EXISTENCE AND UNIQUENESS WITH ULAM STABILITY STUDY OF THE SOLUTION FOR A CLASS OF CAPUTO FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION WITH MIXED CONDITIONS

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Abstract. In this article we show the existence, uniqueness and Ulam stability results of the solution for a class of a nonlinear Caputo fractional integro-differential problem with mixed conditions. we use three fixed point theorems to proof the existence and uniqueness results. By the results obtained, the reasons for the Ulam stability are verified. An example proposed to illustrate our main results.

1. Introduction

More than 300 years ago. Scientists have used incorrect random orders to generalize Ordinary differential equations and integrals by means of fractional differential equations. Where the origin of fractional calculus goes to Newton and Leibniz.
Fractional differential equations are known to many model: physical, biological, genetic and even economic phenomena ... etc.
Several recent studies carried out by many researchers in proving the existence and uniqueness of the solution for fractional differential equations with different conditions (boundary , initial , nonlocal and integral conditions ... etc), for more details [12, 9, 7, 10, 2, 1] and the references therein.
In ancient times, the study of the stability of solutions of fractional differential equations was slow, but recently, many researchers have done it in different articles in several ways (asymptotically stable, Ulam stable and generalized Ulam stable...), see [6, 5, 4, 10, 11].
In light of these studies we will prove the existence and uniqueness with the Ulam stability of the following system:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha+\beta} u^*(\tilde{x}) = \omega(\tilde{x}, u^*(\tilde{x})) + {}^{C}D_{0^+}^{\alpha}\psi(\tilde{x}, u^*(\tilde{x})) + \int_0^{\tilde{x}} N(\tilde{x}, s, u^*(s))ds \\ \\ u^*(0) = u_0^*, \qquad (u^*)'(0) = u_1^* \int_0^{\mathring{\gamma}} u^*(s)ds, \quad 0 < \mathring{\gamma} < 1 \end{cases} \tag{1.1}$$

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where u_0^*, u_1^* are real constants, $1 < \alpha + \beta \leq 2$, ${}^C D_{0^+}^{\beta}$ is the Caputo fractional derivative of order β, and ω, ψ, N defined as

$$\begin{array}{ll} \omega & : J \times X \longrightarrow X \\ \psi & : J \times X \longrightarrow X \\ N & : J \times J \times X \longrightarrow X, \end{array} \tag{1.2}$$

are an appropriate functions, X here is a Banach space.

Our study based on three fixed point theorem to proof the results of existence and uniqueness were we find it in section 3, the generalized stability is devoted to show in Section 4. Note that this representation also allows us to generalize the results obtained recently in the literature. The paper is ended by an example illustrating our results.

2. Some preliminary and integral equation

Here, we present definitions of the fractional integral, fractional Caputo derivative , and some auxiliary Lemmas.

We refer [14, 15] to see Some basic preliminary concepts of fractional calculus, and fixed point theory.

Definition 2.1. [8, 13] *Let $\rho > 0$ and $\hbar : \mathbb{R}_+ \longrightarrow \mathbb{R}$. The Riemann-Liouville fractional integral of order ρ of a function $\hbar$ is defined by*

$$I_{0^+}^{\rho} \hbar(\tilde{x}) = \frac{1}{\Gamma(\rho)} \int_0^{\tilde{x}} (\tilde{x} - s)^{\rho - 1}(s) \hbar(s) ds, \qquad \tilde{x} \in \mathbb{R}_+.$$

Definition 2.2. [8, 14] *Let $\rho > 0$, the Caputo fractional derivative of order ρ of a function $\hbar : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is defined by*

$${}^C D_{0^+}^{\rho} \hbar(\tilde{x}) = \frac{1}{\Gamma(n - \rho)} \int_0^{\tilde{x}} (\tilde{x} - s)^{n - \rho - 1} \hbar^{(n)}(s) ds = I_{0^+}^{n-\rho} \hbar^{(n)}(\tilde{x}), \qquad \tilde{x} \in \mathbb{R}_+.$$

where $n = [\rho] + 1$, provided the right side is pointwise defined on $\mathbb{R}_+$.

Lemma 2.3. [8, 14] *For real numbers $\rho > 0$ and appropriate function $\hbar(\tilde{x}) \in C^{n-1}[0, \infty)$ and $\hbar(\tilde{x})$ exists almost everywhere on any bounded interval of $\mathbb{R}_+$.*

$$(I_{0^+}^{\rho}\, {}^C D_{0^+}^{\rho} \hbar)(\tilde{x}) = \hbar(\tilde{x}) - \sum_{k=0}^{n-1} \frac{\hbar^{(k)}(0)}{k!} \tilde{x}^k.$$

Lemma 2.4. *Let $1 < \alpha + \beta < 2$ and $u_1^* \neq \frac{2}{\mathring{\gamma}^2}$. Assume that ω, ψ and N are three continuous functions. If $u^* \in C(J, X)$ then u^* is solution of (1.1) if and only if u^* satisfies the integral equation*

$$\begin{aligned} u^*(\tilde{x}) \quad = \quad & \int_0^{\tilde{x}} \frac{(\tilde{x} - s)^{\alpha + \beta - 1}}{\Gamma(\alpha + \beta)} \left(\omega(s, u^*(s)) + \int_0^s N(s, \tau, u^*(\tau)) d\tau \right) ds \\ & + \int_0^{\tilde{x}} \frac{(\tilde{x} - s)^{\beta - 1}}{\Gamma(\beta)} \psi(s, u^*(s)) ds + u_0^* - \frac{\psi(0, u_0^*)}{\Gamma(\beta + 1)} \tilde{x}^{\beta} \\ & + \frac{\tilde{x} u_1^*}{(1 - \frac{\mathring{\gamma}^2}{2} u_1^*)} \Bigg[\int_0^{\mathring{\gamma}} \frac{(\mathring{\gamma} - \tau)^{\alpha + \beta}}{\Gamma(\alpha + \beta + 1)} \left(\omega(\tau, u^*(\tau)) + \int_0^{\tau} N(\tau, \sigma, u^*(\sigma)) d\sigma \right) d\tau \\ & + \int_0^{\mathring{\gamma}} \frac{(\mathring{\gamma} - \tau)^{\beta}}{\Gamma(\beta + 1)} \psi(\tau, u^*(\tau)) d\tau - \frac{\psi(0, u_0^*)}{\Gamma(\beta + 2)} \mathring{\gamma}^{\beta + 1} + \mathring{\gamma} u_0^* \Bigg]. \end{aligned} \tag{2.1}$$

Proof. Let $u^* \in C(J,X)$ be a solution of (1.1). Firstly, let show that u^* is solution of (2.1).

By Lemma 2.3, we obtain

$$I_{0^+}^{\alpha+\beta C}D_{0^+}^{\alpha+\beta}u^*(\tilde{x}) = u^*(\tilde{x}) - u^*(0) - (u^*)'(0)\tilde{x}. \tag{2.2}$$

In addition, from equation in (1.1) and Definition 2.1, then use the lemma 2.3 we have

$$\begin{aligned} I_{0^+}^{\alpha+\beta C}D_{0^+}^{\alpha+\beta}u^*(\tilde{x}) &= I_{0^+}^{\alpha+\beta}\Big(\omega(\tilde{x},u^*(\tilde{x})) + \int_0^{\tilde{x}} N(\tilde{x},s,u^*(s))ds + {}^CD_{0^+}^{\alpha}\psi(\tilde{x},u^*(\tilde{x}))\Big) \\ &= I_{0^+}^{\alpha+\beta}\Big(\omega(\tilde{x},u^*(\tilde{x})) + \int_0^{\tilde{x}} N(\tilde{x},s,u^*(s))ds\Big) \\ &\quad + I_{0^+}^{\beta}\psi(\tilde{x},u^*(\tilde{x})) - \frac{\psi(0,u_0^*)}{\Gamma(\beta+1)}\tilde{x}^{\beta}. \end{aligned} \tag{2.3}$$

By substituting (2.3) in (2.2) with first condition in (1.1), yields:

$$\begin{aligned} u^*(\tilde{x}) &= I_{0^+}^{\alpha+\beta}\Big(\omega(\tilde{x},u^*(\tilde{x})) + \int_0^{\tilde{x}} N(\tilde{x},s,u^*(s))ds\Big) \\ &\quad + I_{0^+}^{\beta}\psi(\tilde{x},u^*(\tilde{x})) - \frac{\psi(0,u_0^*)}{\Gamma(\beta+1)}\tilde{x}^{\beta} + u_0^* + (u^*)'(0)\tilde{x}. \end{aligned} \tag{2.4}$$

but, we have

$$\begin{aligned} \frac{(u^*)'(0)}{u_1^*} &= \int_0^{\mathring{\gamma}} u^*(s)ds \\ &= \int_0^{\mathring{\gamma}} \Big[I_{0^+}^{\alpha+\beta}\big[\omega(s,u^*(s)) + \int_0^s N(s,\tau,u^*(\tau))d\tau\big] \\ &\quad + I_{0^+}^{\beta}\psi(s,u^*(s)) + (u^*)'(0)s + u_0^* - \frac{\psi(0,u_0^*)}{\Gamma(\beta+1)}s^{\beta}\Big]ds. \\ &= \int_0^{\mathring{\gamma}} \Big[I_{0^+}^{\alpha+\beta}\big[\omega(s,u^*(s)) + \int_0^s N(s,\tau,u^*(\tau))d\tau\big] + I_{0^+}^{\beta}\psi(s,u^*(s))\Big]ds \\ &\quad - \frac{\psi(0,u_0^*)}{\Gamma(\beta+2)}\mathring{\gamma}^{\beta+1} + \mathring{\gamma}u_0^* + \frac{\mathring{\gamma}^2}{2}(u^*)'(0). \\ &= I_{0^+}^{\alpha+\beta+1}\big[\omega(\mathring{\gamma},u^*(\mathring{\gamma})) + \int_0^{\mathring{\gamma}} N(\mathring{\gamma},\tau,u^*(\tau))d\tau\big] + I_{0^+}^{\beta+1}\psi(\mathring{\gamma},u^*(\mathring{\gamma})) \\ &\quad - \frac{\psi(0,u_0^*)}{\Gamma(\beta+2)}\mathring{\gamma}^{\beta+1} + \mathring{\gamma}u_0^* + \frac{\mathring{\gamma}^2}{2}(u^*)'(0). \end{aligned}$$

then, we find

$$\begin{aligned} (u^*)'(0) &= \frac{u_1^*}{(1-\frac{\mathring{\gamma}^2}{2}u_1^*)}\Big[I_{0^+}^{\alpha+\beta+1}\big[\omega(\mathring{\gamma},u^*(\mathring{\gamma})) + \int_0^{\mathring{\gamma}} N(\mathring{\gamma},\tau,u^*(\tau))d\tau\big] \\ &\quad + I_{0^+}^{\beta+1}\psi(\mathring{\gamma},u^*(\mathring{\gamma})) - \frac{\psi(0,u_0^*)}{\Gamma(\beta+2)}\mathring{\gamma}^{\beta+1} + \mathring{\gamma}u_0^*\Big]. \end{aligned}$$

therefore, we get

$$u^*(\tilde{x}) = I_{0^+}^{\alpha+\beta}\Big(\omega(\tilde{x},u^*(\tilde{x})) + \int_0^{\tilde{x}} N(\tilde{x},s,u^*(s))ds\Big)$$

$$
\begin{aligned}
&+I_{0^+}^{\beta}\psi(\tilde{x},u^*(\tilde{x}))-\frac{\psi(0,u_0^*)}{\Gamma(\beta+1)}\tilde{x}^{\beta}+u_0^*\\
&+\frac{\tilde{x}u_1^*}{(1-\frac{\mathring{\gamma}^2}{2}u_1^*)}\Big[I_{0^+}^{\alpha+\beta+1}\big[\omega(\mathring{\gamma},u^*(\mathring{\gamma}))+\int_0^{\mathring{\gamma}}N(\mathring{\gamma},\tau,u^*(\tau))d\tau\big]\\
&+I_{0^+}^{\beta+1}\psi(\mathring{\gamma},u^*(\mathring{\gamma}))-\frac{\psi(0,u_0^*)}{\Gamma(\beta+2)}\mathring{\gamma}^{\beta+1}+\mathring{\gamma}u_0^*\Big].
\end{aligned}
$$

Finally, we get the integral equivalent equation (2.1).

Conversely, applying ${}^CD_{0^+}^{\alpha+\beta}$ on both sides of (2.1), we find

$$
\begin{aligned}
{}^CD_{0^+}^{\alpha+\beta}u^*(\tilde{x}) &= {}^CD_{0^+}^{\alpha+\beta}I_{0^+}^{\alpha+\beta}\omega(s,u^*(s))+{}^CD_{0^+}^{\alpha+\beta}I_{0^+}^{\beta+\alpha}\int_0^s N(s,\tau,u^*(\tau))d\tau\\
&\quad +{}^CD_{0^+}^{\alpha+\beta}I_{0^+}^{\beta}\psi(s,u^*(s))+{}^CD_{0^+}^{\alpha+\beta}\Big(u^*(0)+(u^*)'(0)\tilde{x}\Big).\\
&= \omega(\tilde{x},u^*(\tilde{x}))+{}^CD_{0^+}^{\alpha}\psi(\tilde{x},u^*(\tilde{x}))+\int_0^{\tilde{x}}N(\tilde{x},s,u^*(s))ds \qquad (2.5)
\end{aligned}
$$

because,

$$
{}^CD_{0^+}^{\alpha+\beta}\Big(u^*(0)+(u^*)'(0)\tilde{x}\Big)=I^{2-\alpha-\beta}\Big[\frac{d^2}{dx^2}\big(u^*(0)+(u^*)'(0)\tilde{x}\big)\Big]=0
$$

this means that u^* satisfies the equation in problem (1.1). Furthermore, by substituting $\tilde{x}$ by 0 in integral equation (2.1), we have clearly that the nonlocal condition in (1.1) holds. Therefore, u^* is solution of problem (1.1), which completes the proof. □

3. Existence Results

Let transform the system (1.1) into fixed point problem as $u^*=\mathcal{F}u^*$, where $F: C(J,X)\longrightarrow C(J,X)$ is an operator defined by following:

$$
\begin{aligned}
\mathcal{F}u^*(\tilde{x}) &= \int_0^{\tilde{x}}\frac{(\tilde{x}-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big(\omega(s,u^*(s))+\int_0^s N(s,\tau,u^*(\tau))d\tau\Big)ds\\
&\quad+\int_0^{\tilde{x}}\frac{(\tilde{x}-s)^{\beta-1}}{\Gamma(\beta)}\psi(s,u^*(s))ds+u_0^*-\frac{\psi(0,u_0^*)}{\Gamma(\beta+1)}\tilde{x}^{\beta}\\
&\quad+\frac{u_1^*\tilde{x}}{(1-\frac{\mathring{\gamma}^2}{2}u_1^*)}\Big[\mathring{\gamma}u_0^*-\frac{\psi(0,u_0^*)}{\Gamma(\beta+2)}\mathring{\gamma}^{\beta+1}\Big]\\
&\quad+\frac{u_1^*\tilde{x}}{(1-\frac{\mathring{\gamma}^2}{2}u_1^*)}\Big[\int_0^{\mathring{\gamma}}\frac{(\mathring{\gamma}-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big(\omega(\tau,u^*(\tau))+\int_0^{\tau}N(\tau,\sigma,u^*(\sigma))d\sigma\Big)d\tau\\
&\quad+\int_0^{\mathring{\gamma}}\frac{(\mathring{\gamma}-\tau)^{\beta}}{\Gamma(\beta+1)}\psi(\tau,u^*(\tau))d\tau\Big].
\end{aligned}
$$

In order to simplify the computations, we offer the following notations:

$$
\Delta = \frac{\|\mu_1\|_{L^\infty}+\|\mu_3\|_{L^\infty}}{2+\beta+\alpha}\Big(\frac{1+\beta+\alpha}{2+\beta+\alpha}+\frac{|u_1^*|\mathring{\gamma}^{\alpha+\beta+1}}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\Big)
$$

$$+\frac{\|\mu_2\|_{L^\infty}}{2+\beta}\left(\frac{2+\beta}{\beta+1}+\frac{|u_1^*|\mathring{\gamma}^{\beta+1}}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\right). \tag{3.1}$$

$$\Delta_1 = \frac{|u_1^*|L}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\left[\frac{2\mathring{\gamma}^{\alpha+\beta+1}}{2+\alpha+\beta}+\frac{\mathring{\gamma}^{\beta+1}}{2+\beta}\right]. \tag{3.2}$$

$$\begin{aligned}\delta &= \frac{2}{\Gamma(\alpha+\beta+1)}+\frac{1}{\Gamma(\beta+1)}\\ &\quad+\frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\left(\frac{2\mathring{\gamma}^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)}+\frac{\mathring{\gamma}^{\beta+1}}{\Gamma(\beta+2)}\right).\end{aligned} \tag{3.3}$$

$$\delta_1 = \frac{|\psi(0,u_0^*)|}{\Gamma(\beta+1)}+|u_0^*|+\frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\left(\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+2)}\mathring{\gamma}^{\beta+1}+\mathring{\gamma}|u_0^*|\right). \tag{3.4}$$

$$\begin{aligned}\delta_2 &= \frac{1}{\alpha+\beta+1}+\frac{2}{\alpha+\beta+2}+\frac{1}{\beta+1}\\ &\quad+\frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\left(\frac{\mathring{\gamma}^{\alpha+\beta+1}}{\alpha+\beta+2}+\frac{\mathring{\gamma}^{\beta+1}}{\beta+2}+\frac{2\mathring{\gamma}^{\alpha+\beta+2}}{\alpha+\beta+3}\right).\end{aligned} \tag{3.5}$$

3.1. **Existence Result by using Leray-Schauder Nonlinear Alternative.**

Theorem 3.1. *Let $\omega,\psi\in C(J\times\mathbb{R},\mathbb{R})$ and $N\in C(J\times J\times\mathbb{R},\mathbb{R})$ be continuous functions. Assume that*

(H1): There exist functions $f_1,f_2\in C(J,\mathbb{R}^+)$, $f_3\in C(J\times J,\mathbb{R}^+)$, with $f=\max\{f_1,f_2,f_3\}$ and nondecreasing functions

$$g_1,g_2,g_3:\mathbb{R}^+\to\mathbb{R}^+,$$

with $g=\max\{g_1,g_2,g_3\}$ such that

$$\begin{aligned}|\omega(\tilde{x},u^*(\tilde{x}))| &\le f_1(\tilde{x})g_1(\|u^*\|),\\ |\psi(\tilde{x},u^*(\tilde{x}))| &\le f_2(\tilde{x})g_2(\|u^*\|)\end{aligned}$$

and

$$|N(\tilde{x},s,u^*(s))|\le f_3(s)g_3(\|u^*\|),$$

for all $\tilde{x}\in[0,1], s\in[0,1], u^*\in\mathbb{R}$.

(H2): There exist a constant $M>0$ such that $\frac{M}{\|f\|g(M)\delta_2+\delta_1}>1$.

Then the problem (1.1) admits at least one solution on J.

Proof. For $r>0$, let

$$B_r=\{\tilde{x}\in C([0,1],\mathbb{R}):\|u^*\|\le r\},$$

be a bounded set in $C([0,1],\mathbb{R})$. We will show that F maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. Then, by (H1), we have

$$\begin{aligned}\|\mathcal{F}u^*(\tilde{x})\| &\le \int_0^{\tilde{x}}\frac{(\tilde{x}-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left(f_1(s)g_1(\|u^*(s)\|)+\int_0^s f_3(s)g_3(\|u^*(\tau)\|)d\tau\right)ds\\ &\quad+\int_0^{\tilde{x}}\frac{(\tilde{x}-s)^{\beta-1}}{\Gamma(\beta)}f_2(s)g_2(\|u^*(s)\|)ds+\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+1)}+|u_0^*|\\ &\quad+\frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\left[\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+2)}\mathring{\gamma}^{\beta+1}+\mathring{\gamma}|u_0^*|\right]\\ &\quad+\frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\left[\int_0^{\mathring{\gamma}}\frac{(\mathring{\gamma}-\tau)^{\beta+1}}{\Gamma(\beta+2)}f_2(\tau)g_2(\|u^*(\tau)\|)d\tau\right.\end{aligned}$$

$$
\begin{aligned}
& + \int_0^{\mathring{\gamma}} \frac{(\mathring{\gamma}-\tau)^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)} \Big(f_1(\tau) g_1(\|u^*(\tau)\|) + \int_0^{\tau} f_3(\tau) g_3(\|u^*(\sigma)\|) d\sigma \Big) d\tau \Big] \\
\leq \quad & \|f\| g(\|u^*\|) \Big[\int_0^{\tilde{x}} \frac{(\tilde{x}-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \big(1+s\big) ds + \int_0^{\tilde{x}} \frac{(\tilde{x}-s)^{\beta-1}}{\Gamma(\beta)} ds \\
& + \frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2} u_1^*|} \Big(\int_0^{\mathring{\gamma}} \frac{(\mathring{\gamma}-\tau)^{\beta+1}}{\Gamma(\beta+2)} d\tau + \int_0^{\mathring{\gamma}} \frac{(\mathring{\gamma}-\tau)^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)} \big(1+\tau\big) \Big) d\tau \Big] \\
& + \frac{|\psi(0,u_0^*)|}{\Gamma(\beta+1)} + |u_0^*| + \frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2} u_1^*|} \Big[\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+2)} \mathring{\gamma}^{\beta+1} + \mathring{\gamma} |u_0^*| \Big] \\
\leq \quad & \|f\| g(\|u^*\|) \Big[\frac{1}{\alpha+\beta+1} + \frac{2}{\alpha+\beta+2} + \frac{1}{\beta+1} \\
& + \frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2} u_1^*|} \Big(\frac{\mathring{\gamma}^{\alpha+\beta+1}}{\alpha+\beta+2} + \frac{\mathring{\gamma}^{\beta+1}}{\beta+2} + \frac{2\mathring{\gamma}^{\alpha+\beta+2}}{\alpha+\beta+3} \Big) \Big] \\
& + \frac{|\psi(0,u_0^*)|}{\Gamma(\beta+1)} + |u_0^*| + \frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2} u_1^*|} \Big(\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+2)} \mathring{\gamma}^{\beta+1} + \mathring{\gamma} |u_0^*| \Big) \\
= \quad & \|f\| g(\|u^*\|) \delta_2 + \delta_1 < +\infty. \qquad (3.6)
\end{aligned}
$$

Let $\tilde{x}_1, \tilde{x}_2 \in [0,1]$ with $\tilde{x}_1 < \tilde{x}_2$ and $u^* \in B_r$, where B_r is a bounded set of $C([0,1],\mathbb{R})$. Then we have

$$
\begin{aligned}
& \|(\mathcal{F}u^*)(\tilde{x}_2) - (\mathcal{F}u^*)(\tilde{x}_1)\| \\
\leq \quad & \int_{\tilde{x}_1}^{\tilde{x}_2} \frac{(\tilde{x}_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big(\|\omega(s,u^*(s))\| + \int_0^s \|N(s,\tau,u^*(\tau))\| d\tau \Big) ds \\
& + \int_{\tilde{x}_1}^{\tilde{x}_2} \frac{(\tilde{x}_2-s)^{\beta-1}}{\Gamma(\beta)} \|\psi(s,u^*(s))\| ds \\
& + \int_0^{\tilde{x}_1} \frac{(\tilde{x}_1-s)^{\alpha+\beta-1} - (\tilde{x}_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big(\|\psi(s,u^*(s))\| + \int_0^s \|N(s,\tau,u^*(\tau))\| d\tau \Big) ds \\
& + \int_0^{\tilde{x}_1} \frac{(\tilde{x}_1-s)^{\beta-1} - (\tilde{x}_2-s)^{\beta-1}}{\Gamma(\beta)} \|\psi(s,u^*(s))\| ds + \frac{|\psi(0,u_0^*)|}{\Gamma(\beta+1)} (\tilde{x}_2^{\beta} - \tilde{x}_1^{\beta}) \\
& + \frac{|u_1^*|(\tilde{x}_2-\tilde{x}_1)}{|1-\frac{\mathring{\gamma}^2}{2} u_1^*|} \Big[\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+2)} \mathring{\gamma}^{\beta+1} + \mathring{\gamma} |u_0^*| + \int_0^{\mathring{\gamma}} \frac{(\mathring{\gamma}-\tau)^{\beta}}{\Gamma(\beta+1)} \psi(\tau,u^*(\tau)) d\tau \\
& + \int_0^{\mathring{\gamma}} \frac{(\mathring{\gamma}-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \Big(\omega(\tau,u^*(\tau)) + \int_0^{\tau} N(\tau,\sigma,u^*(\sigma)) d\sigma \Big) d\tau \Big]. \\
\leq \quad & \|f\| g(\|u^*\|) \Big[\frac{2(\tilde{x}_2-\tilde{x}_1)^{\alpha+\beta} + \big|\tilde{x}_1^{\alpha+\beta} - \tilde{x}_2^{\alpha+\beta}\big|}{\Gamma(\alpha+\beta+1)} + \frac{2(\tilde{x}_2-\tilde{x}_1)^{\beta} + \big|\tilde{x}_1^{\beta} - \tilde{x}_2^{\beta}\big|}{\Gamma(\beta+1)} \\
& + (\tilde{x}_2-\tilde{x}_1) \frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2} u_1^*|} \Big[\frac{\mathring{\gamma}^{\alpha+\beta+1}}{\alpha+\beta+2} + \frac{\mathring{\gamma}^{\beta+1}}{\beta+2} + \frac{2\mathring{\gamma}^{\alpha+\beta+2}}{\alpha+\beta+3} \Big] \Big] \\
& + (\tilde{x}_2^{\beta} - \tilde{x}_1^{\beta}) \frac{|\psi(0,u_0^*)|}{\Gamma(\beta+1)} + (\tilde{x}_2-\tilde{x}_1) \frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2} u_1^*|)} \Big[\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+2)} \mathring{\gamma}^{\beta+1} + \mathring{\gamma} |u_0^*| \Big].
\end{aligned}
$$

If $(\tilde{x}_2 - \tilde{x}_1) \longrightarrow 0$, then the RHS of the above inequality tends to zero independently of $u^* \in B_r$. That is implies: $\|\mathcal{F}u^*(\tilde{x}_2) - \mathcal{F}u^*(\tilde{x}_1)\| \longrightarrow 0$, if $\tilde{x}_2 - \tilde{x}_1 \longrightarrow 0$ then F maps bounded sets into equi-continuous sets of C.

By Arzela-Ascoli theorem, we have $\mathcal{F} : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ is completely continuous.
We will apply the Leray-schauder nonlinear alternative once we establish the boundedness of the set of all solutions to equation

$$u^* = \epsilon \mathcal{F} u^* \quad \textit{for some} \quad \epsilon \in (0,1).$$

Let u^* be a solution of (1.1), then, by (3.6) we have

$$|u^*(\tilde{x})| \quad \leq \quad \|f\| g(\|u^*\|)\delta_2 + \delta_1.$$

which implies:

$$\frac{\|u^*\|}{\|f\| g(\|u^*\|)\delta_2 + \delta_1} \leq 1.$$

Then by (H2), there exist $M > 0$ such that $M \neq \|u^*\|$. Let us define a set

$$Y = \{u^* \in C([0,1],\mathbb{R}) / \ \ \|u^*\| < M\},$$

and then

$$\mathcal{F} : \bar{Y} \longrightarrow C([0,1],\mathbb{R}),$$

is completely continuous. From the choice of Y, there is no $\tilde{x} \in \partial Y$ such that

$$u^* = \epsilon \mathcal{F} u^* \qquad \text{for} \ \ \epsilon \in (0,1),$$

then by the nonlinear Leray-Schauder type, we conclude that the operator $\mathcal{F}$ has a fixed point $u^* \in \bar{Y}$ which is solution of the BVP (1.1). □

3.2. **Existence result by Krasnoselskii's Fixed Point.**

Theorem 3.2. *Let* $\omega, \psi : [0,1] \times X \longrightarrow X$ *and* $N : [0,1] \times [0,1] \times X \longrightarrow X$ *be continuous functions satisfying*

(1) *(H1) The inequalities*

$$\|\omega(\tilde{x}, u^*(\tilde{x})) - \omega(\tilde{x}, v^*(\tilde{x}))\| \leq L_1 \|u^*(\tilde{x}) - v^*(\tilde{x})\|, \quad \tilde{x} \in [0,1], \ \ u^*, v^* \in X$$
$$\|\psi(\tilde{x}, u^*(\tilde{x})) - \psi(\tilde{x}, v^*(\tilde{x}))\| \leq L_2 \|u^*(\tilde{x}) - v^*(\tilde{x})\|, \quad \tilde{x} \in [0,1], \ \ u^*, v^* \in X$$
$$\|N(\tilde{x}, s, u^*(s)) - N(\tilde{x}, s, v^*(s))\| \leq L_3 \|u^*(s) - v^*(s)\|, \quad (\tilde{x}, s) \in G, \ \ u^*, v^* \in X$$

hold where $L_1, L_2, L_3 \geq 0$ *with* $L = \max\{L_1, L_2, L_3\}$ *and* $G = \{(\tilde{x}, s) : 0 \leq s \leq x \leq 1\}$.

(2) *(H2) There exist three functions* $\mu_1^*, \mu_2, \mu_3 \in L^\infty([0,1],\mathbb{R}^+)$ *such that*

$$\|\omega(\tilde{x}, u^*(\tilde{x}))\| \leq \mu_1^*(\tilde{x}) \|u^*(\tilde{x})\|, \quad \tilde{x} \in [0,1], \ \ u^* \in X$$
$$\|\psi(\tilde{x}, u^*(\tilde{x}))\| \leq \mu_2(\tilde{x}) \|u^*(\tilde{x})\|, \quad \tilde{x} \in [0,1], \ \ u^* \in X$$
$$\|N(\tilde{x}, s, u^*(s))\| \leq \mu_3(\tilde{x}) \|u^*(s)\|, \quad (\tilde{x}, s) \in G, \ \ u^* \in X.$$

If $\Delta \leq 1$ *and* $L\Delta_1 \leq 1$, *then the problem (1.1) has at a least one solution on* $[0,1]$.

Proof. For any function $u^* \in C(J, X)$ we define the norm

$$\|u^*\|_1 = \max\{e^{-x} \|u^*(\tilde{x})\| : \ \ \tilde{x} \in [0,1]\}$$

and consider the closed ball

$$B_r = \{u^* \in C(J, X) : \|u^*\|_1 \leq r\}.$$

Next, let us define the operators $\mathcal{F}_1, \mathcal{F}_2$ on B_r as follows

$$\mathcal{F}_1 u^*(\tilde{x}) \quad = \quad \int_0^{\tilde{x}} \frac{(\tilde{x} - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \left(\omega(s, u^*(s)) + \int_0^s N(s, \tau, u^*(\tau)) d\tau \right) ds$$

$$
\begin{aligned}
&+\int_0^{\tilde{x}} \frac{(\tilde{x}-s)^{\beta-1}}{\Gamma(\beta)}\psi(s,u^*(s))ds + u_0^* - \frac{\psi(0,u_0^*)}{\Gamma(\beta+1)}\tilde{x}^{\beta} \\
&+\frac{u_1^* x}{(1-\frac{\mathring{\gamma}^2}{2}u_1^*)}\left[\mathring{\gamma}u_0^* - \frac{\psi(0,u_0^*)}{\Gamma(\beta+2)}\mathring{\gamma}^{\beta+1}\right]. \qquad (3.7)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{F}_2 u^*(\tilde{x}) \quad = \quad & \frac{u_1^* x}{(1-\frac{\mathring{\gamma}^2}{2}u_1^*)}\left[\int_0^{\mathring{\gamma}} \frac{(\mathring{\gamma}-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(\omega(\tau,u^*(\tau)) + \int_0^{\tau} N(\tau,\sigma,u^*(\sigma))d\sigma\right)d\tau\right. \\
&\left. + \int_0^{\mathring{\gamma}} \frac{(\mathring{\gamma}-\tau)^{\beta}}{\Gamma(\beta+1)}\psi(\tau,u^*(\tau))d\tau\right]. \qquad (3.8)
\end{aligned}
$$

For $u, v^* \in B_r$, $\tilde{x} \in [0,1]$, by fixed $r \geq \frac{\delta_1}{1-\Delta}$ and by the assumption (H2) we find:

$$
\begin{aligned}
e^{\tilde{x}}\|\mathcal{F}_1 u^*(\tilde{x}) + \mathcal{F}_2 v^*(\tilde{x})\|_1 \quad \leq \quad & \int_0^{\tilde{x}} \frac{(\tilde{x}-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left(\mu_1(s)\|u^*(s)\| + \int_0^{s} \mu_3(s)\|u^*(\tau)\|d\tau\right)ds \\
&+\int_0^{\tilde{x}} \frac{(\tilde{x}-s)^{\beta-1}}{\Gamma(\beta)}\mu_2(s)\|u^*(s)\|ds + \frac{|\psi(0,u_0^*)|}{\Gamma(\beta+1)} + |u_0^*| \\
&+\frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\left[\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+2)}\mathring{\gamma}^{\beta+1} + \mathring{\gamma}|u_0^*|\right] \\
&+\frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\left[\int_0^{\mathring{\gamma}} \frac{(\mathring{\gamma}-\tau)^{\beta}}{\Gamma(\beta+1)}\mu_2(\tau)\|v^*(\tau)\|d\tau\right. \\
&\left.+\int_0^{\mathring{\gamma}} \frac{(\mathring{\gamma}-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(\mu_1(\tau)\|v^*(\tau)\| + \int_0^{\tau} \mu_3(\tau)\|v^*(\sigma)\|d\sigma\right)d\tau\right] \\
\leq \quad & \int_0^{\tilde{x}} \frac{(\tilde{x}-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left(\|\mu_1\|_{L^\infty}\|u^*\|_1 e^s + \|\mu_3\|_{L^\infty}\|u^*\|_1(e^s-1)\right)ds \\
&+\|\mu_2\|_{L^\infty}\|u^*\|_1\int_0^{\tilde{x}} \frac{(\tilde{x}-s)^{\beta-1}}{\Gamma(\beta)}e^s ds + \frac{|\psi(0,u_0^*)|}{\Gamma(\beta+1)} + |u_0^*| \\
&+\frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\left[\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+2)}\mathring{\gamma}^{\beta+1} + \mathring{\gamma}|u_0^*|\right] \\
&+\frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\left[\int_0^{\mathring{\gamma}} \frac{(\mathring{\gamma}-\tau)^{\beta}}{\Gamma(\beta+1)}\|\mu_2\|_{L^\infty}\|v^*\|_2 e^{\tau}d\tau\right. \\
&\left.+\int_0^{\mathring{\gamma}} \frac{(\mathring{\gamma}-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(\|\mu_1\|_{L^\infty}\|v^*\|_1 e^{\tau} + \|\mu_3\|_{L^\infty}\|v^*\|_3(e^{\tau}-1)\right)d\tau\right]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|\mathcal{F}_1 u \quad + \quad & \mathcal{F}_2 v^*\|_1 \\
\leq \quad & \frac{\tilde{x}^{\alpha+\beta}}{\alpha+\beta+1}\left(\|\mu_1\|_{L^\infty}\|u^*\|_1 + \|\mu_3\|_{L^\infty}\|u^*\|_1\right) + \frac{\tilde{x}^{\beta}}{\beta+1}\|\mu_2\|_{L^\infty}\|u^*\|_1 \\
&+e^{-\tilde{x}}\left[\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+1)} + |u_0^*| + \frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\left(\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+2)}\mathring{\gamma}^{\beta+1} + \mathring{\gamma}|u_0^*|\right)\right] \\
&+\frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\left[\frac{\mathring{\gamma}^{\beta+1}}{\beta+2}\|\mu_2\|_{L^\infty}\|v^*\|_1 + \frac{\mathring{\gamma}^{\alpha+\beta+1}}{\alpha+\beta+2}\left(\|\mu_1\|_{L^\infty}\|v^*\|_1 + \|\mu_3\|_{L^\infty}\|v^*\|_3\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq r\Bigg[\frac{\|\mu_1\|_{L^\infty}+\|\mu_3\|_{L^\infty}}{2+\beta+\alpha}\Bigg(\frac{1+\beta+\alpha}{2+\beta+\alpha}+\frac{|u_1^*|\mathring{\gamma}^{\alpha+\beta+1}}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\Bigg)+\frac{\|\mu_2\|_{L^\infty}}{2+\beta}\Bigg(\frac{2+\beta}{\beta+1}+\frac{|u_1^*|\mathring{\gamma}^{\beta+1}}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\Bigg)\Bigg] \\
&\quad +\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+1)}+|u_0^*|+\frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\Bigg(\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+2)}\mathring{\gamma}^{\beta+1}+\mathring{\gamma}|u_0^*|\Bigg) \\
&= r\Delta+\delta_1\leq r.
\end{aligned}
$$

This implies that $(\mathcal{F}_1u+\mathcal{F}_2v)\in B_r$. We use the estimations:

$$
\frac{e^s}{e^{\tilde{x}}}\leq 1,\quad \frac{e^\tau}{e^{\tilde{x}}}\leq 1,\quad \frac{e^\tau-1}{e^{\tilde{x}}},\quad \frac{e^s-1}{e^{\tilde{x}}}\leq 1.
$$

Now, we establish that $\mathcal{F}_2$ is a contraction mapping. For $u^*,v^*\in X$ and $\tilde{x}\in[0,1]$, we have:

$$
\begin{aligned}
e^x\|\mathcal{F}_2u^*(\tilde{x})-\mathcal{F}_2v^*(\tilde{x})\|_1 &\leq \frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\Bigg[\int_0^{\mathring{\gamma}}\frac{(\mathring{\gamma}-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Bigg(\|\omega(\tau,u^*(\tau))-\omega(\tau,v^*(\tau))\| \\
&\quad +\int_0^\tau\|N(\tau,\sigma,u^*(\sigma))-N(\tau,\sigma,v^*(\sigma))\|d\sigma\Bigg)d\tau \\
&\quad +\int_0^{\mathring{\gamma}}\frac{(\mathring{\gamma}-\tau)^{\beta}}{\Gamma(\beta+1)}\|\psi(\tau,u^*(\tau))-\psi(\tau,v^*(\tau))\|d\tau\Bigg] \\
&\leq \frac{|u_1^*|L}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\Bigg[\int_0^{\mathring{\gamma}}\frac{(\mathring{\gamma}-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Bigg(\|u^*-v^*\|_1e^\tau+\|u^*-v^*\|_1(e^\tau-1)\Bigg) \\
&\quad +\int_0^{\mathring{\gamma}}\frac{(\mathring{\gamma}-\tau)^{\beta}}{\Gamma(\beta+1)}\|u^*-v^*\|_1e^\tau d\tau\Bigg].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|\mathcal{F}_2u-\mathcal{F}_2v^*\|_1 &\leq \frac{|u_1^*|L}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\Bigg[\frac{2\mathring{\gamma}^{\alpha+\beta+1}}{2+\alpha+\beta}+\frac{\mathring{\gamma}^{\beta+1}}{2+\beta}\Bigg]\|u^*-v^*\|_1. \\
&= L\Delta_1\|u^*-v^*\|_1.
\end{aligned}
$$

Then since $L\Delta_1\leq 1,\mathcal{F}_2$ is a contraction mapping. The continuity of the functions h,f and K implies that the operator $\mathcal{F}_1$ is continuous. Also, $\mathcal{F}_1B_r\subset B_r$, for each $u^*\in B_r$, i.e. $\mathcal{F}_1$ is uniformly bounded on B_r as

$$
\begin{aligned}
e^x\|\mathcal{F}_1u^*(\tilde{x})\|_1 &\leq \int_0^{\tilde{x}}\frac{(\tilde{x}-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Bigg(\|\mu_1\|_{L^\infty}\|u^*\|_1e^s+\|\mu_3\|_{L^\infty}\|u^*\|_1(e^s-1)\Bigg)ds \\
&\quad +\|\mu_2\|_{L^\infty}\|u^*\|_1\int_0^{\tilde{x}}\frac{(\tilde{x}-s)^{\beta}}{\Gamma(\beta)}e^sds+\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+1)}\tilde{x}^\beta+|u_0^*| \\
&\quad +\frac{|u_1^*|x}{(1-\frac{\mathring{\gamma}^2}{2}|u_1^*|)}\Bigg[\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+2)}\mathring{\gamma}^{\beta+1}+\mathring{\gamma}|u_0^*|\Bigg].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|\mathcal{F}_1u\|_1 &\leq r\Bigg[\frac{\|\mu_1\|_{L^\infty}+\|\mu_3\|_{L^\infty}}{\alpha+\beta+1}+\frac{\|\mu_2\|_{L^\infty}}{1+\beta}\Bigg] \\
&\quad +\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+1)}+|u_0^*|+\frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\Bigg(\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+2)}\mathring{\gamma}^{\beta+1}+\mathring{\gamma}|u_0^*|\Bigg) \\
&= r\Delta+\delta_1\leq r.
\end{aligned}
$$

Finally, we will show that $\overline{(\mathcal{F}_1 B_r)}$ is equicontinuous. For this end, we define

$$\overline{\omega}=\sup_{(s,u)\in[0,1]\times B_r}\|\omega(s,u)\|,\overline{\psi}=\sup_{(s,u)\in[0,1]\times B_r}\|\psi(s,u)\|,\overline{N}=\sup_{(s,\tau,u)\in G\times B_r}\int_0^s\|N(\tilde{x},s,u)\|d\tau.$$

Let for any $u^*\in B_r$ and for each $\tilde{x}_1,\tilde{x}_2\in[0,1]$ with $\tilde{x}_1\leq\tilde{x}_2$, we have:

$$\begin{aligned}
&\|(\mathcal{F}_1u)(\tilde{x}_2)-(\mathcal{F}_1u)(\tilde{x}_1)\|\\
\leq\ &\frac{\overline{\omega}+\overline{N}}{\Gamma(\alpha+\beta+1)}\Big[2\big|\tilde{x}_2-\tilde{x}_1\big|^{\alpha+\beta}+\big|\tilde{x}_1^{\alpha+\beta}-\tilde{x}_2^{\alpha+\beta}\big|\Big]+\frac{\overline{\psi}}{\Gamma(\beta+1)}\Big[2\big|\tilde{x}_2-\tilde{x}_1\big|^{\beta}+\big|\tilde{x}_1^{\beta}-\tilde{x}_2^{\alpha}\big|\Big]\\
&+(\tilde{x}_2^{\beta}-\tilde{x}_1^{\beta})\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+1)}+(\tilde{x}_2-\tilde{x}_1)\frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|)}\left[\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+2)}\mathring{\gamma}^{\beta+1}+\mathring{\gamma}|u_0^*|\right].
\end{aligned}$$

The right hand side of the last inequality is independent of u^* and tends to zero when $|\tilde{x}_2-\tilde{x}_1|\longrightarrow 0$, this means that $|\mathcal{F}_1u^*(\tilde{x}_2)-\mathcal{F}_1u^*(\tilde{x}_1)|\longrightarrow 0$, which implies that $\overline{\mathcal{F}_1 B_r}$ is equicontinuous, then $\mathcal{F}_1$ is relatively compact on B_r. Hence by Arzela-Ascoli theorem, $\mathcal{F}_1$ is compact on B_r. Now, all hypothesis of Theorem3.2 hold, therefore the operator $\mathcal{F}_1+\mathcal{F}_2$ has a fixed point on B_r. So the problem (1.1) has at least one solution on $[0,1]$. This proves the theorem. □

3.3. **Existence and Uniqueness Result.**

Theorem 3.3. *Assume that (H1) holds. If $L\Delta<1$, then the BVP (1.1) has a unique solution on* $[0,1]$.

Proof. Define $M=\max\{M_1,M_2,M_3\}$, where M_1,M_2,M_3 are positive numbers such that:

$$M_1=\sup_{\tilde{x}\in[0,1]}\|\omega(\tilde{x},0)\|,\qquad M_2=\sup_{\tilde{x}\in[0,1]}\|\psi(\tilde{x},0)\|,\qquad M_3=\sup_{(\tilde{x},s)\in G}\|\int_0^{\tilde{x}}N(\tilde{x},s,0)ds\|.$$

We fix $r_1\geq\frac{M\delta+\delta_1}{1-L\delta}$ and we consider

$$D_{r_1}=\{\tilde{x}\in C([0,1],X):\quad \|u^*\|\leq r_1\}.$$

where Then, in view of the assumption (H1), we have

$$\|\omega(\tilde{x},u^*(\tilde{x}))\|=\|\omega(\tilde{x},u^*(\tilde{x}))-\omega(\tilde{x},0)+\omega(\tilde{x},0)\|\leq\leq L_1\|u^*\|+M_1,$$

$$\|\psi(\tilde{x},u^*(\tilde{x}))\|\leq L_2\|u^*\|+M_2,\ \textbf{and}\ \|\int_0^{\tilde{x}}N(\tilde{x},s,u^*(s))\|\leq L_3\|u^*\|+M_3.$$

First step: We show that $TD_r\subset D_r$. For each $t\in[0,1]$ and for any $u^*\in D_r$,

$$\begin{aligned}
\|(\mathcal{F}u^*)(\tilde{x})\|\ \leq\ &\int_0^{\tilde{x}}\frac{(\tilde{x}-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left(\|\omega(s,u^*(s))\|+\int_0^s\|N(s,\tau,u^*(\tau))\|d\tau\right)ds\\
&+\int_0^{\tilde{x}}\frac{(\tilde{x}-s)^{\beta-1}}{\Gamma(\beta)}\|\psi(s,u^*(s))\|ds+\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+1)}+|u_0^*|\\
&+\frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\left[\int_0^{\mathring{\gamma}}\frac{(\gamma-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(\|\omega(\tau,u^*(\tau))\|+\int_0^{\tau}\|N(\tau,\lambda,u^*(\lambda))\|d\lambda\right)d\tau\right.\\
&\left.+\int_0^{\mathring{\gamma}}\frac{(\mathring{\gamma}-\tau)^{\beta}}{\Gamma(\beta+1)}\|\psi(\tau,u^*(\tau))\|d\tau+\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+2)}\mathring{\gamma}^{\beta+1}+\mathring{\gamma}|u_0^*|\right]\\
\leq\ &(Lr+M)\Bigg[\frac{1}{\alpha+\beta+1}+\frac{2}{\alpha+\beta+2}+\frac{1}{\beta+1}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\left(\frac{\mathring{\gamma}^{\alpha+\beta+1}}{\alpha+\beta+2}+\frac{\mathring{\gamma}^{\beta+1}}{\beta+2}+\frac{2\mathring{\gamma}^{\alpha+\beta+2}}{\alpha+\beta+3}\right)\Bigg]\\
&+\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+1)}+|u_0^*|+\frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\left(\frac{|\psi(0,u_0^*)|}{\Gamma(\beta+2)}\mathring{\gamma}^{\beta+1}+\mathring{\gamma}|u_0^*|\right)\\
=\;&(Lr_1+M)\delta+\delta_1\le r_1.
\end{aligned}
$$

Hence, $\mathcal{F}D_{r_1}\subset D_{r_1}$.

Second step: We shall show that $\mathcal{F}:D_{r_1}\longrightarrow D_r$ is a contraction. From the assumption $(H1)$, we have for any $u,v^*\in D_{r_1}$ and for each $\tilde{x}\in[0,1]$

$$
\begin{aligned}
&\|(\mathcal{F}u^*)(\tilde{x})-(\mathcal{F}v)(\tilde{x})\|\\
\le\;&\int_0^{\tilde{x}}\frac{(\tilde{x}-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[\|\omega(s,u^*(s))-\omega(s,v^*(s))\|+\int_0^s\|N(s,\tau,u^*(\tau))-N(s,\tau,v^*(\tau))\|d\tau\right]ds\\
&+\int_0^{\tilde{x}}\frac{(\tilde{x}-s)^{\beta-1}}{\Gamma(\beta)}\|\psi(s,u^*(s))-\psi(s,v^*(s))\|ds\\
&+\frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\Bigg[\int_0^{\mathring{\gamma}}\frac{(\mathring{\gamma}-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Bigg(\|\omega(\tau,u^*(\tau))-\omega(\tau,v^*(\tau))\|\\
&+\int_0^{\tau}\|N(\tau,\sigma,u^*(\sigma))-N(\tau,\sigma,v^*(\sigma))\|d\sigma\Bigg)d\tau+\int_0^{\mathring{\gamma}}\frac{(\mathring{\gamma}-\tau)^{\beta}}{\Gamma(\beta+1)}\|\psi(\tau,u^*(\tau))-\psi(\tau,v^*(\tau))\|d\tau\Bigg]d\tau\\
\le\;&L\Bigg[\frac{1}{\alpha+\beta+1}+\frac{2}{\alpha+\beta+2}+\frac{1}{\beta+1}\\
&+\frac{|u_1^*|}{|1-\frac{\mathring{\gamma}^2}{2}u_1^*|}\left(\frac{\mathring{\gamma}^{\alpha+\beta+1}}{\alpha+\beta+2}+\frac{\mathring{\gamma}^{\beta+1}}{\beta+2}+\frac{2\mathring{\gamma}^{\alpha+\beta+2}}{\alpha+\beta+3}\right)\Bigg]\|u^*-v^*\|\\
=\;&L\delta\|u^*-v^*\|.
\end{aligned}
$$

Since $L\delta<1$, it follows that $\mathcal{F}$ is a contraction. All assumptions of Banach fixed point theorem are satisfied, then there exists $u^*\in C(J,X)$ such that $\mathcal{F}u^*=u^*$ which is the unique solution of the problem (1.1) in $C(J,X)$. □

4. Generalized Ulam Stabilities

To discus the Ulam stability for (1.1), using the integration $v^*(\tilde{x})=Fv^*(\tilde{x})$. Let define the nonlinear continuous operator

$$\mathcal{Q}:C([0,1],X\longrightarrow C([0,1],X),$$

as follows

$$\mathcal{Q}v^*(\tilde{x})={}^C D^{\alpha+\beta}v^*(\tilde{x})-{}^C D^{\alpha}_{0+}\omega(\tilde{x},v^*(\tilde{x}))-\psi(\tilde{x},v^*(\tilde{x}))-\int_0^{\tilde{x}}N(\tilde{x},s,v^*(s))ds.$$

Definition 4.1. *For each $\epsilon>0$ and for each solution v of (1.1), such that*

$$\|\mathcal{Q}v^*\|\le\epsilon, \tag{4.1}$$

the problem (1.1), is said to be Ulam-Hyers stable if we can find a positive real number ν and a solution $u\in C([0,1],X)$ of (1.1), satisfying the inequality:

$$\|u^*-v^*\|\le\nu\epsilon^*, \tag{4.2}$$

where ϵ^* is a positive real number depending on ϵ.

Definition 4.2. *Let $m \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that for each solution v of (1.1), we can find a solution $u \in C([0,1], X)$ of (1.1) such that*

$$\|u^*(\tilde{x}) - v^*(\tilde{x})\| \leq m(\epsilon), x \in [0,1]. \tag{4.3}$$

Then the problem (1.1), is said to be generalized Ulam-Hyers stable

Definition 4.3. *For each $\epsilon > 0$ and for each solution v of (1.1), the problem (1.1) is called Ulam-Hyers-Rassias stable with respect to $\theta \in C([0,1], \mathbb{R}^+)$ if*

$$\|\mathcal{Q}v^*(\tilde{x})\| \leq \epsilon\theta(\tilde{x}), x \in [0,1], \tag{4.4}$$

and there exist a real number $\nu > 0$ and a solution $v \in C([0,1], X)$ of (1.1) such that

$$\|u^*(\tilde{x}) - v^*(\tilde{x})\| \leq \nu\epsilon_*\theta(\tilde{x}), \quad x \in [0,1], \tag{4.5}$$

where ϵ_* is a positive real number depending on ϵ.

Theorem 4.4. *Under assumption (H1) in Theorem 3.1, with $L\delta < 1$. The problem (1.1), is both Ulam-Hyers and generalized Ulam-Hyers stable.*

Proof. Let $u^* \in C([0,1], X)$ be a solution of (1.1), satisfying (2.1) in the sense of Theorem 3.3, and any solution v^* satisfying (4.1). Then, we obtain:

$$\begin{aligned}
\|v^*(\tilde{x}) - u^*(\tilde{x})\| &= \|v^*(\tilde{x}) - \mathcal{F}v^*(\tilde{x}) + \mathcal{F}v^*(\tilde{x}) - u^*(\tilde{x})\| \\
&= \|v^*(\tilde{x}) - \mathcal{F}v^*(\tilde{x}) + \mathcal{F}v^*(\tilde{x}) - \mathcal{F}u^*(\tilde{x})\| \\
&\leq \|\mathcal{F}v^*(\tilde{x}) - \mathcal{F}u^*(\tilde{x})\| + \|Fv^*(\tilde{x}) - Id(v^*(\tilde{x}))\| \\
&\leq \|\mathcal{F}v^*(\tilde{x}) - \mathcal{F}u^*(\tilde{x})\| + \|\mathcal{Q}v^*\| \\
&\leq L\delta\|u^* - v^*\| + \epsilon,
\end{aligned} \tag{4.6}$$

because $L\delta < 1$ and $\epsilon > 0$, we find

$$\|u^* - v^*\| \leq \frac{\epsilon}{1 - L\delta}.$$

Fixing $\epsilon_* = \frac{\epsilon}{1-L\delta}$ and $\nu = 1$, we obtain the Ulam-Hyers stability condition. In addition, the generalized Ulam-Hyers stability follows by taking $m(\epsilon) = \frac{\epsilon}{1-L\delta}$. □

Theorem 4.5. *Assume that (H1) holds with $L < \delta^{-1}$, and there exists a function $\theta \in C([0,1], \mathbb{R}^+)$ satisfying the condition (4.4). Then the problem (1.1), is Ulam-Hyers-Rassias stable with respect to θ.*

Proof. We have from the proof of Theorem 4.4,

$$\|u^*(\tilde{x}) - v^*(\tilde{x}\| \leq \epsilon_*\theta(\tilde{x}), \quad \tilde{x} \in [0,1],$$

where $\epsilon_* = \frac{\epsilon}{1-L\delta}$. □

Example 4.6. *Consider the following fractional integro-differential problem*

$$\begin{cases}
{}^C D_{0^+}^{\frac{12}{7}} u^*(\tilde{x}) = \omega(\tilde{x}, u^*(\tilde{x})) + {}^C D_{0^+}^{\frac{6}{7}} \psi(\tilde{x}, u^*(\tilde{x})) + \displaystyle\int_0^{\tilde{x}} N(\tilde{x}, s, u^*(s)) ds \\
u^*(0) = 1, \quad (u^*)'(0) = 6 \displaystyle\int_0^{\frac{2}{3}} u^*(s) ds, \quad 0 < \mathring{\gamma} < 1
\end{cases} \tag{4.7}$$

Where

$$\alpha = \beta = \frac{6}{7}, \quad u_0^* = 1, \quad u_1^* = 6, \quad \mathring{\gamma} = \frac{2}{3}.$$

To illustrate our results: Theorem 3.1, and Theorem 4.4, we take for $u^*, v^* \in X = \mathbb{R}^+$ *and* $\tilde{x} \in [0,1]$ *the following continuous functions:*

$$\begin{aligned}
\omega(\tilde{x}, u^*(\tilde{x})) &= \frac{u^*(\tilde{x})(7-x^2)}{245}, \\
\psi(\tilde{x}, u^*(\tilde{x})) &= \frac{2-\ln(\tilde{x}+1)}{86} u^*(\tilde{x}), \\
N(\tilde{x}, s, u^*(s)) &= \frac{2+e^{-(s^2+x^2)}}{93} u^*(s).
\end{aligned}$$

Thus, $L_1 = \frac{8}{245},\ L_2 = \frac{2+\ln(2)}{86},\ L_3 = \frac{1}{31},\ \psi(0, u_0^*) = \psi(0,1) = \frac{1}{35}$, *Moreover,*

$$\mu_1^*(\tilde{x}) = \frac{7-x^2}{245}, \quad \mu_2(\tilde{x}) = \frac{2-\ln(\tilde{x}+1)}{86}, \quad \mu_3(\tilde{x}) = \frac{2+e^{-(s^2+x^2)}}{93}.$$

Obviously, $\|\mu_1^*\|_{L_\infty} = \frac{1}{35}, \quad \|\mu_2\|_{L_\infty} = \frac{1}{43}, \quad \|\mu_3\|_{L_\infty} = \frac{1}{31}$,
and

$$L = \max\{L_1, L_2, L_3\} = \frac{1}{31}.$$

Using the above data, we get :

(1) *:* $\Delta = 0.2285, \quad \Delta_1 = 7.6401, \quad L\Delta_1 = 0.2465 < 1, \quad \delta_1 = 13.1677.$
fixing $r \geq \frac{\delta_1}{1-\Delta} = 17.0673$ *then, there exists at least one solution for the problem (4.7) on* $[0,1]$ *by application of Theorem 3.1*

(2) *:* $\delta = 9.0338, \quad L\delta_1 = 0.4248 < 1,$ *fixing* $r_1 \geq \frac{M\delta+\delta_1}{1-L\delta} = 18.5826.$
then, by Theorem 3.3 there exists a unique solution of (4.7) on $[0,1]$.

(3) *:* $L = \frac{1}{31}, \qquad L\delta_1 = 0.0.2914.$
then, the solution of (4.7) is generalized Ulam-Hyers stable. .

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