Some advances on essential spectra of one sided operator matrix with application

Marwa Belguith, Nedra Moalla and Ines Walha

Faculty of Sciences of Sfax, Department of Mathematics

BP1171, Sfax 3000 Tunisia

e.mail: maroua.belghith@yahoo.com, Nedra.moalla@ipeis.rnu.tn, ines_walha@yahoo.fr

Abstract

This paper deals with a new description of the one sided operator matrix form, as a generalization of the case of the unbounded operator matrix with the non diagonal domain, to investigate some advances in the analysis of some essential spectra under weaker hypotheses then the one provided in the works of [17, 33]. An example of differential equations is tested to ensure the validity of the abstract results.

Keywords. One sided coupled operator matrix, Fredholm perturbations theory, essential spectra, differential equation.

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1 Introduction

The theory of some linear operators acting in Banach space related to their corresponding essential spectra is a modern section of the spectral analysis in operator theory domain. Such kind of theory provide an immense use in many branches such us: in the mathematical, control theory and physical sense, when resolving a number of application that can be formulated in terms of many linear operators. Recently, to the best of our knowledge up, a few results have been appeared on the spectral analysis of unbounded block operators matrices theory with mixed order (we refer the readers to look the papers [2, 3, 4, 8, 14, 17, 21, 22, 25, 26, 27, 29, 30, 33] and among others). This work has its origin and motivation in the analysis of stability problem likewise: in fluid mechanics, transport operators, ordinary differential equations and magnetohydrodynamics ([3, 5, 6, 8, 14, 17, 21, 22, 29, 30, 33]).

The theory of operator matrices, dates backs to 1985, was introduced in the papers of R. Nagel [26, 27] in the context of unbounded operator matrix given by the following form that acting in the product of Banach spaces $X \times Y$

$$\mathcal{A} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

defined with diagonal domain

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times \mathcal{D}(D),$$

later by F. V. Atkinson et al. [2] in 1994, A. A. Shkalikov [30] in 1995, N. Moalla [21] in 2006 and A. Jeribi [14] in 2009 for maximal domain case. However, such kind of domain fail not usually conserved especially when dealing with the study of some physical phenomena. Such case occurs in some examples such us in Delay differential equations, in Population equation with spatial diffusion, a cell equation (see the paper of R. Nagel for more details [25]) and among others. Keeping to this interest, A. Batkai et al. in [3] has spurred this case of non maximality domain in the study of the unbounded operator matrix with non maximal domain. That is, in the case when the domain of operator matrix contains one additional condition of the form $\Phi_X x = \Psi_Y y$ between each components of its elements and for linear operators Φ_X and Ψ_Y from X and Y into Banach space Z, respectively, expressed as the form:

$$\mathcal{D}(\mathcal{A}) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in (\mathcal{D}(A) \cap \mathcal{D}(C)) \times (\mathcal{D}(B) \cap \mathcal{D}(D)) : \ \ \Phi_X x = \Psi_Y y \right\}. \tag{1.1}$$

Since this study, the invention of the notion of the essential spectra is becoming increasingly important in many applications especially in transport equations. Particularly, some progress on the study of this kind of unbounded operator matrix form (1) having the domain (1.1) have been introduced and developed by I. Walha et al. in numerous papers as [7, 8, 17, 33]. Our approach allows us investigate a fine description of some essential spectra of the closure of the matrix form (1), denoted by $\overline{\mathcal{A}}$, defined in (1.1) as the form:

$$\sigma_{ek}(\overline{\mathcal{A}}) := \sigma_{ek}(A_1) \cup \sigma_{ek}(\overline{D - C[-K_\mu \Psi_Y + (\mu I - A_1)^{-1} B]}),$$

where $A_1 := A|_{\mathcal{D}(A) \cap \mathcal{N}(\Phi_X)}$ and K_μ represents the inverse of the restriction of the operator Φ_X on $\mathcal{N}(\mu I - A)$.

Recently, S. Charfi et al. in [8] have paid attention to the research of the description of the essential spectra of the above defined operator matrix $\overline{\mathcal{A}}$. New approaches to old techniques of computation have led to significant advances in the spectral analysis from the theory of operators matrices expressed as follows:

$$\sigma_{ek}(\overline{\mathcal{A}}) := \sigma_{ek}(A_1) \cup \sigma_{ek}(D - \overline{C[-K_\mu \Psi_Y + (\mu I - A_1)^{-1} B]}),$$

for $\mu \in \rho(A_1)$. We shall emphasize on the fact that the above spectral description should be uses a full of assumptions and the Frobenuis-Schur factorization.

Among this direction, we had the idea to deal with general model of unbounded operator matrix, so called the one sided operator matrix defined in the product of Banach spaces $E \times F$ having the domain

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{D}(A_m) \times \mathcal{D}(D) : \quad \phi(f) = \psi(g) \right\},$$

for which

$$\mathcal{A} \begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} A_m f + Bg \\ Cf + Dg \end{pmatrix}, \quad \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{D}(\mathcal{A}).$$

Based on this kind matrix form, we find a weaker assumptions then the one used by S. Charfi et al. in [8] on their entries assuring the analysis problem of its resolvent expression. In that meaning, the obtain resolvent expression are used to describe the constraints of the eigenvalues corresponding of to such kind of operator matrix that is studied. The procedure of the proof is explained in details by the use of some stability criteria on the theory of Fredholm perturbations techniques which allow us providing an amelioration in the characterization of some essential spectra of this kind of model of operator matrix and in terms of his diagonal operators entries. Thus, this result appears as natural in view of scientific progress in this field.

To this interest, we consider an illustrative example of integro-differential equation on $X \times X$-space, where

$$X := L_1((0,1) \times K, dxd\xi), \quad x \in (0,1), \quad \xi = (\xi_1, \xi_2, \xi_3) \in K,$$

(K is the unit sphere of $\mathbf{R}^3$) as follows:

$$\mathcal{A}_{\mathcal{H}} := \begin{pmatrix} T & K_{12} \\ K_{21} & T_H + K_{22} \end{pmatrix}.$$

defined with non maximal domain as:

$$\mathcal{D}(\mathcal{A}_{\mathcal{H}}) := \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{W} \times \mathcal{W} : \quad \begin{pmatrix} f \\ g \end{pmatrix}^i = \mathcal{H} \begin{pmatrix} f \\ g \end{pmatrix}^o \right\},$$

where:

$*$ $\begin{pmatrix} f \\ g \end{pmatrix}^o$ and $\begin{pmatrix} f \\ g \end{pmatrix}^i$ represent the outgoing and the incoming fluxes related by the bounded boundaries operator $\mathcal{H}$ which is expressed as:

$$\begin{cases} \mathcal{H} : X^o \times X^o & \longrightarrow X^i \times X^i \\ \begin{pmatrix} f \\ g \end{pmatrix} & \longmapsto \begin{pmatrix} 0 & H \\ 0 & H \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \end{cases}$$

where $H \in \mathcal{L}(X^0, X^i)$ and the boundary spaces X^0 and X^i are identified as:

$$X^0 := L_1(D^0, |\xi_3| d\xi) \quad \text{and} \quad X^i := L_1(D^i, |\xi_3| d\xi)$$

where the sets of the incoming D^i and the outgoing D^0 boundary of the phase space $D := (0,1) \times K$ are given by:

$$D^i = D_1^i \cup D_2^i = (\{0\} \times K^1) \cup (\{1\} \times K^0) \ \text{ and } \ D^0 = D_1^0 \cup D_2^0 = (\{0\} \times K^0) \cup (\{1\} \times K^1),$$

for

$$K^0 = K \cap \{\xi_3 < 0\} \ \text{ and } \ K^1 = K \cap \{\xi_3 > 0\}.$$

$*$ each closed linear operator T is defined by:

$$\begin{cases} T : \mathcal{D}(T) \subseteq X \longrightarrow X \\ \qquad\qquad\qquad f \longmapsto Tf : (x,\xi) \longmapsto -\xi_3 \dfrac{\partial f}{\partial x}(x,\xi) - \sigma_1(x,\xi) f(x,\xi) \\ \\ \mathcal{D}(T) := \left\{ f \in X : \xi_3 \dfrac{\partial f}{\partial x} \in X \right\} = \mathcal{W}, \end{cases}$$

$*$ the streaming operator T_H is defined by:

$$\begin{cases} T_H : \mathcal{D}(T_H) \subseteq X \longrightarrow X \\ \qquad\qquad\qquad g \longmapsto T_H g : (x,\xi) \longmapsto -\xi_3 \dfrac{\partial g}{\partial x}(x,\xi) - \sigma_2(x,\xi) g(x,\xi) \\ \\ \mathcal{D}(T_H) := \left\{ g \in \mathcal{W} : g^i = H g^o \right\}, \end{cases}$$

$*$ the bounded linear collision operators K_{ij}, $(i,j) = \{(1,2),(2,1),(2,2)\}$, are defined on X by:

$$\begin{cases} K_{ij} : X \longrightarrow X \\ \qquad f \longmapsto K_{ij}\ f : (x,\xi) \longmapsto \displaystyle\int_K \kappa_{ij}(x,\xi,\xi')\ f(x,\xi')\ d\xi' \end{cases}$$

where the frequency $\sigma_j(.,.) \in \mathcal{L}^\infty(-1,1), j = \{1,2\}$, is considered as a positive bounded function on D.

Physically, the function $(x,\xi) \longmapsto \begin{pmatrix} f \\ g \end{pmatrix}(x,\xi)$ represents the number density of neutrons having the position x and the direction cosine of propagation ξ.

Clearly, this model of transport operator may be written as one sided coupled operator matrix as the following form:

$$\mathcal{A} := \begin{pmatrix} T & K_{12} \\ K_{21} & T_H + K_{22} \end{pmatrix} := \begin{pmatrix} A_m & B \\ C & D \end{pmatrix},$$

defined on:

$$\mathcal{D}(\mathcal{A}) := \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{D}(A_m) \times \mathcal{D}(T_H + K_{22}) : \quad \phi(f) = \psi(g) \right\},$$

(for appropriate operators ϕ and ψ introduced in details in Section 4). Taking advantage of the results of Section 3 with a specific choice of the boundary and collision operators, we guarantee the stability of the essential spectra between the operators $\mathcal{A}$ and

$$\mathcal{T}_{\mathcal{H}} := \begin{pmatrix} A_0 & 0 \\ 0 & T_H + K_{22} \end{pmatrix},$$

in other terms, we find a weaker conditions based on the regularity definition of the collision operator which introduced by B. Lods in [19] as a generalization of then defined by M. Moktar-Karroubi in [23] to enrich a fine description of the picture of the eigenvalues of this model of transport operators as:

$$\rho_{ek}(\mathcal{A}) = \rho_{ek}(A_0) \cap \rho_{ek}(T_H).$$

Physically, A_0 corresponds to the transport operator with vacuum boundaries condition as $A_0 := T$, $\mathcal{D}(A_0) := \{f \in \mathcal{D}(T) : f^i = 0\}$.

Our paper is organized as follows:

$-$ Section 2 is devoted to gather some basic definitions about theory of operators and presents its fundamental properties.

$-$ Section 3 is concentrated to describe our model of operator matrix $\mathcal{A}$. Some general hypotheses on the different components of $\mathcal{A}$ are introduced in details in order to characterize its essential spectra.

$-$ Section 4 focuss on details description of the theoretical results to a physical model of transport operators with specific boundaries condition on the nuclear space.

2 Preliminaries results

We start this section by giving some basic definitions and notations that we will need in the sequel.

Let X and Y be two Banach spaces. We denote by $\mathcal{L}(X, Y)$ (resp. $\mathcal{C}(X, Y)$) the set of all bounded (resp. closed, densely defined) linear operators from X into Y. The subset of all compact operators of $\mathcal{L}(X, Y)$ is designated by $\mathcal{K}(X, Y)$. For $A \in \mathcal{C}(X, Y)$, we write $\mathcal{D}(A) \subset X$ for the domain, $N(A) \subset X$ for the null space and $R(A) \subset Y$ for the range of A. The nullity, $\alpha(A)$, of A is defined as the dimension of $N(A)$ and the deficiency, $\beta(A)$, of A is defined as the codimension of $R(A)$ in Y.

Now, we introduce the following important classes of Fredholm operators:

Definition 2.1
(i) The set of upper semi-Fredholm operators from X into Y is defined by:

$$\Phi_+(X, Y) := \{A \in \mathcal{C}(X, Y) : \alpha(A) < \infty, \mathcal{R}(A) \text{ is closed in Y }\}.$$

(ii) The set of lower semi-Fredholm operators from X into Y is defined by:

$$\Phi_-(X, Y) := \{A \in \mathcal{C}(X, Y) : \beta(A) < \infty \}.$$

(iii) The set of Fredholm (resp. semi-Fredholm) operators from X into Y is defined by:

$$\Phi(X, Y) := \Phi_+(X, Y) \cap \Phi_-(X, Y) \ \ (\text{resp.} \Phi_\pm(X, Y) := \Phi_+(X, Y) \cup \Phi_-(X, Y)). \qquad \diamondsuit$$

The set of bounded upper (resp. lower) semi-Fredholm operators from X into Y is defined as:

$$\Phi_+^b(X, Y) = \Phi_+(X, Y) \cap \mathcal{L}(X, Y) \ (\text{resp. } \Phi_-^b(X, Y) = \Phi_-(X, Y) \cap \mathcal{L}(X, Y))$$

while the set of bounded Fredholm operators from X into Y is defined as:

$$\Phi^b(X, Y) = \Phi(X, Y) \cap \mathcal{L}(X, Y).$$

It is should be noted that while $X = Y$, then the sets $\mathcal{L}(X, Y)$, $\mathcal{K}(X, Y)$, $\mathcal{C}(X, Y)$, $\Phi(X, Y)$, $\Phi_+(X, Y)$, $\Phi_-(X, Y)$, $\Phi^b(X, Y)$, $\Phi_+^b(X, Y)$ and $\Phi_-^b(X, Y)$ are replaced, respectively, by $\mathcal{L}(X)$, $\mathcal{K}(X)$, $\mathcal{C}(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$, $\Phi^b(X)$, $\Phi_+^b(X)$ and $\Phi_-^b(X)$. A complex number λ is in Φ_A,

Φ_{+A} or Φ_{-A}, that is, $\lambda - A$ is in $\Phi(X)$, $\Phi_+(X)$ or $\Phi_-(X)$ respectively. The index of an operator $A \in \Phi_\pm(X)$ is defined by: $i(A) := \alpha(A) - \beta(A)$.

Sets of right and left Fredholm operators are defined as:

Definition 2.2
(i) The set of right Fredholm operators from X into Y is defined by:

$$\Phi_r(X,Y) := \{A \in \mathcal{C}(X,Y) : \ \exists A_r \in \mathcal{L}(Y,X_A) \ \text{ such that } \ \widehat{A}A_r - I \in \mathcal{K}(Y)\}$$

(ii) The set of left Fredholm operators from X into Y is defined by:

$$\Phi_l(X,Y) := \{A \in \mathcal{C}(X,Y) : \exists A_l \in \mathcal{L}(Y,X_A) \ \text{ such that } \ A_l\widehat{A} - I \in \mathcal{K}(X_A)\}. \qquad \diamondsuit$$

Nothing that these sets of Fredholm operators satisfying the following inclusions:

$$\Phi(X,Y) \subset \Phi_l(X,Y) \subset \Phi_+(X,Y) \quad \text{ and } \quad \Phi(X,Y) \subset \Phi_r(X,Y) \subset \Phi_-(X,Y).$$

We recall some basic definitions for bounded linear operators in Banach spaces that are meaningful in the study of the stability problem of some essential spectra.

Definition 2.3 Let X be a Banach space.
(i) An operator $A \in \mathcal{L}(X)$ is said to be weakly compact if $A(B)$ is relatively weakly compact in X for every bounded $B \subset X$.

The class of weakly compact operators on X, denoted by $\mathcal{WC}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$ (see [11]).

(ii) An operator $A \in \mathcal{L}(X)$ is said to be strictly singular if the restriction of A to any infinite-dimensional subspace of X is not an homeomorphism.
Let $S(X)$ denotes the set of strictly singular operators on X. $\diamondsuit$

Remark 2.1 (i) Note that the class of strictly singular operators is not compact in general (see [11]), but in a separable Hilbert space X, we have:

$$S(X) = \mathcal{K}(X),$$

we refer the readers to [11, 20, 31, 32], for more properties of this kind of class operators.

(ii) Let X_p denotes the space $L_p(\Omega, d\mu)$ $(1 \leq p \leq \infty)$, where (Ω, Σ, μ) stands for a positive measure space.
Keeping into Theorem 1 in [28], for $X_1 = L_1$-space (res. $C(\Omega)$-spaces, with Ω is a compact Hausdorff space), we have:

$$\mathcal{WC}(X_1) = \mathcal{S}(X_1).$$

However, if $1 < p < \infty$, X_p is reflexive and then

$$\mathcal{L}(X_p) = \mathcal{WC}(X_p).$$

On the other hand, following Theorem 5.2 in [11], we infer that:

$$\mathcal{K}(X_p) \subset_{\neq} \mathcal{S}(X_p) \subset_{\neq} \mathcal{WC}(X_p), \quad \text{for} \quad p \neq 2.$$

For $p = 2$, we have:

$$\mathcal{K}(X_p) = \mathcal{S}(X_p) = \mathcal{WC}(X_p). \qquad \diamondsuit$$

As a generalization of the class of compact operators, we will introduce the class of polynomially compact operators which denoted by $\mathcal{PK}(X)$ and defined as:

$$\mathcal{PK}(X) = \{\ A \in \mathcal{L}(X) \text{ such that there exists a nonzero complex polynomial}$$
$$P(z) = \sum_{k=0}^{n} a_k z^k \text{ satisfying } P(A) \in \mathcal{K}(X)\}.$$

Remark 2.2 (i) Obviously, every compact operator belongs to the class of polynomially compact operators, that is, we mainly have the following inclusion:

$$\mathcal{K}(X) \subset \mathcal{PK}(X).$$

(ii) Let (Ω, μ) be a σ-finite measure space.
Since $X = L_1(\Omega, d\mu)$ (respectively $X = C(\Omega)$-spaces with Ω is a compact Hausdorff space), then we obtain:

$$\mathcal{W}(X) \subset \mathcal{PK}(X),$$

for more details, we infer the readers to Remark 2.3 in [6]. $\diamondsuit$

When dealing with closed, densely defined linear operator, A, on a Banach space, various notions of essential spectra appear. We are concerned with some of them:

$$\sigma_{ek}(A) := \{\mu \in \mathbb{C} :\ \mu - A \notin \Phi_*(X)\} \quad \text{and} \quad \sigma_{ej}(A) := \mathbb{C} \setminus \rho_{ej}(A),$$

where:

$$(\sigma_{ek}(.), \Phi_*(.)) \in \{(\sigma_{er}(.), \Phi_r(.)), (\sigma_{el}(.), \Phi_l(.)), (\sigma_{ew}(.), \Phi(.))\},$$

$$(\sigma_{ek}(), \rho_{ek}()) \in \{(\sigma_{ess}(), \rho_{ess}()), (\sigma_{eb}(), \rho_{eb}())\},$$

$$\rho_{ess}(A) := \{\mu \in \Phi_A, \quad i(\mu - A) = 0\}$$

and

$$\rho_{eb}(A) := \{\mu \in \rho_{ess}(A) :\ \text{all scalars near } \mu \text{ are in } \ \rho(A)\}.$$

Obviously, we can check the following inclusions for each notion of essential spectra:

$$\sigma_{ew}(A) \subseteq \sigma_{ess}(A) \subseteq \sigma_{eb}(A),$$

$$\sigma_{eg}(A) \subset \sigma_{el}(A) \subset \sigma_{ew}(A), \tag{2.1}$$

$$\sigma_{ed}(A) \subset \sigma_{er}(A) \subset \sigma_{ew}(A) \tag{2.2}$$

where $\sigma_{eg}(A) := \{\mu \in \mathbb{C} : \mu - A \notin \Phi_+(X)\}$ and $\sigma_{ed}(A) := \{\mu \in \mathbb{C} : \mu - A \notin \Phi_-(X)\}$.

In this work, we are interested to characterize some essential spectra of unbounded one sided coupled operator matrix involving the theory of Fredholm perturbations. For this purpose, the following definition required.

Definition 2.4
(i) The set of right Fredholm perturbation from X into Y is defined by:

$$\mathcal{P}(\Phi_r(X, Y)) = \{F \in \mathcal{L}(X, Y) : A + F \in \Phi_r(X, Y),\ \forall A \in \Phi_r(X, Y)\}.$$

(ii) The set of left Fredholm perturbation from X into Y is defined by:

$$\mathcal{P}(\Phi_l(X,Y)) = \{F \in \mathcal{L}(X,Y) : A+F \in \Phi_l(X,Y),\ \forall A \in \Phi_l(X,Y)\}.$$

(iii) The set of upper semi-Fredholm perturbation from X into Y is defined by:

$$\mathcal{P}(\Phi_+(X,Y)) = \{F \in \mathcal{L}(X,Y) : A+F \in \Phi_+(X,Y),\ \forall A \in \Phi_+(X,Y)\}.$$

(iv) The set of lower semi-Fredholm perturbation from X into Y is defined by:

$$\mathcal{P}(\Phi_-(X,Y)) = \{F \in \mathcal{L}(X,Y) : A+F \in \Phi_-(X,Y),\ \forall A \in \Phi_-(X,Y)\}.$$

(v) The set of Fredholm perturbation from X into Y is defined by:

$$\mathcal{P}(\Phi(X,Y)) = \{F \in \mathcal{L}(X,Y) : A+F \in \Phi(X,Y),\ \forall A \in \Phi(X,Y)\}. \qquad \diamondsuit$$

Sets of Fredholm perturbations from X into Y can be ordered as:

$$\mathcal{P}(\Phi(X,Y)) \subseteq \mathcal{P}(\Phi_l(X,Y)) \subseteq \mathcal{P}(\Phi_+(X,Y))$$

and

$$\mathcal{P}(\Phi(X,Y)) \subseteq \mathcal{P}(\Phi_r(X,Y)) \subseteq \mathcal{P}(\Phi_-(X,Y)).$$

Remark 2.3

(i) Sets of Fredholm perturbations $\mathcal{P}(\Phi^b(X,Y))$, $\mathcal{P}(\Phi^b_+(X,Y))$, $\mathcal{P}(\Phi^b_-(X,Y))$, $\mathcal{P}(\Phi^b_l(X,Y))$ and $\mathcal{P}(\Phi^b_r(X,Y))$, respectively, may be defined in the same ways as Definition 2.4 if we replace $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, $\Phi_l(X,Y)$ and $\Phi_r(X,Y)$ by $\Phi^b(X,Y)$, $\Phi^b_+(X,Y)$, $\Phi^b_-(X,Y)$, $\Phi^b_l(X,Y)$ and $\Phi^b_r(X,Y)$.

(ii) Following [10], it is shown that $\mathcal{P}(\Phi^b(X,Y))$, $\mathcal{P}(\Phi^b_+(X,Y))$ and $\mathcal{P}(\Phi^b_-(X,Y))$ are closed subset of $\mathcal{L}(X,Y)$ and if $X=Y$, then $\mathcal{P}(\Phi^b(X)) := \mathcal{P}(\Phi^b(X,X))$, $\mathcal{P}(\Phi^b(X)) := \mathcal{P}(\Phi^b_+(X,X))$ and $\mathcal{P}(\Phi^b_-(X)) := \mathcal{P}(\Phi^b_-(X,X))$ are closed two-sided ideals of $\mathcal{L}(X)$.

In [16], it is shown that if $X=Y$, then $\mathcal{P}(\Phi^b_l(X)) := \mathcal{P}(\Phi^b_l(X,X))$, $\mathcal{P}(\Phi^b_r(X)) := \mathcal{P}(\Phi^b_r(X,X))$ are two-sided ideals of $\mathcal{L}(X)$, satisfying:

$$\mathcal{K}(X,Y) \subseteq \mathcal{W}(X,Y) \subseteq \mathcal{P}(\Phi^b_+(X,Y)) \subseteq \mathcal{P}(\Phi^b_l(X,Y)) \subseteq \mathcal{P}(\Phi^b(X,Y)) \tag{2.3}$$

and

$$\mathcal{K}(X,Y) \subseteq \mathcal{W}(X,Y) \subseteq \mathcal{P}(\Phi^b_-(X,Y)) \subseteq \mathcal{P}(\Phi^b_r(X,Y)) \subseteq \mathcal{P}(\Phi^b(X,Y)). \tag{2.4}$$

$\diamondsuit$

The interaction between the study of the property of Fredholm perturbations of the block operator matrix and their component entries provides a significant subject of study in spectral theory and developed by A. Jeribi et al. in [16].

Theorem 2.1 [16, Theorem 3.1-3.2] Let X_i, for $i=\{1,2\}$, be a Banach space and $\mathbb{P} := (P_{ij})_{1\le i,j\le 2}$, where P_{ij} denote a bounded linear operator from X_j into X_i, for $1 \le i,j \le 2$. Then, we have

$$\mathbb{P} \in \mathcal{E}(X_1 \times X_2) \iff P_{ij} \in \mathcal{E}(X_j, X_i), \quad \forall i,j = 1,2,$$

where $(\mathcal{E}(X_1\times X_2), \mathcal{E}(X_j,X_i)) \in \left\{ \substack{(\mathcal{P}(\Phi^b_l(X_1\times X_2)),\mathcal{P}(\Phi^b_l(X_j,X_i))),(\mathcal{P}(\Phi^b_r(X_1\times X_2)),\mathcal{P}(\Phi^b_r(X_j,X_i))),\\ (\mathcal{P}(\Phi^b(X_1\times X_2)),\mathcal{P}(\Phi^b(X_j,X_i)))} \right\}$.

Before owing to formulate our aim, we state the following theorem who is focuss on the stability analysis of some essential spectra of unbounded operator originated by S. Charfi et al. in [8].

Theorem 2.2 [8] Let X be a Banach space, A and B are two closed densely defined linear operators on X. Then, we get:
(i) For some $\mu \in \rho(A) \cap \rho(B)$, we obtain:

$$(\mu I - A)^{-1} - (\mu I - B)^{-1} \in \mathcal{P}(\Phi_l^b(X)) \quad \text{implies that} \quad \sigma_{el}(A) = \sigma_{el}(B).$$

(ii) For some $\mu \in \rho(A) \cap \rho(B)$, we obtain:

$$(\mu I - A)^{-1} - (\mu I - B)^{-1} \in \mathcal{P}(\Phi_r^b(X)) \quad \text{implies that} \quad \sigma_{er}(A) = \sigma_{er}(B). \qquad \diamondsuit$$

3 Main results

The main purpose of this section is to discuss the essential spectra of one sided coupled operator matrix $\mathcal{A}$, that is for one sided operator matrix (with domain consisting of one condition between their components entries).

Let X, E and F be Banach spaces. We consider linear operators:

$$A_m \text{ in } E, \;\; D \text{ in } F, \;\; C \text{ from } \mathcal{D}(A_m) \text{ into } F, \;\; B \text{ from } \mathcal{D}(D) \text{ into } E$$

and the continuous linear operators:

$$\phi \text{ from } \mathcal{D}(A_m) \subset E \text{ into } X,$$

$$\psi \text{ from } \mathcal{D}(D) \subset F \text{ into } X$$

with the following properties:

$(\mathcal{H}_1)$ The operator A_m (resp. D) is densely defined and closed linear operator.

$(\mathcal{H}_2)$ The operator ϕ is surjective.

$(\mathcal{H}_3)$ The operator B (resp. C) is bounded as a mapping from $\mathcal{D}(D)$ (resp. $\mathcal{D}(A_m)$) into E (resp. F).

These assumptions allow to collect some results established by G. Greiner in [12].

Lemma 3.1 *(i) The operator $A_0 := A_m|_{\ker \phi}$ is closed.*

(ii) For $\mu \in \rho(A_0)$ the following decomposition holds:

$$\mathcal{D}(A_m) = \mathcal{D}(A_0) \oplus \ker(\mu - A_m)$$

(iii) Let $\mu \in \rho(A_0)$. Then,

$$\phi_\mu := \phi|_{\ker(\mu - A_m)}$$

is continuous bijection from $\ker(\mu - A_m)$ onto X. $\diamondsuit$

As a direct consequence of the above Lemma, for $\mu \in \rho(A_0) \cap \rho(D)$, the inverse of ϕ_μ will play an important role to define the bounded operators K_μ as follows:

$$\begin{cases} K_\mu : \mathcal{D}(D) \longrightarrow & \mathcal{D}(A_m) \\ \\ \qquad\quad g \quad\;\; \longmapsto & K_\mu(g) = \phi_\mu^{-1} \circ \psi(g). \end{cases}$$

4 Description of the one sided coupled operator

Throughout this hypotheses $(\mathcal{H}_1)$-$(\mathcal{H}_3)$, we define in the product of Banach spaces $E \times F$, the one sided coupled operator matrix $\mathcal{A}$ as follows:

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{D}(A_m) \times \mathcal{D}(D) : \quad \phi(f) = \psi(g) \right\},$$

$$\mathcal{A} \begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} A_m f + Bg \\ Cf + Dg \end{pmatrix}, \quad \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{D}(\mathcal{A}).$$

Our aim is to describe some essential spectra of this kind of operator matrix. For this purpose, we need to decompose it into the following form:

Lemma 4.1 [5] For $\mu \in \rho(A_0) \cap \rho(D)$, we have on $\mathcal{D}(\mathcal{A})$:

$$\mu - \mathcal{A} := \begin{pmatrix} \mu - A_0 & 0 \\ 0 & \mu - D \end{pmatrix} \begin{pmatrix} I & -\Xi_B(\mu) \\ -\Xi_C(\mu) & I \end{pmatrix},$$

where:

$$\Xi_C(\mu) := (\mu - D)^{-1} C \quad \text{and} \quad \Xi_B(\mu) := K_\mu + (\mu - A_0)^{-1} B. \qquad \diamondsuit$$

The following obtained results will be essential to prove the main theorem of the present paper.

Theorem 4.1 Let $\mu \in \rho(A_0) \cap \rho(D)$.
(i) If the operator $\Delta_\mu(C, B) := I - \Xi_C(\mu)\Xi_B(\mu)$ is invertible in $\mathcal{D}(D)$, then $\mu - \mathcal{A}$ is invertible. Moreover,

$$1 \in \rho(\Xi_C(\mu)\Xi_B(\mu)) \quad \Longrightarrow \quad \mu \in \rho(\mathcal{A}),$$

with inverse given by:

$$(\mu - \mathcal{A})^{-1} := \mathcal{Q}_\mu(A_0, D)^{-1} + \Gamma(B, C),$$

where:

- $\mathcal{Q}_\mu(A_0, D) := \begin{pmatrix} \mu - A_0 & 0 \\ 0 & \mu - D \end{pmatrix}$

$$\bullet\ \Gamma(B,C) := \begin{pmatrix} \Xi_B(\mu)(\Delta_\mu(C,B))^{-1}\Xi_C(\mu)(\mu-A_0)^{-1} & \begin{array}{c}\Xi_B(\mu)(\mu-D)^{-1} \\ \Xi_B(\mu)(\Delta_\mu(C,B))^{-1}\Xi_C(\mu)\Xi_B(\mu)(\mu-D)^{-1}\end{array} \\ (\Delta_\mu(C,B))^{-1}\Xi_C(\mu)(\mu-A_0)^{-1} & (\Delta_\mu(C,B))^{-1}\Xi_C(\mu)\Xi_B(\mu)(\mu-D)^{-1} \end{pmatrix}$$

(ii) If the operator $\Delta_\mu(B,C) := I - \Xi_B(\mu)\Xi_C(\mu)$ is invertible in $\mathcal{D}(A_m)$, then $\mu - \mathcal{A}$ is invertible. Moreover,

$$1 \in \rho(\Xi_B(\mu)\Xi_C(\mu)) \quad \Longrightarrow \quad \mu \in \rho(\mathcal{A}),$$

with inverse given by:

$$(\mu - \mathcal{A})^{-1} := \mathcal{Q}_\mu(A_0, D)^{-1} + \widetilde{\Gamma}(B,C),$$

where:

$\widetilde{\Gamma}(B,C) :=$

$$\begin{pmatrix} (\Delta_\mu(B,C))^{-1}\Xi_B(\mu)\Xi_C(\mu)(\mu-A_0)^{-1} & (\Delta_\mu(B,C))^{-1}\Xi_B(\mu)(\mu-D)^{-1} \\ \begin{array}{c}\Xi_C(\mu)(\mu-A_0)^{-1} \\ \Xi_C(\mu)(\Delta_\mu(B,C))^{-1}\Xi_B(\mu)\Xi_C(\mu)(\mu-A_0)^{-1}\end{array} & \Xi_C(\mu)(\Delta_\mu(B,C))^{-1}\Xi_B(\mu)(\mu-D)^{-1} \end{pmatrix}.$$

$\diamondsuit$

Proof. Let $\mu \in \rho(A_0) \cap \rho(D)$.
Based on the Fobenuis-Schur factorization of the matrix operator form $\mathbb{M}_\mu := \begin{pmatrix} I & \Xi_B(\mu) \\ -\Xi_C(\mu) & I \end{pmatrix}$, we infer its decomposition as follows:

$$\mathbb{M}_\mu := \begin{pmatrix} I & 0 \\ -\Xi_C(\mu) & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I - \Xi_C(\mu)\Xi_B(\mu) \end{pmatrix} \begin{pmatrix} I & -\Xi_B(\mu) \\ 0 & I \end{pmatrix} \tag{4.1}$$

or

$$\mathbb{M}_\mu := \begin{pmatrix} I & -\Xi_B(\mu) \\ 0 & I \end{pmatrix} \begin{pmatrix} I - \Xi_B(\mu)\Xi_C(\mu) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -\Xi_C(\mu) & I \end{pmatrix} \tag{4.2}$$

for which, the first and last factors of Eqs. (4.1)-(4.2) are bounded and bounded invertible. Hence, we deduce from the fact that $\Delta_\mu(C,B) := I - \Xi_C(\mu)\Xi_B(\mu)$ (resp. $\Delta_\mu(B,C) := I - \Xi_B(\mu)\Xi_C(\mu)$) is invertible that it is too for the operator matrix $\mathbb{M}_\mu$. Consequently, $\mu - \mathcal{A}$ become invertible while the operator $\begin{pmatrix} \mu - A_0 & 0 \\ 0 & \mu - D \end{pmatrix}$ is invertible too for $\mu \in \rho(A_0) \cap \rho(D)$.
Moreover, assumption $1 \in \rho(\Xi_C(\mu)\Xi_B(\mu))$ (resp. $1 \in \rho(\Xi_B(\mu)\Xi_C(\mu))$) asserts that $I - \Xi_C(\mu)\Xi_B(\mu)$ (resp. $I - \Xi_B(\mu)\Xi_C(\mu)$) has bounded inverse. From what proceed, we can deduce that $\mu - \mathcal{A}$ is bounded with bounded inverse.
Therefore, the explicit resolvent of $\mathcal{A}$ follows from the computation of the product between

$$\mathbb{M}_\mu^{-1} := \begin{pmatrix} I + \Xi_B(\mu)(I - \Xi_C(\mu)\Xi_B(\mu))^{-1}\Xi_C(\mu) & \Xi_B(\mu)(I - \Xi_C(\mu)\Xi_B(\mu))^{-1} \\ (I - \Xi_C(\mu)\Xi_B(\mu))^{-1}\Xi_C(\mu) & (I - \Xi_C(\mu)\Xi_B(\mu))^{-1} \end{pmatrix}$$

$$\Big(\text{resp. } \mathbb{M}_\mu^{-1} := \begin{pmatrix} (I-\Xi_B(\mu)\Xi_C(\mu))^{-1} & (I-\Xi_B(\mu)\Xi_C(\mu))^{-1}\Xi_B(\mu) \\ \Xi_C(\mu)(I-\Xi_B(\mu)\Xi_C(\mu))^{-1} & \Xi_C(\mu)(I-\Xi_B(\mu)\Xi_C(\mu))^{-1}\Xi_B(\mu)+I \end{pmatrix}\Big)$$

and $\mathcal{Q}_\mu(A_0,D)^{-1} := \begin{pmatrix} (\mu-A_0)^{-1} & 0 \\ 0 & (\mu-D)^{-1} \end{pmatrix}$, respectively. Q.E.D.

Now, we are in the position to express the first main results of this section. In the following, we will denote by ${}^C\Omega$ the complement of a subset $\Omega \subset \mathcal{C}$.

Theorem 4.2 Let $\mu \in \rho(A_0)\cap\rho(D)$ and $1 \in \rho(\Xi_C(\mu)\Xi_B(\mu))$.
Then, we have:
(i) If $\Xi_B(\mu)(\mu-D)^{-1} \in \mathcal{P}(\Phi_r^b(F,\mathcal{D}(A_m)))$ and $\Xi_C(\mu)(\mu-A_0)^{-1} \in \mathcal{P}(\Phi_r^b(E,\mathcal{D}(D)))$, then:

$$(\mu-\mathcal{A})^{-1} - \mathcal{Q}_\mu(A_0,D)^{-1} \in \mathcal{P}(\Phi_r^b(E\times F)),$$

in particular,

$$\sigma_{er}(\mathcal{A}) = \sigma_{er}(A_0)\cup\sigma_{er}(D).$$

(ii) If $\Xi_B(\mu)(\mu-D)^{-1} \in \mathcal{P}(\Phi_l^b(F,\mathcal{D}(A_m)))$ and $\Xi_C(\mu)(\mu-A_0)^{-1} \in \mathcal{P}(\Phi_l(E,\mathcal{D}(D)))$, then:

$$(\mu-\mathcal{A})^{-1} - \mathcal{Q}_\mu(A_0,D)^{-1} \in \mathcal{P}(\Phi_l^b(E\times F)),$$

in particular,

$$\sigma_{el}(\mathcal{A}) = \sigma_{el}(A_0)\cup\sigma_{el}(D).$$

(iii) If $\Xi_B(\mu-D)^{-1} \in \mathcal{P}(\Phi^b(F,\mathcal{D}(A_m)))$ and $\Xi_C(\mu-A_0)^{-1} \in \mathcal{P}(\Phi^b(E,\mathcal{D}(D)))$, then:

$$(\mu-\mathcal{A})^{-1} - \mathcal{Q}_\mu(A_0,D)^{-1} \in \mathcal{P}(\Phi^b(E\times F)),$$

with

$$i(\mu-\mathcal{A}) = i(\mu-A_0)+i(\mu-D)$$

in particular,

$$\sigma_{ew}(\mathcal{A}) = \sigma_{ew}(A_0)\cup\sigma_{ew}(D) \qquad \text{and} \qquad \sigma_{ess}(\mathcal{A}) \subseteq \sigma_{ess}(A_0)\cup\sigma_{ess}(D).$$

Moreover,
(iv) If ${}^C\sigma_{ew}(A_0)$ is connected, then

$$\sigma_{ess}(\mathcal{A}) = \sigma_{ess}(A_0)\cup\sigma_{ess}(D).$$

(v) If ${}^C\sigma_{ess}(\mathcal{A})$ and ${}^C\sigma_{ess}(D)$ are connected with $\rho(\mathcal{A}) \neq \emptyset$, then

$$\sigma_{eb}(\mathcal{A}) = \sigma_{eb}(A_0)\cup\sigma_{eb}(D).$$

$\diamondsuit$

Proof: (i) Following the assumptions for $\mu \in \rho(A_0)\cap\rho(D)$ with the fact that $I-\Xi_C(\mu)\Xi_B(\mu)$ is invertible with bounded inverse, we derive from the use of Theorem 2.1 and Theorem 4.1 that:

$$\Gamma(B,C) \in \mathcal{P}(\Phi_r^b(E\times F)).$$

Consequently, Theorem 2.2 asserts that:

$$\mu - \mathcal{A} \in \Phi_r^b(E \times F) \quad \Longleftrightarrow \quad \mu - A_0 \in \Phi_r^b(E) \quad \text{and} \quad \mu - D \in \Phi_r^b(F).$$

(ii) Obviously, the result of the right Fredholm perturbation and its relative essential spectrum follows in a similar ways as in the item (i).
(iii) A derivative consequence from assertions (i) and (ii) revels that:

$$(\mu - \mathcal{A})^{-1} - \mathcal{Q}_\mu(A_0, D)^{-1} \in \mathcal{P}(\Phi_l^b(E \times F)) \cap \mathcal{P}(\Phi_r^b(E \times F)) = \mathcal{P}(\Phi^b(E \times F)),$$

with

$$i(\mu - \mathcal{A}) = i(\mathcal{Q}_\mu(A_0, D)) = i(\mu - A_0) + i(\mu - D),$$

as $\mathcal{Q}_\mu(A_0, D)$ is diagonal operator matrix.

Hence, we obtain:

$$\mu \in \rho_{ess}(A_0) \cap \rho_{ess}(D) \implies \mu \in \rho_{ess}(\mathcal{A}).$$

(iv) The derive the equality result between the Schechter essential spectrum of $\mathcal{A}$ and the union of the Schechter essential spectrum of A_0 and D, we add to the previous inclusion the fact that ${}^C\sigma_{ew}(A_0)$ is connected. Which yields:

$$\mu \in \rho_{ew}(A_0) \quad \Longleftrightarrow \quad \mu \in \rho_{ess}(A_0).$$

Therefore, $\mu \in \rho_{ess}(D)$ and this shows that:

$$\rho_{ess}(\mathcal{A}) \subset \rho_{ess}(A_0) \cap \rho_{ess}(D).$$

(v) The Browder essential spectrum of $\mathcal{A}$ may be computed in terms of the Browder essential spectrum of A_0 and D, since we apply Lemma 2.1 in [15] under some connected arguments.

Q.E.D.

The theory of Fredholmness perturbations play a crucial role in spectral theory. The importance of this kind of theory is tested for two-group of transport equations and it is applicable to propose an abstract framework for the computation in easier manner of some essential spectra of a problem of transport operator.

5 Transport equations in slab geometry

To apply the present results to an example of two-group of transport equations with specific boundaries condition in order to validate the developed methods, we state the Banach space X as:

$$X := L_1((0,1) \times K, dxd\xi), \quad x \in (0,1) \;\; \xi = (\xi_1, \xi_2, \xi_3) \in K$$

where K is the unit sphere of $\mathbf{R}^3$.

We consider the following operator matrix with entries of integro-differential equation form:

$$\mathcal{A} := \begin{pmatrix} T & K_{12} \\ K_{21} & T_H + K_{22} \end{pmatrix}, \tag{5.1}$$

defined with non diagonal domain as:

$$\mathcal{D}(\mathcal{A}) := \left\{ \left(\begin{array}{c} f \\ g \end{array} \right) \in \mathcal{D}(T) \times \mathcal{D}(T_H) : \quad f^i = g^i \right\},$$

where we denote by f^i (resp. g^i) as the incoming fluxes on the boundaries space X^i

$$X^i := L_1(D^i, |\xi_3|d\xi) := L_1(D_1^i, |\xi_3|d\xi) \oplus L_1(D_2^i, |\xi_3|d\xi) := X_1^i \oplus X_2^i$$

associate to this physical model of transport and defined as:

$$f_{|D^i} := f^i \in X^i, \quad (\text{resp. } g_{|D^i} := g^i \in X^i)$$

where the set D^i represents the incoming boundaries of the phase space D as:

$$D^i := D_1^i \cup D_2^i = (\{0\} \times K^1) \cup (\{1\} \times K^0),$$

for

$$K^0 = K \cap \{\xi_3 < 0\} \text{ and } K^1 = K \cap \{\xi_3 > 0\}.$$

On the sobolev space $\mathcal{W}$ defined as:

$$\mathcal{W} := \left\{ \psi \in X : \quad \xi_3 \frac{\partial \psi}{\partial x} \in X \right\},$$

we consider the closed linear operator T as:

$$\left\{ \begin{array}{l} T : \mathcal{D}(T) \subseteq X \longrightarrow X \\ \qquad\qquad f \longrightarrow Tf, \quad (x, \xi) \longmapsto (Tf)(x, \xi) := -\xi_3 \dfrac{\partial f}{\partial x}(x, \xi) - \sigma_1(x, \xi) f(x, \xi) \end{array} \right.$$

The streaming operator T_H is defined as:

$$\left\{ \begin{array}{l} T_H : \mathcal{D}(T_H) \subseteq X \longrightarrow X \\ \qquad\qquad g \longrightarrow T_H g, \quad (x, \xi) \longmapsto (T_H g)(x, \xi) := -\xi_3 \dfrac{\partial g}{\partial x}(x, \xi) - \sigma_2(x, \xi) g(x, \xi) \\ \\ \mathcal{D}(T_H) = \{ g \in \mathcal{W} : \ g^i = H(g^o) \} \end{array} \right.$$

where:
$*$ the incoming and the outgoing fluxes g^i and $g^o := g_{|D^0}$, respectively, are related with the boundary operator H from the boundaries space X^0 into X^i, defined as a off-diagonal operator as the form:

$$H := \left(\begin{array}{cc} 0 & H_{12} \\ H_{21} & 0 \end{array} \right)$$

with entries are given by:

$$\left\{ \begin{array}{c} H_{12} : X_2^0 \longrightarrow X_1^i \\ f(1, .) \longmapsto H_{12} f(\xi) = f(0, \xi) \end{array} \right. \qquad \left\{ \begin{array}{c} H_{21} : X_1^0 \longrightarrow X_2^i \\ f(0, .) \longmapsto H_{21} f(\xi) = f(1, \xi) \end{array} \right.$$

$*$ the outgoing boundaries of the phase space D, denoted by D^0 and given by:

$$D^0 := D_1^0 \cup D_2^0 := (\{0\} \times K^0) \cup (\{1\} \times K^1),$$

$*$ the boundaries space X^0 is defined by the following ways:

$$X^0 := L_1(D^0, |\xi_3| d\xi) := L_1(D_1^0, |\xi_3| d\xi) \oplus L_1(D_2^0, |\xi_3| d\xi) := X_1^0 \oplus X_2^0$$

$*$ the collision frequency $\sigma_j(.,.)$, for $j = \{1, 2\}$, is considered as a positive bounded function on D.

The bounded collision operator matrix $\mathcal{K}$ with bounded linear collision operators K_{ij}, for $(i,j) \in \{(1,2), (2,1), (2,2)\}$, is considered as well:

$$\mathcal{K} = \begin{pmatrix} 0 & K_{12} \\ K_{21} & K_{22} \end{pmatrix},$$

where each collision linear operators $K_{ij}, (i,j) \in \{(1,2),(2,1),(2,2)\}$ are bounded on X and defined by:

$$\begin{cases} K_{ij} : X \longrightarrow X \\ \qquad \psi \longrightarrow K_{ij}\ \psi, \quad (x,\xi) \longmapsto (K_{ij}\ \psi)(x,\xi) := \displaystyle\int_K \kappa_{ij}(x,\xi,\xi')\ \psi(x,\xi')\ d\xi', \end{cases}$$

with measurable kernels $\kappa_{ij} : (0,1) \times K \times K \longrightarrow \mathbf{R}$.

Obviously, keeping with the above indication, this kind of operator matrix $\mathcal{A}$ with integro-differential operators may be regard as one sided operator matrix form of the theoretical result as:

$$\mathcal{A} := \begin{pmatrix} A_m & B \\ C & D \end{pmatrix}$$

with non maximal domain

$$\mathcal{D}(\mathcal{A}) := \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{W} \times \mathcal{D}(T_H) : \quad \phi(f) = \psi(g) \right\}.$$

Precisely, we identify the Banach spaces X, E and F by $L_1(D, dxd\xi)$, the closed operators A_m and D by T and $T_H + K_{22}$, defined on $\mathcal{D}(A_m) = \mathcal{W}$ and $\mathcal{D}(D) = \mathcal{D}(T_H)$ the operators B and C by the collision operators K_{12} and K_{21}, respectively. The domain of this kind of operator matrix considered non diagonal or non maximal, that is, with domain consists with one additional condition modeled by the boundaries condition $f^i = g^i$ which satisfies the following diagram:

$$\begin{tikzcd}
X \supset \mathcal{D}(A_m) := \mathcal{W} \arrow[r, "\phi"] & X^i \\
X \supset \mathcal{D}(D) := \mathcal{D}(T_H) \arrow[ur, "\psi"'] &
\end{tikzcd}$$

where the functions ϕ and ψ are identified as well:

$$\left\{\begin{array}{l}\phi : \mathcal{W} \longrightarrow X^i \\ \qquad f \longmapsto \phi(f) = f^i\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l}\psi : X \longrightarrow X^i \\ \qquad g \longmapsto \psi(g) = Hg^o.\end{array}\right.$$

Physically, the example of integro-differential operator $\mathcal{A}$ considered as Eq. (5.1) modeled the case of two group of transport operator with general boundaries condition. This means that is considered on $\mathcal{D}(\mathcal{A})$ as follows:

$$\mathcal{D}(\mathcal{A}) := \left\{\vartheta := \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{W} \times \mathcal{W} : \quad \vartheta^i = \mathcal{H}\vartheta^o\right\},$$

for specific bounded operator $\mathcal{H}$ given by the form:

$$\left\{\begin{array}{l}\mathcal{H} : X^o \longrightarrow X^i \\ \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \longmapsto \mathcal{H}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} := \begin{pmatrix} 0 & H \\ 0 & H \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}\end{array}\right.$$

Remark 5.1 (i) The operator T_H satisfies the assumption $(\mathcal{H}_1)$, due to Remark 4.1 in [17].

(ii) Obviously, the hypothesis $(\mathcal{H}_2)$ is fulfilled one has the trace mapping ϕ and ψ are continuous and surjective due to Theorem 1 p. 252 in [9].

(iii) The hypothesis $(\mathcal{H}_3)$ still valid since K_{ij} is a bounded operator on X, for $(i,j) \in \{(1,2),(2,1)\}$.$\diamondsuit$

Since we deals with the case of operator matrix with non diagonal domain or non maximal domain, to formulate our significant advances on this theory, we define an associate operator A_0 as follows:

$$\left\{\begin{array}{l}A_0 f(x,\xi) := Tf(x,\xi) := -\xi_3 \dfrac{\partial f}{\partial x}(x,\xi) - \sigma_1(x,\xi) f(x,\xi) \\ \mathcal{D}(A_0) = \{f \in \mathcal{D}(T) : f^i = 0\},\end{array}\right.$$

which corresponds in mathematical physics to the model of transport operator with vacuum boundaries condition.
Consider the real number μ_j^*, for $j = \{1,2\}$, defined in terms of the frequency of the collision by:

$$\mu_j^* := \text{ess-}\inf\{\sigma_j(x,\xi), \ \ (x,\xi) \in D\}.$$

Before moving the picture of the eigenvalues of this physical model of transport operator, we start to express the bounded operator K_μ corresponding to the theoretical part of this paper. To do this, the following terminology will be required.

Lemma 5.1 Let $\mu \in \rho(A_0) \cap \rho(D)$.
K_μ is a bounded operator which is expressed as the mapping:

$$\left\{\begin{array}{l}K_\mu : \mathcal{W} \longrightarrow \mathcal{W} \\ \qquad g \longmapsto K_\mu g, \ \ (x,\xi) \longmapsto (K_\mu g)(x,\xi) = \chi_{K^0}(\xi) H_{12} g(1,\xi) e^{-\int_x^1 \frac{\sigma_1(s,\xi)+\mu}{|\xi_3|}\, ds} \\ \qquad\qquad\qquad\qquad\qquad + \chi_{K^1}(\xi) H_{21} g(0,\xi)\, e^{-\int_0^x \frac{\sigma_1(s,\xi)+\mu}{|\xi_3|}\, ds}.\end{array}\right. \qquad \diamondsuit$$

Proof: Let $\mu \in \rho(A_0) \cap \rho(D)$. Note that the expression of K_μ may be checked by steps:
$*$ We start firstly to revels the expression of $\ker(\mu - A_m)$.
For this, we consider $\varphi \in \mathcal{D}(A_m)$. A short computation revels that:

$$\varphi \in \ker(\lambda - A_m) \quad \text{means that} \quad \varphi(x,\xi) := \begin{cases} \varphi(0,\xi)\, e^{-\int_0^x \frac{\sigma_1(s,\xi)+\mu}{|\xi_3|}\,ds}, & \xi \in K^1 \\ \varphi(1,\xi) e^{-\int_x^1 \frac{\sigma_1(s,\xi)+\mu}{|\mu_3|}\,ds}, & \xi \in K^0 \end{cases}$$

$*$ Secondly, taking into account from the boundaries condition $\phi(f) = \psi(g)$, for $f \in \mathcal{D}(A_m)$ and $g \in \mathcal{D}(D)$, a short computation revels the expansion of $\varphi(.,\xi)$ as follows:

$$\begin{cases} \varphi(0,\xi) = H_{12}g(1,\xi), & \xi \in K^1, \\ \varphi(1,\xi) = H_{21}g(0,\xi), & \xi \in K^0 \end{cases}$$

which makes an explicit formula of K_μ as:

$$(K_\mu g)(x,\xi) := \chi_{K^0}(\xi) H_{21} g(0,\xi) e^{-\int_x^1 \frac{\sigma_1(s,\xi)+\mu}{|\xi_3|}\,ds} + \chi_{K^1}(\xi) H_{12} g(1,\xi) e^{-\int_0^x \frac{\sigma_1(s,\xi)+\mu}{|\xi_3|}\,ds}. \text{ Q.E.D.}$$

Before moving to the picture of the eigenvalues of this kind of transport operator model, we will proof the weak compactness arguments of some operators expressed in terms of the collision operators based on the regularity definition explained by B. Lods in [19].

Definition 5.1 Each collision operator K_{ij} is said to be regular if

$$\{\kappa_{ij}(x,.,\xi'),\ (x,\xi') \in (0,1) \times K\} \in \mathcal{W}(L_1(K,d\xi)),$$

for $(i,j) \in \{(1,2),(2,1),(2,2)\}$. $\diamondsuit$

Lemma 5.2 Assume that the collision operators K_{21} and K_{12} are non-negative, the boundary operators H_{12} and H_{21} are weakly compact on X. Then, for $\mu \in \rho(A_0) \cap \rho(T_H)$ with spectral raduis $r_\sigma(K_{22}(\mu - T_H)^{-1}) < 1$, we have:

(i)The relative weak compact subset $\left\{\dfrac{\kappa_{21}(x,.,\xi')}{|\xi_3'|},\ (x,\xi') \in (0,1) \times K\right\}$ of $L_1(K,d\xi)$ implies that :

$$\Xi_C(\mu)(\mu - A_0)^{-1} := (\mu - T_H - K_{22})^{-1} K_{21} (\mu - T)^{-1} \in \mathcal{W}(X),$$

for $Re\mu > -\mu_1^*$.

(i) If the subset $\left\{\dfrac{\kappa_{12}(x,.,\xi')}{|\xi_3'|},\ (x,\xi') \in (0,1) \times K\right\}$ is relatively weakly compact of $L_1(K,d\xi)$, then we obtain:

$$\Xi_B(\mu)(\mu - T_H - K_{22})^{-1} := \left[K_\mu + (\mu - T)^{-1} K_{12}\right] (\mu - T_H - K_{22})^{-1} \in \mathcal{W}(X),$$

for $Re\mu > -\mu_2^*$. $\diamondsuit$

Proof: (i) The result may be obvious derived from the use of Lemma 4.2 in [17] with the fact that the subset $\mathcal{W}(X)$ is two sided closed ideal of $\mathcal{L}(X)$.

(ii) Let $\mu \in \rho(T_H)$ such that $r_\sigma(K_{22}(\mu - T_H)^{-1}) < 1$ and $Re\mu > -\mu_2^*$. For such μ, the operator $\Xi_B(\mu)(\mu - T_H - K_{22})^{-1}$ may be written as:

$$\Xi_B(\mu)(\mu - T_H - K_{22})^{-1} := \left[K_\mu + (\mu - T)^{-1} K_{12}\right] (\mu - T_H)^{-1} \sum_{n \geq 0} (K_{22}(\mu - T_H)^{-1})^n.$$

Therefore, following Lemma 4.4-(i) in [1], one has $K_{12}(\mu - T_H)^{-1}$ is a weakly compact operator on X. Hence, we deduce from the weak compactness argument of H and the fact that the set $\mathcal{W}(X)$ is a closed two sided-ideal of $\mathcal{L}(X)$, that:

$$\left[K_\mu + (\mu - T)^{-1}K_{12}\right](\mu - T_H)^{-1}(K_{22}(\mu - T_H)^{-1})^n \in \mathcal{W}(X), \ \ \forall n \in \mathbb{N}.$$

Thus, allows us conclude the desired results. Q.E.D.

The following lemma may be essential to derive our advances.

Lemma 5.3 For $\mu \in \rho(A_0) \cap \rho(T_H)$ such that $r_\sigma((\mu - T_H)^{-1}K_{22}) < 1$, we suppose that:

(i) K_{ij} is a non negative and regular collision operator, for $(i, j) \in \{(1, 2), (2, 1)\}$.

(ii) the boundary operators H_{12} and H_{21} are weakly compact on X.

Then, we have:

$$\Xi_C(\mu)\Xi_B(\mu) \in P\mathcal{K}(X).$$

Consequently, the operator $I - \Xi_C(\mu)\Xi_B(\mu)$ is invertible with bounded inverse on X. $\diamondsuit$

Proof: Consider $\mu \in \rho(A_0) \cap \rho(T_H)$ with $r_\sigma((\mu - T_H)^{-1}K_{22}) < 1$. Thus, the following equation

$$(\mu - T_H - K_{22})f = g, \ \text{ for } g \in X$$

may be solved as:

$$(\mu - T_H - K_{22})^{-1} := \sum_{k\geq 0}((\mu - T_H)^{-1}K_{22})^k(\mu - T_H)^{-1}. \tag{5.2}$$

That is means that the fact that $\mu \in \rho(A_0) \cap \rho(T_H)$ satisfying $r_\sigma((\mu - T_H)^{-1}K_{22}) < 1$, asserts that $\mu \in \rho(T_H + K_{22})$.
Therefore, the weak compactness argument of $(\mu - T_H)^{-1}K_{21}$ on X derived from Lemma 4.4 in [1] yields:

$$\Xi_C(\mu) := (\mu - T_H - K_{22})^{-1}K_{21} \in \mathcal{W}(X).$$

On the other hand, keeping with the fact that K_{12} defines a non-negative collision operator, we infer from Lemma 4.8 in [17], that $(\mu - A_0)^{-1}K_{12}$ is weakly compact operator on X. Thus, we conclude that is also for the operator $\Xi_B(\mu)$, while H_{12} and H_{21} are weakly compact operators on X.
Consequently, Remark 2.2 revels that:

$$\Xi_C(\mu)\Xi_B(\mu) \in P\mathcal{K}(X).$$

From that proceed, for $\mu \in \rho(A_0) \cap \rho(T_H)$ with $r_\sigma((\mu - T_H)^{-1}K_{22}) < 1$, we deduce that:

$\mu \in \rho(A_0) \cap \rho(T_H) \cap \rho(T_H + K_{22})$.Then, we get:

$$\begin{aligned}
& N(I - \Xi_C(\mu)\Xi_B(\mu)) \\
= & \Big\{z: \ \ (I - \Xi_C(\mu)\Xi_B(\mu))z = 0\Big\} \\
= & \Big\{z: \ \ (\mu - T_H - K_{22})^{-1}K_{21}\left[K_\mu + (\mu - A_0)^{-1}K_{12}\right]z = z\Big\} \\
= & \Big\{z: \ \ (\mu - T_H - K_{22})(\mu - T_H - K_{22})^{-1}K_{21}\left[K_\mu + (\mu - A_0)^{-1}K_{12}\right]z = (\mu - T_H - K_{22})z\Big\} \\
= & \Big\{z: \ \ K_{21}(\mu - A_0)^{-1}[(\lambda - A_0)K_\mu + K_{12}]z = (\mu - T_H - K_{22})z\Big\} \\
= & \Big\{z: \ \ K_{21}(\mu - A_0)^{-1}K_{12}z = (\mu - T_H - K_{22})z, \text{ while } \ K_\mu \in \ker(\mu - A_m)\Big\} \\
= & \Big\{z: \ \ [\mu - T_H - K_{22} - K_{21}(\mu - A_0)^{-1}K_{12}]z = 0\Big\}.
\end{aligned}$$

Following Remark in [14], we infer for $\mu \in \rho(A_0) \cap \rho(T_H + K_{22})$, that there exist $\mu \in \rho(A_0) \cap \rho(T_H + K_{22}) \cap \rho(T_H + K_{22} + K_{21}(\mu - A_0)^{-1}K_{12})$. In what follows, we deduce that $\mu - T_H - K_{22} - K_{21}(\mu - A_0)^{-1}K_{12}$ is invertible. Thus, asserts that:

$$N(\mu - T_H - K_{22} - K_{21}(\mu - A_0)^{-1}K_{12}) = \{0\}.$$

Consequently,

$$N(I - \Xi_C(\mu)\Xi_B(\mu)) = \{0\}.$$

Therefore, according with the fact that $\Xi_C(\mu)\Xi_B(\mu)$ is polynomially compact on X, we derive from Theorem 2.2 in [13], that $I - \Xi_C(\mu)\Xi_B(\mu)$ is invertible with bounded inverse on X.Q.E.D.

We are now in the position to express the description of the essential spectra of this physical model of transport.

Theorem 5.1 Assume that:

(i) the operator H_{ij} is weakly compact for $(i,j) \in \{(1,2),(2,1)\}$,

(ii) K_{22} defines a regular operator.

(iii) the collision operator K_{ij} is non negative and regular, for $(i,j) \in \{(1,2),(2,1)\}$,

(iv) the subset $\Big\{\dfrac{\kappa_{12}(x,.,\xi')}{|\xi'_3|}, \ (x,\xi') \in (0,1) \times K\Big\}$ (resp. $\Big\{\dfrac{\kappa_{21}(x,.,\xi')}{|\xi'_3|}, \ (x,\xi') \in (0,1) \times K\Big\}$) is relatively weakly compact of $L_1(K, d\xi)$.

Then, we get:

$$\begin{aligned}
\sigma_{ei}(\mathcal{A}) &= \sigma_{ei}(A_0) \cup \sigma_{ei}(T_H + K_{22}) \\
&= \{\mu \in \mathbf{C}: \ \ Re\mu \leq -\min(\mu_1^*, \mu_2^*)\},
\end{aligned}$$

for $\sigma_h(.) \in \{\sigma_{er}(.), \sigma_{el}(.), \sigma_{ew}(.), \sigma_{ess}(.), \sigma_{eb}(.).\}$ $\diamondsuit$

Proof: The Theorem follows from Theorem 4.2, Lemmas 5.2 and 5.3 and the use of Eqs. (2.1), (2.2) and (4.13) in [17]. Q.E.D.

Conclusion: This paper deals with a new model of unbounded block 2×2 operator matrix, called the one sided operator matrix. Therefore, this kind of operator matrix plays a really strong and fruitful role in computation of their essential spectra. Precisely, we develop innovative ways leading to a rigorous study of spectral properties of matrix operator with non diagonal domain (see Theorem 4.2) independently of its Schur complement and under less hypotheses of many earlier works of [2, 3, 8, 17, 21, 22, 33]. Furthermore, to clarify better contribution, we consider an example of transport operator with specific boundaries condition.

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