Uniform a priori estimates for solutions of higher order fractional system

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Abstract: In this paper, we study the positive solutions of higher order Lane-Emden system with Navier exterior conditions

$$\begin{cases} (-\Delta)^s u(x) = v^p(x),\ u(x) > 0, & x \in \Omega, \\ (-\Delta)^s v(x) = u^q(x),\ v(x) > 0, & x \in \Omega, \\ u(x) = -\Delta u(x) = \cdots = (-\Delta)^m u(x) = 0, & x \in \Omega^c, \\ v(x) = -\Delta v(x) = \cdots = (-\Delta)^m v(x) = 0, & x \in \Omega^c, \end{cases}$$

where $1 < p < \infty, 1 < q < \infty, n \geq 3, s = m + \sigma$ with integer m and $\sigma \in (0,1)$, and Ω^c is the complementary of Ω in $\mathbb{R}^n$. We establish the monotonicity of solutions in the inward normal direction near the boundary by using the method of moving plane in a local way. Furthermore, we establish uniform a priori estimates for positive solutions to higher critical order fractional Lane-Emden system for all large exponents by using the combination of Green's representations with suitable cut-off functions. It is well-known that, with such a prior estimate, one will be able to obtain the existence of solutions via topological degree or continuation arguments.

Key words: Critical order; Lane-Emden system; Monotonicity; Method of moving plane in a local way; Green's function and Re-scaling; Uniform a priori estimates.

Mathematics Subject Classification(2020): 35R11, 35B45, 35J30.

1 Introduction

In this paper, we consider the following higher order fractional system in bounded domains with Navier exterior conditions:

$$
\begin{cases}
(-\Delta)^s u(x) = v^p(x),\ u(x) > 0, & x \in \Omega, \\
(-\Delta)^s v(x) = u^q(x),\ v(x) > 0, & x \in \Omega, \\
u(x) = -\Delta u(x) = \cdots = (-\Delta)^m u(x) = 0,\ x \in \Omega^c, \\
v(x) = -\Delta v(x) = \cdots = (-\Delta)^m v(x) = 0,\ x \in \Omega^c.
\end{cases} \tag{1.1}
$$

where $1 < p < \infty, 1 < q < \infty, n \geq 3, s = m + \sigma$ with integer m and $\sigma \in (0,1)$, and Ω^c is the complementary of Ω in $\mathbb{R}^n$.

It follows from [21], the high order fractional Laplacian $(-\Delta)^s$ is defined by

$$
(-\Delta)^s u(x) = (-\Delta)^\sigma \circ (-\Delta)^m u(x). \tag{1.2}
$$

By the definition of the higher order fractional Laplacian in (1.2), we can see that $(-\Delta)^s$ is a nonlocal pseudo-differential operator like the fraction Laplacian. This kind of nonlocality makes the fraction Laplacians different from the regular Laplacians, and poses a strong barrier in the generalization of many fine results from the reular Laplacians to the fraction Laplacians.

We say that operator is in critical order if $s = \frac{n}{2}$, and obviously it is a fractional one when n is odd. This is the focus of the present paper. Let $[2\sigma]$ be the integer part of 2σ, and 2σ be the fractional part of 2σ.

Denote

$$
\mathcal{L}_{2\sigma} = \{u \in L^1_{loc} | \int_{\mathbb{R}^n} \frac{|1 + u(x)|}{1 + |x|^{n+2\sigma}} dx < \infty\}.
$$

For $u \in C^{[2\sigma], 2\sigma+\varepsilon}_{loc}(\Omega) \cap \mathcal{L}_{2\sigma}$ with arbitrarily small $\varepsilon > 0$, the fractional Laplacian in $\mathbb{R}^n$ is a nonlocal operator defined by

$$
\begin{aligned}
(-\Delta)^\sigma u(x) &= C_{n,\sigma} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2\sigma}} dy \\
&= C_{n,\sigma} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x - y|^{n+2\sigma}} dy,
\end{aligned}
$$

where $P.V.$ stands for the Cauchy principle value, $C_{n,\sigma}$ is a normalization constant. Then it is easy to verify that in order (1.2) to make sence, we require that

$$
u \in C^{[2\sigma], 2\sigma+\varepsilon}_{loc}(\Omega),\ (-\Delta)^m u \in \mathcal{L}_{2\sigma}.
$$

The definition of the fractional Laplacian can be further extended to the distribution u in the space $\mathcal{L}_{2\sigma}$ by

$$
\langle\ (-\Delta)^\sigma u, \phi\ \rangle = \int_{\mathbb{R}^n} u(-\Delta)^\sigma \phi dx, \quad \forall \phi \in C_0^\infty(\mathbb{R}^n).
$$

In recent years, the fractional Laplacian has attracted much attention from the mathematical community due to its nonlocality and widespread applications. It can be used to model diverse physical phenomena. For instance, it has various application in probability and finance, in which this operator is define as the generator of α-stable Lévy process that represent random motions, such as the Poisson process and Brownian (see [13, 20]). In the diffusion process, the operator was used to derive heat kernel estimates for many symmetric jump-type processes and to study the acoustic wave equation (see [3, 12]). For more related references, we refer the readers to see the references therein [2,4,5]. Therefore, many authors investigated qualitative properties (existence, regularity, symmetry and monotonicity) of the elliptic equations involving the fractional Laplacion operator (see [1,9,10,14,16,17,24–26]).

The non-locality of the fractional Laplacion makes it difficult to investigate. In order to overcome this difficulty, the extension method was firstly introduced by Caffarelli and Silvestre in [4]. The main idea of the extension method is to reduced the nonlocal problem into a local one in higher dimensions. Specifically, for a function $u : \mathbb{R}^n \to \mathbb{R}$, we consider the extension $U : \mathbb{R}^n \to [0, \infty)$ that satisfies

$$div(y^{1-\gamma}\nabla U) = 0,\ (x, y) \in \mathbb{R}^n \times [0, \infty),\ U(0, x) = u(x).$$

Then we have

$$(-\Delta)^{\frac{\gamma}{2}} u(x) = -C_{n,\gamma} \lim_{y\to 0^+} y^{1-\gamma}\frac{\partial U}{\partial y},\ x \in \mathbb{R}^n.$$

The other method is the integral equations method, such as method of moving planes in integral forms to study their equivalent corresponding integrals ([22, 23]). That is, if we choose the integral equations method to study the well-known nonlinear partical differential equation:

$$(-\Delta)^{\frac{\alpha}{2}} u = u^{\frac{n+p}{n-p}},$$

we need an equaivalent integral form:

$$u(x) = \int_{\mathbb{R}^n} \frac{u^{\frac{n+p}{n-p}}(y)}{|x-y|^{n-\alpha}} dy.$$

We show that the solutions to (1.1) are strictly monotone increasing along the inward normal direction near the boundary of Ω. To this end, we split the high order equation in (1.1) into a fractional equations and m integer order equations, then these $2m + 2$ equations together with the Navier conditions constitute a system. Applying a direct method of moving planes (see [6,9,10]) and iteration technique on this system, we are able to derive the following monotonicity result.

Theorem 1.1. Let $\Omega \subset \mathbb{R}^n$ be a strictly convex bounded domain, $s = m + \sigma$ with integer m and $\sigma \in (0, 1)$. Assume that $u, v \in C^{2m+[2\sigma],2\sigma+\varepsilon}_{loc}(\Omega)$ and $u, v, -\Delta u, -\Delta v, \cdots, (-\Delta)^m u, (-\Delta)^m v$ are lower semi-continuous on $\bar{\Omega}$. Let u, v be a pair of solution of (1.1). Then for any $x^o \in \partial\Omega$, there exists a $\delta_o > 0$ depending only on x^o and Ω such that, $u(x), v(x)$ and $(-\Delta)^i u, (-\Delta)^i v,\ i = 1, \cdots, m$ are monotone increasing along the inward normal direction ν^o in the region

$$\tilde{\Sigma}_{\delta_o} = \{x \in \bar{\Omega} \mid 0 \le (x - x^o) \cdot \nu^o \le \delta_o\}.$$

Moreover, either $u(x), v(x)$ and $(-\Delta)^i u, (-\Delta)^i v,\ i = 1, \cdots, m$ are strictly monotone increasing along the inward normal direction in $\tilde{\Sigma}_{\delta_o}$, or $u(x), v(x)$ are constant in $\tilde{\Sigma}_{\delta_o}$ and $(-\Delta)^i u, (-\Delta)^i v,\ i = 1, \cdots, m$ are constants in $\mathbb{R}^n$.

Remark 1.2. To better illustrate the main ideas, we only consider the simple example as in system (1.1). The methods developed here are also applicable to deal with more general nonlinearities, for example, one can replace s in (1.1) by s and t, i.e.

$$\begin{cases} (-\Delta)^s u(x) = v^p(x), \ u(x) > 0, & x \in \Omega, \\ (-\Delta)^t v(x) = u^q(x), \ v(x) > 0, & x \in \Omega, \\ u(x) = -\Delta u(x) = \cdots = (-\Delta)^{m_1} u(x) = 0, & x \in \Omega^c, \\ v(x) = -\Delta v(x) = \cdots = (-\Delta)^{m_2} v(x) = 0, & x \in \Omega^c, \end{cases}$$

where $1 < p < \infty, 1 < q < \infty, n \geq 3, s = m_1 + \sigma, t = m_2 + \sigma$ with integer m_1, m_2 and $\sigma \in (0,1)$, and Ω^c is the complementary of Ω in $\mathbb{R}^n$. We can also obtain the monotonicity of positive solutions.

The above monotonicity results implies that the maxima of $u(x), v(x)$ and $(-\Delta)^i u$, $(-\Delta)^i v$, $i = 1, \cdots, m$ are attained in the interior of Ω away from the boundary, namely in the set

$$\{x \in \Omega \mid dist(x, \partial\Omega) \geq \delta\}.$$

It follows that the Lebesgue integral of $u(x), v(x)$ and $(-\Delta)^i u, (-\Delta)^i v$, $i = 1, \cdots, m$ on the boundary layer of Ω can be controlled by the corresponding interior integral in Ω, and therefore, the Lebesgue integral in Ω is dominated by the interior values in Ω. Using the moving planes to obtain the estimates to solutions near the boundary for elliptic equations where early appeared in [18].

In order to derive a uniform a priori estimate, we need a uniform Lebesgue estimate, which is often obtained by using the eigenvalues and eigenfunctions in the case of integer order elliptic equations (see [15, 19]). However, for higher order fractional operators, so far as we know, there have not seen any results on the corresponding eigenvalues and eigenfunctions. To go around this obstacle, we use a idea (see [11]) of constructing suitable cut-off functions to obtain the basic uniform Lebesgue estimate.

Combining with the integration by parts formula for higher order fractional operators, we obtain the following uniform Lebesgue estimate for all positive solutions of (1.1) and for all exponents $p \geq p_0 > 1, q \geq q_0 > 1$, there exists C not depend on p, q such that

$$\int_\Omega u^q(x)dx \leq C, \tag{1.3}$$

$$\int_\Omega v^p(x)dx \leq C, \tag{1.4}$$

Then we consider system (1.1) in critical fractional order when $s = \frac{n}{2}$ with odd integer $n \geq 3$, i.e.

$$\begin{cases} (-\Delta)^{\frac{n}{2}} u(x) = v^p(x), & x \in \Omega, \\ (-\Delta)^{\frac{n}{2}} v(x) = u^q(x), & x \in \Omega, \\ u(x) = -\Delta u(x) = \cdots = (-\Delta)^{\frac{n-1}{2}} u(x) = 0, & x \in \Omega^c, \\ v(x) = -\Delta v(x) = \cdots = (-\Delta)^{\frac{n-1}{2}} v(x) = 0, & x \in \Omega^c. \end{cases} \tag{1.5}$$

We know that blowing up and rescaling techniques are used to derive a priori estimates (see [7, 8]) which are usually not uniform and depend on the exponent p, q. In order to obtain a uniform estimate, we use a approach: the combination of a rescaling and the Green's representation of solutions.

Let E, F denote the maximum of u, v in Ω respectively, and from Theorem 1.1 we know that it is attained at

$$\Omega_\delta = \{x \in \Omega \mid dist(x, \partial\Omega) \geq \delta\},$$

which is at least at a distance δ away from the boundary $\partial\Omega$. Without loss of generality, we assume that $0 \in \Omega_\delta$ and $u(0) = E, v(0) = F$.

Let $M = \max\{E, F\}$, make s rescaling

$$\mu(x) = \frac{1}{M^{p+1}} u(M^{-\frac{pq-1}{n}} x), \quad \kappa(x) = \frac{1}{M^{q+1}} v(M^{-\frac{pq-1}{n}} x).$$

Then $\mu(x)$ and $\kappa(x)$ satisfies the equation and the exterior Navier conditions in the rescaled domain $\Omega' = M^{\frac{pq-1}{n}}\Omega$, and it is easy to see that

$$\begin{cases} (-\Delta)^{\frac{n}{2}} \mu(x) = \kappa^p(x), & x \in \Omega', \\ (-\Delta)^{\frac{n}{2}} \kappa(x) = \mu^q(x), & x \in \Omega', \\ \mu(x) = -\Delta\mu(x) = \cdots = (-\Delta)^{\frac{n-1}{2}} \mu(x) = 0, & x \in (\Omega')^c, \\ \kappa(x) = -\Delta\kappa(x) = \cdots = (-\Delta)^{\frac{n-1}{2}} \kappa(x) = 0, & x \in (\Omega')^c. \end{cases}$$

We establish the following uniform a priori estimate.

Theorem 1.3. Assume that $n \geq 3$ is odd, $\Omega \subset \mathbb{R}^n$ is strictly convex and $p_0 q_0 > 1$. Then there exists a constant C depending only on p_0, q_0, n and Ω such that for all $p_0 \leq p < +\infty$, $q_0 \leq q < +\infty$, and for all solutions $u, v \in C^{n,\varepsilon}_{loc}(\Omega) \cap C^{n-2}_0(\Omega)$ to the system (1.5), we have the uniform estimate

$$\|u\|_{L^\infty(\bar{\Omega})}, \|v\|_{L^\infty(\bar{\Omega})} \leq C.$$

The paper is organized as follows: In Section 2, we given some necessary lemmas. In Section 3, we complete the proof of Theorem 1.1 by the moving plane. In Section 4, we establish the uniform a priori estimate and prove Theorem 1.3.

2 Preliminaries

In this section, we will given some necessary lemmas to proof of Theorem 1.1 and Theorem 1.2.

Recall that for the usual Laplacian operator, we have the following well-known *maximum principle*.

Lemma 2.1 ([10]) Assume that $u \in C^2(\bar{\Omega})$ is a solution of

$$\begin{cases} -\Delta u(x) \geq 0, & x \in \Omega, \\ u(x) \geq 0, & x \in \Omega^c, \end{cases}$$

then

$$u(x) \geq 0, \ x \in \Omega.$$

Due to the non-locality of the fractional Laplacian, a correct version of the *maximum principle* is

Lemma 2.2 ([9]) Assume that Ω is a bounded domain in $\mathbb{R}^n$, $u \in \mathcal{L}_{2\sigma}$ is lower semi-continuous on $\bar{\Omega}$ and satisfies

$$\begin{cases} (-\Delta)^\sigma u(x) \geq 0, \ x \in \Omega, \\ \quad u(x) \geq 0, \quad x \in \Omega^c, \end{cases}$$

in the sense of distribution, then

$$u(x) \geq 0, \ x \in \Omega.$$

In [25], a simple *maximum principle* for *anti-symmetric functions* was proved.

Lemma 2.3. Let Ω be a bounded domain in Σ_λ. Suppose that $\omega_\lambda(x) = -\omega_\lambda(x^\lambda)$ is a *anti-symmetric functions*. Assume that $\omega_\lambda(x) \in \mathcal{L}_{2\sigma} \cap C^{1,1}_{loc}(\Omega)$ and is lower *anti-symmetric functions* on $\bar{\Omega}$. If

$$\begin{cases} (-\Delta)^\sigma \omega_\lambda(x) \geq 0, \ x \in \Omega, \\ \quad \omega_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda \backslash \Omega, \end{cases}$$

then

$$\omega_\lambda(x) \geq 0, \ x \in \Omega.$$

Furthermore, if $\omega_\lambda = 0$ at some point in Ω, then

$$\omega_\lambda(x) = 0, \ \textit{almost everywhere in } \mathbb{R}^n.$$

These conclusions hold for unbounded region Ω if we further assume that

$$\liminf_{|x| \to \infty} \omega_\lambda(x) \geq 0.$$

We use a idea of constructing suitable cut-off functions to obtain the basic uniform Lebesgue estimate. In this process, we need the following key ingredient-the integration by parts formula for higher order fractional operators.

Lemma 2.4 ([11]) Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain, let $s = m + \delta$ with integer m and $\delta \in (0,1)$. Suppose that $u \in C^{2m+[2\sigma],2\sigma+\varepsilon}_{loc}(\Omega)$ and $(-\Delta)^i u(x) \in \mathcal{L}_{2\delta}$, then

$$\int_\Omega (-\Delta)^s u(x)\varphi(x)dx = \int_\Omega (-\Delta)^s \varphi(x)u(x)dx, \ \forall \ \varphi(x) \in C_0^\infty(\Omega).$$

3 Monotonicity

We prove Theorem 1.1 by a direct method of moving plane. To better illustrate the idea, we first prove the case for $m = 1$ and then state the proof for $m > 1$.

3.1 The case m=1

Proof. Let $w = \Delta u(x)$, $l = \Delta v(x)$, then

$$\begin{cases} (-\Delta)^\sigma w(x) = v^p(x), & x \in \Omega, \\ (-\Delta)^\sigma l(x) = u^q(x), & x \in \Omega, \\ -\Delta u(x) = w(x), \ -\Delta v(x) = l(x), & x \in \Omega, \\ u(x) = v(x) = w(x) = l(x) = 0, & x \in \Omega^c. \end{cases} \tag{3.1}$$

Since $w(x)$ and $l(x)$ satisfies

$$\begin{cases} (-\Delta)^\sigma w(x) > 0, x \in \Omega, \\ w(x) = 0, \quad x \in \Omega^c. \end{cases} \tag{3.2}$$

By strong maximum principle implies that

$$w(x) > 0,\ x \in \Omega.$$

Similarly, we have

$$l(x) > 0,\ x \in \Omega.$$

For any $x^o \in \partial\Omega$, let ν^o be the unit inward normal vector of $\partial\Omega$ at x^o. We will show that there exits a constant $\delta_o > 0$ depending only on x^o and Ω such that $u(x)$ is monotone increasing along the inward normal direction in the region

$$\tilde{\Sigma}_{\delta_o} = \{x \in \bar{\Omega} \mid 0 \le (x - x^o) \cdot \nu^o \le \delta_o\}. \tag{3.3}$$

To this end, we defined the moving plane by

$$T_\lambda = \{x \in \mathbb{R}^3 \mid (x - x^o) \cdot \nu^o = \lambda\},$$

and the region to the left of the plane

$$\Sigma_\lambda = \{x \in \mathbb{R}^3 \mid (x - x^o) \cdot \nu^o < \lambda\},$$

for $\lambda > 0$, and let x^λ be the reflection of the point x about the plane T_λ. Let

$$u_\lambda(x) = u(x^\lambda),\ v_\lambda(x^\lambda) = v(x),$$

$$w_\lambda(x) = w(x^\lambda),\ l_\lambda(x) = l(x^\lambda),$$

and

$$U^\lambda(x) = u_\lambda(x) - u(x),\ V^\lambda(x) = v_\lambda(x) - v(x),$$
$$W^\lambda(x) = w_\lambda(x) - w(x),\ L^\lambda(x) = l_\lambda(x) - l(x).$$

Then we deduce from (3.1) that, for any λ such that the reflection of $\Sigma_\lambda \cap \Omega$ is contained in Ω,

$$\begin{cases} (-\Delta)^\sigma W^\lambda(x) = v_\lambda^p(x) - v^p(x) = p\xi_\lambda^{p-1}(x) V^\lambda(x),\ x \in \Sigma_\lambda \cap \Omega, \\ (-\Delta)^\sigma L^\lambda(x) = u_\lambda^q(x) - u^q(x) = q\eta_\lambda^{q-1}(x) U^\lambda(x),\ x \in \Sigma_\lambda \cap \Omega, \\ -\Delta U^\lambda(x) = W^\lambda(x),\ -\Delta V^\lambda(x) = L^\lambda(x), \quad x \in \Sigma_\lambda \cap \Omega, \\ U^\lambda(x), V^\lambda(x), W^\lambda(x), L^\lambda(x) \ge 0, \quad x \in \Sigma_\lambda \backslash (\Sigma_\lambda \cap \Omega). \end{cases} \tag{3.4}$$

where $\xi_\lambda(x)$ is valued between $v_\lambda(x)$ and $v(x)$, $\eta_\lambda(x)$ is valued between $v_\lambda(x)$ and $v(x)$.

Step 1. In this step, we prove that there exists some $\delta > 0$ sufficiently small such that

$$U^\lambda(x) \ge 0, V^\lambda(x) \ge 0, W^\lambda(x) \ge 0, L^\lambda(x) \ge 0,\ x \in \Sigma_\lambda \cap \Omega, \tag{3.5}$$

for $0 < \lambda \le \delta$. This actually is a narrow region principle for the system.

$$U^\lambda(x) > 0, V^\lambda(x) > 0, W^\lambda(x) > 0, L^\lambda(x) > 0,\ x \in \Sigma_\lambda \cap \Omega, \tag{3.6}$$

or

$$U^\lambda(x) \equiv 0, V^\lambda(x) \equiv 0, x \in \Sigma_\lambda \cap \Omega,\ W^\lambda(x) \equiv 0, L^\lambda(x) \equiv 0, x \in \mathbb{R}^3. \tag{3.7}$$

We first prove (3.5).

Suppose (3.5) is false, then we may assume that there exists $0 < \lambda \leq \delta$ such that

$$U^\lambda(x) < 0,\ somewhere\ in\ \Sigma_\lambda \cap \Omega. \tag{3.8}$$

Otherwise, if $U^\lambda(x) \geq 0$ in $\Sigma_\lambda \cap \Omega$ for $0 < \lambda \leq \delta$, then we can derive that

$$V^\lambda(x) \geq 0, W^\lambda(x) \geq 0, L^\lambda(x) \geq 0.$$

Indeed, if

$$U^\lambda(x) \geq 0,\ x \in \Sigma_\lambda \cap \Omega,\ for\ 0 < \lambda \leq \delta.$$

By (3.4), we have

$$\begin{cases} (-\Delta)^\sigma L^\lambda(x) \geq 0, & x \in \Sigma_\lambda \cap \Omega, \\ L^\lambda(x) = 0, & x \in \Sigma_\lambda \backslash (\Sigma_\lambda \cap \Omega). \end{cases}$$

By the maximum principle for the anti-symmetric function, we derive

$$L^\lambda(x) \geq 0,\ x \in \Sigma_\lambda \cap \Omega,\ for\ 0 < \lambda \leq \delta. \tag{3.9}$$

Furthermore, we have

$$\begin{cases} -\Delta V^\lambda(x) \geq 0, & x \in \Sigma_\lambda \cap \Omega, \\ V^\lambda(x) = 0, & x \in \Sigma_\lambda \backslash (\Sigma_\lambda \cap \Omega), \end{cases}$$

a maximum principle implies that

$$V^\lambda(x) \geq 0,\ x \in \Sigma_\lambda \cap \Omega,\ for\ 0 < \lambda \leq \delta.$$

Similar to (3.9), we have

$$W^\lambda(x) \geq 0,\ x \in \Sigma_\lambda \cap \Omega,\ for\ 0 < \lambda \leq \delta.$$

Let

$$\phi(x) = cos \frac{(x - x^o) \cdot \nu^o}{\delta}.$$

It follows that $\phi(x) \in [cos1, 1]$, $x \in \Sigma_\lambda$ and

$$-\frac{\Delta \phi(x)}{\phi(x)} = \frac{1}{\delta^2}.$$

Define

$$\overline{U^\lambda}(x) = \frac{U^\lambda(x)}{\phi(x)},\ \overline{V^\lambda}(x) = \frac{V^\lambda(x)}{\phi(x)}.$$

We obtain from (3.7) that there exists a point $x_0 \in \Sigma_\lambda \cap \Omega$ such that

$$\overline{U^\lambda}(x_0) = \min_{\Sigma_\lambda} \overline{U^\lambda}(x) < 0.$$

At the negative minimum point of $\overline{U^\lambda}$, we have

$$
\begin{aligned}
-\Delta U^\lambda(x_0) &= -\Delta\overline{U^\lambda}(x_0)\phi(x_0) - 2\nabla\overline{U^\lambda}(x_0)\cdot\nabla\phi(x_0) - \overline{U^\lambda}(x_0)\Delta\phi(x_0) \\
&= -\Delta\overline{U^\lambda}(x_0)\phi(x_0) - \overline{U^\lambda}(x_0)\Delta\phi(x_0) \\
&\leq -\overline{U^\lambda}(x_0)\Delta\phi(x_0) \\
&= \frac{U^\lambda(x_0)}{\delta^2}.
\end{aligned}
\tag{3.10}
$$

On the other hand, by (3.4) we have

$$
W^\lambda(x_0) = -\Delta U^\lambda(x_0) \leq \frac{U^\lambda(x_0)}{\delta^2}. \tag{3.11}
$$

It thus implies that there exists some $x_1 \in \Sigma_\lambda \cap \Omega$ such that

$$
W^\lambda(x_1) = \min_{\Sigma_\lambda} W^\lambda(x) < 0.
$$

By the definition of the fractional Laplacian and (3.11), we have

$$
\begin{aligned}
(-\Delta)^\sigma W^\lambda(x_1) &= C_{n,\sigma} P.V. \int_{\mathbb{R}^3} \frac{W^\lambda(x_1) - W^\lambda(y)}{|x_1 - y|^{3+2\sigma}} dy \\
&= C_{n,\sigma} P.V. \Big(\int_{\Sigma_\lambda} \frac{W^\lambda(x_1) - W^\lambda(y)}{|x_1 - y|^{3+2\sigma}} dy + \int_{\Sigma_\lambda^c} \frac{W^\lambda(x_1) - W^\lambda(y)}{|x_1 - y|^{3+2\sigma}} dy\Big) \\
&\leq C_{n,\sigma} P.V. \Big(\int_{\Sigma_\lambda} \frac{W^\lambda(x_1) - W^\lambda(y)}{|x_1 - y|^{3+2\sigma}} dy + \int_{\Sigma_\lambda} \frac{W^\lambda(x_1) + W^\lambda(y)}{|x_1 - y^\lambda|^{3+2\sigma}} dy\Big) \\
&\leq C W^\lambda(x_1) \int_{\Sigma_\lambda} \frac{1}{|x_1 - y|^{3+2\sigma}} dy \\
&\leq \frac{C W^\lambda(x_1)}{\delta^{2\delta}} \\
&\leq \frac{C U^\lambda(x_0)}{\delta^{2+2\delta}}.
\end{aligned}
\tag{3.12}
$$

Combining (3.4) and (3.12), we have

$$
V^\lambda(x_1) < 0.
$$

Similar to (3.10), there exists a point $x_2 \in \Sigma_\lambda \cap \Omega$ such that

$$
-\Delta V^\lambda(x_2) \leq \frac{V^\lambda(x_2)}{\delta^2}. \tag{3.13}
$$

Furthermore, there exists some $x_3 \in \Sigma_\lambda \cap \Omega$ such that

$$
L^\lambda(x_3) = \min_{\Sigma_\lambda} L^\lambda(x) < 0,
$$

and

$$
(-\Delta)^\sigma L^\lambda(x_3) \leq \frac{C V^\lambda(x_2)}{\delta^{2+2\delta}}.
$$

Combining (3.4) with (3.12), we obtain

$$
\begin{aligned}
\frac{CU^\lambda(x_0)}{\delta^{2+2\delta}} &\geq (-\Delta)^\sigma W^\lambda(x_1) \\
&= p\xi_\lambda^{p-1}(x_1)\overline{V^\lambda}(x_1)\phi(x_1) \\
&\geq p\xi_\lambda^{p-1}(x_1)\overline{V^\lambda}(x_2)\phi(x_1) \\
&= p\xi_\lambda^{p-1}(x_1)V^\lambda(x_2)\frac{\phi(x_1)}{\phi(x_2)} \\
&\geq C\|v\|^{p-1}_{L^\infty(\bar{\Omega})}V^\lambda(x_2),
\end{aligned}
\tag{3.14}
$$

i.e.

$$
U^\lambda(x_0) \geq C\delta^{2+2\delta}\|v\|^{p-1}_{L^\infty(\bar{\Omega})}V^\lambda(x_2).
$$

Similarly, we have

$$
\frac{CV^\lambda(x_2)}{\delta^{2+2\delta}} \geq C\|u\|^{q-1}_{L^\infty(\bar{\Omega})}U^\lambda(x_0).
$$

Therefore, we obtain

$$
\frac{1}{\delta^{4+4\delta}} \leq C\|u\|^{q-1}_{L^\infty(\bar{\Omega})}\|v\|^{p-1}_{L^\infty(\bar{\Omega})},
$$

which is a contradiction if we choose $\delta > 0$ small enough such that

$$
0 < \delta < C(\|u\|^{q-1}_{L^\infty(\bar{\Omega})}\|v\|^{p-1}_{L^\infty(\bar{\Omega})})^{-\frac{1}{4+4\delta}}.
\tag{3.15}
$$

Therefore, there exists $\delta > 0$ such that (3.5) is holds.

Second, we show (3.6) and (3.7).

Suppose for some $\xi \in \Sigma_\lambda \cap \Omega$ such that $U^\lambda(\xi) = 0$. Then ξ is the minimum point of U^λ. Thus

$$
0 \geq -\Delta U^\lambda(\xi) = W^\lambda(\xi).
$$

Meanwhile, by (3.5), we have

$$
W^\lambda(\xi) \geq 0.
$$

Hence,

$$
W^\lambda(\xi) = 0 = \min_{\Sigma_\lambda} W^\lambda(x),
$$

and

$$
\begin{aligned}
(-\Delta)^\sigma W^\lambda(\xi) &= C_{n,\sigma}P.V.\int_{\mathbb{R}^3}\frac{-W^\lambda(y)}{|x_1-y|^{3+2\sigma}}dy \\
&= C_{n,\sigma}P.V.(\int_{\Sigma_\lambda}\frac{-W^\lambda(y)}{|\xi-y|^{3+2\sigma}}dy + \int_{\Sigma_\lambda^c}\frac{-W^\lambda(y)}{|\xi-y|^{3+2\sigma}}dy) \\
&= C_{n,\sigma}P.V.\int_{\Sigma_\lambda}(\frac{1}{|\xi-y^\lambda|^{3+2\sigma}} - \frac{1}{|\xi-y|^{3+2\sigma}})W^\lambda(y)dy.
\end{aligned}
\tag{3.16}
$$

If $W^\lambda(x) \not\equiv 0$, then (3.16) implies that

$$
(-\Delta)^\sigma W^\lambda(\xi) < 0.
$$

Together with (3.4), it shows that

$$
V^\lambda(\xi) < 0.
$$

This is contradiction with (3.5). Hence

$$W^\lambda(x) \equiv 0,\ x \in\ \Sigma_\lambda.$$

Since

$$W^\lambda(x^\lambda) = -W^\lambda(x),\ x \in\ \Sigma_\lambda.$$

Therefore,

$$W^\lambda(x) \equiv 0,\ x \in \mathbb{R}^3.$$

Again with (3.4), one can easily deduce that

$$L^\lambda(x) \equiv 0,\ x \in \mathbb{R}^3.$$

$$U^\lambda(x) \equiv 0,\ V^\lambda(x) \equiv 0,\ x \in \Sigma_\lambda \cap \Omega.$$

If $W^\lambda(x) \not\equiv 0$ in $\mathbb{R}^3$ and there exists a point $\zeta \in \Sigma_\lambda \cap \Omega$ such that $W^\lambda(\zeta) = 0$, then

$$W^\lambda(\zeta) = \min_{\Sigma_\lambda} W^\lambda(x) = 0,$$

and

$$(-\Delta)^\sigma W^\lambda(\zeta) < 0,$$

which contradicts with (3.4). Therefore,

$$W^\lambda(x) > 0,\ x \in \Sigma_\lambda.$$

By (3.4) and strong maximum principle implies that

$$U^\lambda(x) > 0, V^\lambda(x) > 0, L^\lambda(x) > 0,\ x \in \Sigma_\lambda.$$

Therefore, we have shown (3.6) and (3.7).

Step 2. In this step, we keep moving the plane continuously along the inward normal direction at x^o to the limiting position as long as the inequality

$$U^\lambda(x), V^\lambda(x), W^\lambda(x), L^\lambda(x) \geq 0,\ x \in \Sigma_\lambda, \tag{3.17}$$

holds. We show that this process can be continued as long as the reflection of $\overline{\Sigma_\lambda \cap \Omega}$ is still contained in Ω.

Suppose in the contrary, there exists a small $\rho > 0$ such that the reflection of $\overline{\Sigma_{\lambda_0+\rho} \cap \Omega}$ about $T_{\lambda_0+\rho}$ is still contained in Ω with

$$\lambda_0 = \sup\{\lambda \mid U^\rho(x), V^\rho(x), W^\rho(x), L^\rho(x) \geq 0,\ x \in \Sigma_\rho,\ \rho \leq \lambda\}.$$

By (3.6) and (3.7), we derive

$$U^{\lambda_0}(x) > 0, V^{\lambda_0}(x) > 0, W^{\lambda_0}(x) > 0, L^{\lambda_0}(x) > 0,\ x \in \Sigma_{\lambda_0} \cap \Omega, \tag{3.18}$$

or

$$U^{\lambda_0}(x) \equiv 0, V^{\lambda_0}(x) \equiv 0, x \in \Sigma_{\lambda_0} \cap \Omega,\ W^{\lambda_0}(x) \equiv 0, L^{\lambda_0}(x) \equiv 0, x \in \mathbb{R}^3. \tag{3.19}$$

From (3.19), we know that

$$0 = u(x^o) = u_{\lambda_0}(x^o) > 0.$$

This is a contradiction and we obtain (3.18).

It follows that there exists a constant $C_\tau > 0$ such that

$$U^{\lambda_0}(x), V^{\lambda_0}(x), W^{\lambda_0}(x), L^{\lambda_0}(x) \geq C_\tau > 0,\ x \in \overline{\Sigma_{\lambda_0-\tau} \cap \Omega}.$$

From the continuity of $U^\lambda(x), V^\lambda(x), W^\lambda(x)$, and $L^\lambda(x)$ with respect to λ, we obtain that there exists a $0 < \epsilon < \min\{\rho, \tau\}$ such that

$$U^\lambda(x), V^\lambda(x), W^\lambda(x), L^\lambda(x) \geq 0,\ x \in \overline{\Sigma_{\lambda_0-\tau} \cap \Omega}, \lambda \in (\lambda_0, \lambda_0+\epsilon].$$

Denote

$$D = (\Sigma_\lambda \backslash \overline{\Sigma_{\lambda_0} - \delta}) \cap \Omega.$$

Then it is a narrow region, similar to step 1, we have

$$U^\lambda(x), V^\lambda(x), W^\lambda(x), L^\lambda(x) \geq 0,\ \forall x \in D.$$

Therefore

$$U^\lambda(x), V^\lambda(x), W^\lambda(x), L^\lambda(x) \geq 0,\ x \in \Sigma_\lambda \cap \Omega, \lambda \in (\lambda_0, \lambda_0+\epsilon].$$

This implies that the plane T_λ can still be moved inward a litter bit from T_{λ_0}

In conclusion, there exists a constant $\delta_o > 0$ depending only on x^o and Ω such that $u(x), v(x), -\Delta u(x)$ and $-\Delta v(x)$ are monotone increasing along the inward normal direction in the region

$$\tilde{\Sigma}_{\delta_0} = \{x \in \bar{\Omega} \mid 0 \leq (x - x^o) \cdot \nu^o \leq \delta_o\}.$$

Moreover, either $u(x), v(x), -\Delta u(x)$ and $-\Delta v(x)$ are strictly monotone increasing in $\tilde{\Sigma}_{\delta_0}$ or $u(x)$ and $v(x)$ are constants in $\tilde{\Sigma}_{\delta_0}$, $-\Delta u(x)$ and $-\Delta v(x)$ are constants in $\mathbb{R}^3$. □

3.2 The case $m > 1$

Proof. Let $w_i = (-\Delta)^i u$, $l_i = (-\Delta)^i v$ and

$$W_i^\lambda(x) = w_i^\lambda(x) - w_i(x),\ L_i^\lambda(x) = l_i^\lambda(x) - l_i(x)$$

for $i = 1, \cdots, m$. Then we can deduce from (1.1) that for any λ with the reflection of $\Sigma_\lambda \cap \Omega$ containing in Ω

$$\begin{cases} (-\Delta)^\sigma W_m^\lambda(x) = v_\lambda^p(x) - v^p(x) = p\xi_\lambda^{p-1}(x)V^\lambda(x),\ x \in \Sigma_\lambda \cap \Omega, \\ (-\Delta)^\sigma L_m^\lambda(x) = u_\lambda^q(x) - u^q(x) = q\eta_\lambda^{q-1}(x)U^\lambda(x),\ x \in \Sigma_\lambda \cap \Omega, \\ -\Delta U^\lambda(x) = W_1^\lambda(x),\ -\Delta V^\lambda(x) = L_1^\lambda(x),\ x \in \Sigma_\lambda \cap \Omega, \\ -\Delta W_1^\lambda(x) = W_2^\lambda(x),\ -\Delta L_1^\lambda(x) = L_2^\lambda(x),\ x \in \Sigma_\lambda \cap \Omega, \\ \cdots \\ -\Delta W_{m-1}^\lambda(x) = W_m^\lambda(x),\ -\Delta L_{m-1}^\lambda(x) = L_m^\lambda(x),\ x \in \Sigma_\lambda \cap \Omega, \\ U^\lambda(x), V^\lambda(x), W_i^\lambda(x), L_i^\lambda(x) \geq 0,\ i = 1, \cdots, m,\ x \in \Sigma_\lambda \backslash (\Sigma_\lambda \cap \Omega), \end{cases} \tag{3.20}$$

where $\xi_\lambda(x)$ is valued between $v_\lambda(x)$ and $v(x)$, $\eta_\lambda(x)$ is valued between $v_\lambda(x)$ and $v(x)$.

Define

$$\overline{U^\lambda}(x)=\frac{U^\lambda(x)}{\phi(x)},\ \overline{V^\lambda}(x)=\frac{V^\lambda(x)}{\phi(x)},$$

$$\overline{W_i^\lambda}(x)=\frac{W_i^\lambda(x)}{\phi(x)},\ \overline{L_i^\lambda}(x)=\frac{L_i^\lambda(x)}{\phi(x)},$$

for $i=1,\cdots,m$ and $x\in\Sigma_\lambda\cap\Omega$.

Similarly, we can obtain from (3.20) that there exists a point $x_0\in\Omega$ such that

$$\overline{U^\lambda}(x_0)=\min_{\Sigma_\lambda}\overline{U^\lambda}(x)<0.$$

Through a similar argument as in obtaining (3.11), we derive that there exists some $x_1\in\Sigma_\lambda\cap\Omega$ such that

$$\overline{W^\lambda}(x_1)=\min_{\Sigma_\lambda}\overline{W^\lambda}(x)<0.$$

Then

$$W_2^\lambda(x_1)=-\Delta W_1^\lambda(x_1)\le\frac{W_1^\lambda(x_1)}{\delta^2}<0.$$

Continuing this way, we derive that there exist

$$x_i\in\Sigma_\lambda\cap\Omega,$$

such that

$$\overline{W_i^\lambda}(x_i)=\min_{\Sigma_\lambda}\overline{W_i^\lambda}(x)<0.$$

for $i=2,\cdots,m-1$.

Finally, we have

$$\overline{W_m^\lambda}(x_m)=\min_{\Sigma_\lambda}\overline{W_m^\lambda}(x)<0.$$

and

$$(-\Delta)^\sigma W_m^\lambda(x_m)\le\frac{CU^\lambda(x_0)}{\delta^{2s}}. \tag{3.21}$$

By (3.20), we have

$$V^\lambda(x_m)<0.$$

Then there exists a point $\bar{x}_0\in\Sigma_\lambda\cap\Omega$ such that

$$\overline{V^\lambda}(\bar{x}_0)=\min_{\Sigma_\lambda}\overline{V^\lambda}(x)<0.$$

Similarly, for $i=1,\cdots,m$, there exist

$$\bar{x}_i\in\Sigma_\lambda\cap\Omega,$$

such that

$$\overline{L_i^\lambda}(\bar{x}_i)=\min_{\Sigma_\lambda}\overline{L_i^\lambda}(x)<0.$$

Finally, we have

$$(-\Delta)^\sigma L_m^\lambda(\bar{x}_m)\le\frac{CV^\lambda(\bar{x}_0)}{\delta^{2s}}. \tag{3.22}$$

Arguing similarly as in deriving (3.15), we prove that for

$$0<\delta<C(\|u\|^{q-1}_{L^\infty(\bar\Omega)}\|v\|^{p-1}_{L^\infty(\bar\Omega)})^{-\frac{1}{4s}}, \tag{3.23}$$

it holds

$$U^\lambda(x), V^\lambda(x), W_i^\lambda(x), L_i^\lambda(x) \geq 0,\ x \in \Sigma_\lambda,\ for\ \ 0 < \lambda \leq \delta,$$

for $i = 1, \cdots, m$.

Similar to the case of $m = 1$, we obtain that there exists a constant $\delta_o > 0$ depending only on x^o and Ω such that $u(x), v(x), (-\Delta)^i u(x)$ and $(-\Delta)^i v(x)$ are monotone increasing along the inward normal direction in the region

$$\tilde{\Sigma}_{\delta_o} = \{x \in \bar{\Omega} \mid 0 \leq (x - x^o) \cdot \nu^o \leq \delta_o\}.$$

Moreover, either $u(x), v(x), -\Delta u(x)$ and $-\Delta v(x)$ are strictly monotone increasing in $\tilde{\Sigma}_{\delta_0}$ or $u(x)$ and $v(x)$ are constants in $\tilde{\Sigma}_{\delta_0}$, $(-\Delta)^i u(x)$ and $(-\Delta)^i v(x)$ are constants in $\mathbb{R}^n$.This completes the proof of Theorem 1.1. $\square$

4 Uniform a priori estimates

In this section, we consider the higher critical order fractional equations when $s = \frac{n}{2}$ with odd integer n. We obtain the uniform a priori estimates and thus prove Theorem 1.3 by *Green's* representations and a *rescaling* technique.

4.1 The Green's function

To begin with, we first introduce some notations and derive relevant properties of the Green's function for $(-\Delta)^{\frac{n}{2}}$.

Denote

$$\Omega_\delta = \{x \in \Omega \mid dist(x, \partial\Omega) \geq \delta\}.$$

For arbitrary fixed $x \in \Omega_\delta$, let $G(x, y)$ be the Green's function for $(-\Delta)^{\frac{n}{2}}$ with pole at x associated with Navier exterior conditions. Then

$$G(x, y) = C_n \ln \frac{1}{|x - y|} - h(x, y),\ \forall\ y \in \bar{\Omega},$$

where the $\frac{n}{2}$-harmonic function $h(x, y)$ satisfies

$$\begin{cases} (-\Delta)^{\frac{n}{2}} h(x, y) = 0, & y \in \Omega, \\ (-\Delta)^i h(x, y) = (-\Delta)^i (C_n \ln \frac{1}{|x-y|}),\ i = 0, 1, \cdots, \frac{n-1}{2}, & y \in \Omega^c. \end{cases}$$

The expression of the remainder $h(x, y)$ is unknown, Chen and Li use the equations and the exterior conditions it satisfied to derive the needed properties (see [11]).

Lemma 4.1. The harmonic part $h(x, y)$ in the Green's function is bounded from above for any $x \in \Omega_\delta$ and $y \in \Omega$.

4.2 The uniform estimates

Let E and F denote of the maximum of u and v respectively, i.e.

$$E = \max_{\bar{\Omega}} u, \quad F = \max_{\bar{\Omega}} v.$$

From Theorem 1.1, we know that the maximum E and F are attained at

$$\Omega_\delta = \{x \in \Omega \mid dist(x, \partial\Omega) \geq \delta\},$$

which is at least at a distance δ away from the boundary $\partial\Omega$. Without loss of generality, we may assume that $0 \in \Omega_\delta$ and

$$u(0) = E, \quad v(0) = F.$$

Let $M = \max\{E, F\}$, we first rescale $u(x)$ and $v(x)$ by letting

$$\mu(x) = \frac{1}{M^{p+1}} u(M^{-\frac{pq-1}{n}} x), \quad \kappa(x) = \frac{1}{M^{q+1}} v(M^{-\frac{pq-1}{n}} x). \tag{4.1}$$

It is easy to see that

$$\begin{cases} (-\Delta)^{\frac{n}{2}} \mu(x) = \kappa^p(x), & x \in \Omega', \\ (-\Delta)^{\frac{n}{2}} \kappa(x) = \omega \mu^q(x), & x \in \Omega', \\ \mu(x) = -\Delta\omega(x) = \cdots = (-\Delta)^{\frac{n-1}{2}} \mu(x) = 0, & x \in (\Omega')^c \\ \kappa(x) = -\Delta\kappa(x) = \cdots = (-\Delta)^{\frac{n-1}{2}} \kappa(x) = 0, & x \in (\Omega')^c \end{cases} \tag{4.2}$$

with

$$\mu(0) = \max_{\Omega'} \mu(x) \leq 1, \ \kappa(0) = \max_{\Omega'} \kappa(x) \leq 1,$$

where $\Omega' = M^{\frac{pq-1}{n}} \Omega$.

Denote

$$\Omega'_\delta = \{x \in \Omega' \mid dist(x, \partial\Omega') \geq M^{\frac{pq-1}{n}} \delta\},$$

For arbitrarily $x \in \Omega'_\delta$, by re-scaling, the Green's function for $(-\Delta)^{\frac{n}{2}}$ for (4.2) with pole at x is

$$G'(x, y) = G(\frac{x}{M^{\frac{pq-1}{n}}}, \frac{y}{M^{\frac{pq-1}{n}}}) = C_n \ln \frac{M^{\frac{pq-1}{n}}}{|x-y|} - h(\frac{x}{M^{\frac{pq-1}{n}}}, \frac{y}{M^{\frac{pq-1}{n}}}),$$

where the $\frac{n}{2}$-harmonic function $h(\frac{x}{M^{\frac{pq-1}{n}}}, \frac{y}{M^{\frac{pq-1}{n}}})$ satisfies

$$\begin{cases} (-\Delta)^{\frac{n}{2}} h(\frac{x}{M^{\frac{pq-1}{n}}}, \frac{y}{M^{\frac{pq-1}{n}}}) = 0, & y \in \Omega, \\ (-\Delta)^i h(\frac{x}{M^{\frac{pq-1}{n}}}, \frac{y}{M^{\frac{pq-1}{n}}}) = (-\Delta)^i (C_n \ln \frac{M^{\frac{pq-1}{n}}}{|x-y|}), \ i = 0, 1, \cdots, \frac{n-1}{2}, & y \in \Omega^c. \end{cases}$$

We can derive from Lemma 4.1 that

$$h(\frac{x}{M^{\frac{pq-1}{n}}}, \frac{y}{M^{\frac{pq-1}{n}}}) \leq C, \ x \in \Omega'_\delta, \ y \in \Omega'. \tag{4.3}$$

By Green's representation formula and (4.3), for any $x \in \Omega'_\delta$ and $p \geq p_0, q \geq q_0$, we have

$$\begin{aligned} 1 \geq \mu(x) &= \int_{\Omega'} G'(x, y) \kappa^p(y) dy \\ &= C_n \int_{\Omega'} \ln \frac{M^{\frac{pq-1}{n}}}{|x-y|} \kappa^p(y) dy - \int_{\Omega'} h(\frac{x}{M^{\frac{pq-1}{n}}}, \frac{y}{M^{\frac{pq-1}{n}}}) \kappa^p(y) dy \\ &\geq C_n \int_{\Omega'} \ln \frac{M^{\frac{pq-1}{n}}}{|x-y|} \kappa^p(y) dy - C \int_{\Omega'} \kappa^p(y) dy. \end{aligned} \tag{4.4}$$

Similarly, we have

$$1 \geq \kappa(x) \geq C_n \int_{\Omega'} \ln \frac{M^{\frac{pq-1}{n}}}{|x-y|} \mu^q(y)dy - C \int_{\Omega'} \mu^q(y)dy. \tag{4.5}$$

In order to obtain a uniform a priori estimate for $u(x)$ and $v(x)$, we need to estimate the bounds for $\mu(x)$ and $\kappa(x)$. Therefore, we give the following two lemmas, the first is a uniform Lebesgue estimate and the second is the pointwise bounds.

Lemma 4.2. Assume that $n \geq 3$ is odd, $\Omega \subset \mathbb{R}^n$ is strictly convex and $p_0q_0 > 1$. Let $u, v \in C^{n,\varepsilon}_{loc}(\Omega) \cap C^{n-2}_0(\Omega)$ be a pair of solution of critical order problem (1.5). Then there exists a constant C depending only on p_0, q_0, n and Ω such that for all $p_0 \leq p < +\infty$, $q_0 \leq q < +\infty$, we have

$$\int_{\Omega} u^q(x)dx \leq C, \quad \int_{\Omega} v^p(x)dx \leq C,$$

and therefore, for any solution to (4.2), we have

$$\int_{\Omega'} \mu^q(x)dx \leq \frac{C}{M^{q+1}}, \quad \int_{\Omega'} \kappa^p(x)dx \leq \frac{C}{M^{p+1}}.$$

Proof. Denote

$$\Omega_\delta = \{x \in \Omega \mid dist(x, \partial\Omega) \geq \delta\},$$

Let $\eta(x) \in C_0^\infty(\Omega)$, $\eta(x) \in [0,1]$, $x \in \Omega$ and

$$\eta(x) = \begin{cases} 1, & x \in \Omega_\delta, \\ 0, & x \in \Omega \backslash \Omega_{\frac{\delta}{2}}. \end{cases}$$

First, by Theorem 1.1 and a similar argument as in [15], we have

$$\int_{\Omega} u^q(x)dx \leq C \int_{\Omega_\delta} u^q(x)dx, \quad \int_{\Omega} v^p(x)dx \leq C \int_{\Omega_\delta} v^p(x)dx, \tag{4.6}$$

where C depends only on n and Ω.

Second, applying Theorem 1.2, we have

$$\int_{\Omega} (-\Delta)^{\frac{n}{2}} u(x)\eta(x)dx = \int_{\Omega} u(x)(-\Delta)^{\frac{n}{2}}\eta(x)dx.$$

By (1.5), we have

$$\int_{\Omega} v^p(x)\eta(x)dx = \int_{\Omega} u(x)(-\Delta)^{\frac{n}{2}}\eta(x)dx. \tag{4.7}$$

Combing this with (4.6), we have

$$\begin{aligned} \int_{\Omega} v^p(x)dx &\leq C \int_{\Omega_\delta} v^p(x)dx \\ &\leq C \int_{\Omega} v^p(x)\eta(x)dx \\ &= C \int_{\Omega} u(x)(-\Delta)^{\frac{n}{2}}\eta(x)dx \\ &\leq C \int_{\Omega} u(x)dx \\ &\leq C|\Omega|^{1-\frac{1}{q}} (\int_{\Omega} u^q(x)dx)^{\frac{1}{q}}. \end{aligned} \tag{4.8}$$

Similarly, we obtain

$$\int_\Omega u^q(x)dx \le C|\Omega|^{1-\frac{1}{p}}(\int_\Omega v^p(x)dx)^{\frac{1}{p}}. \tag{4.9}$$

Combing (4.8) and (4.9), we have

$$\int_\Omega v^p(x)dx \le C|\Omega|^{1-\frac{1}{pq}}(\int_\Omega v^p(x)dx)^{\frac{1}{pq}},$$

i.e.

$$\int_\Omega v^p(x)dx \le C.$$

Similarly, we obtain

$$\int_\Omega u^q(x)dx \le C.$$

It follows from (4.1), we have

$$\int_{\Omega'} \mu^q(x)dx \le \frac{C}{M^{q+1}}, \quad \int_{\Omega'} \kappa^p(x)dx \le \frac{C}{M^{p+1}}.$$

This completes the proof of Lemma 4.3. □

Lemma 4.3. Assume that $n \ge 3$ is odd, $\Omega \subset \mathbb{R}^n$ is strictly convex and let $p_0q_0 > 1$. Let $u, v \in C^{n,\varepsilon}_{loc}(\Omega)$ be a pair of solution of critical order problem (1.5), and μ, κ be as defined in (4.1). Then we have

$$\max_{\overline{\Omega'}} |-\Delta\mu(x)| \le \frac{C}{(pq)^{1-\frac{2}{n}}},$$

$$\max_{\overline{\Omega'}} |-\Delta\kappa(x)| \le \frac{C}{(pq)^{1-\frac{2}{n}}},$$

and for any $p \ge p_0$, $q \ge q_0$

$$0 \le 1-\mu(x) \le \frac{C}{pq}, \quad |x| \le \frac{\delta}{(pq)^{\frac{1}{n}}},$$

$$0 \le 1-\kappa(x) \le \frac{C}{pq}, \quad |x| \le \frac{\delta}{(pq)^{\frac{1}{n}}}.$$

Proof. One can easily see from (4.2) and the maximum principle that

$$-\Delta\mu(x) \ge 0, \; x \in \Omega'.$$

By Theorem 1.1, the maximum of $-\Delta\mu(x)$ in Ω' can only be attained at some point $x_1 \in \Omega'_\delta$, i.e.

$$-\Delta\mu(x_1) = \max_{\overline{\Omega'}}(-\Delta\mu(x)).$$

By Lemma 4.1, we know that

$$-\Delta h(\frac{x}{M^{\frac{pq-1}{n}}}, \frac{y}{M^{\frac{pq-1}{n}}}) \le 0.$$

Therefore, it follows by the Green's representation that

$$
\begin{aligned}
-\Delta\mu(x_1) &= C\int_{\Omega'}\frac{\kappa^p(y)}{|x_1-y|^2}dy - C\int_{\Omega'}[-\Delta h(\frac{x_1}{M^{\frac{pq-1}{n}}},\frac{y}{M^{\frac{pq-1}{n}}})]\kappa^p(y)dy \\
&\le C\int_{\Omega'}\frac{\kappa^p(y)}{|x_1-y|^2}dy.
\end{aligned}
\tag{4.10}
$$

We only need to discuss that $M \ge 1$ with guarantees $B_\delta(0) \subset \Omega'$ (δ is small). If $M < 1$, we are done.

Since $\kappa(x) \le 1$, for any $p \ge p_0$, we have

$$
\begin{aligned}
\int_{|x_1-y|\le\frac{\delta}{(pq)^{\frac{1}{n}}}}\frac{\kappa^p(y)}{|x_1-y|^2}dy &\le C\int_{|x_1-y|\le\frac{\delta}{(pq)^{\frac{1}{n}}}}\frac{1}{|x_1-y|^2}dy \\
&\le \frac{C}{(pq)^{1-\frac{2}{n}}},
\end{aligned}
\tag{4.11}
$$

and by Lemma 4.2, we obtain

$$
\begin{aligned}
\int_{\Omega'\cap|x_1-y|\ge\frac{\delta}{(pq)^{\frac{1}{2}-\frac{1}{n}}}}\frac{\kappa^p(y)}{|x_1-y|^2}dy &\le (\frac{1}{(pq)^{\frac{1}{2}-\frac{1}{n}}\delta})^2\int_{\Omega'}\kappa^p(y)dy \\
&= \frac{1}{(pq)^{1-\frac{2}{n}}\delta^2}\frac{C}{M^{p+1}} \\
&\le \frac{C}{(pq)^{1-\frac{2}{n}}},
\end{aligned}
\tag{4.12}
$$

and by Lemma 4.2 and (4.4), for any $x \in \Omega'_\delta$ and $p \ge p_0, q \ge q_0$, we have

$$
\begin{aligned}
1 &\ge C_n\int_{\Omega'}\ln(\frac{M^{\frac{pq-1}{n}}}{|x-y|})\kappa^p(y)dy - \int_{\Omega'}h(\frac{x}{M^{\frac{pq-1}{n}}},\frac{y}{M^{\frac{pq-1}{n}}})\kappa^p(y)dy \\
&\ge C_n\int_{\Omega'}\ln(\frac{M^{\frac{pq-1}{n}}}{|x-y|})\kappa^p(y)dy - \frac{C}{M^{p+1}}.
\end{aligned}
$$

Therefore, we have

$$
\int_{\Omega'}\ln(\frac{M^{\frac{pq-1}{n}}}{|x-y|})\kappa^p(y)dy \le C. \tag{4.13}
$$

It follow that

$$
\begin{aligned}
&\int_{\frac{\delta}{(pq)^{\frac{1}{n}}}\le|x_1-y|\le\delta(pq)^{\frac{1}{2}-\frac{1}{n}}}\frac{\kappa^p(y)}{|x_1-y|^2}dy \\
&\le \int_{\Omega'}\ln(\frac{M^{\frac{pq-1}{n}}}{|x-y|})\kappa^p(y)dy \max_{\frac{\delta}{(pq)^{\frac{1}{n}}}\le|x_1-y|\le\delta(pq)^{\frac{1}{2}-\frac{1}{n}}}\{\frac{1}{|x_1-y|^2}\frac{1}{ln(\frac{M^{\frac{pq-1}{n}}}{|x-y|})}\} \\
&\le C\frac{(pq)^{\frac{2}{n}}}{\delta^2}\frac{1}{\frac{pq-1}{n}\ln M - \ln((pq)^{\frac{1}{2}-\frac{1}{n}}\delta)} \\
&\le \frac{C}{(pq)^{1-\frac{2}{n}}}.
\end{aligned}
\tag{4.14}
$$

Combining (4.10)-(4.14), we have

$$0 \le -\Delta\mu(x_1) = \max_{\overline{\Omega'}}(-\Delta\mu(x)) \le \frac{C}{(pq)^{1-\frac{2}{n}}}.$$

Since $B_\delta(0) \subset \Omega'$ and $\mu(0) \le 1$, by applying the Harnack inequality, we derive

$$\sup_{B_r(0)}(1-\mu(x)) \le C(\inf_{B_r(0)}(1-\mu(x)) + r\|\Delta\mu\|_{L^n(B_{2r}(0))}),$$

for all $r \in [0, \frac{\delta}{4}]$.

Therefore, we have

$$0 \le 1-\mu(x) \le -\Delta\mu(x_1)r^2 \le \frac{Cr^2}{(pq)^{1-\frac{2}{n}}},\ |x| \le r,$$

which implies that for any $p \ge p_0$ and $q \ge q_0$,

$$0 \le 1-\mu(x) \le \frac{C}{pq}, \quad |x| \le \frac{\delta}{(pq)^{\frac{1}{n}}},$$

Similarly, we have

$$\max_{\overline{\Omega'}}|-\Delta\kappa(x)| \le \frac{C}{(pq)^{1-\frac{2}{n}}},$$

$$0 \le 1-\kappa(x) \le \frac{C}{pq}, \quad |x| \le \frac{\delta}{(pq)^{\frac{1}{n}}}.$$

This completes the proof of Lemma 4.3. □

Proof of Theorem 1.3. By Lemma 4.3 and (4.13), we obtain

$$\begin{aligned}
C &\ge \int_{\Omega'} \ln(\frac{M^{\frac{pq-1}{n}}}{|x-y|})\kappa^p(y)dy \\
&\ge \int_{B_{\frac{\delta}{(pq)^{\frac{1}{n}}}}(0)} \ln(\frac{M^{\frac{pq-1}{n}}}{|x-y|})(1-\frac{C}{pq})^p dy \\
&\ge C\int_0^{\frac{\delta}{(pq)^{\frac{1}{n}}}} \ln(\frac{M^{\frac{pq-1}{n}}}{r})r^{n-1}dr \\
&\ge \frac{C}{pq}\ln(\frac{M^{\frac{pq-1}{n}}(pq)^{\frac{1}{n}}}{\delta}) \\
&\ge C\ln M.
\end{aligned}$$

It follow that

$$M \le C,$$

where C depends only on n, p_0, q_0 and Ω. This completes the proof of Theorem 1.3. □

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