AN ADAPTIVE DISCONTINUOUS GALERKIN METHOD FOR A STOKES/BIOT FLUID-POROELASTIC STRUCTURE INTERACTION MODEL

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Abstract. The paper presents an a posteriori error estimator for a (piecewise linear) nonconforming finite element approximation of the problem defining the interaction between a free fluid and poroelastic structure. The free fluid is governed by the Stokes equations, while the flow in the poroelastic medium is modeled using the Biot poroelasticity system. Equilibrium and kinematic conditions are imposed on the interface. The approach utilizes the same nonconforming Crouzeix-Raviart element discretization on the entire domain [Houédanou Koffi Wilfrid, Results in Applied Mathematics 7 (2020) 100127, Elsevier]. For this discretization, we derive a residual indicator based on the jumps of normal derivative of the nonconforming approximation. Lower and upper bounds form the main results with minimal assumptions on the mesh.

Mathematics Subject Classification [MSC]: 74S05,74S10,74S15, 74S20,74S25,74S30.
Key Words: Stokes-Biot model; Nonconforming finite element method; A posteriori error analysis.

Contents

1. Introduction

In this paper, we develop an a posteriori error analysis for solving the interaction of a free incompressible viscous Newtonian fluid with a fluid within a poroelastic medium. This is a challenging multiphysics problem with applications to predicting and controlling processes arising in groundwater flow in fractured aquifers, oil and gas extraction, arterial flows, and industrial filters. In these applications, it is important to model properly the interaction between the free fluid with the fluid within the porous medium, and to take into account the effect of the deformation of the medium. For example, geomechanical effects play an important role in hydraulic fracturing, as well as in modeling phenomena such as subsidence and compaction.

We adopt the Stokes equations to model the free fluid and the Biot system [**1**] for the fluid in the poroelastic media. In the latter, the volumetric deformation of the elastic porous matrix is complemented with the Darcy equation that describes the average velocity of the fluid in the pores. The model features two different kinds of coupling across the interface: Stokes-Darcy coupling [**2**–**10**] and fluid-structure interaction (FSI) [**11**–**15**].

The well-posedness of the mathematical model based on the Stokes-Biot system for the coupling between a fluid and a poroelastic structure is studied in [**16**]. A numerical study of the problem, using a Navier-Stokes equations for the fluid, is presented in [**11**, **17**], utilizing a variational multiscale approach to stabilized the finite element spaces. The problem is solved using both a monolithic and a partitioned approach, with the latter requiring subiterations between the two problems.

Nonphysical pressure oscillations are observed in finite element calculations of Biot's poroelastic equations in low-permeable media. These pressure oscillations may be understood as a failure of compatibility between the finite element spaces, rather than elastic locking. In [**18**], Joachim Berdal Haga et al. have presented evidence to support this view by comparing and contrasting the pressure oscillations in low-permeable porous media with those in low-compressible porous media. As a consequence, it is possible to use established families of stable mixed elements as candidates for choosing finite element spaces for Biot's equations. Through comparison with the displacement-solid pressure mixed formulation of linear elasticity, they identify the spurious pressure modes as a specific consequence of a vanishing Brezzi inf-sup constant. Since the Brezzi inf-sup condition for the poroelastic equations takes on a similar form as in, e.g., the mixed linear elasticity or Stokes problem, this identification opens up the field to a plethora of stable element candidates. These can be used directly for the basic solid displacement-fluid pressure two-field formulation of poroelasticity, or in combinations for the various three- and four-field formulations involving solid pressure and/or fluid velocity [**18**].

Finite element analysis of an arbitrary Lagrangian-Eulerian method for Stokes/parabolic moving interface problem with jump coefficients has been studied in [**19**]. The authors in [**20**] study a numerical solution of the coupled system of the time-dependent Stokes and fully dynamic Biot equations. They establish stability of the scheme and derive error estimates for the fully discrete coupled scheme. Numerical errors and convergence rates for smooth problems as well as tests on realistic material parameters have been presented. In [**21**], Jing Wen and Yinnian He consider a strongly conservative discretization for the rearranged Stokes-Biot model based on interior penalty discontinuous Galerkin method and mixed finite element method. The existence and uniqueness of solution of the numerical scheme have been presented. Then, the analysis of stability and priori error estimates have been derived. The numerical examples under uniform meshes, which well validate the analysis of convergence and the strong mass conservation are presented. A staggered finite element procedure for the coupled Stokes-Biot system with fluid entry resistance has been studied by Bergkamp et al. in [**22**] while Ambartsumyan et al. study in [**23**] flow and transport in fractured poroelastic media using Stokes flow in the fractures and the Biot model in the porous media. In [**24**], semidiscrete continuous-in-time approximation has been proposed for the weak coupled mixed formulation. For the discretization of the fluid velocity and pressure the authors have used the finite elements which include the MINI-elements, the Taylor-Hood elements and the conforming Crouzeix-Raviart elements. For the discretization of the porous medium problem they choose the spaces that include Raviart-Thomas and Brezzi-Douglas-Marini elements. An a priori error analysis is performed with some numerical tests confirming the convergence rates.

A posteriori error estimators are computable quantities, expressed in terms of the discrete solution and of the data that measure the actual discrete errors without the knowledge of the exact solution. They are essential to design adaptive mesh refinement algorithms which aqui- distribute the computational effort and optimize the approximation efficiency. Since the pioneering work of Babuška and Rheinboldt [**25**–**28**], adaptive finite element methods based on a posteriori error estimates have been extensively investigated.

In the article [**29**], the author studies a stabilized nonconforming mixed finite element method using the Crouzeix-Raviart element for the Stokes-Biot problem. Considering mixed formulation of the Darcy problem, the fluid velocity and pressure are treated as functions defined in the entire domain. Existence, uniqueness of the finite element solution of the corresponding discrete problem and a priori estimates have been shown. The proofs use the standard theory for mixed problems. The approach presented is independent of the normal vectors of the interior edges in both regions, thus making the resulting finite element matrix sparser.

We use a nonconforming finite element method that has so many advantages for the velocities and piecewise constant for the pressures in both the Stokes and Biot regions, and apply a stabilization term penalizing the jumps over the element edges of the piecewise continuous velocities. Indeed, one can construct finite element methods where the incompressibility condition is exactly satisfied (cf. Fortin [**30**]) but this leads to the use of complex elements of limited applicability (e.g. oil and gas extraction for conforming case). Thus, in the work [**29**], Houédanou has constructed and studies finite element method using simpler elements where the incompressibility condition is only approximatively satisfied [cf. definition of operator div_h (22)].

So, in this paper we have found it very convenient to use nonconforming finite elements which violate the interelement continuity condition of the velocities. To our best knowledge, there is no a-posteriori error estimation for the strongly coupled mixed formulation (19) [**29**, Section 3] of the coupled Stokes-Biot problem where a nonconforming finite element method is used.

The paper is organized as follows. Some preliminaries and notation are given in Section 2. In the section 3 the a posteriori error estimates are derived. The efficiency result is derived using the technique of bubble function introduced by R. Verfürth [**31**] and used in similar context by C. Carstensen [**32**, **33**]. The main results are given in the Section 5. We offer our conclusion and the further works in Section 6.

2. Preliminaires and notation

2.1. Model problem. We consider a multiphysics model problem for free fluid's interaction with a flow in a deformable porous media, where the simulation domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a union of non-overlapping regions Ω_f and Ω_p. Here Ω_f is a free fluid region with flow governed by the Stokes equations and Ω_p is a poroelastic material governed by the Biot system. For simplicity of notation, we assume that each region is connected. The extension to non-connected regions is straightforward. The two regions are separated by an interface $\Gamma_{fp} = \partial\Omega_f \cap \partial\Omega_p$. Let $\Gamma_\star = \partial\Omega_\star \setminus \Gamma_{fp}$, $\star = f, p$. Each interface and boundary is assumed to be polygonal ($d = 2$) or polyhedral ($d = 3$). We denote by $\mathbf{n}_f$ (resp. $\mathbf{n}_p$) the unit outward normal vector along $\partial\Omega_f$ (resp. Ω_p). Note that on the interface Γ_{fp}, we have $\mathbf{n}_f = -\mathbf{n}_p$. Figure 1 gives a schematic representation of the geometry. For any function v defined in Ω, since its restriction to Ω_f or Ω_p could play a

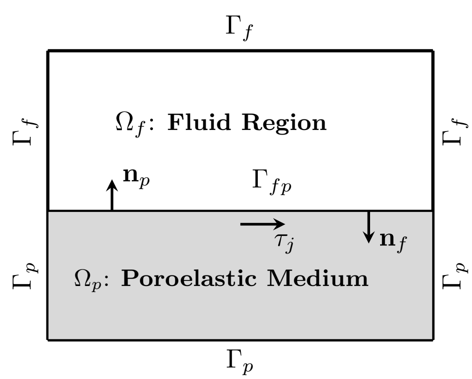

FIGURE 1. Global domain Ω consisting of the fluid region Ω_f and the poroelastic media region Ω_p separated by the interface Γ_{fp}.

different mathematical roles (for instance their traces on Γ_{fp}), we will set $v_f = v_{|\Omega_f}$ and $v_p = v_{|\Omega_p}$. In Ω, we denote by $\mathbf{u}$ the fluid velocity and by p the pressure, and let η_p be the displacement in

Ω_p. Let $\mu > 0$ be the fluid viscosity, let $\mathbf{f} \in [L^2(\Omega)]^d$ be the body force terms, and let g be external source or sink terms satisfing the compatibility condition $\int_\Omega g(x)dx = 0$.
Let $\mathbf{D}(\mathbf{u})$ and $\sigma_f(\mathbf{u}, p)$ denote, respectively, the deformation rate tensor and the stress tensor:

$$\mathbf{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right), \text{ and } \sigma_f(\mathbf{u}, p) = -p\mathbf{I} + 2\mu\mathbf{D}(\mathbf{u}).$$

In the free fluid region Ω_f, $(\mathbf{u}, p)$ satisfy the Stokes equations:

$$\begin{aligned} -\nabla \cdot \sigma_f(\mathbf{u}, p) &= \mathbf{f} \text{ in } \Omega_f & (1)\\ \nabla \cdot \mathbf{u} &= g \text{ in } \Omega_f & (2)\\ \mathbf{u} &= \mathbf{0} \text{ on } \Gamma_f. & (3) \end{aligned}$$

Let $\sigma_e(\eta_p)$ and $\sigma_p(\eta_p, p_p)$ be the elastic and poroelastic stress tensors, respectively:

$$\sigma_e(\eta_p) = \lambda_p\left(\nabla \cdot \eta_p\right)\mathbf{I} + 2\mu_p\mathbf{D}(\eta_p), \;\; \sigma_p(\eta_p, p_p) = \sigma_e(\eta_p) - \alpha p_p\mathbf{I}, \tag{4}$$

where $0 < \lambda_{\min} \leq \lambda_p(\mathbf{x}) \leq \lambda_{\max}$ and $0 < \mu_{\min} \leq \mu_p(\mathbf{x}) \leq \mu_{\max}$ are the Lamé parameters, and $0 < \alpha \leq 1$ is the Biot-Willis constant. The poroelasticity region Ω_p is governed by the modified static Biot system [**24**]:

$$\begin{aligned} -\nabla \cdot \sigma_p(\eta_p, p_p) &= \mathbf{f} \text{ in } \Omega_p & (5)\\ \mu\mathbf{K}^{-1}\mathbf{u} + \nabla p &= 0 \text{ in } \Omega_p, & (6)\\ \alpha\nabla \cdot \eta_p + \nabla \cdot \mathbf{u} &= g \text{ in } \Omega_p, & (7)\\ \mathbf{u} \cdot \mathbf{n}_d &= 0 \text{ on } \Gamma_p & (8)\\ \eta_p &= \mathbf{0} \text{ on } \Gamma_p. & (9) \end{aligned}$$

$\mathbf{K}$ the symmetric and uniformly positive definite rock permeability tensor, satisfying, for some constants $0 < k_{\min} \leq k_{\max}$,

$$\forall \xi \in \mathbb{R}^d, k_{\min}\xi^T\xi \leq \xi^T\mathbf{K}(\mathbf{x})\xi \leq k_{\max}\xi^T\xi, \forall \mathbf{x} \in \Omega_p.$$

Following [**1**], the interface conditions on the fluid-poroelasticity interface Γ_{fp} are mass conservation, balance of stresses, and the Beavers-Joseph-Saffman (BJS) condition [**34**] modeling slip with friction:

$$\begin{aligned} \mathbf{u}_f \cdot \mathbf{n}_f + \mathbf{u}_p \cdot \mathbf{n}_p &= 0 \text{ on } \Gamma_{fp}, & (10)\\ \sigma_f\mathbf{n}_f + \sigma_p\mathbf{n}_p &= 0 \text{ on } \Gamma_{fp} & (11)\\ -(\sigma_f\mathbf{n}_f) \cdot \mathbf{n}_f &= p_p, \text{ on } \Gamma_{fp} & (12)\\ -(\sigma_f\mathbf{n}_f) \cdot \tau_{f,j} &= \mu\alpha_{BJF}\sqrt{K_j^{-1}}\,(\mathbf{u}_f) \cdot \tau_{f,j} \text{ on } \Gamma_{fp}, & (13) \end{aligned}$$

where $\tau_{f,j}$, $1 \leq j \leq d-1$, is an orthogonal system of unit tangent vectors on Γ_{fp}, $K_j = (\mathbf{K}\tau_{f,j})\cdot\tau_{f,j}$, and $\alpha_{BJS} \geq 0$ is an experimentally determined friction coefficient. We note that continuity of flux constraints the normal velocity of the solid skeleton, while the BJS condition accounts for its tangential velocity.

Equations (1)-(13) consist of the model of the coupled Stokes and Biot flows problem that we will study below.

2.2. Strongly coupled weak formulation. We begin this subsection by introducing some useful notation. We first introduce some Sobolev spaces [**35**] and norms. If W is a bounded domain of $\mathbb{R}^d$ and m is a non negative integer, the Sobolev space $H^m(W) = W^{m,2}(W)$ is defined in the usual way with the usual norm $\|\cdot\|_{m,W}$ and semi-norm $|.|_{m,W}$. In particular, $H^0(W) = L^2(W)$ and we write $\|\cdot\|_W$ for $\|\cdot\|_{0,W}$. Similarly we denote by $(\cdot,\cdot)_W$ the $L^2(W)$ $[L^2(W)]^d$ or $[L^2(W)]^{d\times d}$ inner product. For shortness if W is equal to Ω, we will drop the index Ω, while for any $m \geq 0$, $\|\cdot\|_{m,\star} = \|\cdot\|_{m,\Omega_\star}$, $|.|_{m,\star} = |.|_{m,\Omega_\star}$ and $(.,.)_\star = (\cdot,\cdot)_{\Omega_\star}$, for $\star = f, p$. The space $H_0^m(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$. Let $[H^m(\Omega)]^d$ be the space of vector valued functions $\mathbf{v} = (v_1, \ldots, v_d)$ with components v_i in $H^m(\Omega)$. The norm and the seminorm on $[H^m(\Omega)]^d$ are given by

$$\| \mathbf{v} \|_{m,\Omega} := \left(\sum_{i=0}^{d} \| v_i \|_{m,\Omega}^2 \right)^{1/2} \quad \text{and } |\mathbf{v}|_{m,\Omega} := \left(\sum_{i=0}^{d} |v_i|_{m,\Omega}^2 \right)^{1/2} . \tag{14}$$

For a connected open subset of the boundary $E \subset \partial\Omega_f \cup \partial\Omega_p$, we write $\langle ., . \rangle_E$ for the $L^2(E)$ inner product (or duality pairing), that is, for scalar valued functions λ, σ one defines:

$$\langle \lambda, \sigma \rangle_E := \int_E \lambda \sigma ds \tag{15}$$

For a open subset F of the entire domain Ω, i.e. $F \subseteq \Omega$, we define the space $H(\text{div}; F)$ by:

$$H(\text{div}; F) := \left\{ \mathbf{v} \in [L^2(F)]^d : \quad \text{div}\, \mathbf{v} \in L^2(F) \right\}, \tag{16}$$

with a norm:

$$\| \mathbf{v} \|_{H(\text{div};F)} := \left(\| \mathbf{v} \|_{[L^2(F)]^d}^2 + \| \text{div}\, \mathbf{v} \|_{L^2(F)}^2 \right)^{\frac{1}{2}}, \quad \forall \mathbf{v} \in H(\text{div}; F). \tag{17}$$

To present a variational form of the coupled problem we define the following three spaces for the velocity $\mathbf{u}$, the structure displacement η_p and the pressure:

$$\mathbf{H} := \left\{ \mathbf{v} \in \mathbf{H}(\text{div}; \Omega) : \mathbf{v}_f \in [H^1(\Omega_f)]^d, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_f, \mathbf{v} \cdot \mathbf{n}_p = 0 \text{ on } \Gamma_p \right\},$$

equipped with the norm:

$$\| \mathbf{v} \|_{\mathbf{H}} := \left(|\mathbf{v}|_{1,f}^2 + \| \mathbf{v} \|_{H(\text{div};\Omega_p)}^2 \right)^{\frac{1}{2}},$$

$$\mathbf{X}_p := \left\{ \xi_p \in [H^1(\Omega_p)]^d : \xi_p = \mathbf{0} \text{ on } \Gamma_p \right\},$$

with the norm

$$\| \xi_p \|_{\mathbf{X}_p} := |\xi_p|_{1,p},$$

and

$$\mathbb{M} := L_0^2(\Omega) \times L_0^2(\Omega_p),$$

equipped with the norm $\| \mathbf{Q} \|_{\mathbb{M}} := \left(\| Q_1 \|_{0,\Omega}^2 + \| Q_2 \|_{0,\Omega_p}^2 \right)^{1/2}, \forall \mathbf{Q} = (Q_1, Q_2) \in \mathbb{M}$.

Note that the vector valued functions in $\mathbf{H}$ have (weakly) continuous normal components on Γ_{fp} (consequence of Theorem I.2.5 of [**36**, p. 27]).

We set $\mathbb{H} = \mathbf{H} \times \mathbf{X}_p$ equipped with the product norm

$$\| \mathbf{V} \|_{\mathbb{H}} := \| \mathbf{v} \|_{\mathbf{H}} + \| \xi_p \|_{\mathbf{X}_p}, \quad \forall \mathbf{V} = (\mathbf{v}, \xi_p) \in \mathbb{H}. \tag{18}$$

Let us further introduce two bilinear forms:

$$\mathbf{A} : \mathbb{H} \times \mathbb{H} \to \mathbb{R}, (\mathbf{U}, \mathbf{V}) \mapsto \mathbf{A}(\mathbf{U}, \mathbf{V}) \text{ define by,}$$

$$\begin{aligned} \mathbf{A}(\mathbf{U}, \mathbf{V}) \quad := \quad & (2\mu \mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}))_{\Omega_f} + \left(\mu \mathbf{K}^{-1} \mathbf{u}, \mathbf{v} \right)_{\Omega_p} \\ + \quad & (2\mu_p \mathbf{D}(\eta_p), \mathbf{D}(\xi_p))_{\Omega_p} + (\lambda_p \nabla \cdot \eta_p, \nabla \cdot \xi_p)_{\Omega_p} \\ + \quad & \sum_{j=1}^{d-1} \langle \mu \alpha_{BJS} \sqrt{K_j^{-1}} \mathbf{u}_f \cdot \tau_{f,j}, \mathbf{v}_f \cdot \tau_{f,j} \rangle_{\Gamma_{fp}}, \end{aligned}$$

$$\mathbf{B} : \mathbb{H} \times \mathbb{M} \to \mathbb{R}, (\mathbf{V}, \mathbf{Q}) \mapsto \mathbf{B}(\mathbf{V}, \mathbf{Q}) \text{ with}$$

$$\mathbf{B}(\mathbf{V}, \mathbf{Q}) := -(Q_1, \text{div}\, \mathbf{v})_\Omega - \alpha (Q_2, \text{div}\, \xi_p)_{\Omega_p}, \text{ where } \mathbf{Q} = (Q_1, Q_2),$$

and two linear forms

$$\mathbf{L} : \mathbb{H} \to \mathbb{R}, \mathbf{V} \mapsto \mathbf{L}(\mathbf{V}) := (\mathbf{f}, \mathbf{v})_\Omega$$

and

$$\mathbf{G} : \mathbb{M} \to \mathbb{R}, \mathbf{Q} = (Q_1, Q_2) \mapsto \mathbf{G}(\mathbf{Q}) := -(g, Q_1)_\Omega.$$

The weak formulation of the coupled problem (1)-(13) can be stated as follows: find $(\mathbf{U},\mathbf{P}) \in \mathbb{H}\times\mathbb{M}$ with $\mathbf{U}=(\mathbf{u},\eta_p)$ and $\mathbf{P}=(p,p_p)$ such that:

$$\left\{\begin{array}{ccll} \mathbf{A}(\mathbf{U},\mathbf{V})+\mathbf{B}(\mathbf{V},\mathbf{P}) & = & \mathbf{L}(\mathbf{V}) & \forall \mathbf{V}=(\mathbf{v},\xi_p)\in\mathbb{H} \\ \mathbf{B}(\mathbf{U},\mathbf{Q}) & = & \mathbf{G}(\mathbf{Q}) & \forall \mathbf{Q}=(Q_1,Q_2)\in\mathbb{M}. \end{array}\right. \tag{19}$$

Note that if $\mathbf{f}$ and g are of mean zero, (19) directly implies that (1)-(11) hold (the differential equations being understood in the distributional sense), while the interface conditions (12) and (13) are imposed in a weak sense.

This problem has a unique solution as proved in [**29**, Theorem 3.1].

THEOREM 2.1. *If $\boldsymbol{f}\in[L^2(\Omega)]^d$ and $g\in L^2_0(\Omega)$, then there exists a unique solution $(\boldsymbol{U},P)\in\mathbb{H}\times\mathbb{M}$ to the problem (19).*

2.3. Discontinuous Galerkin Discretization. In this section, we will use the nonconforming Crouzeix-Raviart piecewise linear finite element approximation for velocity and piecewise constant approximation for pressure and establish the existence and uniqueness of a finite element solution of the discrete problem.

Let $\mathcal{T}_h$ be a family of triangulations of $\bar{\Omega}$ with nondegenerate elements (i.e. triangles for $d=2$ and tetrahedrons for $d=3$). For any $K\in\mathcal{T}_h$, we denote by h_K the diameter of K and ρ_K the diameter of the largest ball inscribed into K.

We set:

$$h=\max_{K\in\mathcal{T}_h} h_K, \text{ and } \sigma_h=\max_{T\in\mathcal{T}_h}\frac{h_K}{\rho_K} \tag{20}$$

We assume that the family of triangulations is regular, in the sense that there exists $\sigma_0>0$ such that $\sigma_h\leqslant\sigma_0$, for all $h>0$. We also assume that the triangulation is conforming with respect to the partition of Ω into Ω_f and Ω_p, namely each $K\in\mathcal{T}_h$ is either in Ω_f or in Ω_p (see Figs. 2, 3,4 for illustration).

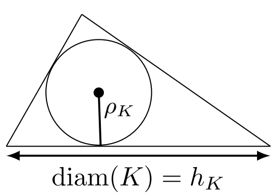

FIGURE 2. Isotropic element K in $\mathbb{R}^2$.

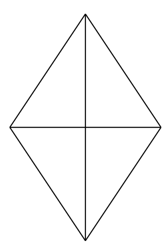

FIGURE 3. Example of conforming mesh in $\mathbb{R}^2$

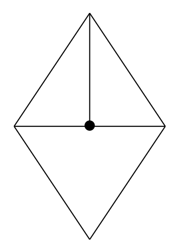

FIGURE 4. Example of nonconforming mesh in $\mathbb{R}^2$

Let $\mathcal{T}_h^f$ and $\mathcal{T}_h^p$ be the corresponding induced triangulations of Ω_f and Ω_p. For any $K\in\mathcal{T}_h$, we denote by $\mathcal{E}(K)$ (resp. $\mathcal{N}(K)$) the set of its edges ($d=2$) or faces ($d=3$) (resp. vertices) and set $\mathcal{E}_h=\displaystyle\bigcup_{K\in\mathcal{T}_h}\mathcal{E}(K)$, $\mathcal{N}_h=\displaystyle\bigcup_{K\in\mathcal{T}_h}\mathcal{N}(K)$. For $\mathcal{A}\subset\overline{\Omega}$ we define

$$\mathcal{E}_h(\mathcal{A})=\{E\in\mathcal{E}_h : E\subset\mathcal{A}\}.$$

Notice that $\mathcal{E}_h$ can be split up in the form

$$\mathcal{E}_h=\mathcal{E}_h(\Omega_f^+)\cup\mathcal{E}_h(\Omega_p)\cup\mathcal{E}_h(\partial\Omega_p) \tag{21}$$

where $\Omega_f^+=\Omega_f\cup\Gamma_f$. Note that $\mathcal{E}_h(\Gamma_{fp})$ is included in $\mathcal{E}_h(\partial\Omega_p)$.

With every edges $E \in \mathcal{E}_h$, we associate a unit vector $\mathbf{n}_E$ such that $\mathbf{n}_E$ is orthogonal to E and equals to the unit exterior normal vector to $\partial\Omega$ if $E \subset \partial\Omega$. For any $E \in \mathcal{E}_h$ and any piecewise continuous function φ, we denote by $[\varphi]_E$ its jump across E in the direction of $\mathbf{n}_E$:

$$[\varphi]_E(x) := \begin{cases} \lim\limits_{t\to 0+} \varphi(x + t\mathbf{n}_E) - \lim\limits_{t\to 0+} \varphi(x - t\mathbf{n}_E) & \text{for an interior edge/face } E, \\ -\lim\limits_{t\to 0+} \varphi(x - t\mathbf{n}_E) & \text{for a boundary edge/face } E \end{cases}$$

Based on the above notations, we introduce a variant of the nonconforming Crouzeix-Raviart piecewise linear finite element space:

$$\mathbf{H}_h := \Big\{ \mathbf{v}_h \in [L^2(\Omega)]^d : \mathbf{v}_{h|K} \in [\mathbb{P}^1(K)]^d \ \forall K \in \mathcal{T}_h, ([\mathbf{v}_h]_E, \mathbf{1})_E = 0 \ \forall E \in \mathcal{E}_h(\Omega_f^+) \ , \\ ([\mathbf{v}_h \cdot \mathbf{n}_E]_E, 1)_E = 0 \ \ \forall E \in \mathcal{E}_h(\Omega_p) \cup \mathcal{E}_h(\partial\Omega_p) \} \ ,$$

$$\mathbf{X}_{ph} := \Big\{ \xi_{ph} \in [L^2(\Omega_p)]^d : \xi_{ph|K} \in [\mathbb{P}^1(K)]^d \ \forall K \in \mathcal{T}_h^p, ([\xi_{ph}]_E, \mathbf{1})_E = 0 \ \forall E \in \mathcal{E}_h(\bar{\Omega}_p) \Big\} .$$

For $X \subseteq \Omega$, we set

$$E_h(X) \ := \ \Big\{ q_h \in L_0^2(X) : q_{h|K} \in \mathbb{P}^0(K) \ \forall K \subset X, K \in \mathcal{T}_h \Big\} .$$

and we define

$$\begin{aligned} \mathbb{M}_h &:= E_h(\Omega) \times E_h(\Omega_p) \subset \mathbb{M} \\ \mathbb{H}_h &:= \mathbf{H}_h \times \mathbf{X}_{ph} \not\subseteq \mathbb{H}. \end{aligned}$$

Where $\mathbb{P}^m(K)$ is the space of the restrictions to K of all polynomials of degree less than or equal to m.

The space $\mathbb{M}_h$ is equipped with the norm $\| \cdot \|_{\mathbb{M}}$ while the norm on $\mathbb{H}_h$ will be specified later on. The choice of $\mathbf{H}_h$ is more natural since the space $\mathbf{H}_h$ approximates only $\mathbf{H}(\text{div}; \Omega_p)$ and not $[H^1(\Omega_p)]^d$, while our a-posteriori error analysis is only valid in this larger space.

Let us introduce the discrete divergence operator $\text{div}_h \in \mathcal{L}(\mathbf{H}_h; E_h(\Omega)) \cap \mathcal{L}(\mathbf{H}; L_0^2(\Omega))$ by

$$(\text{div}_h \, \mathbf{v}_h)_{|K} = \text{div}(\mathbf{v}_{h|K}), \forall K \in \mathcal{T}_h, \tag{22}$$

or $\text{div}_h \in \mathcal{L}(\mathbf{X}_{ph}; E_h(\Omega_p)) \cap \mathcal{L}(\mathbf{X}_p; L_0^2(\Omega_p))$ by

$$(\text{div}_h \, \xi_{ph})_{|K} = \text{div}(\xi_{ph|K}), \forall K \in \mathcal{T}_h^p. \tag{23}$$

Then, for $\mathbf{U}_h = (\mathbf{u}_h, \eta_{ph}) \in \mathbb{H}_h$, $\mathbf{V}_h = (\mathbf{v}_h, \xi_{ph}) \in \mathbb{H}_h$ and $\mathbf{Q}_h = (Q_{1h}, Q_{2h}) \in \mathbb{M}_h$, we can introduce two bilinear forms:

$$\begin{aligned} \mathbf{A}_h(\mathbf{U}_h, \mathbf{V}_h) \ &:= \ \sum_{K \in \mathcal{T}_h^f} (2\mu \mathbf{D}(\mathbf{u}_h), \mathbf{D}(\mathbf{v}_h))_K + \left(\mu \mathbf{K}^{-1} \mathbf{u}_h, \mathbf{v}_h\right)_{\Omega_p} \\ &+ \ \sum_{K \in \mathcal{T}_h^p} (2\mu_p \mathbf{D}(\eta_{ph}), \mathbf{D}(\xi_{ph}))_K + (\lambda_p \, \text{div}_h \, \eta_{ph}, \text{div}_h \, \xi_{ph})_{\Omega_p} \\ &+ \ \sum_{j=1}^{d-1} \langle \mu \alpha_{BJS} \sqrt{K_j^{-1}} \mathbf{u}_{fh} \cdot \tau_{f,j}, \mathbf{v}_{fh} \cdot \tau_{f,j} \rangle_{\Gamma_{fp}}, \end{aligned}$$

$$\mathbf{B}_h(\mathbf{V}_h, \mathbf{Q}_h) := -(Q_{1h}, \text{div}_h \, \mathbf{v}_h)_\Omega - \alpha(Q_{2h}, \text{div}_h \, \xi_{ph})_{\Omega_p}.$$

Then, we propose the following discrete problem: find $(\mathbf{U}_h, \mathbf{P}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$ with $\mathbf{U}_h = (\mathbf{u}_h, \eta_{ph}) \in \mathbf{H}_h \times \mathbf{X}_{ph}$ and $\mathbf{P}_h = (p_h, p_{ph}) \in E_h(\Omega) \times E_h(\Omega_p)$ such that:

$$\begin{cases} \mathbf{A}_h(\mathbf{U}_h, \mathbf{V}_h) + \mathbf{B}_h(\mathbf{V}_h, \mathbf{P}_h) + \mathbf{J}(\mathbf{U}_h, \mathbf{V}_h) \ = \ \mathbf{L}(\mathbf{V}_h) \ \forall \mathbf{V}_h \in \mathbb{H}_h \\ \mathbf{B}_h(\mathbf{U}_h, \mathbf{Q}_h) \ = \ \mathbf{G}(\mathbf{Q}_h) \ \forall \mathbf{Q}_h \in \mathbb{M}_h. \end{cases} \tag{24}$$

This is the natural discretization of the weak formulation (19) only with the penalizing term $\mathbf{J}(\mathbf{U}_h, \mathbf{V}_h)$ added, where $\mathbf{V}_h = (\mathbf{v}_h, \xi_{ph})$. We define the bilinear form $\mathbf{J}(\cdot, \cdot)$ following the decomposition of $\mathcal{E}_h$:

$$\mathbf{J}(\mathbf{U}_h, \mathbf{V}_h) = \mathbf{J}_{\Omega_f^+}(\mathbf{u}_h, \mathbf{v}_h) + \mathbf{J}_{\Omega_p}(\mathbf{u}_h, \mathbf{v}_h) + \mathbf{J}_{\partial\Omega_p}(\mathbf{u}_h, \mathbf{v}_h) + \mathbf{J}_{\Omega_p^+}(\eta_{ph}, \xi_{ph}),$$

where

$$\begin{aligned}
\mathbf{J}_{\Omega_f^+}(\mathbf{u}_h,\mathbf{v}_h) &:= (1+2\mu)\sum_{E\in\mathcal{E}_h(\Omega_f^+)} h_E^{-1}\int_E [\mathbf{u}_h]_E\cdot[\mathbf{v}_h]_E ds,\\
\mathbf{J}_{\Omega_p}(\mathbf{u}_h,\mathbf{v}_h) &:= \sum_{E\in\mathcal{E}_h(\Omega_p)} h_E^{-1}\int_E [\mathbf{u}_h]_E\cdot[\mathbf{v}_h]_E ds,\\
\mathbf{J}_{\partial\Omega_p}(\mathbf{u}_h,\mathbf{v}_h) &:= \sum_{E\in\mathcal{E}_h(\partial\Omega_p)} h_E^{-1}\int_E [\mathbf{u}_h\cdot\mathbf{n}_E]_E[\mathbf{v}_h\cdot\mathbf{n}_E]_E ds \quad \text{and}\\
\mathbf{J}_{\Omega_p^+}(\eta_{ph},\xi_{ph}) &:= \sum_{E\in\mathcal{E}_h(\Omega_p^+)} h_E^{-1}\int_E (1+2\mu_p)[\eta_{ph}]_E\cdot[\xi_{ph}]_E ds.
\end{aligned}$$

Here, h_E is the length ($d=2$) or diameter ($d=3$) of E. Note that each element of $\mathcal{E}_h$ only contributes with one jump term in $\mathbf{J}(\mathbf{U}_h,\mathbf{V}_h)$.

We are now able to define the norm on $\mathbb{H}_h$:

$$\| \mathbf{V}_h \|_{\mathbb{H}_h} := \Big[\| \mathbf{v}_h \|_{\mathbf{H}_h}^2 + \| \xi_{ph} \|_{\mathbf{X}_{ph}}^2 + \mathbf{J}(\mathbf{V}_h,\mathbf{V}_h)\Big]^{1/2}, \tag{25}$$

where

$$\| \mathbf{v}_h \|_{\mathbf{H}_h} := \left(\sum_{K\in\mathcal{T}_h^f} |\mathbf{v}_h|_{1,K}^2 + \sum_{j=1}^{d-1}\langle \mathbf{v}_{fh}\cdot\tau_j, \mathbf{v}_{fh}\cdot\tau_j\rangle_{\Gamma_{fp}} + \| \mathbf{v}_h \|_{\Omega_p}^2 + \| \operatorname{div}_h \mathbf{v}_h \|_{\Omega_p}^2\right)^{1/2},$$

and

$$\| \xi_{ph} \|_{\mathbf{X}_{ph}} := \left(\sum_{K\in\mathcal{T}_h^p} |\xi_{ph}|_{1,K}^2\right)^{1/2}.$$

The following results holds [**29**, Theorems 4.1 and 5.1]:

THEOREM 2.2. *There exists a unique solution $(\boldsymbol{U}_h,\boldsymbol{P}_h)\in\mathbb{H}_h\times\mathbb{M}_h$ to discrete problem (24) and if the solution $(\boldsymbol{U},\boldsymbol{P})\in\mathbb{H}\times\mathbb{M}$ of the continuous problem (19) is smooth enough, then we have:*

$$\| \boldsymbol{U}-\boldsymbol{U}_h \|_{\mathbb{H}_h\cup\mathbb{H}} + \| \boldsymbol{P}-\boldsymbol{P}_h \|_{\mathbb{M}} \lesssim h\left(|\boldsymbol{u}|_{2,\Omega_f} + |\boldsymbol{u}|_{2,\Omega_p} + |\eta_p|_{2,\Omega_p} + |p|_{1,\Omega_f} + |p|_{1,\Omega_p}\right).$$

Here and below, in order to avoid excessive use of constants, the abbreviation $x\lesssim y$ stand for $x\le cy$, with c a positive constant independent of x, y and $\mathcal{T}_h$.

3. Error estimators

In order to solve the Stokes-Biot coupled problem by efficient adaptive finite element methods, reliable and efficient a posteriori error analysis is important to provide appropriated indicators. In this section, we first define the local and global indicators and then the lower and upper error bounds are derived (see Sects. 5.2.1 and 5.2.2).

3.1. Residual error estimators. The general philosophy of residual error estimators is to estimate an appropriate norm of the correct residual by terms that can be evaluated easier, and that involve the data at hand. To this end denote the exact element residuals by:

$$
\begin{aligned}
\mathbf{R}_1 &= \mathbf{f} + \nabla \cdot \sigma_f(\mathbf{u}_h, p_h) \quad \text{in } K \in \mathcal{T}_h^f && (26)\\
\mathbf{R}_2 &= g - \nabla \cdot \mathbf{u}_h \quad \text{in } K \in \mathcal{T}_h^f && (27)\\
\mathbf{R}_3 &= \mu \mathbf{K}^{-1}\mathbf{u}_h + \nabla p_h \quad \text{in } K \in \mathcal{T}_h^p && (28)\\
\mathbf{R}_4 &= \mathbf{f} + \nabla \cdot \sigma_p(\eta_{ph}, p_{ph}) \quad \text{in } K \in \mathcal{T}_h^p && (29)\\
\mathbf{R}_5 &= g - \alpha \nabla \cdot \eta_{ph} - \nabla \cdot \mathbf{u}_h \quad \text{in } K \in \mathcal{T}_h^p && (30)\\
\mathbf{R}_6(j) &= \mu \alpha_{BJF} \sqrt{K_j^{-1}}(\mathbf{u}_{fh}) \cdot \tau_{f,j} + (\sigma_f(\mathbf{u}_h, p_h)\mathbf{n}_f) \cdot \tau_{f,j} \quad \text{on } E \in \mathcal{E}_h(\partial K \cap \bar{\Gamma}_{fp}) && (31)\\
\mathbf{R}_7 &= p_{ph} + (\sigma_f(\mathbf{u}_{fh}, p_h)\mathbf{n}_f) \cdot \mathbf{n}_f \quad \text{on } E \in \mathcal{E}_h(\partial K \cap \bar{\Gamma}_{fp}). && (32)
\end{aligned}
$$

As it is common, these exact residuals are replaced by some finite-dimensional approximation called approximate element residual $\mathbf{r}_{i,K}$, $i \in \{1, 4\}$:

$$\mathbf{r}_{i,K} \in [\mathbb{P}^m(K)]^d \quad \text{on } K \in \mathcal{T}_h^l, \quad l \in \{f, p\}.$$

This approximation is here achieved by projecting $\mathbf{f}$ on the space of piecewise constant functions in Ω_l, more precisely for all $K \in \mathcal{T}_h$, we take

$$\mathbf{f}_{K,l} = \frac{1}{|K|}\int_K \mathbf{f}(x)dx, \quad l \in \{f, p\}, \quad \forall K \in \mathcal{T}_h^l.$$

Finally the global function $\mathbf{f}_h$ is defined by:

$$\mathbf{f}_{h,l} = \mathbf{f}_{K,l} \quad \text{in } K, \quad \forall K \in \mathcal{T}_h^l.$$

Hence

$$
\begin{aligned}
\mathbf{r}_{1,K} &:= \mathbf{f}_{K,f} + \nabla \cdot \sigma_f(\mathbf{u}_h, p_h) \quad \text{in } K \in \mathcal{T}_h^f, && (33)\\
\mathbf{r}_{4,K} &:= \mathbf{f}_{K,p} + \nabla \cdot \sigma_p(\eta_{ph}, p_{ph}) \quad \text{in } K \in \mathcal{T}_h^p, && (34)
\end{aligned}
$$

with $\mathbf{u}_{l,h} := \mathbf{u}_{h|\Omega_l}$ and $p_{l,h} := p_{h|\Omega_l} \quad l = f, p$.

Next, introduce the gradient jump in normal direction by

$$\mathbf{J}_{E,\mathbf{n}_E} := \begin{cases} [\sigma_f(\mathbf{u}_h, p_h) \cdot \mathbf{n}_E]_E & \text{for an interior edge/face } E \in \mathcal{E}_h(\Omega_f), \\ \mathbf{0} & \text{for a boundary edge/face } E \in \mathcal{E}_h(\Gamma_f). \end{cases}$$

and

$$\mathbf{G}_{E,\mathbf{n}_E} := \begin{cases} [\sigma_p(\eta_{p,h}, p_{p,h}) \cdot \mathbf{n}_E]_E & \text{for an interior edge/face } E \in \mathcal{E}_h(\Omega_p), \\ \mathbf{0} & \text{for a boundary edge/face } E \in \mathcal{E}_h(\Gamma_p). \end{cases}$$

DEFINITION 3.1. *(**Residual error estimator**) The residual error estimator is locally defined by:*

$$\Upsilon_K := \left(\sum_{i=1}^{11} \Upsilon_{i,K}^2\right)^{\frac{1}{2}}, \quad \textit{for each } K \in \mathcal{T}_h, \tag{35}$$

where

$$\Upsilon_{1,K}^2 := \begin{cases} h_K^2 \parallel \boldsymbol{r}_{1,K} \parallel_K^2 & \text{if } K \in \mathcal{T}_h^f, \\ h_K^2 \parallel \boldsymbol{r}_{4,K} \parallel_K^2 & \text{if } K \in \mathcal{T}_h^p \end{cases}, \tag{36}$$

$$\Upsilon_{2,K}^2 := \begin{cases} \parallel \boldsymbol{R}_3 \parallel_K^2 & \text{if } K \in \mathcal{T}_h^p, \\ 0 & \text{if } K \in \mathcal{T}_h^f \end{cases}, \tag{37}$$

$$\Upsilon_{3,K}^2 := \begin{cases} \parallel \operatorname{curl}(\boldsymbol{R}_3) \parallel_K^2 & \text{if } K \in \mathcal{T}_h^p, \\ 0 & \text{if } K \in \mathcal{T}_h^f \end{cases}, \tag{38}$$

$$\Upsilon_{4,K}^2 := \begin{cases} \parallel \boldsymbol{R}_2 \parallel_K^2 & \text{if } K \in \mathcal{T}_h^f, \\ \parallel \boldsymbol{R}_5 \parallel_K^2 & \text{if } K \in \mathcal{T}_h^p \end{cases}, \tag{39}$$

$$\Upsilon_{5,K}^2 := \sum_{E \in \mathcal{E}_h(\partial K \cap \bar{\Gamma}_{fp})} h_E \left\{ \sum_{j=1}^{d-1} \parallel \boldsymbol{R}_6(j) \parallel_E^2 \right\}, \tag{40}$$

$$\Upsilon_{6,K}^2 := \sum_{E \in \mathcal{E}_h(\partial K \cap \bar{\Gamma}_{fp})} h_E \parallel \boldsymbol{R}_7 \parallel_E^2 \tag{41}$$

$$\Upsilon_{7,K}^2 := \begin{cases} \displaystyle\sum_{E \in \mathcal{E}_h(\partial K \cap \bar{\Omega}_f)} h_E \parallel \boldsymbol{J}_{E,\boldsymbol{n}_E} \parallel_E^2 & \text{if } K \in \mathcal{T}_h^f, \\ \displaystyle\sum_{E \in \mathcal{E}_h(\partial K \cap \bar{\Omega}_p)} h_E \left(\parallel \boldsymbol{G}_{E,\boldsymbol{n}_E} \parallel_E^2 + \parallel [p_h]_E \parallel_E^2 \right) & \text{if } K \in \mathcal{T}_h^p \end{cases}, \tag{42}$$

$$\Upsilon_{8,K}^2 := \sum_{E \in \mathcal{E}_h(\partial K \cap \Omega_p)} h_E^{-1} \parallel [\boldsymbol{u}_h]_E \parallel_E^2, \tag{43}$$

$$\Upsilon_{9,K}^2 := \sum_{E \in \mathcal{E}_h(\partial K \cap \partial\Omega_p)} h_E^{-1} \parallel [\boldsymbol{u}_h \cdot \boldsymbol{n}_E]_E \parallel_E^2, \tag{44}$$

$$\Upsilon_{10,K}^2 := \sum_{E \in \mathcal{E}_h(\partial K \cap \Omega_f^+)} h_E^{-1}(1+2\mu) \parallel [\boldsymbol{u}_h]_E \parallel_E^2 \tag{45}$$

$$\Upsilon_{11,K} := \sum_{E \in \mathcal{E}_h(\partial K \cap \Omega_p^+)} h_E^{-1} \parallel (1+2\mu_p)[\eta_{ph}]_E \parallel_E^2 . \tag{46}$$

The global residual error estimator is given by:

$$\Upsilon := \left(\sum_{K \in \mathcal{T}_h} \Upsilon_K^2 \right)^{\frac{1}{2}}. \tag{47}$$

Furthermore denote the local and global approximation terms by

$$\Psi_K := h_K \parallel \mathbf{f} - \mathbf{f}_h \parallel_K, \forall K \in \mathcal{T}_h,$$

and

$$\Psi := \left(\sum_{K \in \mathcal{T}_h} \Psi_K^2 \right)^{1/2}. \tag{48}$$

4. Analytical tools

4.1. Some technical results. Our a posteriori analysis requires some analytical results that are recalled.

The first one concerns a sort of Helmholtz decomposition of elements of $\mathbf{H}$. Recall first that if $d = 3$,

$$H_0(\operatorname{curl}, \Omega_p) = \{\psi \in L^2(\Omega_p)^3 : \operatorname{curl} \psi \in L^2(\Omega_p)^3 \text{ and } \psi \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega_p\}.$$

THEOREM 4.1. *(Ref. [**2**, Page 708]) Any* $\boldsymbol{v} \in \boldsymbol{H}$ *admits the Helmholtz type decomposition*

$$\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_1, \tag{49}$$

where $\boldsymbol{v}_0, \boldsymbol{v}_1 \in \boldsymbol{H}$ *but satisfying* $\boldsymbol{v}_0 \in H^1(\Omega)^d$,

$$\boldsymbol{v}_1 = \begin{cases} \boldsymbol{0} & in \quad \Omega_f, \\ \operatorname{curl} \beta_p & in \quad \Omega_p, \end{cases} \tag{50}$$

where $\beta_p \in H_0^1(\Omega_p)$ *if* $d = 2$, *while* $\beta_p \in H^1(\Omega_p)^3 \cap H_0(\operatorname{curl}, \Omega_p)$ *if* $d = 3$, *with the estimate*

$$\|\boldsymbol{v}_0\|_{1,\Omega} + \|\beta_p\|_{1,\Omega_p} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{H}}. \tag{51}$$

The second result that we need is a regularity result for the solution $\mathbf{u} \in \mathbf{H}$ of (19):

THEOREM 4.2. *([**2**, Page 710]) Let* $(\boldsymbol{U}, \boldsymbol{P}) \in \mathbb{H} \times \mathbb{M}$ *be the unique solution of (19) with* $\boldsymbol{U} = (\boldsymbol{u}, \eta_p) \in \boldsymbol{H} \times \boldsymbol{X}_p$. *If* $\boldsymbol{K} \in [C^{0,1}(\bar{\Omega}_p)]^{d\times d}$, *then there exists* $\delta > 0$ *such that:*

$$\boldsymbol{u}_{|\Omega_p} \in [H^{\frac{1}{2}+\delta}(\Omega_p)]^d.$$

Note that the regularity of $\mathbf{u} \in [H^{\frac{1}{2}+\delta}(\Omega_p)]^d$, with $\delta > 0$ allows to give a meaning to $\mathbf{J}_{\Omega_p}(\mathbf{u}, \mathbf{w}) + \mathbf{J}_{\partial\Omega_p}(\mathbf{u}, \mathbf{w})$ for all $\mathbf{w} \in \mathbf{H} \cup \mathbf{H}_h$ and hence to show that $\mathbf{J}(\mathbf{U}, \mathbf{W}) = 0$ for all $\mathbf{W} = (\mathbf{w}, \xi_p) \in \mathbb{H} \cup \mathbb{H}_h$.

Let us finish this section by an estimation of the non conformity error (see [**2**, Theorem 3.3]):

THEOREM 4.3. *For any* $\boldsymbol{U}_h = (\boldsymbol{u}_h, \eta_{ph}) \in \mathbb{H}_h$ *we have*

$$\inf_{\boldsymbol{W}_h \in \mathbb{H}_h \cap \mathbb{H}} \| \boldsymbol{U}_h - \boldsymbol{W}_h \|^2_{\mathbb{H}_h} \lesssim \boldsymbol{J}(\boldsymbol{U}_h, \boldsymbol{U}_h). \tag{52}$$

4.2. Clément interpolation operator. In order to derive the upper error bounds, we introduce the Clément interpolation operator $\mathrm{I}^0_{\mathrm{Cl}} : H^1_0(\Omega) \longrightarrow \mathcal{P}^b_c(\mathcal{T}_h)$ that approximates optimally non-smooth functions by continuous piecewise linear functions:

$$\mathcal{P}^b_c(\mathcal{T}_h) := \left\{ v \in C^0(\overline{\Omega}) : \ v_{|K} \in \mathbb{P}^1(K), \ \forall K \in \mathcal{T}_h \text{ and } v = 0 \text{ on } \partial\Omega \right\}$$

In addition, we will make use of a vector valued version of $\mathrm{I}^0_{\mathrm{Cl}}$, that is, $\mathbf{I}^0_{\mathrm{Cl}} : [H^1_0(\Omega)]^d \longrightarrow [\mathcal{P}^b_c(\mathcal{T}_h)]^d$, which is defined componentwise by $\mathrm{I}^0_{\mathrm{Cl}}$. The following lemma establishes the local approximation properties of $\mathrm{I}^0_{\mathrm{Cl}}$ (and hence of $\mathbf{I}^0_{\mathrm{Cl}}$), for a proof see [**37**, Section 3].

LEMMA 4.1. *There exist constants* $C_1, C_2 > 0$, *independent of* h, *such that for all* $v \in H^1_0(\Omega)$ *there hold*

$$\begin{aligned} \| v - I^0_{Cl}(v) \|_K &\leq C_1 h_K \| v \|_{1,\Delta(K)} \ \ \forall K \in \mathcal{T}_h, \quad and \qquad (53) \\ \| v - I^0_{Cl}(v) \|_E &\leq C_2 h_E^{1/2} \| v \|_{1,\Delta(E)} \ \ \forall E \in \mathcal{E}_h, \qquad (54) \end{aligned}$$

where $\Delta(K) := \cup\{K' \in \mathcal{T}_h : K' \cap K \neq \emptyset\}$ *and* $\Delta(E) := \cup\{K' \in \mathcal{T}_h : K' \cap E \neq \emptyset\}$.

4.3. Inverse inequalities. In order to derive the lower error bounds, we proceed similarly as in [**32**] and [**33**], by applying inverse inequalities, and the localization technique based on simplex-bubble and face-bubble functions. To this end, we recall some notation and introduce further preliminary results. Given $K \in \mathcal{T}_h$, and $E \in \mathcal{E}(K)$, we let b_K and b_E be the usual simplexe-bubble and face-bubble functions respectively (see (1.5) and (1.6) in [**38**]). In particular, b_K satisfies $b_K \in \mathbb{P}^3(K)$, $supp(b_K) \subseteq K$, $b_K = 0$ on ∂K, and $0 \leq b_K \leq 1$ on K. Similarly, $b_E \in \mathbb{P}^2(K)$, $supp(b_E) \subseteq \omega_E := \{K' \in \mathcal{T}_h : E \in \mathcal{E}(K')\}$, $b_E = 0$ on $\partial K \smallsetminus E$ and $0 \leq b_E \leq 1$ in ω_E. We also recall from [**31**] that, given $k \in \mathbb{N}$, there exists an extension operator $L : C(E) \longrightarrow C(K)$ that satisfies $L(p) \in \mathbb{P}^k(K)$ and $L(p)_{|E} = p, \forall p \in \mathbb{P}^k(E)$. A corresponding vectorial version of L, that is, the componentwise application of L, is denoted by $\mathbf{L}$. Additional properties of b_K, b_E and L are collected in the following lemma (see [**31**]):

LEMMA 4.2. *Given $k \in \mathbb{N}^*$, there exist positive constants depending only on k and shape-regularity of the triangulations (minimum angle condition), such that for each simplexe K and $E \in \mathcal{E}(K)$ there hold*

$$\begin{aligned}
\| q \|_K &\lesssim \| q b_K^{1/2} \|_K \lesssim \| q \|_K, \forall q \in \mathbb{P}^k(K) && (55)\\
|q b_K|_{1,K} &\lesssim h_K^{-1} \| q \|_K, \forall q \in \mathbb{P}^k(K) && (56)\\
\| p \|_E &\lesssim \| b_E^{1/2} p \|_E \lesssim \| p \|_E, \forall p \in \mathbb{P}^k(E) && (57)\\
\| L(p) \|_K + h_E |L(p)|_{1,K} &\lesssim h_E^{1/2} \| p \|_E \ \forall p \in \mathbb{P}^k(E) && (58)
\end{aligned}$$

LEMMA 4.3. *(Continuous trace inequality) There exists a positive constant $\beta_1 > 0$ depending only on σ_0 such that*

$$\| \boldsymbol{v} \|_{\partial K}^2 \leqslant \beta_1 \| \boldsymbol{v} \|_K \| \boldsymbol{v} \|_{1,K}, \ \forall K \in \mathcal{T}_h, \forall \boldsymbol{v} \in [H^1(K)]^d. \qquad (59)$$

5. Main results

We set $\mathbb{X} := \mathbb{H} \times \mathbb{M}$ and $\mathbb{X}_h := \mathbb{H}_h \times \mathbb{M}_h$ and define on $\mathbb{X}$, the continuous bilinear form $\mathbb{B}$ by:

$$\mathbb{B}(\mathbb{U}, \mathbb{W}) := \mathbf{A}(\mathbf{U}, \mathbf{V}) + \mathbf{B}(\mathbf{V}, \mathbf{P}) + \mathbf{B}(\mathbf{U}, \mathbf{Q}) \quad \text{for } \mathbb{U} = (\mathbf{U}, \mathbf{P}) \quad \text{and for } \mathbb{W} = (\mathbf{V}, \mathbf{Q}).$$

We also define on the discrete space $\mathbb{H}_h$, the form,

$$\mathbb{B}_h(\mathbb{U}_h, \mathbb{W}_h) := \mathbf{A}_h(\mathbf{U}_h, \mathbf{V}_h) + \mathbf{B}_h(\mathbf{V}_h, \mathbf{P}_h) + \mathbf{B}_h(\mathbf{U}_h, \mathbf{Q}_h) + \mathbf{J}(\mathbf{U}_h, \mathbf{V}_h) \text{ for } \mathbb{U}_h = (\mathbf{U}_h, \mathbf{P}_h) \text{ and } \mathbb{W}_h = (\mathbf{V}_h, \mathbf{Q}_h).$$

The spaces $\mathbb{X}$ and $\mathbb{X}_h$ are equipped with the product-norms:

$$|||(\mathbf{U}, \mathbf{P})||| = \| \mathbf{U} \|_{\mathbb{H}} + \| \mathbf{P} \|_{\mathbb{M}} \quad \text{and } |||(\mathbf{U}_h, \mathbf{P}_h)|||_h = \| \mathbf{U}_h \|_{\mathbb{H}_h \cup \mathbb{H}} + \| \mathbf{P}_h \|_{\mathbb{M}} \quad \text{respectively.}$$

To prove local efficiency for $\omega \subset \Omega$, let us denote by

$$\begin{aligned}
\| (\mathbf{W}, \mathbf{Q}) \|_{h,\omega}^2 &= \sum_{K \subset \bar{\omega} \cap \bar{\Omega}_f} |\mathbf{v}|_{1,K}^2 + \sum_{K \subset \bar{\omega} \cap \bar{\Omega}_p} |\xi_p|_{1,K}^2 \\
&+ \sum_{K \subset \bar{\omega} \cap \bar{\Omega}_p} (\| \mathbf{v} \|_K^2 + \| \operatorname{div}_h \mathbf{v} \|_K^2) \\
&+ \|\mathbf{v}_f \times \mathbf{n}\|_{\Gamma_{fp} \cap \bar{\omega}}^2 + \sum_{K \subset \bar{\omega}} \mathbf{J}_K(\mathbf{W}, \mathbf{W}) \\
&+ \| \mathbf{Q} \|_\omega, \quad \forall (\mathbf{W}, \mathbf{Q}) = (\mathbf{v}, \xi_p, \mathbf{Q}) \in \mathbf{H}_h \cup \mathbf{H} \times \mathbf{X}_p \cup \mathbf{X}_{ph} \times \mathbb{M},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{J}_K(\mathbf{W}, \mathbf{W}) &= (1 + 2\mu) \sum_{E \in \mathcal{E}_h(\Omega_f^+) \cap \mathcal{E}(K)} h_E^{-1} \|[\mathbf{v}]_E\|_E^2 \\
&+ \sum_{E \in \mathcal{E}_h(\Omega_p) \cap \mathcal{E}(K)} h_E^{-1} \|[\mathbf{v}]_E\|_E^2 + \sum_{E \in \mathcal{E}_h(\partial\Omega_p) \cap \mathcal{E}(K)} h_E^{-1} \|[\mathbf{v} \cdot \mathbf{n}_E]_E\|_E^2 \\
&+ \sum_{E \in \mathcal{E}_h(\partial\Omega_p) \cap \mathcal{E}(K)} h_E^{-1} \| [(1 + \mu_p)\xi_p]_E \|_E^2
\end{aligned}$$

5.1. Optimality result. The main result of this paper can be stated as follows:

(1) **Reliability of $\{\Upsilon_K\}_{K \in \mathcal{T}_h}$**: The a posteriori error estimator Υ is consider reliable if it satisfies:

$$|||(\mathbf{U} - \mathbf{U}_h, \mathbf{P} - \mathbf{P}_h)|||_h \lesssim \Upsilon + \Psi. \qquad (60)$$

(2) **Efficiency of $\{\Upsilon_K\}_{K \in \mathcal{T}_h}$**:
Under the assumptions of Theorem 4.2, the following lower error bound holds:

$$\Upsilon_K \lesssim |||(\mathbf{U} - \mathbf{U}_h, \mathbf{P} - \mathbf{P}_h)|||_{h, \tilde{w}_K} + \sum_{K' \in \tilde{w}_K} \Psi_{K'}, \qquad (61)$$

where $\tilde{w}_K$ is a finite union of neighboring elements of K.

5.2. Proof of the a-posteriori error estimate.

5.2.1. ***Proof of the Reliability Estimate.*** In this subsection, we shall prove the estimate (60).
Let us start with the following result.

LEMMA 5.1. *Let the assumptions of Theorem 4.2 be satisfied. Then for all* $\mathbb{W} = (\boldsymbol{V}, \boldsymbol{P}) \in \mathbb{H} \times \mathbb{M}$*, we have the estimate:*

$$\mathbb{B}_h(\mathbb{U} - \mathbb{U}_h, \mathbb{W}) \lesssim (\tilde{\Upsilon} + \Psi)|||\mathbb{W}|||_h, \tag{62}$$

where the estimator $\tilde{\Upsilon}$ *is defined by:*

$$\tilde{\Upsilon} \quad := \quad \left\{ \sum_{K \in \mathcal{T}_h} \left(\sum_{i=1}^{7} \Upsilon_{i,K}^2 \right) \right\}^{\frac{1}{2}}. \tag{63}$$

PROOF. Let $\mathbb{W} = (\mathbf{V}, \mathbf{Q}) \in \mathbb{H} \times \mathbb{M}$ with $\mathbf{V} = (\mathbf{v}, \eta_p)$ and $\mathbf{Q} = (Q_1, Q_2)$. By Theorem 4.1 $\mathbf{v}$ admits the decomposition (49) with $\mathbf{v}_0, \mathbf{v}_1 \in \mathbf{H}$ and satisfying the properties stated in Theorem 4.1. Then we take $\mathbb{W}_h = (\mathbf{V}_h, \mathbf{0}) \in \mathbb{H}_h \times \mathbb{M}_h$ where $\mathbf{V}_h = (\mathbf{v}_h, \xi_{ph})$ with $\mathbf{v}_h = \mathbf{v}_{0,h} + \mathbf{v}_{1,h}$, where $\mathbf{v}_{0,h} = \mathbf{I}^0_{\mathrm{Cl}} \mathbf{v}_0$ and

$$\mathbf{v}_{1,h} = \left\{ \begin{array}{ccc} \mathbf{0} & \text{in} & \Omega_f, \\ \operatorname{curl} \mathbf{I}^0_{\mathrm{Cl}} \psi & \text{in} & \Omega_p, \end{array} \right. \qquad \text{and } \xi_{ph} = \mathbf{I}^0_{\mathrm{Cl}} \xi_p. \tag{64}$$

In $2D$, $\mathbf{I}^0_{\mathrm{Cl}} \psi$ is the standard Clément interpolant of ψ, while in $3D$, we take the vectorial Clément interpolant from [**2**] that satisfies the same estimate as the standard one (see [**2**]). Note that $\mathbf{v}_{0,h}$ belongs to $\mathbf{H}_h \cap [H^1_0(\Omega)]^d$ while $\mathbf{v}_{1,h}$ simply belongs to $\mathbf{H} \cap \mathbf{H}_h$ ($\mathbf{I}^0_{\mathrm{Cl}} \psi$ being in $H^1_0(\Omega_p)$ if $d = 2$ and $\mathbf{I}^0_{\mathrm{Cl}} \psi \in H^1(\Omega_p)^3 \cap H_0(\mathrm{curl}, \Omega_p)$ if $d = 3$), its curl belongs to $H_0(\mathrm{div}, \Omega_p)$, hence $\mathbf{v}_{1,h}$, its extension by zero in Ω_f, stays in $H_0(\mathrm{div}, \Omega)$. With these definitions and noticing that

$\mathrm{div}(\mathbf{v}-\mathbf{v}_h)=\mathrm{div}(\mathbf{v}_0-\mathbf{v}_{0,h})$ and that $\mathbf{J}(\mathbf{U}_h,\mathbf{V}_h)=0$, we may write:

$$
\begin{aligned}
\mathbb{B}_h(\mathbb{U}-\mathbb{U}_h,\mathbb{W}) &= \mathbb{B}_h(\mathbb{U}-\mathbb{U}_h,\mathbb{W}-\mathbb{W}_h)\\
&= \mathbf{A}_h(\mathbf{U}-\mathbf{U}_h,\mathbf{V}-\mathbf{V}_h)+\mathbf{B}_h(\mathbf{V}-\mathbf{V}_h,\mathbf{P}-\mathbf{P}_h)+\mathbf{B}_h(\mathbf{U}-\mathbf{U}_h,\mathbf{Q})\\
&= \mathbf{A}_h(\mathbf{U},\mathbf{V}-\mathbf{V}_h)+\mathbf{B}_h(\mathbf{V}-\mathbf{V}_h,\mathbf{P})+\mathbf{B}_h(\mathbf{U},\mathbf{Q})\\
&- [\mathbf{A}_h(\mathbf{U}_h,\mathbf{V}-\mathbf{V}_h)+\mathbf{B}_h(\mathbf{V}-\mathbf{V}_h,\mathbf{P}_h)+\mathbf{B}_h(\mathbf{U}_h,\mathbf{Q})]\\
&= \mathbf{L}(\mathbf{V}-\mathbf{V}_h)+\mathbf{G}(\mathbf{Q})\\
&- [\mathbf{A}_h(\mathbf{U}_h,\mathbf{V}-\mathbf{V}_h)+\mathbf{B}_h(\mathbf{V}-\mathbf{V}_h,\mathbf{P}_h)+\mathbf{B}_h(\mathbf{U}_h,\mathbf{Q})]\\
&= (\mathbf{f},\mathbf{v}-\mathbf{v}_h)_\Omega-(g,Q_1)_\Omega\\
&- [\mathbf{A}_h(\mathbf{U}_h,\mathbf{V}-\mathbf{V}_h)+\mathbf{B}_h(\mathbf{V}-\mathbf{V}_h,\mathbf{P}_h)+\mathbf{B}_h(\mathbf{U}_h,\mathbf{Q})]\\
&= \sum_{K\in\mathcal{T}_h}\{(\mathbf{f},\mathbf{v}-\mathbf{v}_h)_K-(g,Q_1)_K\}\\
&- \mathbf{A}_h(\mathbf{U}_h,\mathbf{V}-\mathbf{V}_h)-\mathbf{B}_h(\mathbf{V}-\mathbf{V}_h,\mathbf{P}_h)-\mathbf{B}_h(\mathbf{U}_h,\mathbf{Q})\\
&= \sum_{K\in\mathcal{T}_h}\{(\mathbf{f},\mathbf{v}-\mathbf{v}_h)_K\}-\sum_{K\in\mathcal{T}_h^f}(g,Q_1)_K\\
&- \sum_{K\in\mathcal{T}_h^f}(2\mu\mathbf{D}(\mathbf{u}_h),\mathbf{D}(\mathbf{v}-\mathbf{v}_h))_K-\sum_{K\in\mathcal{T}_h}\left(\mu\mathbf{K}^{-1}\mathbf{u}_h,\mathbf{v}-\mathbf{v}_h\right)_K\\
&- \sum_{K\in\mathcal{T}_h^p}(2\mu_p\mathbf{D}(\eta_{ph}),\mathbf{D}(\xi-\xi_{ph}))_K-\sum_{K\in\mathcal{T}_h^p}(\lambda_p\,\mathrm{div}_h\,\eta_{ph},\mathrm{div}(\xi_p-\xi_{ph}))_K\\
&- \sum_{j=1}^{d-1}\langle\mu\alpha_{BJS}\sqrt{K_j^{-1}}\mathbf{u}_{fh}\cdot\tau_{f,j},(\mathbf{v}_f-\mathbf{v}_{fh})\cdot\tau_{f,j}\rangle_{\Gamma_{fp}}\\
&+ \sum_{K\in\mathcal{T}_h}(p_h,\mathrm{div}(\mathbf{v}_0-\mathbf{v}_{0h}))_K+\sum_{K\in\mathcal{T}_h^p}\alpha(p_{ph},\mathrm{div}(\xi_p-\xi_{ph}))_K\\
&+ \sum_{K\in\mathcal{T}_h}(\mathbf{Q},\mathrm{div}\,\mathbf{u}_h)_K+\sum_{K\in\mathcal{T}_h^p}\alpha(Q_2,\mathrm{div}(\eta_{ph}))_K
\end{aligned}
$$

Integrate by parts element by element and add boundary (resp. internal) terms that appear on the same edge (resp. at the same element) and reminding that $\mathbf{v}=\mathbf{v}_0$ and $\mathbf{v}_h=\mathbf{v}_{0,h}$ in Ω_f, we obtain:

$$
\begin{aligned}
\mathbb{B}_h(\mathbb{U}-\mathbb{U}_h,\mathbb{W}) &= \sum_{K\in\mathcal{T}_h^f}(\mathbf{R}_1,\mathbf{v}-\mathbf{v}_h)_K-\sum_{K\in\mathcal{T}_h^p}(\mathbf{R}_3,\mathbf{v}-\mathbf{v}_h)_K+\sum_{K\in\mathcal{T}_h^p}(\mathbf{R}_4,\xi_p-\xi_{ph})_K\\
&- \sum_{K\in\mathcal{T}_h^f}(\mathbf{R}_2,Q_1)_K-\sum_{K\in\mathcal{T}_h^p}(\mathbf{R}_5,Q_2)_K+\sum_{E\in\mathcal{E}_h(\Gamma_{fp})}(\mathbf{R}_7,(\mathbf{v}_0-\mathbf{v}_{0h})\cdot\mathbf{n}_f)_E\\
&- \sum_{E\in\mathcal{E}_h(\Gamma_{fp})}\sum_{j=1}^{d-1}(\mathbf{R}_6(j),(\mathbf{v}_0-\mathbf{v}_{0h})\cdot\tau_j)_E-\sum_{E\in\mathcal{E}_h(\Omega_f^+)}(\mathbf{J}_{E,\mathbf{n}_E},\mathbf{v}_0-\mathbf{v}_{0h})_E\\
&- \sum_{E\in\mathcal{E}_h(\Omega_p^+)}(\mathbf{G}_{E,\mathbf{n}_E},\xi_p-\xi_{ph})_E+\sum_{E\in\mathcal{E}_h(\Omega_p)}([p_{ph}]_E,(\mathbf{v}_0-\mathbf{v}_{0,h})\cdot\mathbf{n}_E)_E
\end{aligned}
$$

We now introduce the approximation $\mathbf{f}_h$ of $\mathbf{f}$ for appropriated terms and we have:

$$\begin{aligned}
\mathbb{B}_h(\mathbb{U}-\mathbb{U}_h,\mathbb{W}) &= \sum_{K\in\mathcal{T}_h^f}(\mathbf{r}_1,\mathbf{v}-\mathbf{v}_h)_K - \sum_{K\in\mathcal{T}_h^p}(\mathbf{R}_3,\mathbf{v}_0-\mathbf{v}_{0,h})_K + \sum_{K\in\mathcal{T}_h^p}(\mathbf{r}_4,\xi_p-\xi_{ph})_K \\
&- \sum_{K\in\mathcal{T}_h^f}(\mathbf{R}_2,Q_1)_K - \sum_{K\in\mathcal{T}_h^p}(\mathbf{R}_5,Q_2)_K + \sum_{E\in\mathcal{E}_h(\Gamma_{fp})}(\mathbf{R}_7,(\mathbf{v}_0-\mathbf{v}_{0h})\cdot\mathbf{n}_f)_E \\
&- \sum_{E\in\mathcal{E}_h(\Gamma_{fp})}\sum_{j=1}^{d-1}(\mathbf{R}_6(j),(\mathbf{v}_0-\mathbf{v}_{0h})\cdot\tau_j)_E - \sum_{E\in\mathcal{E}_h(\Omega_f^+)}(\mathbf{J}_{E,\mathbf{n}_E},\mathbf{v}_0-\mathbf{v}_{0h})_E \\
&- \sum_{E\in\mathcal{E}_h(\Omega_p^+)}(\mathbf{G}_{E,\mathbf{n}_E},\xi_p-\xi_{ph})_E + \sum_{E\in\mathcal{E}_h(\Omega_p)}([p_{ph}]_E,(\mathbf{v}_0-\mathbf{v}_{0,h})\cdot\mathbf{n}_E)_E \\
&+ \sum_{K\in\mathcal{T}_h}(\mathbf{f}-\mathbf{f}_h,\mathbf{v}-\mathbf{v}_h)_K - \sum_{K\in\mathcal{T}_h^p}(\mathbf{R}_3,\mathbf{v}_1-\mathbf{v}_{1,h})_K
\end{aligned}$$

Now for a triangle $K\in\mathcal{T}_h^p$, we recall that

$$\mathbf{v}_1-\mathbf{v}_{1,h}=\operatorname{curl}(\psi-\mathbf{I}^0_{\mathrm{Cl}}\psi)\quad\text{ in } K,$$

and use Green's formula to get

$$\begin{aligned}
\sum_{K\in\mathcal{T}_h^p}(\mathbf{R}_3,\mathbf{v}_1-\mathbf{v}_{1,h})_K &= \sum_{K\in\mathcal{T}_h^p}\big(\mathbf{R}_3,\operatorname{curl}(\psi-\mathbf{I}^0_{\mathrm{Cl}}\psi)\big)_K \\
&- \sum_{K\in\mathcal{T}_h^p}\Big[\big(\operatorname{curl}\mathbf{R}_3,\psi-\mathbf{I}^0_{\mathrm{Cl}}\psi\big)_K \\
&+ \big(\operatorname{curl}\mathbf{R}_3\times\mathbf{n},\psi-\mathbf{I}^0_{\mathrm{Cl}}\psi\big)_{\partial K}\Big]
\end{aligned}$$

We deduce the error equation,

$$\begin{aligned}
\mathbb{B}_h(\mathbb{U}-\mathbb{U}_h,\mathbb{W}) &= \sum_{K\in\mathcal{T}_h^f}(\mathbf{r}_1,\mathbf{v}-\mathbf{v}_h)_K - \sum_{K\in\mathcal{T}_h^p}(\mathbf{R}_3,\mathbf{v}_0-\mathbf{v}_{0,h})_K + \sum_{K\in\mathcal{T}_h^p}(\mathbf{r}_4,\xi_p-\xi_{ph})_K \\
&- \sum_{K\in\mathcal{T}_h^f}(\mathbf{R}_2,Q_1)_K - \sum_{K\in\mathcal{T}_h^p}(\mathbf{R}_5,Q_2)_K + \sum_{E\in\mathcal{E}_h(\Gamma_{fp})}(\mathbf{R}_7,(\mathbf{v}_0-\mathbf{v}_{0h})\cdot\mathbf{n}_f)_E \\
&- \sum_{E\in\mathcal{E}_h(\Gamma_{fp})}\sum_{j=1}^{d-1}(\mathbf{R}_6(j),(\mathbf{v}_0-\mathbf{v}_{0h})\cdot\tau_j)_E - \sum_{E\in\mathcal{E}_h(\Omega_f^+)}(\mathbf{J}_{E,\mathbf{n}_E},\mathbf{v}_0-\mathbf{v}_{0h})_E \\
&- \sum_{E\in\mathcal{E}_h(\Omega_p^+)}(\mathbf{G}_{E,\mathbf{n}_E},\xi_p-\xi_{ph})_E + \sum_{E\in\mathcal{E}_h(\Omega_p)}([p_{ph}]_E,(\mathbf{v}_0-\mathbf{v}_{0,h})\cdot\mathbf{n}_E)_E \\
&+ \sum_{K\in\mathcal{T}_h}(\mathbf{f}-\mathbf{f}_h,\mathbf{v}-\mathbf{v}_h)_K \\
&+ \sum_{K\in\mathcal{T}_h^p}\Big[\big(\operatorname{curl}\mathbf{R}_3,\psi-\mathbf{I}^0_{\mathrm{Cl}}\psi\big)_K \\
&- \big(\operatorname{curl}\mathbf{R}_3\times\mathbf{n},\psi-\mathbf{I}^0_{\mathrm{Cl}}\psi\big)_{\partial K}\Big]
\end{aligned}$$

Cauchy-Schwarz inequality and the approximation properties of Lemma 4.1 imply the required estimate and finish the proof.

□

The second result of this subsection is given by the following lemma:

LEMMA 5.2. *Under the assumptions of Theorem 4.2, the following estimation holds:*

$$|||(\boldsymbol{U}-\boldsymbol{U}_h,\boldsymbol{P}-\boldsymbol{P}_h)|||_h \quad\lesssim\quad \tilde{\Upsilon}+\Psi+\inf_{\mathbb{W}_h\in\mathbb{H}\cap\mathbb{H}_h\times\mathbb{M}_h}|||\mathbb{U}_h-\mathbb{W}_h|||_h, \tag{65}$$

where $\mathbb{U}_h = (\boldsymbol{U}_h, \boldsymbol{P}_h)$ *and* $\tilde{\Upsilon}$ *is given by (63).*

Proof. For an arbitrary $\mathbb{W}_h \in \mathbb{H} \cap \mathbb{H}_h \times \mathbb{M}_h$, the inf-sup condition of $\mathbb{B}$ on $\mathbb{H} \times \mathbb{M}$ leads to

$$|||\mathbb{U} - \mathbb{W}_h|||_h \quad \lesssim \sup_{\mathbb{W} \in \mathbb{H} \times \mathbb{M}} \frac{\mathbb{B}(\mathbb{U} - \mathbb{W}_h, \mathbb{W})}{|||\mathbb{W}|||_h}, \tag{66}$$

hence

$$|||\mathbb{U} - \mathbb{W}_h|||_h \quad \lesssim \sup_{\mathbb{W}_h \in \mathbb{H} \times \mathbb{M}} \left\{ \frac{\mathbb{B}_h(\mathbb{U} - \mathbb{U}_h, \mathbb{W}) + \mathbb{B}_h(\mathbb{U}_h - \mathbb{W}_h, \mathbb{W})}{|||\mathbb{W}|||_h} \right\}. \tag{67}$$

Combining the estimates (62) and (67), it comes:

$$|||\mathbb{U} - \mathbb{W}_h|||_h \quad \lesssim \quad \tilde{\Upsilon} + \Psi + \sup_{\mathbb{W} \in \mathbb{H} \times \mathbb{M}} \frac{\mathbb{B}_h(\mathbb{U}_h - \mathbb{W}_h, \mathbb{W})}{|||\mathbb{W}|||_h}. \tag{68}$$

The continuity of the operator $\mathbb{B}_h$ implies that:

$$|||\mathbb{U} - \mathbb{W}_h|||_h \quad \lesssim \quad \tilde{\Upsilon} + \Psi + |||\mathbb{U}_h - \mathbb{W}_h|||_h. \tag{69}$$

Thus, by the triangular inequality we deduce that:

$$|||\mathbb{U} - \mathbb{U}_h|||_h \quad \lesssim \quad \tilde{\Upsilon} + \Psi + |||\mathbb{U}_h - \mathbb{W}_h|||_h, \quad \forall \mathbb{W}_h \in \mathbb{H} \cap \mathbb{H}_h \times \mathbb{M}_h, \tag{70}$$

or equivalently,

$$|||(\mathbf{U} - \mathbf{U}_h, \mathbf{P} - \mathbf{P}_h)|||_h \quad \lesssim \quad \tilde{\Upsilon} + \Psi + \inf_{\mathbb{W}_h \in \mathbb{H} \cap \mathbb{H}_h \times \mathbb{M}_h} |||\mathbb{U}_h - \mathbb{W}_h|||_h. \tag{71}$$

Thus, this lemma holds. $\square$

Combining Theorem 4.3 and estimate (65), we have the main result in this subsection:

Theorem 5.1. *Under the assumptions of Theorem 4.2, the a posteriori error estimator* Υ *satisfies (60).*

5.2.2. ***Proof of the Efficiency Estimate***. In this subsection, we shall prove the estimate (61). We bound each term of the residual separately. Since by theorem4.2 the jump of $\mathbf{U} = (\mathbf{u}, \eta_p) \in \mathbf{H} \times \mathbf{X}_p$ is zero through all the edges of Ω_p, hence for all $i \in \{8, 9, 10, 11\}$, we clearly have:

$$\Upsilon_{i,K}^2 \lesssim \mathbf{J}_K(\mathbf{U}_h, \mathbf{U}_h)) = \mathbf{J}_K(\mathbf{U}_h - \mathbf{U}, \mathbf{U}_h - \mathbf{U}) \lesssim |||(\mathbf{U} - \mathbf{U}_h, \mathbf{P} - \mathbf{P}_h)|||_{h,K} \tag{72}$$

Hence it remains to estimate the local indicators for $i \leq 7$.

(1) **Residual Element $\mathbf{r}_{1,K}$ in Ω_f.** Let $K \in \mathcal{T}_h^f$ and set $\mathbf{w}_K := \mathbf{r}_{1,K} b_K \in [H_0^1(K)]^d$, and consider

$$\int_K \mathbf{r}_{1,K} \cdot \mathbf{w}_K = \int_K [\mathbf{f}_{K,f} + \nabla \cdot \sigma_f(\mathbf{u}_h, p_h)] \cdot \mathbf{w}_K \tag{73}$$

Introduce $\mathbf{f}$ and use the weak formulation (19) with $\mathbf{V} = (\mathbf{w}_K, \mathbf{0}) \in \mathbb{H}$ to get,

$$\begin{aligned}
\int_K \mathbf{r}_{1,K} \cdot \mathbf{w}_K \quad &= \quad \int_K (\mathbf{f}_{K,f} - \mathbf{f}) \cdot \mathbf{w}_K \\
&+ \quad \int_K (2\mu \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{w}_K) - p \operatorname{div} \mathbf{w}_K) \\
&+ \quad \int_K [2\mu \operatorname{div} \mathbf{D}(\mathbf{u}_h) - \nabla p_h] \cdot \mathbf{w}_K
\end{aligned}$$

Integrating by parts in this last term we get

$$\begin{aligned}
\int_K \mathbf{r}_{1,K} \cdot \mathbf{w}_K \quad &= \quad \int_K (\mathbf{f}_{K,f} - \mathbf{f}) \cdot \mathbf{w}_K + 2\mu \int_K \mathbf{D}(\mathbf{u} - \mathbf{u}_h) : \mathbf{D}(\mathbf{w}_K) \\
&- \quad \int_K (p - p_h) \operatorname{div} \mathbf{w}_K.
\end{aligned}$$

Cauchy-Schwarz inequality implies that

$$\int_K \mathbf{r}_{1,K}\cdot\mathbf{w}_K \lesssim \|\mathbf{f}-\mathbf{f}_{K,f}\|_K\|\mathbf{w}_K\|_K + [2\mu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_K + \|p-p_h\|_K]\,\|\nabla\mathbf{w}_K\|_K$$

The inverse inequalities (55), (56) and the obvious relation $\|\mathbf{w}_K\|_K \leqslant \|\mathbf{r}_{1,K}\|_K$ imply

$$\|\mathbf{r}_{1,K}\|_K^2 \lesssim [\|\mathbf{f}-\mathbf{f}_{K,f}\|_K + h_K^{-1}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_K + h_K^{-1}\|p-p_h\|_K]\,\|\mathbf{r}_{1,K}\|_K$$

or equivalently,

$$h_K\|\mathbf{r}_{1,K}\|_K \lesssim \Psi_K + |||(\mathbf{U}-\mathbf{U}_h,\mathbf{P}-\mathbf{P}_h)|||_{h,K} \tag{74}$$

(2) **Residual Element $\mathbf{r}_{4,K}$ in Ω_p.** Let $K\in\mathcal{T}_h^p$ and $\mathbf{w}_K := \mathbf{r}_{4,K}b_K \in [H_0^1(K)]^d$ and we consider

$$\int_K \mathbf{r}_{4,K}\cdot\mathbf{w}_K = \int_K [\mathbf{f}_{K,p} + \nabla\cdot\sigma_p(\eta_{ph},p_{ph})]\cdot\mathbf{w}_K \tag{75}$$

introduce $\mathbf{f}$ and use the weak formulation (19) with $\mathbf{V}=(\mathbf{0},\mathbf{w}_K)\in\mathbb{H}$ to get

$$\begin{aligned}\int_K \mathbf{r}_{4,K}\cdot\mathbf{w}_K &= \int_K(\mathbf{f}_{K,p}-\mathbf{f})\cdot\mathbf{w}_K + \int_K 2\mu_p\mathbf{D}(\eta_p):\mathbf{D}(\mathbf{w}_K)\\ &+ \int_K \lambda_p\nabla\cdot\eta_p\nabla\cdot\mathbf{w}_K - \alpha\int_K p_p\nabla\cdot\mathbf{w}_K\\ &+ \alpha\int_K \nabla p_{ph}\cdot\mathbf{w}_K - \int_K \nabla[\lambda_p(\nabla\cdot\eta_{ph})]\cdot\mathbf{w}_K - \int_K \nabla\cdot[2\mu_p\mathbf{D}(\eta_{ph})]\cdot\mathbf{w}_K\end{aligned}$$

Integrating by parts in these three last terms we get

$$\begin{aligned}\int_K \mathbf{r}_{4,K}\cdot\mathbf{w}_K &= \int_K(\mathbf{f}_{K,p}-\mathbf{f})\cdot\mathbf{w}_K + \int_K 2\mu_p\mathbf{D}(\eta_p-\eta_{ph}):\mathbf{D}(\mathbf{w}_K)\\ &+ \int_K \lambda_p\nabla\cdot(\eta_p-\eta_{ph})\nabla\cdot\mathbf{w}_K - \alpha\int_K(p_p-p_{ph})\nabla\cdot\mathbf{w}_K.\end{aligned}$$

Cauchy-Schwarz inequality, the conditions $0<\lambda_{\min}\le\lambda_p(\mathbf{x})\le\lambda_{\max}$ and $0<\mu_{\min}\le\mu_p(\mathbf{x})\le\mu_{\max}$ for all $\mathbf{x}\in\Omega_p$, and the inverse inequalities (55), (56) lead to

$$h_K\|\mathbf{r}_{4,K}\|_K \lesssim \Psi_K + |||(\mathbf{U}-\mathbf{U}_h,\mathbf{P}-\mathbf{P}_h)|||_{h,K} \tag{76}$$

From (74) and (76), we deduce:

$$\Upsilon_{1,K} \lesssim \Psi_K + |||(\mathbf{U}-\mathbf{U}_h,\mathbf{P}-\mathbf{P}_h)|||_{h,K}. \tag{77}$$

(3) **Residual Element $\mathbf{R}_3$ in Ω_p.** Let $K\in\mathcal{T}_h^p$ and, use the relation $\mu\mathbf{K}^{-1}\mathbf{u}+\nabla p=\mathbf{0}$ in Ω_p to obtain

$$\begin{aligned}\mathbf{R}_3 &= [\mu\mathbf{K}^{-1}\mathbf{u}_h+\nabla p_h]\\ &= -[\mu\mathbf{K}^{-1}\mathbf{u}+\nabla p] + [\mu\mathbf{K}^{-1}\mathbf{u}_h)+\nabla(p_h)]\\ &= -[\mu\mathbf{K}^{-1}(\mathbf{u}-\mathbf{u}_h)+\nabla(p-p_h)]\end{aligned}$$

As before Cauchy-Schwarz inequality leads to

$$\Upsilon_{2,K} = \|\mathbf{R}_3\|_{0,K} \lesssim |||(\mathbf{U}-\mathbf{U}_h,\mathbf{P}-\mathbf{P}_h)|||_{h,K} \tag{78}$$

(4) **Curl Residual Element** $\operatorname{curl}\mathbf{R}_3$ **in** Ω_p: For $K\in\mathcal{T}_h^p$, we use also the relation $\mu\mathbf{K}^{-1}\mathbf{u}+\nabla p=\mathbf{0}$ in Ω_p to obtain

$$\begin{aligned}\operatorname{curl}\mathbf{R}_3 &= \operatorname{curl}(\mu\mathbf{K}^{-1}\mathbf{u}_h+\nabla p_h)\\ &= -\operatorname{curl}[\mu\mathbf{K}^{-1}\mathbf{u}+\nabla p] + \operatorname{curl}[\mu\mathbf{K}^{-1}\mathbf{u}_h)+\nabla(p_h)]\\ &= -\operatorname{curl}[\mu\mathbf{K}^{-1}(\mathbf{u}-\mathbf{u}_h)+\nabla(p-p_h)]\end{aligned}$$

Using Cauchy-Schwarz inequality, we obtain,

$$\Upsilon_{3,K} = \|\operatorname{curl}\mathbf{R}_3\|_{0,K} \lesssim |||(\mathbf{U}-\mathbf{U}_h,\mathbf{P}-\mathbf{P}_h)|||_{h,K} \tag{79}$$

(5) **Residual Element $\mathbf{R}_2$ in Ω_f**: We directly see that

$$g - \operatorname{div} \mathbf{u}_h = \operatorname{div} \mathbf{u} - \operatorname{div} \mathbf{u}_h = \operatorname{div}(\mathbf{u} - \mathbf{u}_h),$$

hence by Cauchy-Schwarz inequality we conclude

$$\| \mathbf{R}_2 \|_K \quad \lesssim \quad |||(\mathbf{U} - \mathbf{U}_h, \mathbf{P} - \mathbf{P}_h)|||_{h,K}. \tag{80}$$

(6) **Residual Element $\mathbf{R}_5$ in Ω_p:** As before, we directly, see that

$$g - \alpha\nabla \cdot \eta_{ph} + \nabla \cdot \mathbf{u}_h = \alpha\nabla \cdot (\eta_p - \eta_{ph}) + \nabla \cdot (\mathbf{u} - \mathbf{u}_h),$$

hence by Cauchy-Schwarz inequality we have,

$$\| \mathbf{R}_5 \|_K \quad \lesssim \quad |||(\mathbf{U} - \mathbf{U}_h, \mathbf{P} - \mathbf{P}_h)|||_{h,K}. \tag{81}$$

The inequalities (80) and (81) lead to:

$$\Upsilon_{4,K} \quad \lesssim \quad |||(\mathbf{U} - \mathbf{U}_h, \mathbf{P} - \mathbf{P}_h)|||_{h,K}. \tag{82}$$

(7) **Normal Jump $\mathbf{J}_{E,\mathbf{n}_E}$ in Ω_f:** For each edge/face $E \in \mathcal{E}_h(\Omega_f)$, we consider $w_E = K_1 \cup K_2$. As $\mathbf{J}_{E,\mathbf{n}_E} \in [\mathbb{P}^0(E)]^d$ we set

$$\mathbf{w}_E := -\mathbf{J}_{E,\mathbf{n}_E} b_E \in [H_0^1(w_E)]^d.$$

First the weak formulation (19) with $\mathbf{V} = (\mathbf{w}_E, \mathbf{0}) \in \mathbb{H}$ yields

$$\mathbf{A}(\mathbf{U}, \mathbf{V}) + \mathbf{B}(\mathbf{V}, \mathbf{P}) = \mathbf{L}(\mathbf{V}),$$

that is equivalent to

$$\int_{w_E} \mathbf{f} \cdot \mathbf{w}_E = \int_{w_E} 2\mu \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{w}_E) - \int_{w_E} p \operatorname{div} \mathbf{w}_E - \int_{\partial w_E} \sigma_f(\mathbf{u}, p)\mathbf{n}_E \cdot \mathbf{w}_E. \tag{83}$$

By elementwise partial integration we further have

$$\begin{aligned} -\int_E \mathbf{J}_{E,\mathbf{n}_E} \cdot \mathbf{w}_E \quad &= \quad \int_{w_E} 2\mu \mathbf{D}(\mathbf{u}_h) : \mathbf{D}(\mathbf{w}_E) - \int_{w_E} p_h \operatorname{div}(\mathbf{w}_E) \\ &- \quad \sum_{i=1}^{2} \int_{K_i} (-2\mu \operatorname{div} \mathbf{D}(\mathbf{u}_h) + \nabla p_h) \cdot \mathbf{w}_E \end{aligned}$$

Hence by the previous identity (83) we get

$$\begin{aligned} -\int_E \mathbf{J}_{E,\mathbf{n}_E} \cdot \mathbf{w}_E \quad &= \quad \sum_{i=1}^{2} \int_{K_i} [\mathbf{f} + 2\mu \operatorname{div} \mathbf{D}(\mathbf{u}_h) + \nabla p_h] \cdot \mathbf{w}_E \\ &- \quad \int_{w_E} \mathbf{D}(\mathbf{u} - \mathbf{u}_h) : \mathbf{D}(\mathbf{w}_E) + \int_{w_E} (p - p_h) \operatorname{div}(\mathbf{w}_E) \end{aligned}$$

We introduce the approximation $\mathbf{f}_h$ of $\mathbf{f}$, use the Cauchy-Schwarz inequality and the inverse inequalities (57)-(58) to get

$$\begin{aligned} \| \mathbf{J}_{E,\mathbf{n}_E} \|_E \quad &\lesssim \quad h_E^{1/2} \left(\sum_{i=1}^{2} (\| \mathbf{f} - \mathbf{f}_h \|_{K_i} + \| \mathbf{r}_{1,K_i} \|_{K_i}) \right) \\ &+ \quad h_E^{-1/2} \left(|\mathbf{u} - \mathbf{u}_h|_{1,\omega_E} + \| p - p_h \|_{\omega_E} \right) \end{aligned}$$

The previous bound (74) of $\mathbf{r}_{1,K_i}$ and the obvious estimate $h_E \leqslant h_K$ imply that

$$h_E^{1/2} \| \mathbf{J}_{E,\mathbf{n}_E} \|_E \quad \lesssim \quad |\mathbf{u} - \mathbf{u}_h|_{1,\omega_E} + \| p - p_h \|_{\omega_E} + \sum_{K' \subset \omega_E} h_{K'} \| \mathbf{f} - \mathbf{f}_h \|_{K'} \ . \tag{84}$$

(8) **Pressure Jump in Ω_p** : For each edge/face $E \in \mathcal{E}_h(\Omega_p)$, we consider $\omega_E = K_1 \cup K_2$. As $[p_h]_E \in \mathbb{P}^0(E)$ we set

$$\mathbf{w}_E := [p_h]_E b_E \mathbf{n}_E \in [H_0^1(\omega_E)]^d.$$

First we notice that as $p \in H^1(\omega_E)$ we have by Green formula

$$\int_{\omega_E} (\nabla p \cdot \mathbf{w}_E + p \operatorname{div} \mathbf{w}_E) = 0.$$

Again by elementwise partial integration we further have

$$\int_E [p_h]_E \mathbf{w}_E . \mathbf{n}_E = \sum_{i=1}^{2} \int_{K_i} (\nabla p_h \cdot \mathbf{w}_E + p_h \operatorname{div} \mathbf{w}_E).$$

Taking the difference of these two identities we obtain

$$\int_E [p_h]_E \mathbf{w}_E . \mathbf{n}_E = \sum_{i=1}^{2} \int_{T_i} (\nabla (p_h - p) \cdot \mathbf{w}_E + (p_h - p) \operatorname{div} \mathbf{w}_E).$$

Recalling that $\nabla p = -\mu \mathbf{K}^{-1}\mathbf{u}$ and introducing the term $\mu \mathbf{K}^{-1}\mathbf{u}_h$, we find

$$\begin{aligned}
\int_E [p_h]_E \mathbf{w}_E . \mathbf{n}_E &= \sum_{i=1}^{2} \int_{K_i} (\nabla p_h + \mu \mathbf{K}^{-1}\mathbf{u}) \cdot \mathbf{w}_E + (p_h - p) \operatorname{div} \mathbf{w}_E) \\
&= \sum_{i=1}^{2} \int_{K_i} (\nabla p_h + \mu \mathbf{K}^{-1}\mathbf{u}_h) \cdot \mathbf{w}_E + (p_h - p) \operatorname{div} \mathbf{w}_E) \\
&+ \sum_{i=1}^{2} \int_{K_i} (\mu \mathbf{K}^{-1}(\mathbf{u} - \mathbf{u}_h)) \cdot \mathbf{w}_E.
\end{aligned}$$

Cauchy-Schwarz inequality and inverse inequalities lead to

$$\begin{aligned}
\|[p_h]_E\|_E &\lesssim \sum_{i=1}^{2} \|\mathbf{R}_3\|_{K_i} h_E^{\frac{1}{2}} + \|p_h - p\|_{K_i} h_E^{-\frac{1}{2}} \\
&+ h_E^{\frac{1}{2}} \sum_{i=1}^{2} \|\mathbf{K}^{-1}(\mathbf{u} - \mathbf{u}_h)\|_{K_i}.
\end{aligned}$$

Since $h_E \leq 1$, then by (78), we deduce that

$$h_E^{\frac{1}{2}} \|[p_h]_E\|_E \lesssim \|p - p_h\|_{\omega_E} + \|\mathbf{K}^{-1}(\mathbf{u} - \mathbf{u}_h)\|_{\omega_E}. \tag{85}$$

(9) **Normal Jump $\mathbf{G}_{E,\mathbf{n}_E}$ in Ω_p:** For each edge/face $E \in \mathcal{E}_h(\Omega_p)$, we consider $w_E = K_1 \cup K_2$. As $\mathbf{G}_{E,\mathbf{n}_E} \in [\mathbb{P}^0(E)]^d$ we set

$$\mathbf{w}_E := -\mathbf{G}_{E,\mathbf{n}_E} b_E \in [H_0^1(w_E)]^d.$$

First the weak formulation (19) with $\mathbf{V} = (\mathbf{0}, \mathbf{w}_E) \in \mathbf{H} \times \mathbf{X}_p$ yields

$$\mathbf{A}(\mathbf{U}, \mathbf{V}) + \mathbf{B}(\mathbf{V}, \mathbf{P}) = \mathbf{L}(\mathbf{V}),$$

that is equivalent to

$$\begin{aligned}
\int_{w_E} \mathbf{f} \cdot \mathbf{w}_E &= \int_{w_E} \mu \mathbf{K}^{-1}\mathbf{u} \cdot \mathbf{w}_E + \int_{w_E} 2\mu_p \mathbf{D}(\eta_p) : \mathbf{D}(\mathbf{w}_E) \\
&+ \int_{w_E} \lambda_p \nabla \cdot \eta_p \nabla \cdot \mathbf{w}_E - \alpha \int_{w_E} p_p \nabla \cdot \mathbf{w}_E - \int_{\partial w_E} [\sigma_p(\eta_p, p_p)\mathbf{n}_E] \cdot \mathbf{w}_E
\end{aligned}$$

By elementwise partial integration we further have

$$\begin{aligned}
-\int_{w_E} \mathbf{G}_{E,\mathbf{n}_E} \cdot \mathbf{w}_E &= \int_{w_E} 2\mu_p \mathbf{D}(\eta_{ph}) : \mathbf{D}(\mathbf{w}_E) + \int_{w_E} \lambda_p \nabla \cdot \eta_{ph} \nabla \cdot \mathbf{w}_E - \alpha \int_{w_E} p_{ph} \nabla \cdot \mathbf{w}_E \\
&- \sum_{i=1}^{2} \int_{K_i} -\sigma_p(\eta_{ph}, p_{ph}) \cdot \mathbf{w}_E.
\end{aligned}$$

By the previous identity we get,

$$
\begin{aligned}
-\int_{w_E} \mathbf{G}_{E,\mathbf{n}_E}\cdot \mathbf{w}_E &= -\int_{w_E} 2\mu_p \mathbf{D}(\eta_p-\eta_{ph}) : \mathbf{D}(\mathbf{w}_E) - \int_{w_E} \lambda_p \nabla\cdot(\eta_p-\eta_{ph})\nabla\cdot \mathbf{w}_E \\
&+ \alpha\int_{w_E} (p_p-p_{ph})\nabla\cdot \mathbf{w}_E + \sum_{i=1}^{2}\int_{K_i} [\mathbf{f}_h + \sigma_p(\eta_{ph},p_{ph})]\cdot \mathbf{w}_E \\
&+ \int_{w_E} \mu \mathbf{K}^{-1}(\mathbf{u}-\mathbf{u}_h)\cdot \mathbf{w}_E + \int_{w_E} [\mu\mathbf{K}^{-1}\mathbf{u}_h + \nabla p_{ph}]\cdot \mathbf{w}_E \\
&+ \sum_{i=1}^{2}\int_{K_i} (\mathbf{f}-\mathbf{f}_h)\cdot \mathbf{w}_E
\end{aligned}
$$

The previous bounds of $\mathbf{r}_{4,K}$, $\mathbf{R}_3$ and the obvious estimate $h_E \leq h_K$ imply that

$$
h_E^{1/2} \parallel \mathbf{G}_{E,\mathbf{n}_E} \parallel_E \lesssim |||(\mathbf{U}-\mathbf{U}_h, \mathbf{P}-\mathbf{P}_h)|||_{h,w_E} + \Psi_{K_1} + \Psi_{K_2}. \tag{86}
$$

From (84), (85) and (86) we deduce the estimation

$$
\Upsilon_{7,K} \lesssim |||(\mathbf{U}-\mathbf{U}_h, \mathbf{P}-\mathbf{P}_h)|||_{h,\tilde{w}_K} + \sum_{K'\subset \tilde{w}_K} \Psi_{K'}. \tag{87}
$$

(10) **Interface Elements on Γ_{fp} ($\Upsilon_{5,K}$ and $\Upsilon_{6,K}$):**
To estimate $\Upsilon_{5,K}$ and $\Upsilon_{6,K}$, we fix an edge E included in Γ_{fp} and for a constant $\mathbf{r}_E$ fixed later on and a unit vector $\mathbf{i}$, we consider

$$
\mathbf{w}_E := \mathbf{r}_E b_E \mathbf{i},
$$

that clearly belongs to $\mathbf{H}$. We take $\mathbf{W} = (\mathbf{w}_E, \mathbf{0})$ and the weak formulation (19) yields:

$$
\int_{w_E} \mathbf{f}\cdot \mathbf{w}_E = \mathbf{A}(\mathbf{U},\mathbf{W}) + \mathbf{B}(\mathbf{W},\mathbf{P}), \tag{88}
$$

that is equivalent to

$$
\begin{aligned}
\int_{w_E} \mathbf{f}\cdot \mathbf{w}_E &= \int_{K_f} [2\mu \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{w}_E) - p\nabla\cdot \mathbf{w}_E] + \int_{K_p} [\mu \mathbf{K}^{-1}\mathbf{u}\cdot \mathbf{w}_E - p\nabla\cdot \mathbf{w}_E] \\
&+ \sum_{j=1}^{d-1}\int_{\Gamma_{fp}} \left[\mu\alpha_{BJS}\sqrt{K_j^{-1}}\mathbf{u}_f\cdot\tau_{f,j}][\mathbf{w}_E\cdot\tau_{f,j}\right]
\end{aligned} \tag{89}
$$

where K_f (resp. K_p) is the unique triangle/tetrahedron included in $\bar{\Omega}_f$ (resp. $\bar{\Omega}_p$) having E as edge/face. On the other hand, integrating by parts in K_f and in K_p yields

$$
\begin{aligned}
&\int_{K_f} (2\mu \mathbf{D}(\mathbf{u}_h) : \mathbf{D}(\mathbf{w}_E) - p_h \operatorname{div} \mathbf{w}_E) + \int_{K_p} (\mu \mathbf{K}^{-1}\mathbf{u}_h\cdot \mathbf{w}_E - p_h \operatorname{div} \mathbf{w}_E) \\
&+ \sum_{j=1}^{d-1}\int_{\Gamma_{fp}} \left[\mu\alpha_{BJS}\sqrt{K_j^{-1}}\mathbf{u}_{f,h}\cdot\tau_{f,j}][\mathbf{w}_{E,f}\cdot\tau_{f,j}\right] \\
&= -\int_{K_f} (2\mu \operatorname{div} \mathbf{D}(\mathbf{u}_h) - \nabla p_h)\cdot \mathbf{w}_E + \int_{K_p} (\mu \mathbf{K}^{-1}\mathbf{u}_h\cdot \mathbf{w}_E + \nabla p_h)\cdot \mathbf{w}_E \\
&+ \sum_{j=1}^{d-1}\int_{\Gamma_{fp}} \left[\mu\alpha_{BJS}\sqrt{K_j^{-1}}\mathbf{u}_{f,h}\cdot\tau_{f,j}][\mathbf{w}_{E,f}\cdot\tau_{f,j}\right] \\
&- \int_{E} ([p_h]_E \mathbf{w}_E\cdot \mathbf{n}_E - 2\mu(D(\mathbf{u}_{f,h}\mathbf{n}_E)\cdot \mathbf{w}_E).
\end{aligned}
$$

Subtracting this identity to (89) we find

$$\int_E([p_h]_E\mathbf{w}_E\cdot\mathbf{n}_E-2\mu(\mathbf{D}(\mathbf{u}_{f,h}\mathbf{n}_E)\cdot\mathbf{w}_E)-\sum_{j=1}^{d-1}\int_{\Gamma_{fp}}\left[\mu\alpha_{BJS}\sqrt{K_j^{-1}}\mathbf{u}_{fh}\cdot\tau_{f,j}][\mathbf{w}_{E,f}\cdot\tau_{f,j}\right]$$
$$=\int_{K_f}(2\mu\mathbf{D}(\mathbf{u}-\mathbf{u}_h):\mathbf{D}(\mathbf{w}_E)-(p-p_h)\operatorname{div}\mathbf{w}_E)+\int_{K_p}(\mu\mathbf{K}^{-1}(\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{w}_E-(p-p_h)\operatorname{div}\mathbf{w}_E)$$
$$+\sum_{j=1}^{d-1}\int_{\Gamma_{fp}}\left[\mu\alpha_{BJS}\sqrt{K_j^{-1}}(\mathbf{u}_f-\mathbf{u}_{fh})\cdot\tau_{f,j}][\mathbf{w}_{E,f}\cdot\tau_{f,j}\right]$$
$$-\int_{K_f}(\mathbf{f}+2\mu\operatorname{div}\mathbf{D}(\mathbf{u}_h)-\nabla p_h)\cdot\mathbf{w}_E-\int_{K_p}(-\mu\mathbf{K}^{-1}\mathbf{u}_h\cdot\mathbf{w}_E-\nabla p_h)\cdot\mathbf{w}_E.$$

In that last terms introducing the element residual $\mathbf{r}_{1,K}$ and $\mathbf{R}_3$, we arrive at

$$\begin{aligned}&\int_E([p_h]_E\mathbf{w}_E\cdot\mathbf{n}_E-2\mu(\mathbf{D}(\mathbf{u}_{f,h})\mathbf{n}_E\cdot\mathbf{w}_E)-\sum_{j=1}^{d-1}\int_{\Gamma_{fp}}\left[\mu\alpha_{BJS}\sqrt{K_j^{-1}}(\mathbf{u}_f-\mathbf{u}_{fh})\cdot\tau_{f,j}][\mathbf{w}_{E,f}\cdot\tau_{f,j}\right]\\
=\;&\int_{K_f}(2\mu\mathbf{D}(\mathbf{u}-\mathbf{u}_h):\mathbf{D}(\mathbf{w}_E)-(p-p_h)\operatorname{div}\mathbf{w}_E)\\
+\;&\int_{K_p}(\mu\mathbf{K}^{-1}(\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{w}_E-(p-p_h)\operatorname{div}\mathbf{w}_E)\\
&+\sum_{j=1}^{d-1}\int_{\Gamma_{fp}}\left[\mu\alpha_{BJS}\sqrt{K_j^{-1}}(\mathbf{u}_f-\mathbf{u}_{fh})\cdot\tau_{f,j}][\mathbf{w}_{E,f}\cdot\tau_{f,j}\right]\\
&-\int_{K_f}(\mathbf{f}-\mathbf{f}_h+\mathbf{r}_{f,K})\cdot\mathbf{w}_E-\int_{K_p}\mathbf{R}_3\cdot\mathbf{w}_E.\end{aligned}\tag{90}$$

(a) To estimate $\Upsilon_{5,K}$, for each $j=1,\dots,d-1$, we take $\mathbf{r}_E=\mathbf{R}_6(j)$ and $\mathbf{i}=\tau_j$. With this choice, the identity (90) and the inverse inequality (57) yield

$$\begin{aligned}\|r_E\|_E^2\;\lesssim\;&\int_{K_f}[(2\mu\mathbf{D}(\mathbf{u}-\mathbf{u}_h):\mathbf{D}(\mathbf{w}_E)-(p-p_h)\operatorname{div}\mathbf{w}_E)]\\
+\;&\int_{K_p}[(\mu\mathbf{K}^{-1}(\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{w}_E-(p-p_h)\operatorname{div}\mathbf{w}_E)]\\
+\;&\sum_{j=1}^{d-1}\int_{\Gamma_{fp}}\left[\mu\alpha_{BJS}\sqrt{K_j^{-1}}(\mathbf{u}_f-\mathbf{u}_{fh})\cdot\tau_{f,j}][\mathbf{w}_{E,f}\cdot\tau_{f,j}\right]\\
-\;&\int_{K_f}(\mathbf{f}-\mathbf{f}_h+\mathbf{r}_{1,K})\cdot\mathbf{w}_E-\int_{K_p}\mathbf{R}_3\cdot\mathbf{w}_E.\end{aligned}$$

Hence Cauchy-Schwarz inequality, the inverse inequalities (58) and the estimates of bounds $\mathbf{r}_{1,K_l}$ and $\mathbf{R}_3$ lead to:

$$h_E^{\frac{1}{2}}\|\mathbf{R}_6(j)\|_E\lesssim|\mathbf{u}-\mathbf{u}_h|_{h,\omega_E}+\|p-p_h\|_{h,\omega_E}+\Psi_{K_f}+\Psi_{K_p},\tag{91}$$

with $\omega_E=K_f\cup K_p$.
Thus,

$$\Upsilon_{5,K}\;\lesssim\;|||(\mathbf{U}-\mathbf{U}_h,\mathbf{P}-\mathbf{P}_h)|||_{h,\tilde{w}_K}+\sum_{K'\subset\tilde{w}_K}\Psi_{K'}.\tag{92}$$

(b) To estimate $\Upsilon_{6,K}$, we take $r_E=\mathbf{R}_7$ and $\mathbf{i}=\mathbf{n}_f$. As before the identity (90), the inverse inequalities (57) and (58) and the estimates of bounds $\mathbf{r}_{K,l}$ and $\mathbf{R}_3$ lead to

$$\Upsilon_{6,K}\;\lesssim\;|||(\mathbf{U}-\mathbf{U}_h,\mathbf{P}-\mathbf{P}_h)|||_{h,\tilde{w}_K}+\sum_{K'\subset\tilde{w}_K}\Psi_{K'}.\tag{93}$$

Combining The estimates (72), (77), (78), (79), (82), (85), (87) and (92) we have the main result of this section:

THEOREM 5.2. *Under the assumptions of Theorem 4.2, the family* $\{\Upsilon_K\}_{K\in\mathcal{T}_h}$ *satisfies (61).*

6. Summary

In this paper we have discussed a posteriori error estimates for a finite element approximation of the Stokes-Biot system. A residual type a posteriori error estimator is provided, that is both reliable and efficient. Many issues remain to be addressed in this area, let us mention other types of a posteriori error estimators or implementation and convergence analysis of adaptive finite element methods. Further it is well known that an internal layer appears at the interface Γ_{fp} as the permeability tensor degenerates, in that case anisotropic meshes have to be used in this layer (see for instance [**8**]). Hence we intend to extend our results to such anisotropic meshes.

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