RESEARCH ARTICLE

Analyzing stability of equilibrium points in impulsive neural network models involving generalized piecewise alternately advanced and retarded argument

Kuo-Shou Chiu

Departamento de Matemática, Facultad de Ciencias Básicas, Universidad Metropolitana de Ciencias de la Educación, José Pedro Alessandri 774, Santiago, Chile

Correspondence
Kuo-Shou Chiu, Departamento de Matemática, Facultad de Ciencias Básicas, Universidad Metropolitana de Ciencias de la Educación. Email: kschiu@umce.cl

Funding Information
This research was supported by the PGI 03-2020 DIUMCE

Summary

In this paper, we investigate the models of the impulsive cellular neural network with piecewise alternately advanced and retarded argument of generalized argument (in short IDEPCAG). To ensure the existence, uniqueness and global exponential stability of the equilibrium state, several new sufficient conditions are obtained, which extend the results of the previous literature. The method is based on utilizing Banach's fixed point theorem and a new IDEPCAG's Gronwall inequality. The criteria given are easy to check and when the impulsive effects do not affect, the results can be extracted from those of the non-impulsive systems. Typical numerical simulation examples are used to show the validity and effectiveness of proposed results. We end the article with a brief conclusion.

KEYWORDS:
Impulsive neural networks, Piecewise constant argument of generalized type, Equilibrium, Global exponential stability, Gronwall integral inequality.

1 | INTRODUCTION

Multi-variable feedback systems can exert the retroactive effect on very different time scales. Exemplifying by the extremes, according to the date of the information that is used to feedback, this action can define: (a) a *continuous process* or (b) one *discrete process*. In case (a), the growth rates of the variables are feed backed at each instant, let's say in real time. While, in case (b) there is a set of isolated dates, for example, a succession of instants in which the information is taken, in order to feedback the period between two consecutive sequence elements.

Normally and for mathematical modeling purposes, in case (a) differential equations are used and in case (b), if there is no other dynamics effect between the feedback times, difference equations can be used to express the essence of the dynamics. There are processes (real-world systems, such as some biotechnology-based ones) that can not be categorized into types (a) or (b), as they combine characteristics of both types of scales among other particular effects. This leads to the use of hybrid type equations, for example the *impulsive differential equations with piecewise constant arguments* (in abbreviation: IDEPCA), that were first considered by Wiener and Lakshmikantham[34] in 2000, and differential equations with piecewise constant argument without impulsive effect (in short, DEPCA) were studied by Shah and Wiener[30] and Wiener[32] in 1983; and has been investigated by many authors. We highlight the book of J. Wiener[33], pioneer of DEPCA, that recollects much of the research done in DEPCA. In the case, DEPCA of generalized type, were discussed extensively in[1,2,8,9,10,11,13,14,15,16,17,29].

When scales are mixed these feedback systems can be visualized as control systems, in that, one scale represents the intrinsic of the process and the other is external intervention. However, based on internal parameters. As an example, mentioned in

Busenberg and Cooke[4], is the example of the stabilization of hybrid control systems with feedback delay, in which a hybrid system is a dynamical system that presents both continuous and discrete dynamic behavior. What scale is the internal and what is the control, whether continuous or discrete? That depends on the attributes and simplifications of modeling on the process, being the most usual, to represent the intrinsic process with the continuous time scale and to reflect the intervention from the external environment to the system with the discrete scale.

Note that, either as a feedback system or as a system under control, the questions of interest usually refer to the behavior of the variables in the long term, in particular looking for specific patterns according to values in the space of feasible parameters. For reasons of practical necessity for the modeled processes, the most recurrently sought behavior is stability, in some sense, for example, seen as convergence to a steady state or towards dynamic cycles.

As far as the present work is concerned, we are interested in systems of n-variables $x(t)$, with hybrid type feedback, *i.e.* in which to the properly continuous retroaction, that is, the differential system $x'(t) = \mathcal{F}(t, x(t))$, is added another action $\mathcal{G}(\cdot, \gamma(\cdot), x(\gamma(\cdot)))$ of constant type during intervals of time $I_\kappa = [t_\kappa, t_{\kappa+1})$, $\kappa \in \mathbb{N}$, whose edge points are a predetermined sequence of times $\{t_\kappa\}$, this from internal information obtained in said sequence and $\mathfrak{J}_\kappa$, are impulsive effects at the moments t_κ.

Hence

$$\begin{cases} x'(t) = \mathcal{F}(t, x(t)) + \mathcal{G}\big(t, \gamma(t), x(\gamma(t))\big), \quad t \neq t_\kappa, & \text{(1a)} \\ \Delta x(t_\kappa) = \mathfrak{J}_\kappa(x(t_\kappa^-)), \quad \kappa \in \mathbb{N}, & \text{(1b)} \end{cases}$$

where the timer is given by $\gamma(t) = \gamma_\kappa$, $t_\kappa \le \gamma_\kappa < t_{\kappa+1}$, if $t \in I_\kappa$.

In 1988, Chua et al.[20] presented a new class of information-processing systems referred to as cellular neural networks (CNNs). It is known that the study of the stability of CNNs, DCNNs (delayed CNNs) and ICNNs (CNNs with impulses) is an important problem in theory and application. Many essential aspects of these networks, such as qualitative features of stability, periodicity, oscillation, and convergence problems have been examined by many other authors (see[3,5,7,14,21,22,23,24,25,26,27,28,31,35,36,37] and the references cited therein.

In 2000, J. Cao[5] proposed the problem of neural networks with transmission delays by using the Lyapunov method. Afterwards, considering theory of M-matrices, some stability criterion were established for delayed Hopfield neural networks[7] and the convergence behavior of a unique equilibrium of ICNNs was derived from[21].

In 2003, in view of Halanay-type inequalities and the Lyapunov methods, Mohamad and Gopalsamy[27] discussed the stability of DCNNs with continuous and discrete time; Zhou and Hu[36] (2008) studied periodic and stability conditions for DCNNs with variable and distributed delays. In 2004, by using Mawhin's coincidence degree theory and Gronwall's inequality, Liu and Liao[25] investigated DCNNs with periodic coefficients.

J. H. Park[28] (2006), B. Wang et al.[31] (2008), Zhang[37] (2009), O.M. Kwona et al.[23] and T. Li[24] (2012) acquired some delay-dependent stability criteria for interval time-varying delays neural networks, by constructing a Lyapunov-Krasovskii functional and linear matrix inequalities. In[26] and[35], some criteria have been derived for high-order neural networks without and with time-varying delays, which were analyzed using the Lyapunov method and analytical technique by linear matrix inequality.

In 2006, Huang et al.[22] were the first in considering a cellular neural network defined by (1a) with $\mathfrak{J}(\cdot, x(\cdot)) = 0$ and where the i-th component of $\mathcal{G}(\cdot, \gamma(\cdot), x(\gamma(\cdot)))$ is given by

$$x_i'(t) = -a_i([t]) + \sum_{j=1}^{m} b_{ij}([t])\, g_j(x_j([t])) + d_i([t]), \tag{2}$$

where $i = 1, 2, ..., m$ and $\gamma(t) = [t]$ is the greatest integer function. In this case, $x'(t)$ depends during all the interval $[n, n+1)$, n an integer number, only of the value of functions defined at instant n. So, equations type (1a), with $\gamma(\cdot)$ a constant delay of generalized type, are named *differential equation with generalized piecewise constant delays* (DEGPCD). The theory of the DEGPCD with impulsive effect (IDEGPCD) has been investigated by few authors. See[1,2,12,19].

We say that a deviation argument is of piecewise alternately advanced and retarded argument, and denote $\gamma(t) = \gamma_\kappa$, $t_\kappa \le \gamma_\kappa < t_{\kappa+1}$, if $t \in I_\kappa$, for all $\kappa \in \mathbb{N}$. One can easily see, the deviation argument $\ell(t) = t - \gamma(t)$ is assumed to be negative for $t_\kappa < t < \gamma_\kappa$ and positive for $\gamma_\kappa < t < t_{\kappa+1}$, $\kappa \in \mathbb{N}$. Therefore, Eq. (1a) is of considerable interest: on each interval $[t_\kappa,\ t_{\kappa+1})$ it is of alternately advanced and retarded type. Eq. (1a) is of advanced type on $I_\kappa^+ = [t_\kappa,\ \gamma_\kappa]$ and retarded type on $I_\kappa^- = (\gamma_\kappa, t_{\kappa+1})$. So, equations type (1a), with $\gamma(\cdot)$ of alternately advanced and retarded type, are named *differential equation with piecewise alternately advanced and retarded argument of generalized type* (DEPCAG). The equations type can represent anticipatory models. Note that the scientific mathematical community around the DEPCAG with impulsive effect (IDEPCAG) is very limited. See[6,18].

In the present work, we will consider a case of the IDEPCAG system (1a)-(1b) of more linear nature, but also combining information of the instant with information of the past, present, future and impulsive effect. This is, (1a)-(1b) with:

$$\begin{cases} x'(t) = -A \cdot x(t) + B \cdot f(x(t)) + C \cdot g(x(\gamma(t))) + D \quad t \neq t_\kappa, & (3a) \\ \Delta x(t_\kappa) = \mathfrak{I}_\kappa(x(t_\kappa^-)), \quad \kappa \in \mathbb{N}, & (3b) \end{cases}$$

where $\mathfrak{F}(t, x(t)) = -A \cdot x(t) + B \cdot f(x(t)) + D$, $\mathcal{G}(t, \gamma(t), x(\gamma(t))) = C \cdot g(x(\gamma(t)))$; $A = \mathrm{diag}\{a_i\}$, $B = \{b_{ij}\}$ and $C = \{c_{ij}\}$ are real $n \times n$-constant matrices and $D = \{d_i\}$ is real $n \times 1$-constant matrix, $\mathfrak{I}_k = \{\mathfrak{I}_{i\kappa}\}$ represents the impulsive effects.

Notice that, to know information about the behavior of solutions of (3a)-(3b), as a mathematical problem, has an historical evolution, we can point out that:

(1) In 2010, M. U. Akhmet et al.[1] applied linearization method and established stability criterion for the equilibrium and the periodic solution of the IDEGPCD system.

(2) In 2013, K.-S. Chiu[11] obtained some sufficient conditions for the equilibrium of the IDEPCA system with the particular argument $m\left[\frac{t+l}{m}\right]$, where l and m are positive real numbers such that $l < m$.

(2) In 2021, K.-S. Chiu[19] obtained some sufficient conditions for the equilibrium of the IDEGPCD system with the linear approximation method.

The novelty of our work is to present new and simple sufficient conditions ensuring existence, uniqueness and global exponential stability of the equilibrium state for impulsive neural network models with piecewise alternately advanced and retarded argument of generalized type (ICNN models with the IDEPCAG system). The proposed criteria extend the results of the previous literature. The method is given by the traditional and tailored route of a: IDEPCAG's Gronwall inequality and Banach contraction principle.

The rest of the paper is organized as follows. Firstly, we will introduce some preliminary concepts and propositions. Then by using a new IDEPCAG's Gronwall inequality and the contraction mapping principle, we obtain several criteria for the existence and uniqueness of the equilibrium state of the ICNN models (3a)-(3b). Moreover under some easily verifiable conditions, our unique equilibrium state of the ICNN models (3a)-(3b) is globally exponentially stable. Finally, two examples with the numerical simulations are given to show the effectiveness of our results.

2 | PRELIMINARY NOTES

In this section, we present some preliminary concepts and propositions, which are used to proof the stability of solutions of the ICNN models.

The impulsive system under study is the following ICNN models with IDEPCAG system:

$$\begin{cases} \dfrac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j\big(x_j(t)\big) + \sum_{j=1}^{n} c_{ij} g_j\big(x_j(\gamma(t))\big) + d_i, \quad t \neq t_\kappa, & (4a) \\ \Delta x_i(t_\kappa) = \mathfrak{I}_{i\kappa}(x_i(t_\kappa^-)), \quad \kappa \in \mathbb{N}, & (4b) \end{cases}$$

with $1 \leq i \leq n$, where

- The constant argument of generalized type is determined by a strictly increasing unbounded sequence of times $\{t_\kappa\}$ and the function $\gamma(\cdot)$ defined by $\gamma(t) = \gamma_\kappa$, $t_\kappa \leq \gamma_\kappa < t_{\kappa+1}$, if $t \in I_\kappa = [t_\kappa, t_{\kappa+1})$.
- The positive constant a_i denotes the relative rate with which the i–th unit resets its potential to the resting state when isolated from other units and inputs. So in (4a), it represents an exponential decay.
- The measure of activation of continuous type (resp. piecewise constant type) of the j-th neuron to its incoming potentials is given at any time by the function $f_j(x_j(\cdot))$ (resp.$g_j(x_j(\gamma(\cdot)))$).
- The constant b_{ij} (resp. c_{ij}) represents the weight of continuous type (resp. piecewise type) of the j-th unit on the i-th unit.
- For each neuron, there is an activation flow from outside the system. It is represented by the function d_i for the i-th one.

- $\Delta x_i(t_\kappa) = x_i(t_\kappa) - x_i(t_\kappa^-)$, where $x_i(t_\kappa^-) = \lim_{h\to 0^-} x_i(t_\kappa + h)$ and $\mathfrak{J}_{i\kappa}(x_i(t_\kappa^-))$ at the impulsive moment t_κ.

In this paper, we understand that a function $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T$, T denotes the transpose of a matrix, is a solution of the ICNN models with the IDEPCAG system (4a)-(4b) in $\mathbb{R}^+ = [0, \infty)$, if $x(t)$ is continuous with possible points of discontinuity of the first kind at t_κ, $\kappa \in \mathbb{N}$ such that the derivative $x'(t)$ exists at each point $t \in \mathbb{R}$, with the possible exception of the points $t_\kappa \in \mathbb{R}$, $\kappa \in \mathbb{N}$, where a one-sided derivative exists, and $x(t_\kappa)$ satisfies the impulsive effects (4b), $\kappa \in \mathbb{N}$. Moreover the ICNN models with the IDEPCAG system (4a)-(4b) is satisfied by $x(t)$ on each interval $\left(t_\kappa, t_{\kappa+1}\right)$, $\kappa \in \mathbb{N}$ as well.

For $x \in \mathbb{R}^n$, its norms are defined as

$$\|x\|_1 = \left(\sum_{i=1}^{n} \left|x_i\right|\right) \quad \text{and} \quad \|x\| = \max_{1\le i\le n} \left|x_i\right|.$$

For reasons of convenience, certain assumptions are formulated below, which will be convened when necessary.

(H_1) The functions f_i and g_i with $f_i(0) = 0$, $g_i(0) = 0$, $0 \le i \le n$, satisfy the Lipschitz condition:

$$|f_i(u) - f_i(v)| \le \mathfrak{L}_i^f |u - v|, \quad |g_i(u) - g_i(v)| \le \mathfrak{L}_i^g |u - v|.$$

for some positive constants $\mathfrak{L}_i^f$, $\mathfrak{L}_i^g$ and for all $u, v \in \mathbb{R}^+$.

(H_2) The impulsive operator $J_{i\kappa}$, $0 \le i \le n$, $\kappa \in \mathbb{N}$, satisfies

$$|\mathfrak{J}_{i\kappa}(u) - \mathfrak{J}_{i\kappa}(v)| \le \mathfrak{L}_{i\kappa}^J |u - v|,$$

for the positive constant $\mathfrak{L}_{i\kappa}^J$ and for all $u, v \in \mathbb{R}^+$.

(H_3) For any $\tau > 0$, it is satisfied $\hat{\kappa}(\tau) =: \max\left\{\kappa_1, \kappa_2\right\} < 1$, where

$$\kappa_1 = \max_{1\le i\le n} \left\{ \sup_{1\le\kappa\le i(\tau)} \left(\frac{e^{a_i\cdot\vartheta_\kappa^-} - 1}{a_i}\right) \left[\sum_{j=1}^{n} \mathfrak{L}_j^f |b_{ij}| + \sum_{j=1}^{n} \mathfrak{L}_j^g |c_{ij}|\right] \right\}$$

$$\kappa_2 = \max_{1\le i\le n} \left\{ \sup_{i(\tau)\le\kappa} \left(\frac{e^{a_i\cdot\vartheta_\kappa^+} - 1}{a_i}\right) \left[\sum_{j=1}^{n} \mathfrak{L}_j^f |b_{ij}| + \sum_{j=1}^{n} \mathfrak{L}_j^g |c_{ij}|\right] \right\}$$

here $i(\cdot)$ is an indexer defined by $i(t) = \kappa$ if $t \in I_\kappa = [t_\kappa, t_{\kappa+1})$, and $\vartheta_\kappa^+ = \gamma_\kappa - t_\kappa$, $\vartheta_\kappa^- = t_{\kappa+1} - \gamma_\kappa$, $\kappa \in \mathbb{N}$.

(H_3') For any $\tau > 0$, it is satisfied $\hat{\kappa}(\tau) < 1$, where

$$\hat{\kappa}(\tau) = \max_{1\le i\le n} \left\{ \sup_{1\le\kappa\le i(\tau)} \left(\frac{e^{a_i\cdot\vartheta_\kappa} - 1}{a_i}\right) \left[\sum_{j=1}^{n} \mathfrak{L}_j^f |b_{ij}| + \sum_{j=1}^{n} \mathfrak{L}_j^g |c_{ij}|\right] \right\}$$

and $\vartheta_\kappa = t_{\kappa+1} - t_\kappa$, $\kappa \in \mathbb{N}$.

To study the ICNN models with the IDEPCAG system (4a)-(4b), we need the following proposition.

Proposition 1. Integral Representation: Given a pair $(\tau, \zeta) \in \mathbb{R}^+ \times \mathbb{R}^n$, a function $x = (x_1(\cdot), \cdots, x_n(\cdot))^T : \mathbb{R}^+ \to \mathbb{R}^n$, such that $x(\tau) = (x_1(\tau), x_2(\tau)..., x_n(\tau))^T = \zeta$, is a solution of the ICNN models with the IDEPCAG system (4a)-(4b) if and only if their coordinates satisfy on $\mathbb{R}^+$ the set of integral equations

$$\begin{aligned} x_i(t) =& e^{-a_i(t-\tau)} x_i(\tau) + \int_{\tau}^{t} e^{-a_i(t-s)} \left[\sum_{j=1}^{n} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij} g_j\left(x_j\left(\gamma(s)\right)\right) + d_i\right] ds \\ &+ \sum_{k=i(\tau)+1}^{i(t)} e^{-a_i(t-t_\kappa)} \mathfrak{J}_{i\kappa}(x_i(t_\kappa^-)),\ i \in \{1, \cdots, n\}, \end{aligned} \tag{5}$$

or

$$x(t) = e^{-A\cdot(t-\tau)}\zeta + \int_{\tau}^{t} e^{-A\cdot(t-s)}\Big[B\cdot f(x(s)) + C\cdot g\left(x\left(\gamma(s)\right)\right) + D\Big]ds \\ + \sum_{k=i(\tau)+1}^{i(t)} e^{-A\cdot(t-t_\kappa)}\mathfrak{J}_\kappa(x(t_\kappa^-)),\ t\in\mathbb{R}^+. \tag{6}$$

We do not show the proof of this affirmation, since it can be demonstrated in the same approach as Proposition in [11] and Proposition 2.1 in [14].

The following lemma, which is one of the most important tool will be used in the proofs of our results.

Lemma 1. IDEPCAG's Gronwall Inequality: Let $\upsilon : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-negative piecewise continuous with possible discontinuity points of the first kind at $t = t_\kappa$, $\kappa \in \mathbb{N}$ for which the inequality satisfying

$$\upsilon(t) \le \upsilon(\tau) + \left|\int_{\tau}^{t}\left[\alpha_1\upsilon(s) + \alpha_2\upsilon(\gamma(s))\right]ds\right| + \sum_{\kappa=i(\tau)+1}^{i(t)} \varrho_k\upsilon(t_\kappa^-), \tag{7}$$

where α_1, α_2, ϱ_k are non-negative constants. Then:

1. For $\tau \le t$,
$$\upsilon(t) \le \upsilon(\tau)\left\{\prod_{\kappa=i(\tau)+1}^{i(t)}\left(1+\varrho_\kappa\right)\right\}e^{\left(\alpha_1+\frac{\alpha_2}{1-\eta^+}\right)\cdot(t-\tau)}. \tag{8}$$
2. For $0 \le t \le \tau$,
$$\upsilon(t) \le \upsilon(\tau)\left\{\prod_{\kappa=i(t)+1}^{i(\tau)}\frac{1}{1-\varrho_k}\right\}e^{\left(\alpha_1+\frac{\alpha_2}{1-\eta^-}\right)\cdot(\tau-t)}, \tag{9}$$

   where
$$\eta^+ := \sup_{i(\tau)\le\kappa}\left(\gamma_\kappa - t_\kappa\right)\left(\alpha_1+\alpha_2\right) \le \eta < 1,\quad \eta^- := \sup_{1\le\kappa\le i(\tau)}\left(t_{\kappa+1}-\gamma_\kappa\right)\left(\alpha_1+\alpha_2\right)\le\eta<1 \quad\text{and}\quad \max_{1\le\kappa\le i(\tau)}\varrho_k < 1. \tag{10}$$

Proof. First, consider $\tau \le t$. Suppose that $\psi(t)$ is the right side of the inequality (7). Then $\psi(\tau) = \upsilon(\tau)$, $\upsilon \le \psi$, ψ is a non-decreasing function and piecewise differentiable, and from (7), we have

$$\begin{cases}\psi'(t) \le \alpha_1\psi(t) + \alpha_2\psi\left(\gamma(t)\right), & t \ne t_\kappa,\\ \psi(t_\kappa) \le (1+\varrho_\kappa)\cdot\psi(t_\kappa^-), & \kappa\in\mathbb{N}.\end{cases} \tag{11}$$

If $\tau \le \ell \le t$ with $t, \ell \in I_i$, we obtain

$$\psi(t) - \psi(r) \le \int_{\ell}^{t}\left(\alpha_1\psi(s) + \alpha_2\psi\left(\gamma(s)\right)\right)ds. \tag{12}$$

With $t = \gamma_i$, $\ell = t_i$ in (12) for $t \in I_i$, as ψ is a non-decreasing function, we get

$$\psi(\gamma_i) \le \psi(t_i) + \int_{t_i}^{\gamma_i}(\alpha_1\psi(s) + \alpha_2\psi(\gamma_i))ds \\ \le \psi(t_i) + \int_{t_i}^{\gamma_i}(\alpha_1+\alpha_2)\psi(\gamma_i)ds = \psi(t_i) + (\gamma_i - t_i)(\alpha_1+\alpha_2)\psi(\gamma_i). \tag{13}$$

By (10), we have

$$\psi(\gamma_i) \le \frac{\psi(t_i)}{1-\eta^+}. \tag{14}$$

Take now in (12) with $t \in I_i$ and $\ell = t_i$, we give

$$\begin{aligned} \psi(t) &\le \psi(t_i) + \int_{t_i}^{t} (\alpha_1 \psi(s) + \alpha_2 \psi(\gamma_i)) ds \\ &\le \psi(t_i) + \int_{t_i}^{t} \left(\alpha_1 \psi(s) + \frac{\alpha_2}{1-\eta^+} \psi(t_i) \right) ds \\ &\le \psi(t_i) + \int_{t_i}^{t} \left\{ \left(\alpha_1 + \frac{\alpha_2}{1-\eta^+} \right) \psi(s) \right\} ds. \end{aligned} \tag{15}$$

Then, applying the Gronwall's Lemma, we have:

$$\psi(t) \le \psi(t_i) e^{\left(\alpha_1 + \frac{\alpha_2}{1-\eta^+}\right)\cdot(t-t_i)} \quad \text{for} \quad t \in I_i.$$

By the impulsive condition (11), we obtain:

$$\psi(t_{i+1}) \le \left(1 + \varrho_{i+1}\right) \psi(t_i) e^{\left(\alpha_1 + \frac{\alpha_2}{1-\eta^+}\right)\cdot(t_{i+1}-t_i)}. \tag{16}$$

From (16), recursively we have

$$v(t) \le \psi(t) \le \psi(\tau) \left\{ \prod_{k=i(\tau)+1}^{i(t)} \left(1 + \varrho_k\right) \right\} e^{\left(\alpha_1 + \frac{\alpha_2}{1-\eta^+}\right)\cdot(t-\tau)},$$

by $\psi(\tau) = v(\tau)$, we obtain (8).

Now, if $0 \le t \le \tau$. Suppose that $w(t)$ is the right side of the inequality (7). So $w(\tau) = v(\tau)$, $v \le w$, w is a non-increasing function and piecewise differentiable and from (7), we give

$$\begin{cases} w'(t) \le -\left[\alpha_1 w(t) + \alpha_2 w(\gamma(t))\right], \\ w(t_\kappa^-) \le (1-\varrho_\kappa)^{-1} \cdot w(t_\kappa). \end{cases} \tag{17}$$

If $\tau \ge \ell \ge t \ge 0$ with $t, \ell \in I_j$, we obtain

$$w(t) - w(\ell) \le -\int_{\ell}^{t} \left(\alpha_1 w(s) + \alpha_2 w(\gamma(s))\right) ds. \tag{18}$$

With $t = \gamma_j$, in (18) for $t \in I_j$ and $\ell = t_{j+1}^-$, since w is a non-increasing function, we have

$$\begin{aligned} w(\gamma_j) &\le w(t_{j+1}^-) - \int_{t_{j+1}}^{\gamma_j} (\alpha_1 w(s) + \alpha_2 w(\gamma_j)) ds \\ &\le w(t_{j+1}^-) + w(\gamma_j) \cdot (\alpha_1 + \alpha_2)(t_{j+1} - \gamma_j). \end{aligned}$$

By (10), we have

$$w(\gamma_j) \le \frac{w(t_{j+1}^-)}{1-\eta^-}. \tag{19}$$

Take now (19) in (18) with $t \in I_j$ and $\ell = t_{j+1}^-$, to get

$$\begin{aligned} w(t) &\leq w(t_{j+1}^-) + \int_t^{t_{j+1}} (\alpha_1 w(s) + \alpha_2 w(\gamma_j))ds \\ &\leq w(t_{j+1}^-) + \int_t^{t_{j+1}} \left(\alpha_1 w(s) + \frac{\alpha_2}{1-\eta^-} w(t_{j+1}^-)\right)ds \\ &\leq w(t_{j+1}^-) + \int_t^{t_{j+1}} \left(\alpha_1 + \frac{\alpha_2}{1-\eta^-}\right) w(s)ds \end{aligned}$$

because w is a non-increasing function. Then, applying the Gronwall's Lemma, we have:

$$w(t) \leq w(t_{j+1}^-)e^{\left(\alpha_1 + \frac{\alpha_2}{1-\eta^-}\right)\cdot(t_{j+1}-t)} \quad \text{for} \quad t \in I_j.$$

By (17) and $t = t_j$ we have:

$$w(t_j) \leq (1-\varrho_{j+1})^{-1} w(t_{j+1}) e^{\left(\alpha_1 + \frac{\alpha_2}{1-\eta^-}\right)(t_{j+1}-t_j)}. \tag{20}$$

From (20), recursively we obtain

$$\begin{aligned} v(t) \leq w(t) &\leq (1-\varrho_{j+1})^{-1} w(t_{j+1}) e^{\left(\alpha_1 + \frac{\alpha_2}{1-\eta^-}\right)(t_{j+1}-t)} \\ &\leq (1-\varrho_{j+1})^{-1}(1-\varrho_{j+2})^{-1} w(t_{j+2}) e^{\left(\alpha_1 + \frac{\alpha_2}{1-\eta^-}\right)(t_{j+2}-t)} \\ &\leq \dots \\ &\leq w(\tau) \left\{ \prod_{\kappa=j+1}^{i(\tau)} \left(1-\varrho_\kappa\right)^{-1} \right\} e^{\left(\alpha_1 + \frac{\alpha_2}{1-\eta^-}\right)\cdot(\tau-t)}, \end{aligned} \tag{21}$$

by $w(\tau) = v(\tau)$ we obtain (9). The proof is complete. The IDEPCAG's Gronwall inequality appears to be new. □

We can see that the ICNN models with the IDEPCAG system (4a)-(4b) do not have impulsive condition within the intervals $[t_i, t_{i+1})$, $i \in \mathbb{N}$, which is just like the DEPCAG system. Then applying the identical technique of Gronwall inequality with piecewise constant argument (see[9] and[10]). We have the following Proposition.

Proposition 2. Let the conditions (H_1), (H_2) and (H_3) be fulfilled. Then, given an initial condition $(\tau, \zeta) \in \mathbb{R}^+ \times \mathbb{R}^n$, the ICNN models with the IDEPCAG system (4a)-(4b) on $[t_{i(\tau)}, t_{i(\tau)+1})$ has a unique solution $x(\cdot) = x(\cdot, \tau, \zeta) = (x_1(\cdot), ..., x_n(\cdot))^T$ such that $x(\tau) = (x_1^0, ..., x_n^0)^T = \zeta$.

The previous proposition assures the existence and uniqueness of solutions in a local sense. The following theorem provides the existence of a unique solution when the initial moment is an arbitrary positive real number τ.

Theorem 1. Let the conditions (H_1), (H_2) and (H_3) be fulfilled. Then, given an initial condition $(\tau, \zeta) \in \mathbb{R}^+ \times \mathbb{R}^n$, the ICNN models with the IDEPCAG system (4a)-(4b) has a unique solution $x(\cdot) = x(\cdot, \tau, \zeta) = (x_1(\cdot), ..., x_n(\cdot))^T$ such that $x(\tau) = (x_1^0, ..., x_n^0)^T = \zeta$.

Proof. Let $\tau \in \mathbb{R}^+$, then we can see that $\tau \in [t_{i(\tau)}, t_{i(\tau)+1})$. Using Proposition 2, the ICNN models with the IDEPCAG system (4a)-(4b) has a unique solution $x(\cdot) = x(\cdot, \tau, \zeta) = (x_1(\cdot), ..., x_n(\cdot))^T$ on $[t_{i(\tau)}, t_{i(\tau)+1})$ such that $x(\tau) = (x_1^0, ..., x_n^0)^T = \zeta$.
Applying the condition (4b), we have

$$x(t_{i(\tau)+1}, \tau, \zeta) = x(t_{i(\tau)+1}^-, \tau, \zeta) + \mathfrak{J}_{i(\tau)+1}\left(x(t_{i(\tau)+1}^-, \tau, \zeta)\right).$$

Now, in the following interval $[t_{i(\tau)+1}, t_{i(\tau)+2})$ the solution of the ICNN models with the IDEPCAG system (4a)-(4b) satisfies

$$y'(t) = -A \cdot y(t) + B \cdot f(y(t)) + C \cdot g(y(\gamma(t))) + D,$$

and the ICNN models with the IDEPCAG system (4a)-(4b) admit a unique solution $y(t, t_{i(\tau)+1}, y^0)$ with the initial condition $y^0 = x(t_{i(\tau)+1}, \tau, \zeta)$. By definition of the solution of the ICNN model $x(t, \tau, \zeta) = y(t, t_{i(\tau)+1}, y^0)$ on $[t_{i(\tau)+1}, t_{i(\tau)+2})$. As $\mathbb{R}^+ = \bigcup_{i=1}^{\infty} [t_i, t_{i+1})$, this completes the proof by the mathematical induction.

□

Remark 1. If we consider the deviation argument that is of the constant delay of generalized type, i.e, $\gamma(t) = \gamma_i = t_i$, if $t \in [t_i, t_{i+1})$, $i \in \mathbb{N}$. The ICNN models with the IDEPCAG system (4a)-(4b) can be reduced to the following IDEGPCD system:

$$\begin{cases} \dfrac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j\big(x_j(t)\big) + \sum_{j=1}^{n} c_{ij} g_j\big(x_j(\beta(t))\big) + d_i, \quad t \neq t_\kappa, & (22a) \\ \Delta x_i(t_\kappa) = \mathfrak{J}_{i\kappa}(x_i(t_\kappa^-)), \quad \kappa \in \mathbb{N}, & (22b) \end{cases}$$

with $1 \le i \le n$, where $\beta(t) = t_\kappa$ if $t \in I_\kappa = [t_\kappa, t_{\kappa+1})$. Then we have the following observations.

i) The ICNN models with the IDEGPCD system is neither more nor less than system (1.1) in[1]. Since those works not have a global IDEGPCD's Gronwall-type inequality, the results for this system have more stronger conditions, see[19, Example 1 and Remark 4.1].

ii) The **IDEPCAG's Gronwall Inequality** of this paper reduces to the result of the **IDEGPCD's Gronwall Inequality** in[19, Lemma 2.1].

iii) The condition (H_3) with $\kappa_1 < 1$ reduces to the condition (H_3') which is the same condition **(E)** in[19].

From Theorem 1 and Remark 1, we can conclude the following results.

Corollary 1. Let the conditions (H_1), (H_2) and (H_3') be fulfilled. Then, given an initial condition $(\tau, \zeta) \in \mathbb{R}^+ \times \mathbb{R}^n$, the ICNN models with the IDEGPCD system (22a)-(22b) has a unique solution $x(\cdot) = x(\cdot, \tau, \zeta) = (x_1(\cdot), ..., x_n(\cdot))^T$ such that $x(\tau) = (x_1^0, ..., x_n^0)^T = \zeta$.

Applying our results to CNN models with the DEPCAG system (4a) and CNN models with the DEGPCD system (22a) without impulsive effects, we have:

Corollary 2. Let the conditions (H_1) and (H_3) be fulfilled. Then, given an initial condition $(\tau, \zeta) \in \mathbb{R}^+ \times \mathbb{R}^n$, there exists a unique solution $x(\cdot) = x(\cdot, \tau, \zeta) = (x_1(\cdot), ..., x_n(\cdot))^T$ of the CNN models with the DEPCAG system (4a), such that $x(\tau) = (x_1^0, ..., x_n^0)^T = \zeta$.

Corollary 3. Let the conditions (H_1) and (H_3') be fulfilled. Then, given an initial condition $(\tau, \zeta) \in \mathbb{R}^+ \times \mathbb{R}^n$, there exists a unique solution $x(\cdot) = x(\cdot, \tau, \zeta) = (x_1(\cdot), ..., x_n(\cdot))^T$ of the CNN models with the DEGPCD system (22a), such that $x(\tau) = (x_1^0, ..., x_n^0)^T = \zeta$.

3 | MAIN RESULTS

In this section, we shall establish the sufficient criteria for global exponential stability of the equilibrium state of the ICNN models with the IDEPCAG system (4a)-(4b).

3.1 | Existence of a unique equilibrium state

In this subsection, without asking for the conditions of differentiability, monotonicity or boundedness, we present sufficient conditions that are easily verifiable for the existence and uniqueness of the equilibrium of the ICNN models with the IDEPCAG system (4a)-(4b).

Notice that an equilibrium of the ICNN models with the IDEPCAG system (4a)-(4b) is the vector $x^* = (x_1^*, x_2^* ..., x_n^*)^T \in \mathbb{R}^n$ satisfies

$$a_i x_i^* = \sum_{j=1}^{n} b_{ij} f_j\left(x_j^*\right) + \sum_{j=1}^{n} c_{ij} g_j\left(x_j^*\right) + d_i, \quad i \in \{1, 2, ..., n\}. \tag{23}$$

Moreover, suppose that the impulse functions $\mathfrak{J}_{ik}(\cdot)$ satisfy the condition $\mathfrak{J}_{i\kappa}(x_i^*) = 0$, $i \in \{1, 2, ..., n\}$, $\kappa \in \mathbb{N}$. If we consider $c_{ij} = 0$, the ICNN models with the IDEPCAG system (4a)-(4b) reduces to the ICNN models in[21].

Now, we establish the conditions for the existence and uniqueness of the equilibrium state, $x^* = (x_1^*, x_2^*..., x_n^*)^T$, of the ICNN models with the IDEPCAG system (4a)-(4b).

Theorem 2. Let the conditions (H_1), (H_2) and (H_3) be fulfilled and the constants a_i, b_{ij}, c_{ij}, $\mathfrak{L}_i^f$, $\mathfrak{L}_i^g$ satisfy

$$a_i > \sum_{j=1}^{n} \mathfrak{L}_j^f \left|b_{ij}\right| + \sum_{j=1}^{n} \mathfrak{L}_j^g \left|c_{ij}\right|, \quad i \in \{1, 2, ..., n\}. \tag{24}$$

Then the ICNN models with the IDEPCAG system (4a)-(4b) admit a unique equilibrium state.

Proof. Let a mapping $G : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$G(\upsilon_1, ..., \upsilon_n) = \left(\frac{1}{a_1} \left\{ \sum_{j=1}^{n} b_{1j} f_j\left(\upsilon_j\right) + \sum_{j=1}^{n} c_{1j} g_j\left(\upsilon_j\right) + d_1 \right\}, \cdots, \frac{1}{a_n} \left\{ \sum_{j=1}^{n} b_{nj} f_j\left(\upsilon_j\right) + \sum_{j=1}^{n} c_{nj} g_j\left(\upsilon_j\right) + d_n \right\} \right)^T.$$

We will prove that $G : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction mapping on $\mathbb{R}^n$ with the supremum norm.
For $\upsilon = (\upsilon_1, ..., \upsilon_n)^T \in \mathbb{R}^n$, $\mathrm{v} = (\mathrm{v}_1, ..., \mathrm{v}_n)^T \in \mathbb{R}^n$, we have

$$\begin{aligned}
&||G(\upsilon_1..., \upsilon_n) - G(\mathrm{v}_1..., \mathrm{v}_n)|| \\
&= \max_{1\le i\le n} \left| \frac{1}{a_i} \left[\sum_{j=1}^{n} \left[b_{ij} f_j\left(\upsilon_j\right) + c_{ij} g_j\left(\upsilon_j\right)\right] + d_i \right] - \frac{1}{a_i} \left[\sum_{j=1}^{n} \left[b_{ij} f_j\left(\mathrm{v}_j\right) + c_{ij} g_j\left(\mathrm{v}_j\right)\right] + d_i \right] \right| \\
&\le \max_{1\le i\le n} \left\{ \frac{1}{a_i} \sum_{j=1}^{n} \left[\left|b_{ij}\right| \left|f_j\left(\upsilon_j\right) - f_j\left(\mathrm{v}_j\right)\right| \right] + \frac{1}{a_i} \sum_{j=1}^{n} \left[\left|c_{ij}\right| \left|g_j\left(\upsilon_j\right) - g_j\left(\mathrm{v}_j\right)\right| \right] \right\} \\
&\le \max_{1\le i\le n} \left\{ \frac{1}{a_i} \sum_{j=1}^{n} \left[\mathfrak{L}_j^f \left|b_{ij}\right| \left|\upsilon_j - \mathrm{v}_j\right| \right] + \frac{1}{a_i} \sum_{j=1}^{n} \left[\mathfrak{L}_j^g \left|c_{ij}\right| \left|\upsilon_j - \mathrm{v}_j\right| \right] \right\} \\
&\le \max_{1\le i\le n} \left\{ \frac{1}{a_i} \left[\sum_{j=1}^{n} \mathfrak{L}_j^f \left|b_{ij}\right| + \sum_{j=1}^{n} \mathfrak{L}_j^g \left|c_{ij}\right| \right] \right\} \cdot \max_{1\le j\le n} \left|\upsilon_j - \mathrm{v}_j\right| \\
&\le \rho_1 ||\upsilon - \mathrm{v}||,
\end{aligned}$$

where the number

$$\rho_1 = \max_{1\le i\le n} \left[\frac{\sum_{j=1}^{n} \mathfrak{L}_j^f \left|b_{ij}\right| + \sum_{j=1}^{n} \mathfrak{L}_j^g \left|c_{ij}\right|}{a_i} \right]$$

satisfies $0 < \rho_1 < 1$ by virtue of the condition (24). Then we have

$$||G(\upsilon) - G(\mathrm{v})|| \le \rho_1 ||\upsilon - \mathrm{v}||, \quad \upsilon, \mathrm{v} \in \mathbb{R}^n,$$

which conclude that G is a contraction mapping on $\mathbb{R}^n$. By the Banach fixed-point theorem, the system (23) admit a unique solution x^* such that $G(x^*) = x^*$. Then the ICNN models with the IDEPCAG system (4a)-(4b) has a unique equilibrium state. □

Theorem 3. Suppose that conditions (H_1), (H_2) and (H_3) hold, the constants a_i, b_{ij}, c_{ij}, $\mathfrak{L}_i^f$, $\mathfrak{L}_i^g$ satisfy

$$a_j > \mathfrak{L}_j^f \sum_{i=1}^{n} \left|b_{ij}\right| + \mathfrak{L}_j^g \sum_{i=1}^{n} \left|c_{ij}\right|, \quad j \in \{1, 2, ..., n\}. \tag{25}$$

Then the ICNN models with the IDEPCAG system (4a)-(4b) admit a unique equilibrium state.

Proof. Let $a_i x_i^* = y_i^*$, $i \in \{1, 2, ..., n\}$ in the system (23), we give:

$$y_i^* = \sum_{j=1}^{n} b_{ij} f_j\left(\frac{y_j^*}{a_j}\right) + \sum_{j=1}^{n} c_{1j} g_j\left(\frac{y_j^*}{a_j}\right) + d_i, \qquad i \in \{1, 2, ..., n\}. \tag{26}$$

It is enough to demonstrate the existence of a unique solution of the system (26).
Let a mapping $\mathcal{G} : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\mathcal{G}(v_1, ..., v_n) = \left(\left\{ \sum_{j=1}^{n} b_{1j} f_j \left(\frac{v_j}{a_j} \right) + \sum_{j=1}^{n} c_{1j} g_j \left(\frac{v_j}{a_j} \right) + d_1 \right\}, \cdots, \left\{ \sum_{j=1}^{n} b_{nj} f_j \left(\frac{v_j}{a_j} \right) + \sum_{j=1}^{n} c_{nj} g_j \left(\frac{v_j}{a_j} \right) + d_n \right\} \right)^T.$$

Then, for any $v = (v_1, ..., v_n)^T \in \mathbb{R}^n$, $\mathrm{v} = (\mathrm{v}_1, ..., \mathrm{v}_n)^T \in \mathbb{R}^n$, we have

$$\begin{aligned}
\|\mathcal{G}(v) - \mathcal{G}(\mathrm{v})\|_1 &= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} b_{ij} \left(f_j \left(\frac{v_j}{a_j} \right) - f_j \left(\frac{\mathrm{v}_j}{a_j} \right) \right) + \sum_{j=1}^{n} c_{ij} \left(g_j \left(\frac{v_j}{a_j} \right) - g_j \left(\frac{\mathrm{v}_j}{a_j} \right) \right) \right| \\
&\leq \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} \left(\frac{\mathfrak{L}_j^f}{a_j} \left| b_{ij} \right| \left| v_j - \mathrm{v}_j \right| \right) + \sum_{j=1}^{n} \left(\frac{\mathfrak{L}_j^g}{a_j} \left| c_{ij} \right| \left| v_j - \mathrm{v}_j \right| \right) \right\} \\
&\leq \left[\max_{1 \leq j \leq n} \left(\frac{\mathfrak{L}_j^f}{a_j} \sum_{i=1}^{n} \left| b_{ij} \right| + \frac{\mathfrak{L}_j^g}{a_j} \sum_{i=1}^{n} \left| c_{ij} \right| \right) \right] \sum_{j=1}^{n} \left| v_j - \mathrm{v}_j \right| \\
&:= \rho_2 \|v - \mathrm{v}\|_1,
\end{aligned}$$

By the assumption $\rho_2 < 1$, this implies that the mapping $\mathcal{G} : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction mapping. By Banach fixed-point theorem $\mathcal{G}$ has exactly one fixed point x^* in $\mathbb{R}^n$ such that $\mathcal{G}(x^*) = x^*$. Thus, the ICNN models with the IDEPCAG system (4a)-(4b) has exactly one equilibrium state. The proof is now complete. □

3.2 | Global exponential stability of equilibrium state

In this subsection, we want to discuss the stability of the ICNN models with the IDEPCAG system (4a)-(4b).

Let the following change of variables

$$z_i(t) = x_i(t) - x_i^*, \qquad \tilde{f}(z_i(t)) = f(x_i(t) + x_i^*) - f(x_i^*),$$
$$\tilde{g}\left(z_i(\gamma(t))\right) = g\left(x_i(\gamma(t)) + x_i^*\right) - g(x_i^*), \qquad \tilde{\mathfrak{J}}_{i\kappa}(z_i(t_\kappa^-)) = \mathfrak{J}_{i\kappa}(x_i(t_\kappa^-) + x_i^*),$$

so that the ICNN models with the IDEPCAG system (4a)-(4b) can be rewritten as

$$\begin{cases} z'(t) = -A \cdot z(t) + B \cdot \tilde{f}(z(t)) + C \cdot \tilde{g}\left(z\left(\gamma(t)\right)\right), \quad t \neq t_\kappa, & (27a) \\ \Delta z(t_\kappa) = \tilde{\mathfrak{J}}_k(z(t_\kappa^-)), \ \kappa \in \mathbb{N}, & (27b) \end{cases}$$

where $\tilde{f}(z(t)) = \begin{pmatrix} \tilde{f}_1(z_1(t)) \\ \vdots \\ \tilde{f}_n(z_n(t)) \end{pmatrix}$, $\tilde{g}\left(z\left(\gamma(t)\right)\right) = \begin{pmatrix} \tilde{g}_1\left(z_1\left(\gamma(t)\right)\right) \\ \vdots \\ \tilde{g}_n\left(z_n\left(\gamma(t)\right)\right) \end{pmatrix}$ and $\tilde{\mathfrak{J}}_\kappa(z(t_\kappa^-)) = \begin{pmatrix} \mathfrak{J}_{1\kappa}(z_1(t_\kappa^-)) \\ \vdots \\ \mathfrak{J}_{n\kappa}(z_n(t_\kappa^-)) \end{pmatrix}$.
We can see that $\tilde{f}_i(\cdot)$ and $\tilde{g}_i(\cdot)$, with $\tilde{f}_i(0) = \tilde{g}_i(0) = 0$, satisfy the Lipschitz condition:

$$|\tilde{f}_i(u) - \tilde{f}_i(v)| \leq \mathfrak{L}_i^f |u - v|, \qquad |\tilde{g}_i(u) - \tilde{g}_i(v)| \leq \mathfrak{L}_i^g |u - v|,$$

and $\tilde{\mathfrak{J}}_{i\kappa}$ satisfies

$$\tilde{\mathfrak{J}}_{i\kappa}(0) = 0, \quad |\tilde{\mathfrak{J}}_{i\kappa}(u) - \tilde{\mathfrak{J}}_{i\kappa}(v)| \leq \mathfrak{L}_{i\kappa}^J |u - v|,$$

for $v, v \in \mathbb{R}^+$, $i \in \{1, ..., n\}$, $\kappa \in \mathbb{N}$.

The stability of the trivial solution for the IDEPCAG system (27a)-(27b) is then studied in the same way as that of the equilibrium state x^* of the ICNN models with the IDEPCAG system (4a)-(4b).

The following notations are required in the section:

$$a_* = \min_{1 \leq i \leq n} a_i, \quad \vartheta^- = \sup_{\kappa \in \mathbb{N}} (t_{\kappa+1} - \gamma_\kappa), \quad \vartheta^+ = \sup_{\kappa \in \mathbb{N}} (\gamma_\kappa - t_\kappa), \quad \vartheta_\kappa = t_{\kappa+1} - t_\kappa, \quad \vartheta = \sup_{\kappa \in \mathbb{N}} \vartheta_\kappa,$$
$$\mathfrak{L}_\kappa^J = \max_{1 \leq i \leq n} \mathfrak{L}_{i\kappa}^J, \qquad \mathfrak{L}_{i(t)} = \max_{i(\tau)+1 \leq \kappa \leq i(t)} \frac{\ln(1 + \mathfrak{L}_\kappa^J)}{\vartheta_\kappa}, \qquad \mu^* = \max_{1 \leq i \leq n} \mu_i,$$

and

$$\mu_i = \sum_{j=1}^{n} \mathfrak{L}_j^f |b_{ij}| + \sum_{j=1}^{n} \mathfrak{L}_j^g |c_{ij}| \frac{e^{a_* \cdot \vartheta^-}}{1-\hat{v}}, \qquad \max_{1\le i\le n} \left(\sum_{j=1}^{n} \mathfrak{L}_j^f |b_{ij}| + \sum_{j=1}^{n} \mathfrak{L}_j^g \left|c_{ij}\right| e^{a_* \cdot \vartheta^-} \right) \cdot \vartheta^+ \le \hat{v} < 1.$$

Now, we will introduce the definition and lemma, so as to be used within proof of the stability of the trivial solution for the ICNN models with IDEPCAG system.

Definition 1. The equilibrium state x^* of the ICNN models with the IDEPCAG system (4a)-(4b) is globally exponentially stable, if there exist constants $\alpha, \lambda > 0$ such that

$$|x(t) - x^*| \le \alpha \, |x(\tau) - x^*| \, e^{-\lambda \cdot (t-\tau)}, \quad \tau \le t.$$

Lemma 2. Let the conditions (H_1), (H_2) and (H_3) be fulfilled and ψ, φ are the solutions of the ICNN models with the IDEPCAG system (4a)-(4b). Then the following inequality holds

$$|\psi(t) - \varphi(t)| \le |\psi(\tau) - \varphi(\tau)| \exp\left(-\mathfrak{C}_{i(t)} \cdot (t-\tau)\right), \quad t \ge \tau, \tag{28}$$

where $\mathfrak{C}_{i(t)} = a_* - \mathfrak{L}_{i(t)} - \mu^*$.

Proof. Suppose that $\psi(t) = (\psi_1, .., \psi_n)^T$ and $\varphi(t) = (\varphi_1, .., \varphi_n)^T$ are arbitrary solutions of the ICNN models with the IDEPCAG system (4a)-(4b). Let $z(t) = \psi(t) - \varphi(t)$, by the IDEPCAG system (4a)-(4b), we have

$$\begin{cases} z'(t) = -A \cdot z(t) + B \cdot \Big(f(z(t) + \varphi(t)) - f(\varphi(t)) \Big) \\ \qquad\qquad + C \cdot \Big\{ g\big(z(\gamma(t)) + \varphi(\gamma(t)) \big) - g\big(\varphi(\gamma(t)) \big) \Big\}, \\ \Delta z(t_\kappa) = \mathfrak{J}_\kappa(z(t_\kappa^-) + \varphi(t_\kappa^-)) - \mathfrak{J}_\kappa(\varphi(t_\kappa^-)), \ \kappa \in \mathbb{N}. \end{cases} \tag{29}$$

Using Proposition 1, we obtain the following integral equations

$$\begin{aligned} z(t) = \exp(-A \cdot (t-\tau)) \, z(\tau) &+ \int_\tau^t \exp(-A \cdot (t-s)) \cdot \mathfrak{R}(s, z(s)) ds \\ &+ \sum_{\kappa=i(\tau)+1}^{i(t)} \exp\left(-A \cdot (t-t_\kappa)\right) \cdot \mathfrak{J}_k(z(t_\kappa^-)), \end{aligned} \tag{30}$$

where

$$\begin{aligned} \mathfrak{R}(s, z(s)) :=& B \cdot \Big(f(z(s) + \varphi(s)) - f(\varphi(s)) \Big) \\ &+ C \cdot \Big\{ g\big(z\left(\gamma(t)\right) + \varphi(\gamma(t)) \big) - g\left(\varphi(\gamma(t))\right) \Big\}, \end{aligned}$$

and

$$\mathfrak{J}_\kappa(z(t_\kappa^-)) := \mathfrak{J}_\kappa\big(z(t_\kappa^-) + \varphi(t_\kappa^-)\big) - \mathfrak{J}_\kappa(\varphi(t_\kappa^-)).$$

By the condition (H_1) and (H_2), we have

$$|\mathfrak{R}_i(s, z(s))| \le \left(\sum_{j=1}^{n} \mathfrak{L}_j^f |b_{ij}| |z_j(s)| + \sum_{j=1}^{n} \mathfrak{L}_j^g |c_{ij}| \left| z_j\left(\gamma(s)\right) \right| \right),$$

$$|\mathfrak{R}(s, z(s))| \le \max_{1\le i\le n} \left(\sum_{j=1}^{n} \mathfrak{L}_j^f |b_{ij}| |z(s)| + \sum_{j=1}^{n} \mathfrak{L}_j^g |c_{ij}| \left| z\left(\gamma(s)\right) \right| \right),$$

and

$$\left| \mathfrak{J}_\kappa(z(t_\kappa^-)) \right| \le \mathfrak{L}_\kappa^J \left| z(t_\kappa^-) \right|.$$

Using (30), we can obtain that $u_i(t) = e^{a_* \cdot (t-\tau)} |z_i(t)|$ satisfies

$$\begin{aligned} |u_i(t)| \le |\psi_i(\tau) - \varphi_i(\tau)| &+ \int_\tau^t \left(\sum_{j=1}^{n} \mathfrak{L}_j^f |b_{ij}| |u_j(s)| + \sum_{j=1}^{n} \mathfrak{L}_j^g \left|c_{ij}\right| \left| u_j\left(\gamma(s)\right) \right| e^{a_* \cdot (s-\gamma(s))} \right) ds \\ &+ \sum_{\kappa=i(\tau)+1}^{i(t)} \mathfrak{L}_k^J \left| u_i(t_\kappa^-) \right|, \end{aligned}$$

or

$$
\begin{aligned}
|u(t)| &\le |\psi(\tau)-\varphi(\tau)| + \max_{1\le i\le n}\left\{\int_\tau^t \left(\sum_{j=1}^n \mathfrak{L}_j^f |b_{ij}||u_j(s)| + \sum_{j=1}^n \mathfrak{L}_j^g \left|c_{ij}\right| \left|u_j\left(\gamma(s)\right)\right| e^{a_*\cdot(s-\gamma(s))}\right) ds\right\} \\
&\quad + \sum_{\kappa=i(\tau)+1}^{i(t)} \mathfrak{L}_k^J \left|u_i(t_\kappa^-)\right| \\
&\le |\psi(\tau)-\varphi(\tau)| + \max_{1\le i\le n}\left\{\int_\tau^t \left[\sum_{j=1}^n \left(\mathfrak{L}_j^f |b_{ij}||u(s)| + \mathfrak{L}_j^g \left|c_{ij}\right| e^{a_*\cdot\vartheta^-} \left|u\left(\gamma(s)\right)\right|\right)\right] ds\right\} \\
&\quad + \sum_{\kappa=i(\tau)+1}^{i(t)} \mathfrak{L}_\kappa^J \left|u(t_k^-)\right|,
\end{aligned}
\tag{31}
$$

for $t\in[\tau,\infty)$.

Applying the IDEPCAG's Gronwall Inequality (Lemma 1), we have

$$
|u(t)| \le |\psi(\tau)-\varphi(\tau)| \left\{\prod_{\kappa=i(\tau)+1}^{i(t)} \left(1+\mathfrak{L}_\kappa^J\right)\right\} e^{\max\limits_{1\le i\le n}\left\{\sum\limits_{j=1}^n \mathfrak{L}_j^f |b_{ij}| + \sum\limits_{j=1}^n \mathfrak{L}_j^g |c_{ij}| \frac{e^{a_*\cdot\vartheta^-}}{1-\hat{\vartheta}}\right\}\cdot(t-\tau)}.
$$

Then we have

$$
\begin{aligned}
|\psi(t)-\varphi(t)| &\le |\psi(\tau)-\varphi(\tau)| \left\{\prod_{\kappa=i(\tau)+1}^{i(t)} \left(1+\mathfrak{L}_\kappa^J\right)\right\} e^{\left\{-\left(a_*-\max\limits_{1\le i\le n}\mu_i\right)\cdot(t-\tau)\right\}} \\
&\le |\psi(\tau)-\varphi(\tau)| e^{\left\{-\left(a_*-\max\limits_{1\le i\le n}\mu_i\right)\cdot(t-\tau)+\ln\left(\prod\limits_{\kappa=i(\tau)+1}^{i(t)}\left(1+\mathfrak{L}_\kappa^J\right)\right)\right\}} \\
&\le |\psi(\tau)-\varphi(\tau)| e^{\left\{-\left(a_*-\max\limits_{1\le i\le n}\mu_i\right)\cdot(t-\tau)+\sum\limits_{\kappa=i(\tau)+1}^{i(t)}\frac{\ln\left(1+\mathfrak{L}_\kappa^J\right)}{\vartheta_\kappa}\cdot\vartheta_\kappa\right\}},
\end{aligned}
$$

or

$$
|\psi(t)-\varphi(t)| \le |\psi(\tau)-\varphi(\tau)| \exp\left\{-\left(a_*-\max_{1\le i\le n}\mu_i - \max_{i(\tau)+1\le\kappa\le i(t)} \frac{\ln(1+\mathfrak{L}_\kappa^J)}{\vartheta_\kappa}\right)\cdot(t-\tau)\right\},
$$

and the statement (28) follows. □

Theorem 4. If the hypotheses of Theorem 2 and

$$
a_* - \mathfrak{L}_{i(t)} - \mu^* > 0, \quad \text{for} \quad t\in\mathbb{R}^+ \tag{32}
$$

hold, then the unique equilibrium state of the ICNN models with the IDEPCAG system (4a)-(4b) is globally exponentially stable.

Proof. According to the result of Theorem 2, the ICNN models with the IDEPCAG system (4a)-(4b) has a unique equilibrium state x^*. Now consider that $x(t,\zeta)$ is a solution of (4a)-(4b) with the initial condition ζ and let $\wp(t) = x(t,\zeta) - x^*$. By Lemma 2, we have

$$
|\wp(t)| \le |\wp(\tau)| e^{-\mathfrak{C}_{i(t)}\cdot(t-\tau)},
$$

where $\mathfrak{C}_{i(t)} = a_* - \mu^* - \mathfrak{L}_{i(t)}$. By the condition (32), we can conclude that $|\wp(t)| \to 0$ as $t\to\infty$. Then the trivial solution of the ICNN models with the IDEPCAG system (27) is globally exponentially stable. So, the equilibrium state of the ICNN models with the IDEPCAG system (4a)-(4b) is globally exponentially stable. □

By the same way to proof Theorem 4, we have:

Theorem 5. If the hypotheses of Theorem 3 and (32) hold, then the unique equilibrium state of the ICNN models with the IDEPCAG system (4a)-(4b) is globally exponentially stable.

Remark 2. Theorem 5 reduces to the stability result of [11, Theorem 9] with the classic piecewise alternately advanced and retarded argument, we are able to see that the results obtained in this article extend and improve the results given in [11] .

Remark 3. The existence criterion (23)-(24) and the stability criterion (32) can be easily solved by using some existing software, for example, the MATLAB.

Without impulsive effects, we have the following corollaries of Lemma 2, Theorem 4 and Theorem 5.

Corollary 4. Let the conditions (H_1) and (H_3) be fulfilled and ψ, φ are the solutions of the CNN models with the DEPCAG system (4a). Then the following inequality holds

$$|\psi(t) - \varphi(t)| \le |\psi(\tau) - \varphi(\tau)| \exp\left(-\mathfrak{C} \cdot (t - \tau)\right) \tag{33}$$

where $\mathfrak{C} = a_* - \mu^*$.

Corollary 5. If the hypotheses of Corollary 4 and the conditions (24) (or (25)),

$$a_* - \mu^* > 0 \tag{34}$$

hold. Then the unique equilibrium state of the CNN models with the DEPCAG system (4a) is globally exponentially stable.

If we consider the deviation argument that is of the constant delay of generalized type, i.e, $\gamma(t) = \gamma_i = t_i$, if $t \in [t_i, t_{i+1})$, $i \in \mathbb{N}$. We have the following corollaries.

Corollary 6. If the hypotheses of Corollary 1 and the conditions (24) (or (25)),

$$a_* - \beta^* - \mathfrak{L}_{i(t)} > 0, \quad \text{for} \quad t \in \mathbb{R}^+ \tag{35}$$

hold, where

$$\beta^* = \max_{i \in [1,..,n]} \left(\sum_{j=1}^{n} \mathcal{L}_j^f |b_{ij}| + \sum_{j=1}^{n} \mathcal{L}_j^g |c_{ij}| e^{a_* \cdot \vartheta} \right).$$

Then the unique equilibrium state of the ICNN models with the IDEGPCD system (22a)-(22b) is globally exponentially stable.

Remark 4. Corollary 6 reduces to the stability result of [19, Theorem 3.2]. Moreover this Corollary generalizes corresponding result obtained by [1, Theorem 3.1] under complicated and stronger conditions. See [19, Example 1].

Without impulsive effects, we have the following result.

Corollary 7. If the hypotheses of Corollary 3 and (24) (or (25)) hold, then the unique equilibrium state of the CNN models with the DEGPCD system (22a) is globally exponentially stable.

4 | ILLUSTRATIVE EXAMPLES WITH SIMULATIONS

In this section we should present two illustrative examples with simulations for our proposed results.

Example 1. Consider the following ICNN models with the IDEPCAG system:

$$\begin{cases} x_1' = -a_1 x_1 + b_{11} f_1(x_1) + b_{12} f_2(x_2) + c_{12} g_2(x_2(\gamma(\cdot)))) + c_{13} g_3(x_3(\gamma(\cdot))) + d_1 \\ x_2' = -a_2 x_2 + b_{21} f_1(x_1) + b_{23} f_3(x_3) + c_{21} g_1(x_1(\gamma(\cdot)))) + c_{22} g_2(x_2(\gamma(\cdot))) + d_2 \\ x_3' = -a_3 x_2 + b_{31} f_1(x_1) + b_{32} f_2(x_2) + c_{31} g_1(x_1(\gamma(\cdot)))) + c_{33} g_3(x_3(\gamma(\cdot))) + d_3, \end{cases} \tag{36a}$$

$$\begin{cases} \Delta x_1(t_\kappa) = \mathfrak{J}_{1\kappa}(x_1(t_\kappa^-)) \\ \Delta x_2(t_\kappa) = \mathfrak{J}_{2\kappa}(x_2(t_\kappa^-)), \\ \Delta x_3(t_\kappa) = \mathfrak{J}_{3\kappa}(x_3(t_\kappa^-)), \end{cases} \tag{36b}$$

where

$$\begin{array}{lll} a_1 = 1.2, & a_2 = 0.7, & a_3 = 0.9, \\ b_{11} = 0.25, & b_{12} = 0.45, & b_{21} = 0.15, \\ b_{23} = 0.35, & b_{31} = 0.35, & b_{32} = 0.25, \\ c_{12} = 0.15, & c_{13} = 0.35, & c_{21} = 0.25, \\ c_{22} = 0.35, & c_{31} = 0.45, & c_{33} = 0.25, \\ d_1 = 0.2, & d_2 = 0.1, & d_3 = 0.2, \end{array}$$

and $\gamma(t)=\frac{3\pi}{8}\kappa-\frac{\pi}{4}$, if $\frac{3\pi}{8}(\kappa-1)\le t<\frac{3\pi}{8}\kappa$, $\kappa\in\mathbb{N}$.
The output functions are

$$f_1(x_1(t))=\tanh\left(\frac{x_1(t)}{6}\right),\qquad f_2(x_2(t))=\tanh\left(\frac{x_2(t)}{4}\right),\qquad f_3(x_3(t))=\tanh\left(\frac{x_3(t)}{8}\right)$$
$$g_1(x_1(\gamma(t)))=\tanh\left(\frac{x_1(\gamma(t))}{4}\right),\quad g_2(x_2(\gamma(t)))=\tanh\left(\frac{x_2(\gamma(t))}{8}\right),\ g_3(x_3(\gamma(t)))=\tanh\left(\frac{x_3(\gamma(t))}{3}\right).$$

The impulsive functions are

$$\mathfrak{J}_{1\kappa}(x_1(t_\kappa^-))=\mathfrak{J}_{1\kappa}\left(x_1\left(\frac{3\pi}{8}(\kappa-1)^-\right)\right)=\frac{x_1\left(\frac{3\pi}{8}(\kappa-1)^-\right)-x_1^*}{5},$$
$$\mathfrak{J}_{2\kappa}(x_2(t_\kappa^-))=\mathfrak{J}_{2\kappa}\left(x_2\left(\frac{3\pi}{8}(\kappa-1)^-\right)\right)=\frac{x_2\left(\frac{3\pi}{8}(\kappa-1)^-\right)-x_2^*}{8},$$
$$\mathfrak{J}_{3\kappa}(x_3(t_\kappa^-))=\mathfrak{J}_{3\kappa}\left(x_3\left(\frac{3\pi}{8}(\kappa-1)^-\right)\right)=\frac{x_3\left(\frac{3\pi}{8}(\kappa-1)^-\right)-x_3^*}{6}$$

where $x_1^*=0.22081$, $x_2^*=0.20723$, $x_3^*=0.30335$.

We can easily verify that the point $x^*=\left(x_1^*,x_2^*,x_3^*\right)^T$ satisfies

$$\begin{cases} a_1x_1^*=\sum_{j=1}^{2}b_{1j}f_j(x_j^*)+\sum_{j=1}^{2}c_{1j}g_j\left(x_j^*\right)+d_1,\\ a_2x_2^*=\sum_{j=1}^{2}b_{2j}f_j(x_j^*)+\sum_{j=1}^{2}c_{2j}g_j\left(x_j^*\right)+d_2,\\ a_3x_3^*=\sum_{j=1}^{2}b_{3j}f_j(x_j^*)+\sum_{j=1}^{2}c_{3j}g_j\left(x_j^*\right)+d_3,\end{cases}$$

approximately. And it is clear that $\mathfrak{J}_{i\kappa}\left(x_i^*\right)=0$ for $i=1,2,3$. By simple calculation, we can see that $a_*=0.7$, $\vartheta^+=\vartheta_\kappa^+=\frac{\pi}{8}$, $\vartheta^-=\vartheta_\kappa^-=\frac{\pi}{4}$, $\vartheta=\vartheta_\kappa=\frac{3\pi}{8}$, $\mathfrak{L}_1^f=\mathfrak{L}_{3\kappa}^J=\frac{1}{6}$, $\mathfrak{L}_2^f=\mathfrak{L}_1^g=\frac{1}{4}$, $\mathfrak{L}_3^f=\mathfrak{L}_2^g=\mathfrak{L}_{2\kappa}^J=\frac{1}{8}$, $\mathfrak{L}_3^g=\frac{1}{3}$, $\mathfrak{L}_{1\kappa}^J=\frac{1}{5}$, $\mathfrak{L}_\kappa^J=\frac{1}{5}$ and $\mathfrak{L}_{i(t)}=\ln(1+\mathfrak{L}_\kappa^J)/\vartheta_\kappa\approx 0.15476$.
Then

$$\max_{1\le i\le 3}\left\{\left(\frac{e^{a_i\cdot\vartheta^-}-1}{a_i}\right)\left[\sum_{j=1}^{3}\mathfrak{L}_j^f|b_{ij}|+\sum_{j=1}^{3}\mathfrak{L}_j^g|c_{ij}|\right]\right\}\approx 0.377986<1,$$
$$\max_{1\le i\le 3}\left\{\left(\frac{e^{a_i\cdot\vartheta^+}-1}{a_i}\right)\left[\sum_{j=1}^{3}\mathfrak{L}_j^f|b_{ij}|+\sum_{j=1}^{3}\mathfrak{L}_j^g|c_{ij}|\right]\right\}\approx 0.149165<1,$$

and

$$a_1=1.2>0.289583\approx\sum_{j=1}^{3}\mathfrak{L}_j^f\left|b_{1j}\right|+\sum_{j=1}^{3}\mathfrak{L}_j^g\left|c_{1j}\right|,$$
$$a_2=0.7>0.175=\sum_{j=1}^{3}\mathfrak{L}_j^f\left|b_{2j}\right|+\sum_{j=1}^{3}\mathfrak{L}_j^g\left|c_{2j}\right|,$$
$$a_3=0.9>0.316667\approx\sum_{j=1}^{3}\mathfrak{L}_j^f\left|b_{3j}\right|+\sum_{j=1}^{3}\mathfrak{L}_j^g\left|c_{3j}\right|.$$

By Theorem 2, we can conclude that the ICNN models with the IDEPCAG system (36a)-(36b) has a unique equilibrium state x^*. On the other hand, we have

$$\hat{\nu}=\max_{1\le i\le 3}\left(\sum_{j=1}^{3}\mathfrak{L}_j^f|b_{ij}|+\sum_{j=1}^{3}\mathfrak{L}_j^g\left|c_{ij}\right|e^{a_*\cdot\vartheta^-}\right)\cdot\vartheta^+\approx 0.1807149<1,$$

$$\mu_1=\sum_{j=1}^{3}\mathfrak{L}_j^f|b_{1j}|+\sum_{j=1}^{3}\mathfrak{L}_j^g|c_{1j}|\frac{e^{a_*\cdot\vartheta^-}}{1-\hat{\nu}}\approx 0.431113<0.5452406\approx a_*-\mathfrak{L}_{i(t)},$$

$$\mu_2=\sum_{j=1}^{3}\mathfrak{L}_j^f|b_{2j}|+\sum_{j=1}^{3}\mathfrak{L}_j^g|c_{2j}|\frac{e^{a_*\cdot\vartheta^-}}{1-\hat{\nu}}\approx 0.273166<0.5452406\approx a_*-\mathfrak{L}_{i(t)}$$

and

$$\mu_3 = \sum_{j=1}^{3} \mathfrak{L}_j^f |b_{3j}| + \sum_{j=1}^{3} \mathfrak{L}_j^g |c_{3j}| \frac{e^{a_* \cdot \vartheta^-}}{1-\hat{\nu}} \approx 0.53504 < 0.5452406 \approx a_* - \mathfrak{L}_{i(t)}.$$

Then

$$a_* - \mathfrak{L}_{i(t)} - \mu^* \approx 0.0102002 > 0.$$

One can see that all conditions (H_1), (H_2), (H_3), (24) and (32) in Theorem 4 are satisfied. So, by Theorem 4, the unique equilibrium state of the ICNN models with the IDEPCAG system (36a)-(36b) is globally exponentially stable. The simulation of the unique equilibrium state x^* of the ICNN models (36a)-(36b) with and without impulses, are shown in Figs. 1 and Figs. 2.

For the simulation, the initial states $\left(x_1(0), x_2(0), x_3(0)\right)^T$ are given by the random function. Figs. 1. show that the conditions obtained in this article are valid for the ICNN models with the IDEPCAG system (36a)-(36b).

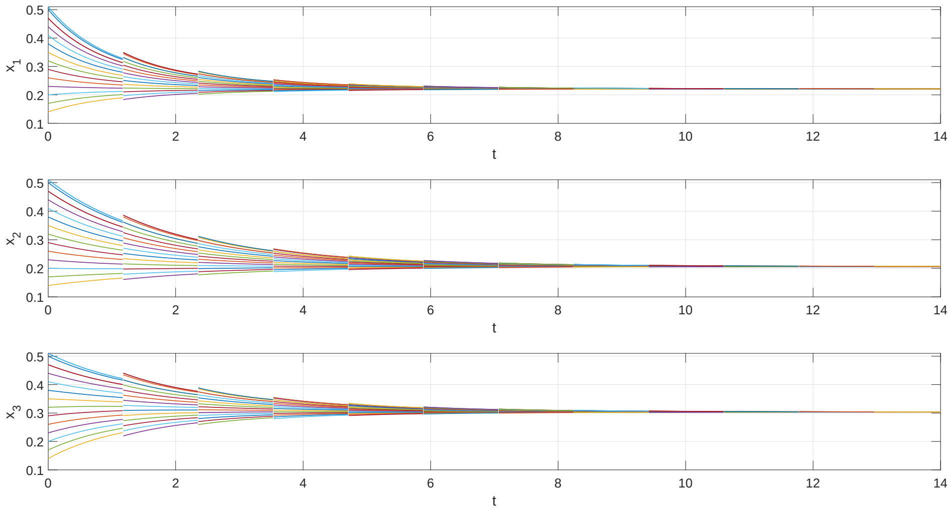

Fig. 1a. Convergence of the unique globally exponentially stable equilibrium state for the ICNN models with the IDEPCAG system (36a)-(36b).

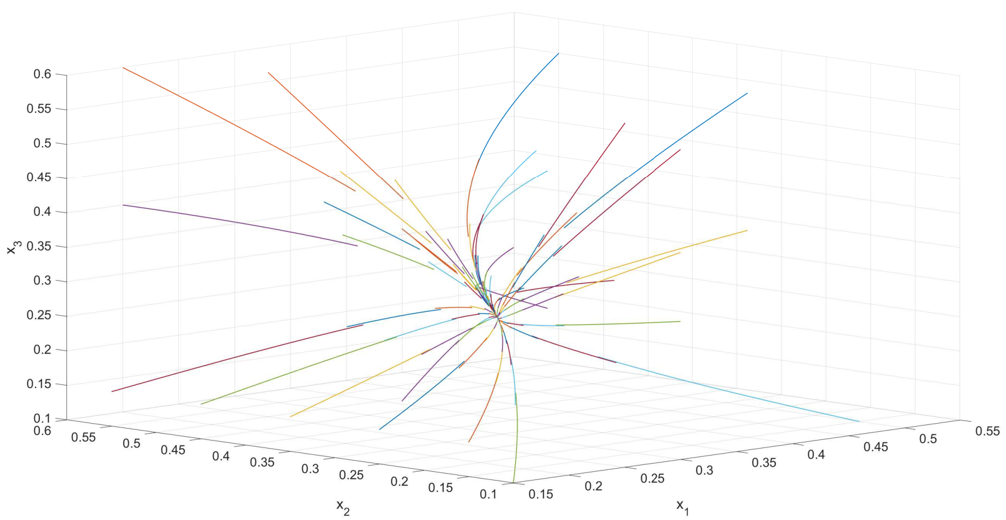

Fig. 1b. Phase portrait of state variables for the ICNN models with the IDEPCAG system (36a)-(36b).

Note that the simulation illustrates that all trajectories uniformly converge to the unique exponentially stable equilibrium point where $x^* = (0.22081; 0.20723; 0.30335)^T$.

The numerical simulation, the initial states is chosen as $(0.5; 0.35; 0.1)^T$, illustrates that the trajectory uniformly converge to the unique equilibrium $x^* = (0.22081; 0.20723; 0.30335)^T$ for the CNN models with the DEPCAG system (36a).

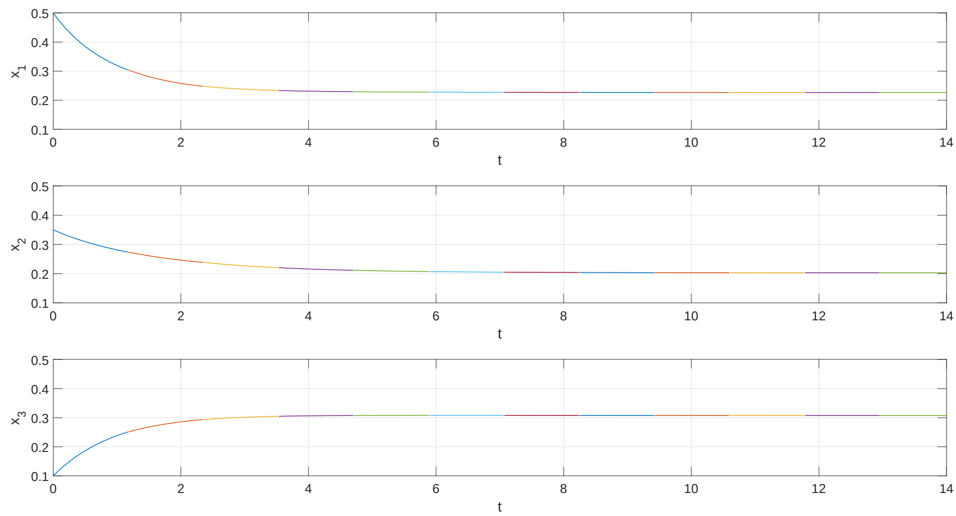

Fig. 2a. Convergence of the unique globally exponentially stable equilibrium state for the CNN models with the DEPCAG system (37a) without impulsive effects.

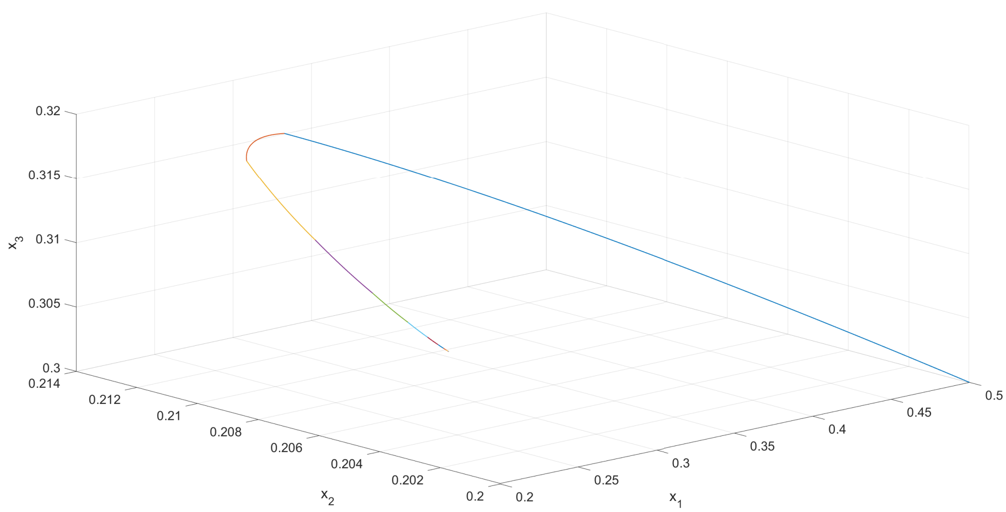

Fig. 2b. Convergence of the unique globally exponentially stable equilibrium state for the CNN models with the DEPCAG system (37a) with the initial conditions $(0.5; 0.2; 0.3)^T$.

Example 2. Consider the following ICNN models with the IDEPCAG system:

$$\frac{dx(t)}{dt} = -\begin{pmatrix} 0.9 & 0 \\ 0 & 0.6 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0.16 & 0.25 \\ 0.25 & 0.18 \end{pmatrix}\begin{pmatrix} \tanh\left(\frac{x_1(t)}{6}\right) \\ \tanh\left(\frac{x_2(t)}{5}\right) \end{pmatrix} + \begin{pmatrix} 0.23 & 0.25 \\ 0.15 & 0.27 \end{pmatrix}\begin{pmatrix} \frac{|x_1(\gamma(t))+1|-|x_1(\gamma(t))-1|}{8} \\ \frac{|x_2(\gamma(t))+1|-|x_2(\gamma(t))-1|}{16} \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \tag{37a}$$

$$\Delta x(t_\kappa) = \begin{pmatrix} \frac{x_1(3(\kappa-1)^-)-x_1^*}{6} \\ \frac{x_2(3(\kappa-1)^-)-x_2^*}{8} \end{pmatrix}, \tag{37b}$$

where $\gamma(t) = 3(\kappa - 1) + 1$, if $3(\kappa - 1) \le t < 3\kappa$, $\kappa \in \mathbb{N}$.
The output functions are

$$f_1(x_1(t)) = \tanh\left(\tfrac{x_1(t)}{6}\right), \qquad f_2(x_2(t)) = \tanh\left(\tfrac{x_2(t)}{5}\right),$$
$$g_1(x_1(\gamma(t))) = \tfrac{|x_1(\gamma(t))+1|-|x_1(\gamma(t))-1|}{8}, \quad g_2(x_2(\gamma(t))) = \tfrac{|x_2(\gamma(t))+1|-|x_2(\gamma(t))-1|}{16}.$$

The impulsive functions are

$$\mathfrak{J}_{1\kappa}(x_1(t_\kappa^-)) = \mathfrak{J}_{1\kappa}\left(x_1(3(\kappa-1)^-)\right) = \tfrac{x_1(3(\kappa-1)^-)-x_1^*}{8},$$
$$\mathfrak{J}_{2\kappa}(x_2(t_\kappa^-)) = \mathfrak{J}_{2\kappa}\left(x_2(3(\kappa-1)^-)\right) = \tfrac{x_2(3(\kappa-1)^-)-x_2^*}{6},$$

where $x_1^* = 3.7103$, $x_2^* = 3.8762$.
We can easily verify that the point $x^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ satisfies

$$\begin{cases} a_1 x_1^* = \sum_{j=1}^{2} b_{1j} f_j(x_j^*) + \sum_{j=1}^{2} c_{1j} g_j\left(x_j^*\right) + d_1, \\ a_2 x_1^* = \sum_{j=1}^{2} b_{2j} f_j(x_j^*) + \sum_{j=1}^{2} c_{2j} g_j\left(x_j^*\right) + d_2, \end{cases}$$

approximately. And it is clear that $\mathfrak{J}_{i\kappa}\left(x_i^*\right) = 0$ for $i = 1, 2$. By simple calculation, we can see that $a_* = 0.6$, $\vartheta^+ = \vartheta_\kappa^+ = 1$, $\vartheta^- = \vartheta_\kappa^- = 2$, $\vartheta = \vartheta_\kappa = 3$, $\mathfrak{L}_1^f = \mathfrak{L}_{1\kappa}^J = \frac{1}{6}$, $\mathfrak{L}_2^f = 0.2$, $\mathfrak{L}_1^g = 0.25$, $\mathfrak{L}_2^g = \mathfrak{L}_{2\kappa}^J = 0.125$, $\mathfrak{L}_\kappa^J = \frac{1}{6}$ and $\mathfrak{L}_{i(t)} = \ln(1 + \mathfrak{L}_\kappa^J)/\vartheta_\kappa \approx 0.05138$.
Then

$$\max_{1\le i\le 2}\left\{\left(\frac{e^{a_i\cdot\vartheta^-}-1}{a_i}\right)\left[\sum_{j=1}^{2}\mathfrak{L}_j^f|b_{ij}| + \sum_{j=1}^{2}\mathfrak{L}_j^g|c_{ij}|\right]\right\} \approx 0.928106 < 1,$$
$$\max_{1\le i\le 2}\left\{\left(\frac{e^{a_i\cdot\vartheta^+}-1}{a_i}\right)\left[\sum_{j=1}^{2}\mathfrak{L}_j^f|b_{ij}| + \sum_{j=1}^{2}\mathfrak{L}_j^g|c_{ij}|\right]\right\} \approx 0.268269 < 1,$$

and

$$a_1 = 0.9 > 0.1883333 \approx \mathfrak{L}_1^f \sum_{j=1}^{2}|b_{1j}| + \mathfrak{L}_1^g \sum_{j=1}^{2}|c_{1j}|,$$
$$a_2 = 0.6 > 0.1385 = \mathfrak{L}_2^f \sum_{j=1}^{2}|b_{2j}| + \mathfrak{L}_2^g \sum_{j=1}^{2}|c_{2j}|.$$

By Theorem 3, we can conclude that the ICNN models with the IDEPCAG system (37a)-(37b) has a unique equilibrium state x^*.
On the other hand, we have

$$\hat{\nu} = \max_{1\le i\le 2}\left(\sum_{j=1}^{2}\mathfrak{L}_j^f|b_{ij}| + \sum_{j=1}^{2}\mathfrak{L}_j^g\left|c_{ij}\right|e^{a_*\cdot\vartheta^-}\right)\cdot\vartheta^+ \approx 0.371327 < 1,$$
$$\mu_1 = \sum_{j=1}^{2}\mathfrak{L}_j^f|b_{1j}| + \sum_{j=1}^{2}\mathfrak{L}_j^g|c_{1j}|\frac{e^{a_*\cdot\vartheta^-}}{1-\hat{\nu}} \approx 0.545368 < 0.548616 \approx a_* - \mathfrak{L}_{i(t)},$$

and

$$\mu_2 = \sum_{j=1}^{2}\mathfrak{L}_j^f|b_{2j}| + \sum_{j=1}^{2}\mathfrak{L}_j^g|c_{2j}|\frac{e^{a_*\cdot\vartheta^-}}{1-\hat{\nu}} \approx 0.422617 < 0.548616 \approx a_* - \mathfrak{L}_{i(t)}.$$

Then

$$a_* - \mathfrak{L}_{i(t)} - \mu^* \approx 0.0032476 > 0.$$

One can see that all conditions (H_1), (H_2), (H_3), (25) and (32) in Theorem 5 are satisfied. So, by Theorem 5, the unique equilibrium state of the ICNN models with the IDEPCAG system (37a)-(37b) is globally exponentially stable. The simulation of the unique equilibrium state x^* of the ICNN models (37a)-(37b) with and without impulses, are shown in Figs. 3 and Figs. 4.

For the simulation, the initial states $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$ are given by the random function. Fig. 3a. show that the conditions obtained in this article are valid for the ICNN models with the IDEPCAG system (37a)-(37b).

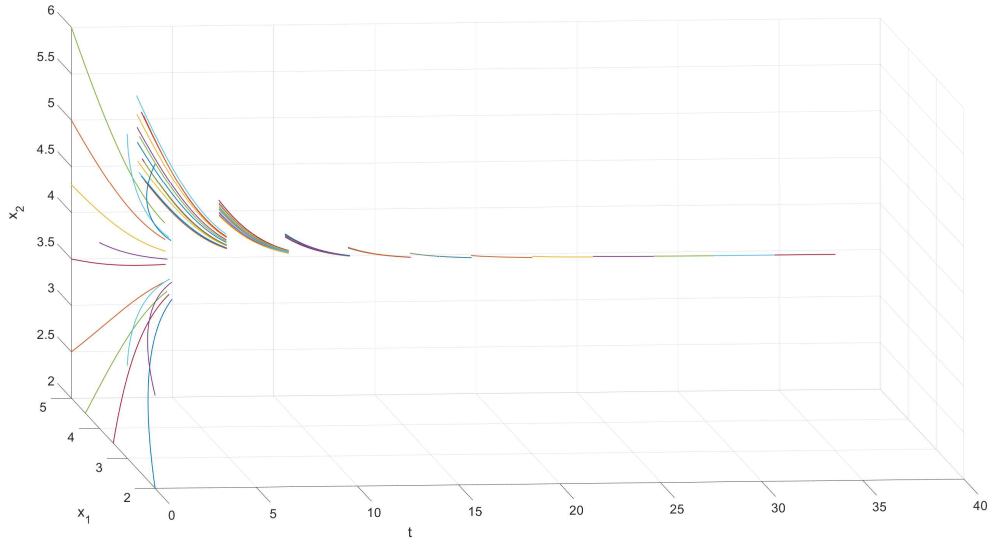

Fig. 3a. Some trajectories uniformly convergent to the unique equilibrium state for the ICNN models with the IDEPCAG system (37a)-(37b).

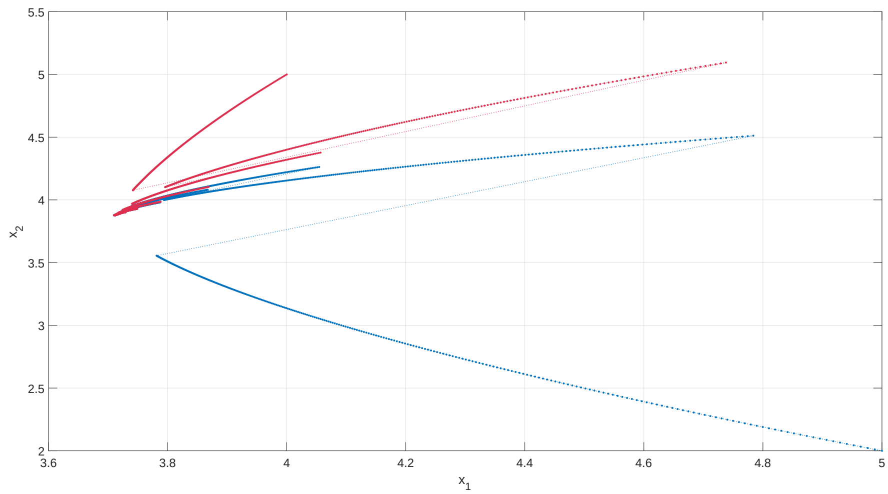

Fig. 3b. Exponential convergence of two trajectories towards the unique equilibrium state for the ICNN models with the IDEPCAG system (37a)-(37b). Initial conditions: (i) (4; 5) in red and (ii) (5; 2) in blue.

Now, for the numerical simulation, we show transient behavior of the CNN models with the DEPCAG system (37a) without impulses.

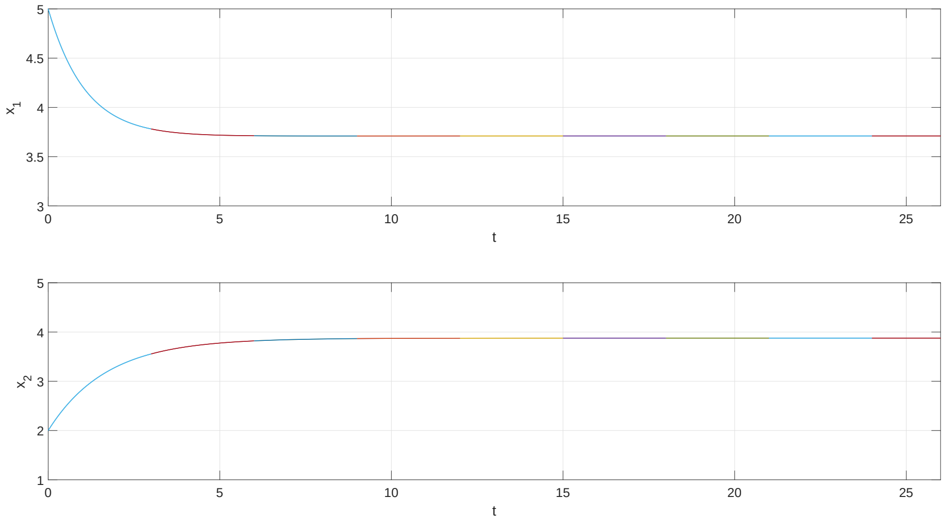

Fig. 4a. Convergence of the unique globally exponentially stable equilibrium state for the CNN models with the DEPCAG system (37a) without impulsive effects.

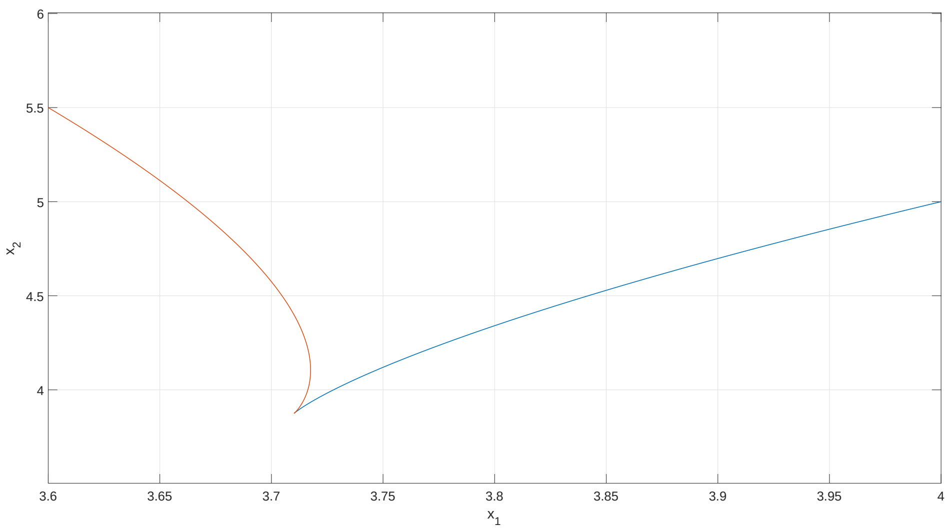

Fig. 4b. Convergence of the unique globally exponentially stable equilibrium state for the CNN models with the DEPCAG system (37a) without impulsive effects. Initial conditions: (i) (3.6; 5.5) in red and (ii) (4; 5) in blue.

Remark 5. Note that the simulation shows that some trajectories converge to the unique equilibrium state $\begin{pmatrix} 3.7103 \\ 3.8762 \end{pmatrix}$ of the CNN models with the DEPCAG system (37a).

5 | CONCLUSIONS

In this paper, the unique globally exponentially stable equilibrium state for the impulsive cellular neural network models with piecewise alternately advanced and retarded argument of generalized type have been investigated. By using the equivalent integral equation, a new IDEPCAG's Gronwall inequality and Banach fixed-point theorem, some new sufficient conditions have been developed to ensure the existence, uniqueness and global exponential stability of the equilibrium state for general non-autonomous ICNN models with the IDEPCAG system. The proposed criteria for the existence and stability theorems are easily tested by analyzing multiple relationships between neural network parameters and Lipschitz constants without asking for the conditions of differentiability, monotonicity or boundedness. Based on the proposed approach, it is unnecessary to utilize

Razumikhin-type technique or construct a Lyapunov function that is applied from the previous literature. Moreover, illustrative simulation examples show that the approach used is more efficient and extend the results of the previous literature [11] and [19].

REFERENCES

References

1. M. Akhmet and E. Yπmaz, Impulsive Hopfield-type neural network system with piecewise constant argument, Nonlinear Anal. Real World Appl. 11 (2010) 2584–2593.
2. M. Akhmet and E. Yπmaz, Equilibria of Neural Networks with Impact Activations and Piecewise Constant Argument. In: Neural Networks with Discontinuous/Impact Activations. Nonlinear Systems and Complexity, vol 9. Springer, New York, NY. 2014.
3. E. Barone and C. Tebaldi, Stability of equilibria in a neural network model, Math. Meth. Appl. Sci., 23 (2000) 1179–1193.
4. S. Busenberg and K. Cooke, Vertically transmitted diseases: models and dynamics in Biomathematics, vol. 23, Springer-Verlag, Berlin, 1993.
5. J. Cao, Global asymptotic stability of neural networks with transmission delays, Int. J. Syst. Sci. 31 (2000) 1313–1316.
6. S. Castillo, M. Pinto, R. Torres, Asymptotic formulae for solutions to impulsive differential equations with piecewise constant argument of generalized type, Electron. J. Differential Equations, Vol. 2019 (2019), No. 40, pp. 1-22.
7. T. Chen, Global exponential stability of delayed Hopfield neural networks, Neural Networks 14 (2001) 977–980.
8. K.-S. Chiu and M. Pinto, Variation of parameters formula and Gronwall inequality for differential equations with a general piecewise constant argument, Acta Math. Appl. Sin. Engl. Ser., 27, No. 4 (2011) 561–568.
9. K.-S. Chiu and M. Pinto, Periodic solutions of differential equations with a general piecewise constant argument and applications, E. J. Qualitative Theory of Diff. Equ., 46. (2010) 1–19.
10. K.-S. Chiu, M. Pinto and J.-Ch. Jeng, Existence and global convergence of periodic solutions in recurrent neural network models with a general piecewise alternately advanced and retarded argument, Acta Appl. Math. 133 (2014) 133–152.
11. K.-S. Chiu, Existence and global exponential stability of equilibrium for impulsive cellular neural network models with piecewise alternately advanced and retarded argument, Abstract and Applied Analysis, vol. 2013, Article ID 196139, 13 pages, 2013. doi:10.1155/2013/196139
12. K.-S. Chiu, On generalized impulsive piecewise constant delay differential equations, Science China Mathematics 58 (2015) 1981–2002.
13. K.-S. Chiu and J.-Ch. Jeng, Stability of oscillatory solutions of differential equations with general piecewise constant arguments of mixed type, Math. Nachr. 288 (2015) 1085–1097.
14. K.-S. Chiu, Exponential stability and periodic solutions of impulsive neural network models with piecewise constant argument, Acta Appl. Math. 151 (2017) 199–226.
15. K.-S. Chiu, Asymptotic equivalence of alternately advanced and delayed differential systems with piecewise constant generalized arguments, Acta Math. Sci. 38 (2018) 220–236.
16. K.-S. Chiu and T. Li, Oscillatory and periodic solutions of differential equations with piecewise constant generalized mixed arguments, Math. Nachr. 292 (2019) 2153–2164.
17. K.-S. Chiu, Green's function for periodic solutions in alternately advanced and delayed differential systems, Acta Math. Appl. Sin. Engl. Ser. 36 (4) (2020) 936–951.

18. K.-S. Chiu, Green's function for impulsive periodic solutions in alternately advanced and delayed differential systems and applications, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 70 (1) (2021) 15–37.

19. K.-S. Chiu, Existence and global exponential stability of equilibrium for impulsive neural network models with generalized piecewise constant delays, Asian-European Journal of Mathematics, to appear.

20. L. O. Chua and L. Yang, Cellular neural networks: Theory, IEEE Trans. Circuits Syst., 35 (1988) 1257–1272.

21. K. Gopalsamy, Stability of artificial neural networks with impulses, Appl. Math. Comput. 154 (2004) 783–813.

22. Z. K. Huang, X. H. Wang and F. Gao, The existence and global attractivity of almost periodic sequence solution of discrete-time neural networks, Phys. Lett. A, 350 (2006) 182–191.

23. O. M. Kwona, S. M. Lee, Ju H. Park and E. J. Cha , New approaches on stability criteria for neural networks with interval time-varying delays, Appl. Math. Comput. 218 (2012) 9953–9964.

24. T. Li, X. Yao, L. Wu and J. Li, Improved delay-dependent stability results of recurrent neural networks, Appl. Math. Comput. 218 (2012) 9983–9991.

25. Z. Liu, and L. Liao, Existence and global exponential stability of periodic solutions of cellular neural networks with time-varying delays, J. Math. Anal. Appl. 290 (2004) 247–262.

26. X. Y. Lou and B. T. Cui, Novel global stability criteria for high-order Hopfield-type neural networks with time-varying delays, J. Math. Anal. Appl. 330 (2007) 144–158.

27. S. Mohamad and K. Gopalsamy, Exponential stability of continuous-time and discrete-time cellular neural networks with delays, Appl. Math. Comput. 135 (2003) 17–38.

28. J. H. Park, Global exponential stability of cellular neural networks with variable delays, Appl. Math. Comput. 183 (2006) 1214–1219.

29. M. Pinto, Cauchy and Green matrices type and stability in alternately advanced and delayed differential systems, J. Difference Equ. Appl. 17 (2) (2011) 235–254.

30. S. M. Shah and J. Wiener, Advanced differential equations with piecewise constant argument deviations, Internat. J. Math.and Math. Sci. 6 (1983) 671–703.

31. B. Wang, S. Zhong and X. Liu, Asymptotical stability criterion on neural networks with multiple time-varying delays, Appl. Math. Comput. 195 (2008) 809–818.

32. J. Wiener, Differential equations with piecewise constant delays, V. Lakshmikantham, Trends in the Theory and Practice of Nonlinear Differential Equations, Marcel Dekker, New York, 1983, 547–580.

33. J. Wiener, Generalized Solutions of Functional Differential Equations, World Scientific, Singapore, 1993.

34. J. Wiener and V. Lakshmikantham, Differential equations with piecewise constant argument and impulsive equations, Nonlinear Stud., 7 (2000) 60–69.

35. B. Xu, X. Liu and X. Liao, Global exponential stability of high order Hopfield type neural networks, Appl. Math. Comput. 174 (2006) 98–116.

36. L. Zhou and G. Hu, Global exponential periodicity and stability of cellular neural networks with variable and distributed delays, Appl. Math. Comput. 195 (2008) 402–411.

37. Y. Zhang, D. Yue and E. Tian, New stability criteria of neural networks with interval time-varying delay: A piecewise delay method, Appl. Math. Comput. 208 (2009) 249–259.