

Proof. Integrating the inequality (3.1) on the interval $(0, T)$ yields

$$
\begin{aligned}
& \int_0^T E(t) dt + V_1(T) - V_1(0) + V_2(T) - V_2(0) \\
\leq & \frac{1}{2}\int_0^T \int_{S_2} \frac{1}{m-\beta\gamma}\eta^2 d\sigma dt + C_1 \int_0^T \int_{S_2} (\frac{\partial u}{\partial \nu})^2 + (u')^2 d\sigma dt.
\end{aligned}
\tag{3.8}
$$

Then we use inequality (3.2) to obtain

$$
\int_0^T E(t) dt \leq C_4 \Big\{ \int_0^T \int_{S_2} ((u_2')^2 + (\frac{\partial u_2}{\partial \nu})^2 + \eta^2) d\sigma dt \Big\} + C_0(E(T) + E(0)),
\tag{3.9}
$$

where $C_4 = \max\{\frac{1}{2(m-\beta\gamma)}, C_1\}$ and $C_0 = 4\max\{C_2, C_3\}$.

We notice that

$$
\begin{aligned}
-\dot{E}(t) =& \beta \int_{S_2} (u_2')^2 d\sigma + \frac{\gamma}{m-\beta\gamma}\int_{S_2} (\eta')^2 d\sigma \\
=& \beta \int_{S_2} (u_2')^2 d\sigma + \frac{\gamma}{m-\beta\gamma}\int_{S_2} (-\frac{1}{\gamma}\eta + \frac{m-\beta\gamma}{\gamma}u_2')^2 d\sigma \\
\leq & C_5 \int_{S_2} ((u_2')^2 + \eta^2) d\sigma,
\end{aligned}
$$

where $C_5 = \max\{\frac{2}{(m-\beta\gamma)\gamma}, \frac{2(m-\beta\gamma)}{\gamma} + \beta\}$. Then, we get

$$\begin{aligned}
&E(0) + C_0(E(T) + E(0)) \\
=& \int_0^{2C_0+1} E(t)dt + \int_0^{2C_0+1} (E(0) - E(t))dt + C_0(E(T) - E(0)) \\
=& \int_0^{2C_0+1} E(t)dt - \int_0^{2C_0+1} (\int_0^t \dot{E}(\tau)d\tau)dt \\
\leq& \int_0^{2C_0+1} E(t)dt + C_5 \int_0^{2C_0+1} \int_{S_2} ((u')^2 + \eta^2)d\sigma dt.
\end{aligned} \tag{3.10}$$

Now set $T_0 = 2C_0 + 1$, substituting (3.9) in (3.10) complete the proof.

Proof of Theorem 1.1. Applying Cauchy inequality, we can make an estimate of $\dot{E}(t)$,

$$\begin{aligned}
-\dot{E}(t) =& -\int_\Omega a(x)\nabla u \cdot \nabla u' + u'u''dx - \int_{S_2} \frac{1}{m-\beta\gamma}\eta\eta' d\sigma \\
=& \int_\Omega a(x)\Delta u u' - u'u''dx - \int_{S_2} a_2 \frac{\partial u_2}{\partial \nu} u_2' d\sigma - \int_{S_2} \frac{1}{m-\beta\gamma}\eta\eta' d\sigma \\
=& -\int_{S_2} a_2 \frac{\partial u_2}{\partial \nu} u_2' - \int_{S_2} \frac{1}{m-\beta\gamma}\eta\eta' d\sigma \\
=& -\int_{S_2} a_2 \frac{\partial u_2}{\partial \nu} u_2' - \int_{S_2} \frac{1}{m-\beta\gamma}\eta(-\frac{1}{\gamma}\eta + \frac{m-\beta\gamma}{\gamma}u')d\sigma \\
=& \int_{S_2} -a_2 \frac{\partial u_2}{\partial \nu} u_2' + \frac{1}{2\gamma m}\eta^2 d\sigma + \int_{S_2} (-\frac{1}{2\gamma m} + \frac{1}{(m-\beta\gamma)\gamma})\eta^2 d\sigma - \int_{S_2} \frac{1}{\gamma}\eta u' d\sigma \\
\geq& \int_{S_2} \frac{\gamma}{2m}(a_2 \frac{\partial u_2}{\partial \nu})^2 + \frac{m}{2\gamma}(u')^2 d\sigma + \int_{S_2} (-\frac{1}{2\gamma m} + \frac{1}{(m-\beta\gamma)\gamma})\eta^2 d\sigma - \int_{S_2} \frac{1}{K_1\gamma}\eta^2 d\sigma \\
&- \int_{S_2} \frac{K_1}{4\gamma}(u')^2 d\sigma \\
=& \int_{S_2} \frac{\gamma}{2m}(a_2 \frac{\partial u_2}{\partial \nu})^2 d\sigma + \int_{S_2} (-\frac{1}{2\gamma m} + \frac{1}{(m-\beta\gamma)\gamma} - \frac{1}{K_1\gamma})\eta^2 d\sigma + \int_{S_2} (\frac{m}{2\gamma} - \frac{K_1}{4\gamma})(u')^2 d\sigma \\
=& \int_{S_2} \frac{\gamma}{2m}(a_2 \frac{\partial u_2}{\partial \nu})^2 d\sigma + \int_{S_2} \frac{\beta(m+\beta\gamma)}{2m^2(m-\beta\gamma)}\eta^2 d\sigma + \int_{S_2} \frac{\beta m}{2(m+\beta\gamma)}(u')^2 d\sigma \\
\geq& C_6 \int_{S_2} ((u_2')^2 + (\frac{\partial u_2}{\partial \nu})^2 + \eta^2)d\sigma.
\end{aligned} \tag{3.11}$$

by choosing $K_1 = \frac{2m^2}{m+\beta\gamma}$ and $C_6 = \max\{\frac{\gamma}{2m}, \frac{\beta(m+\beta\gamma)}{2m^2(m-\beta\gamma)}, \frac{\beta m}{2(m+\beta\gamma)}\}$.

By Lemma 3.4 and the above inequality (3.11), we have

$$E(0) \leq C_T \left\{ \int_0^T \int_{S_2} ((u_2')^2 + (\frac{\partial u_2}{\partial \nu})^2 + \eta^2)d\sigma dt \right\} \leq -\frac{C_T}{C_6}\int_0^T \dot{E}(t)dt = -\frac{C_T}{C_6}(E(T) - E(0)),$$

for all $T > T_0$, which yields

$$E(T) \leq \frac{C_T - C_6}{C_T} E(0).$$

Thus the exponential decay of the energy functional $E(t)$ is obtained. □

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Declaration of conflicting of interest

The Authors declare that there is no conflict of interest.

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