Analysis of long-term solution of chemotactic model with indirect signal consumption in three-dimensional case

Qiaoling Hu*

School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China

Abstract: In this paper, we consider the chemotaxis model

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - vw, & x \in \Omega,\ t > 0, \\ w_t = -\delta w + u, & x \in \Omega,\ t > 0 \end{cases}$$

under homogeneous Neumann boundary conditions in a bounded and convex domain $\Omega \subset \mathbb{R}^3$ with smooth boundary, where $\delta > 0$ is a given parameter. It is shown that for arbitrarily large initial data, this problem admits at least one global weak solution for which there exists $T > 0$ such that the solution (u, v, w) is bounded and smooth in $\Omega \times (T, \infty)$. Furthermore, it is asserted that such solutions approach spatially constant equilibria in the large time limit.

Keywords: Global existence; Boundedness; Eventual smoothness; Indirect signal consumption

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1 Introduction

Chemotaxis is a kind of tendentious response of insects, cells and bacteria to chemical stimulation in the external environment, especially in foraging, courtship and ovulation. It is important for bacteria to look for food, such as glucose, so that they can move to places where food molecules are high and far away from toxic ones.

A pioneering result in the chemotactic model of cell migration obtained by Keller and Segel in [9], they proposed

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

where u and v are expressed as the density of cells or bacteria and the concentration of chemical signals. In the framework of (1.1), the existing result is that the solution will blow up in finite time when the space dimension is two-dimensional or higher-dimensional and it is proved that for each $q > \frac{n}{2}$ and $p > n, n \geq 3$ and the initial data (u_0, v_0) satisfies $\|u_0\|_{L^q(\Omega)} < \varepsilon$ and $\|\nabla v_0\|_{L^p(\Omega)} < \varepsilon$, where n denotes space dimension, the solution is global in time and bounded ([14], [22], [24]). Particularly, when Ω is a ball in $\mathbb{R}^n$ with $n \geq 3$, there exists a radially symmetric positive initial data (u_0, v_0) such that the corresponding solution blows up in finite time [24]. In a word, if $n = 1$, the solution of (1.1) is global in time and bounded [12]. If $n = 2$ and $\int_\Omega u_0 < 4\pi$, the solution will be global and bounded [11]. If $n \geq 3$ and the initial value are small, the solution to (1.1) is global and bounded [22].

However, even more general organic compound tend to move towards the nutrients they consume. We can obtain the following model

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - vu, & x \in \Omega,\ t > 0. \end{cases} \tag{1.2}$$

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*E-mail: hqlmath@163.com

For (1.2), the existing result is that Tao proved in [16] that if the corresponding initial value is sufficiently smooth and satisfies $\|v_0\|_{L^\infty(\Omega)} \leq \frac{1}{6(n+1)}$, there is global classical solution. In the three-dimensional case, it was shown that for arbitrarily large initial data, this problem admits one global weak solution and there exists $T > 0$ such that (u, v) is bounded and smooth in $\Omega \times (T, \infty)$ in [16].

There are also models related to (1.2) such as

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \kappa u - \mu u^2, & x \in \Omega, t > 0, \\ v_t = \Delta v - uv, & x \in \Omega, t > 0. \end{cases} \tag{1.3}$$

The existing result is that there exists a global, bounded and classical solution for suitably large μ and there exists a weak solution for any $\mu > 0$. Moreover, in the case of $\kappa > 0$, the solution convergence to the constant equilibrium $(\frac{\kappa}{\mu}, 0)$ [8]. In addition, there is global and bounded solution to (1.3) with $\kappa = \mu = 0$ is only known under the smallness condition

$$\chi \|v_0\|_{L^\infty(\Omega)} \leq \frac{1}{6(n+1)} \tag{1.4}$$

on the initial data or in a two-dimensional setting ([23],[25],[18]). In three-dimensional bounded domains, weak solution eventually becomes smooth [18]. In addition, chemotaxis-consumption models are embedded into more complex frame-works. Such as coupled chemotaxis-fluid systems with nonlinear diffusion systems, nonlinear chemotactic sensitivity, and with zeroth order terms accounting for logistic growth or competition between species have been analyzed.

In this paper, we deal with a chemotactic model with indirect consumption

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega, \ t > 0, \\ v_t = \Delta v - vw, & x \in \Omega, \ t > 0, \\ w_t = -\delta w + u, & x \in \Omega, \ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded, smooth and convex region, $\delta > 0$ is a given parameter, (u_0, v_0, w_0) is the given initial value and satisfies

$$\begin{cases} u_0 \in C^0(\overline{\Omega}), & x \in \overline{\Omega}, \ u_0 > 0, \\ v_0 \in W^{1,q}(\Omega), & q > 3, \ x \in \Omega, \ v_0 > 0, \\ w_0 \in C^0(\overline{\Omega}), & x \in \overline{\Omega}, \ w_0 > 0. \end{cases} \tag{1.6}$$

The interest of this paper is whether the interaction of chemotactic cross diffusion leads to the singularity of the solution. It has been shown that (1.5) has a global weak solution when $n = 2$ in [23]. It has been proved that if either $n \leq 2$ or $\|v_0\|_{L^\infty(\Omega)} \leq \frac{1}{3n}$, (1.5) has a unique, global and classical solution (u, v, w) and convergence of $(u(\cdot, t), v(\cdot, t), w(\cdot, t))$ towards a spatially constant equilibrium as $t \to \infty$ in [3].

Therefore, a natural question is whether the global weak solution of (1.5) is still bounded and smooth for any large initial value, or if the solution is singular, the solution will blow up in finite time or infinite time, or the solution will eventually disappear. The result of this paper is that the global weak solution of (1.5) will at least eventually become bounded and smooth, and eventually tend to a constant equilibrium state.

The main result can be read as follows.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain with smooth boundary, and assume that u_0, v_0 and w_0 satisfy (1.6). Then in the sense of Definition 2.1 below, (1.5) has a global weak solution. Moreover, there exists $T > 0$ such that the solution (u, v, w) is bounded and belongs to $C^{2,1}(\overline{\Omega} \times [T, \infty)) \times C^{2,1}(\overline{\Omega} \times [T, \infty)) \times C^0(\overline{\Omega} \times [T, \infty))$, and we have*

$$u(x, t) \to \overline{u}_0, \ v(x, t) \to 0 \ and \ w(x, t) \to \overline{w}_0 \quad as \ t \to \infty \tag{1.7}$$

for all $x \in \Omega$, where $\overline{u}_0 := \frac{1}{|\Omega|} \int_\Omega u_0$, $\overline{w}_0 = \frac{\overline{u}_0}{\delta}$.

Remark 1.1 *Theorem 1.1 shows the behavior of the solution in a long time, but does not rules out the possibility of finite-time blow up in a stronger topology.*

The structure of this paper is arranged as follows: In Section 2, the definition of global weak solution of (1.5) is given and a basic energy functional of (1.5) is derived and estimated and some results of boundedness are obtained. The weak stability results of u and the estimate of uniform decay of v are derived in Section 3. In Section 4, the final boundedness and regularity of u are deduced, which depend on the boundedness of $\|u_\varepsilon(\cdot, t)\|_{L^p(\Omega)}$ by using the weight function $\varphi(v_\varepsilon)$. Finally, we can get the stability result of u and the corresponding result of w, so as to complete the proof of Theorem 1.1 in Section 5.

2 Preliminaries

The following concept of weak solutions appears to be natural in the present setting.

Definition 2.1 *By a global weak solution of* (1.5) *we mean a pair* (u, v, w) *of functions*

$$u \in L^1_{loc}\big([0, \infty); L^1(\Omega)\big), \ v \in L^1_{loc}\big([0, \infty); W^{1,1}(\Omega)\big), \ w \in L^1_{loc}\big([0, \infty); L^1(\Omega)\big)$$

such that

$$vw \ and \ u\nabla v \ belong \ to \ L^1_{loc}\big([0, \infty); L^1(\Omega)\big),$$

and such that the identities

$$-\int_0^\infty \int_\Omega u\xi_t - \int_\Omega u_0\xi(\cdot, 0) = -\int_0^\infty \int_\Omega \nabla u \cdot \nabla \xi + \int_0^\infty \int_\Omega u\nabla v \cdot \nabla \xi,$$
$$-\int_0^\infty \int_\Omega v\xi_t - \int_\Omega v_0\xi(\cdot, 0) = -\int_0^\infty \int_\Omega \nabla v \cdot \nabla \xi - \int_0^\infty \int_\Omega vw\xi \tag{2.1}$$

and

$$-\int_0^\infty \int_\Omega w\xi_t - \int_\Omega w_0\xi(\cdot, 0) = -\delta \int_0^\infty \int_\Omega w\xi + \int_0^\infty \int_\Omega u\xi$$

hold for all $\xi \in C_0^\infty\big(\overline{\Omega} \times [0, \infty)\big)$.

A global weak solution of (1.5) in the above sense can be obtained as the limit of a sequence of solutions $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$, $\varepsilon = \varepsilon_j \in (0, 1)$, of the regularized problems

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \nabla \cdot \big(u_\varepsilon F_\varepsilon'(u_\varepsilon)\nabla v_\varepsilon\big), & x \in \Omega, \ t > 0, \\ v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon w_\varepsilon, & x \in \Omega, \ t > 0, \\ w_{\varepsilon t} = -\delta w_\varepsilon + F_\varepsilon(u_\varepsilon), & x \in \Omega, \ t > 0, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = \dfrac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u_\varepsilon(x, 0) = u_0(x), \quad v_\varepsilon(x, 0) = v_0(x), \quad w_\varepsilon(x, 0) = w_0(x), & x \in \Omega, \end{cases} \tag{2.2}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Here,

$$F_\varepsilon(s) := \frac{\ln(1 + \varepsilon s)}{\varepsilon}, \ s \geq 0$$

for $\varepsilon \in (0, 1)$.

Lemma 2.1 *Let initial value* (u_0, v_0, w_0) *satisfy* (1.6) *and* $F_\varepsilon(s)$ *be as defined above, then for all* $\varepsilon \in (0, 1)$, *the system* (2.2) *exists an unique global and classical solution* $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ *and satisfies* $u_\varepsilon > 0, v_\varepsilon > 0, w_\varepsilon > 0$ *in* $\Omega \times [0, \infty)$.
Moreover, there exists a sequence $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)$ *of numbers* $\varepsilon_j \to 0$ *such that*

$$u_\varepsilon \to u; \ v_\varepsilon \to v; \ w_\varepsilon \to w \ in \ L^1_{loc}\big(\overline{\Omega} \times [0, \infty)\big) \ for \ a.e. \ in \ \big(\Omega \times [0, \infty)\big),$$

where the nonnegative function (u, v, w) *is a global weak solution of* (1.5) *in the sense of Definition by* (2.1).

Proof. Firstly, we prove that $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ is a global classical solution of (2.2).
Existence. With R > 0 (R is a constat) and $T \in (0, 1)$ to be specified below, in the Banach space

$$X := L^\infty\big([0, T); C^0(\overline{\Omega})\big) \times L^\infty\big([0, T); W^{1,q}(\Omega)\big),$$

3

where $q > 3$. We consider the closed set

$$S := \left\{(u_\varepsilon, v_\varepsilon) \in X \big| \|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot, t)\|_{W^{1,q}(\Omega)} \leq \mathrm{R} \text{ for a.e. } t \in (0, T) \right\}$$

and introduce a mapping $\phi = (\phi_1, \phi_2)$ on S by defining

$$\phi_1(u_\varepsilon, v_\varepsilon)(\cdot, t) := e^{t\Delta}u_{\varepsilon 0} - \int_0^t e^{(t-s)\Delta}\nabla \cdot \left(u_\varepsilon F_\varepsilon{}'(u_\varepsilon)\nabla v_\varepsilon\right)(\cdot, s)ds$$

and

$$\phi_2(u_\varepsilon, v_\varepsilon)(\cdot, t) := e^{t\Delta}v_{\varepsilon 0} - \int_0^t e^{(t-s)\Delta}\psi(u_\varepsilon)v_\varepsilon(\cdot, s)ds,$$

where $\psi(u_\varepsilon)(\cdot, t) := e^{-\delta t}w_{\varepsilon 0} + \int_0^t F_\varepsilon(u_\varepsilon)e^{\delta(s-t)}(\cdot, s)$ for all $(u_\varepsilon, v_\varepsilon) \in S$ and $t \in (0, T)$. Here and below, $(e^{t\Delta})_{t \geq 0}$ denotes the Neumann heat semigroup.

Then since $q > 3$, we can take $\beta \in (0, 1)$ such that $\frac{3}{2q} < \beta < \frac{1}{2}$, then we can obtain $D(B^\beta) \hookrightarrow C^0(\overline{\Omega})$, where B stands for the sectorial operator $-\Delta + 1$ in $L^q(\Omega)$ with homogeneous Neumann boundary conditions [5]. Using the standard estimate of thermal semigroups ([22], Lemma (iv)) and the properties of the function: $F_\varepsilon(s) = \frac{\ln(1+\varepsilon s)}{\varepsilon}$, $F_\varepsilon(s)$ is nonnegative and satisfies $0 < F_\varepsilon{}'(s) = \frac{1}{1+\varepsilon s} \leq 1$ for all $s \geq 0$ and $F_\varepsilon \in C^2([0, \infty))$. So there exist some positive constants C_1, C_2, $C_3(R)$ such that

$$\begin{aligned}
\|\phi_1(u_\varepsilon, v_\varepsilon)(\cdot, t)\|_{L^\infty(\Omega)} &\leq \|e^{t\Delta}u_{\varepsilon 0}\|_{L^\infty(\Omega)} + C_1 \int_0^t \|B^\beta e^{-(t-s)(B-1)}\nabla \cdot \left(u_\varepsilon F_\varepsilon{}'(u_\varepsilon)\nabla v_\varepsilon\right)\|_{L^q(\Omega)}ds \\
&\leq \|u_{\varepsilon 0}\|_{L^\infty(\Omega)} + C_2 \int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2q}}\|u_\varepsilon F_\varepsilon'(u_\varepsilon)\nabla v_\varepsilon\|_{L^q(\Omega)}ds \\
&\leq \|u_{\varepsilon 0}\|_{L^\infty(\Omega)} + C_2 \int_0^t (t-s)^{-\frac{1}{2}-\beta}\|u_\varepsilon \nabla v_\varepsilon\|_{L^q(\Omega)}ds \\
&\leq \|u_{\varepsilon 0}\|_{L^\infty(\Omega)} + C_2\|u_\varepsilon\|_{L^\infty(\Omega \times (0,T))}\|\nabla v_\varepsilon(\cdot, s)\|_{L^\infty((0,T);L^q(\Omega))} \int_0^t (t-s)^{-\frac{1}{2}-\beta}ds \\
&\leq \|u_{\varepsilon 0}\|_{L^\infty(\Omega)} + C_3(R)T^{\frac{1}{2}-\beta} \quad \text{for all } t \in (0, T).
\end{aligned} \tag{2.3}$$

We note that $q > 3$ and $\alpha > \frac{n}{4} = \frac{3}{4}$ imply that $W^{1,q}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ and $D(B^\alpha) \hookrightarrow C^0(\overline{\Omega})$ in ([5], [13]). Proceeding similarly, we Fix any $\gamma \in (\frac{1}{2}, 1)$ and estimate

$$\begin{aligned}
\|\phi_2(u_\varepsilon, v_\varepsilon)(\cdot, t)\|_{W^{1,q}(\Omega)} &\leq \|e^{t\Delta}v_{\varepsilon 0}\|_{W^{1,q}(\Omega)} + C_4 \int_0^t \|B^\gamma e^{-(t-s)(B-1)}\psi(u_\varepsilon)v_\varepsilon(\cdot, s)\|_{L^q(\Omega)}ds \\
&\leq C_5\|v_{\varepsilon 0}\|_{W^{1,q}(\Omega)} + C_4 \int_0^t (t-s)^{-\gamma}\|\psi(u_\varepsilon)v_\varepsilon(\cdot, s)\|_{L^q(\Omega)}ds \\
&\leq C_5\|v_{\varepsilon 0}\|_{W^{1,q}(\Omega)} + C_6(R) \int_0^t (t-s)^{-\gamma}ds \\
&\leq C_5\|v_{\varepsilon 0}\|_{W^{1,q}(\Omega)} + C_7(R)T^{-\gamma+1} \quad \text{for all } t \in (0, T)
\end{aligned} \tag{2.4}$$

for some $C_4 > 0$, $C_5 > 0$, $C_6(R) > 0$, and $C_7(R) > 0$.
Combined with (2.3) and (2.4), this proves that $\phi = (\phi_1, \phi_2)$ maps S into itself if we take $\mathrm{R} > 0$ large enough and $T > 0$ sufficiently small. Next, we prove that when T is sufficiently small, $\phi = (\phi_1, \phi_2)$ in fact becomes a contraction on S. For all $(u_{\varepsilon 1}, v_{\varepsilon 1}), (u_{\varepsilon 2}, v_{\varepsilon 2}) \in S$, we have

$$\begin{aligned}
\|\phi_1(u_{\varepsilon 1}, v_{\varepsilon 1})(\cdot, t) - \phi_1(u_{\varepsilon 2}, v_{\varepsilon 2})(\cdot, t)\|_{L^\infty(\Omega)} &= \left\|\int_0^t e^{(t-s)\Delta}\nabla \cdot (u_{\varepsilon 1}F_\varepsilon'(u_{\varepsilon 1})\nabla v_{\varepsilon 1} - u_{\varepsilon 2}F_\varepsilon'(u_{\varepsilon 2})\nabla v_{\varepsilon 2})ds\right\|_{L^\infty(\Omega)} \\
&\leq C_8 \int_0^t (t-s)^{-\beta-\frac{1}{2}}\|u_{\varepsilon 1}F_\varepsilon{}'(u_{\varepsilon 1})\nabla v_{\varepsilon 1} - u_{\varepsilon 2}F_\varepsilon{}'(u_{\varepsilon 2})\nabla v_{\varepsilon 2}ds\|_{L^q(\Omega)} \\
&\leq C_8 T^{-\beta+\frac{1}{2}}\|(u_{\varepsilon 1}, v_{\varepsilon 1}) - (u_{\varepsilon 2}, v_{\varepsilon 2})\|_X \quad \text{for all } t \in (0, T).
\end{aligned}$$

Similarly,

$$\|\phi_2(u_{\varepsilon 1}, v_{\varepsilon 1})(\cdot, t) - \phi_2(u_{\varepsilon 2}, v_{\varepsilon 2})(\cdot, t)\|_{W^{1,q}(\Omega)} = \left\| \int_0^t e^{(t-s)\Delta} \big(\psi(u_{\varepsilon 1})v_{\varepsilon 1} - \psi(u_{\varepsilon 2})v_{\varepsilon 2}\big)ds \right\|_{W^{1,q}(\Omega)}$$

$$\leq C_9 \int_0^t (t-s)^{-\gamma} \|\psi(u_{\varepsilon 1})v_{\varepsilon 1} - \psi(u_{\varepsilon 2})v_{\varepsilon 2}ds\|_{L^q(\Omega)}$$

$$\leq C_9 T^{-\gamma+1} \|(u_{\varepsilon 1}, v_{\varepsilon 1}) - (u_{\varepsilon 2}, v_{\varepsilon 2})\|_X \quad \text{for all } t \in (0, T).$$

Therefore, when T is sufficiently small, $\phi = (\phi_1, \phi_2)$ is a contractive mapping from S to itself. From the Banach fixed theorem that there exists $(u_\varepsilon, v_\varepsilon) \in S$ such that $\phi(u_\varepsilon, v_\varepsilon) = (u_\varepsilon, v_\varepsilon)$. Because $v_0 \in W^{1,q}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ and $v_\varepsilon = \phi_2(u_\varepsilon, v_\varepsilon)$, then we can obtain $v_\varepsilon \in C^0([0, T_{\max}); C^0(\overline{\Omega}))$. Moreover, according to the standard regularity theory of parabolic equation and the standard estimate of thermal semigroup ([4], [7]), $(u_\varepsilon, v_\varepsilon)$ actually is a smooth solution of (2.2) and satisfies the following smooth properties:

$$u_\varepsilon \in C^0([0, T); L^2(\Omega)) \cap L^\infty((0, T); C^0(\overline{\Omega})) \cap C^{2,1}((\overline{\Omega}) \times (0, T)),$$

$$v_\varepsilon \in C^0([0, T); L^2(\Omega)) \cap L^\infty((0, T); W^{1,q}(\Omega)) \cap C^{2,1}((\overline{\Omega}) \times (0, T)).$$

From $w_\varepsilon = \psi(u_\varepsilon)$ and the smoothness of u_ε, we can get $w_\varepsilon \in C^0(\overline{\Omega} \times [0, T)) \cap C^{0,1}(\overline{\Omega} \times (0, T))$. Namely, $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ is the classical solution of (2.2) in $\Omega \times (0, T)$. From the above estimates, we can know that the coefficient T of the contractive mapping is only related to the initial value $\|u_0\|_{L^\infty(\Omega)}$ and $\|v_0\|_{W^{1,q}(\Omega)}$. Therefore, the solution $(u_\varepsilon, v_\varepsilon)$ can be extended to $T_{\max} \leq \infty$, then $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ is the global classical solution of (2.2).

Uniqueness. In order to demonstrate uniqueness within the indicated class, we suppose that $(u_{\varepsilon 1}, v_{\varepsilon 1}, w_{\varepsilon 1})$ and $(u_{\varepsilon 2}, v_{\varepsilon 2}, w_{\varepsilon 2})$ are two solutions of (2.2) in $\Omega \times (0, T)$ for some $T > 0$. We fix $T_0 \in (0, T)$ and multiply the difference of the PDEs satisfied by $u_{\varepsilon 1}$ and $u_{\varepsilon 2}$ by $u_{\varepsilon 1} - u_{\varepsilon 2}$ to obtain

$$(u_{\varepsilon 1} - u_{\varepsilon 2})_t = \Delta(u_{\varepsilon 1} - u_{\varepsilon 2}) - \nabla \cdot \big(u_{\varepsilon 1} F_\varepsilon'(u_{\varepsilon 1})\nabla v_{\varepsilon 1} - u_{\varepsilon 2} F_\varepsilon'(u_{\varepsilon 2})\nabla v_{\varepsilon 2}\big)$$

and

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_{\varepsilon 1} - u_{\varepsilon 2})^2 + \int_\Omega |\nabla(u_{\varepsilon 1} - u_{\varepsilon 2})|^2 = \int_\Omega u_{\varepsilon 1} F_\varepsilon'(u_{\varepsilon 1})\nabla v_{\varepsilon 1} \cdot \nabla(u_{\varepsilon 1} - u_{\varepsilon 2})$$

$$- \int_\Omega u_{\varepsilon 2} F_\varepsilon'(u_{\varepsilon 2})\nabla v_{\varepsilon 2} \cdot \nabla(u_{\varepsilon 1} - u_{\varepsilon 2})$$

$$= \int_\Omega (u_{\varepsilon 1} - u_{\varepsilon 2})F_\varepsilon'(u_{\varepsilon 1})\nabla v_{\varepsilon 1} \cdot \nabla(u_{\varepsilon 1} - u_{\varepsilon 2}) \tag{2.5}$$

$$+ \int_\Omega u_{\varepsilon 2}\big(F_\varepsilon'(u_{\varepsilon 1}) - F_\varepsilon'(u_{\varepsilon 2})\big)\nabla v_{\varepsilon 1} \cdot \nabla(u_{\varepsilon 1} - u_{\varepsilon 2})$$

$$+ \int_\Omega u_{\varepsilon 2} F_\varepsilon'(u_{\varepsilon 2})\nabla(v_{\varepsilon 1} - v_{\varepsilon 2}) \cdot \nabla(u_{\varepsilon 1} - u_{\varepsilon 2})$$

$$:= I_1 + I_2 + I_3 \quad \text{for all } t \in (0, T_0).$$

Since $T_0 < T$, there exists $C_{10} > 0$ such that

$$\|u_{\varepsilon 1}\|_{L^\infty(\Omega)} + \|u_{\varepsilon 2}\|_{L^\infty(\Omega)} + \|\nabla v_{\varepsilon 1}\|_{L^q(\Omega)} + \|\nabla v_{\varepsilon 2}\|_{L^q(\Omega)} \leq C_{10} \quad \text{for all } t \in (0, T_0).$$

Now using Young's inequality and Hölder's inequality, we can yield that

$$I_1 \leq \frac{1}{6}\int_\Omega |\nabla(u_{\varepsilon 1} - u_{\varepsilon 2})|^2 + \frac{3}{2}\int_\Omega (u_{\varepsilon 1} - u_{\varepsilon 2})^2 |\nabla v_{\varepsilon 1}|^2$$

$$\leq \frac{1}{6}\int_\Omega |\nabla(u_{\varepsilon 1} - u_{\varepsilon 2})|^2 + \frac{3}{2}\Big(\int_\Omega |\nabla v_{\varepsilon 1}|^q\Big)^{\frac{2}{q}}\Big(\int_\Omega (u_{\varepsilon 1} - u_{\varepsilon 2})^{\frac{2q}{q-2}}\Big)^{\frac{q-2}{q}}$$

$$\leq \frac{1}{6}\int_\Omega |\nabla(u_{\varepsilon 1} - u_{\varepsilon 2})|^2 + \frac{3}{2}C_{10}^2\|u_{\varepsilon 1} - u_{\varepsilon 2}\|^2_{L^{\frac{2q}{q-2}}(\Omega)} \quad \text{for all } t \in (0, T_0).$$

According to Ehring's Lemma and the compactness of the embedding $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^{\frac{2q}{q-2}}(\Omega)$, we have

$$I_1 \leq \frac{1}{6}\int_\Omega |\nabla(u_{\varepsilon 1} - u_{\varepsilon 2})|^2 + C_{11}\|u_{\varepsilon 1} - u_{\varepsilon 2}\|^2_{W^{1,2}(\Omega)}$$

$$\leq \frac{1}{3}\int_\Omega |\nabla(u_{\varepsilon 1} - u_{\varepsilon 2})|^2 + C_{12}\int_\Omega (u_{\varepsilon 1} - u_{\varepsilon 2})^2 \quad \text{for all } t \in (0, T).$$

By similar arguments involving the Lipschitz continuity of F_ε', for some positive constants C_{14} and C_{15}, we have

$$I_2 \le \frac{1}{6} \int_\Omega |\nabla(u_{\varepsilon 1} - u_{\varepsilon 2})|^2 + C_{13} \int_\Omega (u_{\varepsilon 1} - u_{\varepsilon 2})^2 |\nabla v_{\varepsilon 1}|^2$$
$$\le \frac{1}{3} \int_\Omega |\nabla(u_{\varepsilon 1} - u_{\varepsilon 2})|^2 + C_{14} \int_\Omega (u_{\varepsilon 1} - u_{\varepsilon 2})^2$$

and

$$I_3 \le \frac{1}{3} \int_\Omega |\nabla(u_{\varepsilon 1} - u_{\varepsilon 2})|^2 + C_{15} \int_\Omega |\nabla(v_{\varepsilon 1} - v_{\varepsilon 2})|^2$$

for all $t \in (0, T_0)$. We can obtain the following formula by substituting I_1, I_2 and I_3 into (2.5)

$$\frac{d}{dt} \int_\Omega (u_{\varepsilon 1} - u_{\varepsilon 2})^2 \le C_{15} \int_\Omega |\nabla(v_{\varepsilon 1} - v_{\varepsilon 2})|^2 + C_{16} \int_\Omega (u_{\varepsilon 1} - u_{\varepsilon 2})^2$$

for all $t \in (0, T_0)$. Proceeding similarly, we have

$$\frac{1}{2} \frac{d}{dt} \int_\Omega (v_{\varepsilon 1} - v_{\varepsilon 2})^2 + \int_\Omega |\nabla(v_{\varepsilon 1} - v_{\varepsilon 2})|^2$$
$$= -\int_\Omega (w_{\varepsilon 1} - w_{\varepsilon 2}) v_{\varepsilon 1} (v_{\varepsilon 1} - v_{\varepsilon 2}) - \int_\Omega w_{\varepsilon 2} (v_{\varepsilon 1} - v_{\varepsilon 2})^2$$
$$\le \frac{1}{2} \int_\Omega (w_{\varepsilon 1} - w_{\varepsilon 2})^2 + C_{17} \int_\Omega v_{\varepsilon 1}^2 (v_{\varepsilon 1} - v_{\varepsilon 2})^2 + \|w_{\varepsilon 2}\|_{C^0(\overline{\Omega} \times [0,T))} \int_\Omega (v_{\varepsilon 1} - v_{\varepsilon 2})^2$$
$$\le \frac{1}{2} \int_\Omega (w_{\varepsilon 1} - w_{\varepsilon 2})^2 + C_{17} \Big(\int_\Omega v_{\varepsilon 1}^q \Big)^{\frac{2}{q}} \Big(\int_\Omega (v_{\varepsilon 1} - v_{\varepsilon 2})^{\frac{2q}{q-2}} \Big)^{\frac{q-2}{q}} + \|w_{\varepsilon 2}\|_{C^0(\overline{\Omega} \times [0,T))} \int_\Omega (v_{\varepsilon 1} - v_{\varepsilon 2})^2$$
$$\le \frac{1}{2} \int_\Omega (w_{\varepsilon 1} - w_{\varepsilon 2})^2 + C_{18} \int_\Omega (v_{\varepsilon 1} - v_{\varepsilon 2})^2 + \frac{1}{2} \int_\Omega |\nabla(v_{\varepsilon 1} - v_{\varepsilon 2})|^2$$

and

$$C_{15} \frac{d}{dt} \int_\Omega (v_{\varepsilon 1} - v_{\varepsilon 2})^2 + C_{15} \int_\Omega |\nabla(v_{\varepsilon 1} - v_{\varepsilon 2})|^2 \le C_{15} \int_\Omega (w_{\varepsilon 1} - w_{\varepsilon 2})^2 + C_{19} \int_\Omega (v_{\varepsilon 1} - v_{\varepsilon 2})^2$$

for all $t \in (0, T_0)$. Finally, integrating by parts, we find some positive constants C_{20} and C_{21} fulfilling

$$\frac{1}{2} \frac{d}{dt} \int_\Omega (w_{\varepsilon 1} - w_{\varepsilon 2})^2 = -\delta \int_\Omega (w_{\varepsilon 1} - w_{\varepsilon 2})^2 + \int_\Omega \big(F_\varepsilon(u_{\varepsilon 1}) - F_\varepsilon(u_{\varepsilon 2}) \big)(w_{\varepsilon 1} - w_{\varepsilon 2})$$
$$\le C_{20} \int_\Omega (w_{\varepsilon 1} - w_{\varepsilon 2})^2 + C_{21} \int_\Omega (u_{\varepsilon 1} - u_{\varepsilon 2})^2$$

for all $t \in (0, T_0)$. All in all,

$$\frac{d}{dt} \Big\{ \int_\Omega (u_{\varepsilon 1} - u_{\varepsilon 2})^2 + C_{15} \int_\Omega (v_{\varepsilon 1} - v_{\varepsilon 2})^2 + \frac{1}{2} \int_\Omega (w_{\varepsilon 1} - w_{\varepsilon 2})^2 \Big\}$$
$$\le C_{22} \int_\Omega (u_{\varepsilon 1} - u_{\varepsilon 2})^2 + C_{19} \int_\Omega (v_{\varepsilon 1} - v_{\varepsilon 2})^2 + C_{23} \int_\Omega (w_{\varepsilon 1} - w_{\varepsilon 2})^2$$

for all $t \in (0, T_0)$, We infer that $y(t) := \int_\Omega (u_{\varepsilon 1} - u_{\varepsilon 2})^2 + C_{15} \int_\Omega (v_{\varepsilon 1} - v_{\varepsilon 2})^2 + \frac{1}{2} \int_\Omega (w_{\varepsilon 1} - w_{\varepsilon 2})^2$ satisfies $y'(t) \le C_{24} y(t)$ for all $t \in (0, T_0)$ for some $C_{24} > 0$ depending on T_0 only. From Gronwall's inequality and $y(0) = 0$, this yields $y \equiv 0$ in $(0, T_0)$. Therefore, $u_{\varepsilon 1} = u_{\varepsilon 2}$, $v_{\varepsilon 1} = v_{\varepsilon 2}$, $w_{\varepsilon 1} = w_{\varepsilon 2}$ for all $t \in (0, T_0)$ and thereby proves the clam, for $T_0 \in (0, T)$ is arbitrary. Finally, from the comparison principle of parabolic equation, we can obtain the conclusion $u_\varepsilon > 0$, $v_\varepsilon > 0$ and from $\psi(u_\varepsilon) = w_\varepsilon$, then we can yield $w_\varepsilon > 0$. □

Lemma 2.2 *For all $\varepsilon \in (0, 1)$, the solution of (2.2) satisfies*

$$\frac{1}{|\Omega|} \int_\Omega u_\varepsilon(\cdot, t) = \overline{u}_0 \quad \text{for all } t \ge 0.$$

Proof. Integrating the first equation of (2.2) on $\Omega \times (0, t)$, then we can obtain

$$\int_0^t \int_\Omega u_{\varepsilon t} = 0$$

as well as

$$\int_\Omega u_\varepsilon(\cdot, t) = \int_\Omega u_0.$$

Then we have

$$\frac{1}{|\Omega|} \int_\Omega u_\varepsilon(\cdot, t) = \frac{1}{|\Omega|} \int_\Omega u_0 = \overline{u}_0. \quad \square$$

Lemma 2.3 *Let $\varepsilon \in (0, 1)$. Then for the solution of (2.2),*

$$t \to \|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \text{ is not increasing in } [0, \infty).$$

Proof. Since $v_{\varepsilon t} \le \Delta v_\varepsilon$ due to the fact that w_ε and v_ε are nonnegative, the claim results obtained by the application of the maximum principle of parabolic equation. $\quad \square$

Lemma 2.4 *For all $\varepsilon \in (0, 1)$ we have*

$$\int_0^\infty \int_\Omega v_\varepsilon w_\varepsilon \le \int_\Omega v_{\varepsilon 0}.$$

In particular, the limit couple (v, w) defined through Lemma 2.1 fulfills

$$\int_0^\infty \int_\Omega vw \le \int_\Omega v_0.$$

Proof. Integrating the second equation of (2.2) on $\Omega \times (0, t)$, then

$$\int_\Omega v_\varepsilon(\cdot, t) + \int_0^t \int_\Omega v_\varepsilon w_\varepsilon = \int_\Omega v_{\varepsilon 0} \quad \text{for all } t > 0.$$

Since $v_\varepsilon \ge 0$, we can yield

$$\int_0^\infty \int_\Omega v_\varepsilon w_\varepsilon \le \int_\Omega v_{\varepsilon 0}.$$

Particularly,

$$\int_0^\infty \int_\Omega vw \le \int_\Omega v_0$$

can be obtained by using the Fatou's lemma. $\quad \square$

Lemma 2.5 *For each $\varepsilon \in (0, 1)$, the solution of (2.2) satisfies*

$$\frac{d}{dt}\left\{ \int_\Omega u_\varepsilon \ln u_\varepsilon + 2 \int_\Omega |\nabla \sqrt{v_\varepsilon}|^2 \right\} + \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 + \frac{1}{2} \int_\Omega w_\varepsilon \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \tag{2.6}$$
$$+ \int_\Omega \nabla v_\varepsilon \cdot \nabla w_\varepsilon + \int_\Omega F_\varepsilon(u_\varepsilon) \Delta v_\varepsilon \le 0$$

for all $t > 0$.

Proof. Multiplying the two ends of the first equation of (2.2) by $(1 + \ln u_\varepsilon)$ and integrating by parts, we have

$$\int_\Omega u_{\varepsilon t}(1 + \ln u_\varepsilon) = \int_\Omega \Delta u_\varepsilon(1 + \ln u_\varepsilon) - \int_\Omega \nabla \cdot \left(u_\varepsilon F_\varepsilon{}'(u_\varepsilon) \nabla v_\varepsilon \right)(1 + \ln u_\varepsilon)$$

as well as

$$\frac{d}{dt} \int_\Omega u_\varepsilon \ln u_\varepsilon + \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} = \int_\Omega F_\varepsilon{}'(u_\varepsilon) \nabla v_\varepsilon \cdot \nabla u_\varepsilon. \tag{2.7}$$

Through a direct integration by parts, we can obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\psi(c)|^2 = \int_\Omega \nabla\psi(c)\big(\nabla\psi(c)\big)_t = \int_\Omega \psi'(c)\nabla c \cdot \big(\psi''(c)c_t\nabla c + \psi'(c)\nabla c_t\big)$$

$$= \int_\Omega \psi'(c)\psi''(c)|\nabla c|^2 c_t + \int_\Omega \psi'(c)^2 \nabla c \cdot \nabla c_t$$

$$= -\int_\Omega \psi'(c)\psi''(c)|\nabla c|^2 c_t - \int_\Omega \psi'(c)^2 c_t \Delta c$$

for all $\psi,\, c \in C^{2,1}\big(\overline{\Omega}\times[0,t)\big)$. Moreover, we have

$$2\frac{d}{dt}\int_\Omega |\nabla\sqrt{v_\varepsilon}|^2 = \frac{1}{2}\int_\Omega v_\varepsilon^{-2}|\nabla v_\varepsilon|^2 v_{\varepsilon t} - \int_\Omega v_\varepsilon^{-1}\Delta v_\varepsilon . v_{\varepsilon t}$$

$$= \frac{1}{2}\int_\Omega v_\varepsilon^{-2}|\nabla v_\varepsilon|^2 (\Delta v_\varepsilon - v_\varepsilon w_\varepsilon) - \int_\Omega v_\varepsilon^{-1}\Delta v_\varepsilon(\Delta v_\varepsilon - v_\varepsilon w_\varepsilon) \tag{2.8}$$

$$= \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2 \Delta v_\varepsilon}{v_\varepsilon^2} - \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2 w_\varepsilon}{v_\varepsilon} - \int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} + \int_\Omega w_\varepsilon \Delta v_\varepsilon.$$

Now by ([23], Lemma 3.1), we have

$$-\int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} = -\int_\Omega \frac{|D^2 v_\varepsilon|^2}{v_\varepsilon} - \frac{3}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}\Delta v_\varepsilon + \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{2}\int_{\partial\Omega}\frac{1}{v_\varepsilon}\frac{\partial}{\partial\nu}|\nabla v_\varepsilon|^2 \tag{2.9}$$

and

$$\int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 = \int_\Omega \frac{|D^2 v_\varepsilon|^2}{v_\varepsilon} - 2\int_\Omega \frac{D^2 v_\varepsilon \cdot \nabla v_\varepsilon \cdot \nabla v_\varepsilon}{v_\varepsilon^2} + \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}. \tag{2.10}$$

Combining (2.9), (2.10) with (2.8), we can obtain

$$2\frac{d}{dt}\int_\Omega |\nabla\sqrt{v_\varepsilon}|^2 = -\int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 - 2\int_\Omega \frac{D^2 v_\varepsilon \cdot \nabla v_\varepsilon \cdot \nabla v_\varepsilon}{v_\varepsilon^2} + 2\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} - \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}\Delta v_\varepsilon$$

$$- \int_\Omega \nabla v_\varepsilon \cdot \nabla w_\varepsilon - \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2 w_\varepsilon}{v_\varepsilon}\frac{1}{2}\int_{\partial\Omega}\frac{1}{v_\varepsilon}\frac{\partial}{\partial\nu}|\nabla v_\varepsilon|^2 \tag{2.11}$$

and

$$-2\int_\Omega \frac{(D^2 v_\varepsilon \cdot \nabla v_\varepsilon)\cdot\nabla v_\varepsilon}{v_\varepsilon^2} = \int_\Omega \frac{|\nabla v_\varepsilon|^2 \Delta v_\varepsilon}{v_\varepsilon^2} - 2\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}. \tag{2.12}$$

Putting (2.12) into (2.11), we can yield that

$$2\frac{d}{dt}\int_\Omega |\nabla\sqrt{v_\varepsilon}|^2 + \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 + \int_\Omega \nabla v_\varepsilon \cdot \nabla w_\varepsilon + \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2 w_\varepsilon}{v_\varepsilon} \le \frac{1}{2}\int_{\partial\Omega}\frac{1}{v_\varepsilon}\frac{\partial}{\partial\nu}|\nabla v_\varepsilon|^2. \tag{2.13}$$

Combining (2.7) and (2.13), we can obtain that

$$\frac{d}{dt}\Big\{\int_\Omega u_\varepsilon \ln u_\varepsilon + 2\int_\Omega |\nabla\sqrt{v_\varepsilon}|^2\Big\} + \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 + \int_\Omega \nabla v_\varepsilon \cdot \nabla w_\varepsilon$$

$$+ \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2 w_\varepsilon}{v_\varepsilon} + \int_\Omega F_\varepsilon(u_\varepsilon)\Delta v_\varepsilon \le \frac{1}{2}\int_{\partial\Omega}\frac{1}{v_\varepsilon}\frac{\partial}{\partial\nu}|\nabla v_\varepsilon|^2.$$

Since the boundary condition $\frac{\partial v_\varepsilon}{\partial\nu} = 0$ and the convexity of $\partial\Omega$ imply that $\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu} \le 0$ on $\partial\Omega$ in ([19], Lemma 3.2), this immediately yields (2.6). $\square$

Corollary 2.1 *There exists $C > 0$ such that for all $\varepsilon \in (0,1)$ the solution of (2.2) satisfies*

$$\int_0^\infty \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \le C, \tag{2.14}$$

$$\int_0^\infty \int_\Omega |\nabla v_\varepsilon|^4 \le C, \tag{2.15}$$

8

$$\int_0^\infty \int_\Omega |\nabla w_\varepsilon|^2 \le C, \tag{2.16}$$

$$\int_\Omega |\nabla v_\varepsilon|^2 \le C \ \text{ for all } t > 0, \tag{2.17}$$

$$\int_0^\infty \int_\Omega |D^2 v_\varepsilon|^2 \le C \tag{2.18}$$

and

$$\int_0^\infty \int_\Omega w_\varepsilon |\nabla v_\varepsilon|^2 \le C. \tag{2.19}$$

Proof. Integrating (2.6) over $s \in (0, t)$ we obtain

$$
\begin{aligned}
&2\int_\Omega |\nabla \sqrt{v_\varepsilon}|^2 + \int_0^t \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \int_0^t \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 + \frac{1}{2} \int_0^t \int_\Omega w_\varepsilon \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \\
&\le -\int_0^t \int_\Omega \nabla v_\varepsilon \cdot \nabla w_\varepsilon - \int_0^t \int_\Omega F_\varepsilon(u_\varepsilon) \Delta v_\varepsilon \\
&\quad \left\{ -\int_\Omega u_\varepsilon \ln u_\varepsilon(\cdot, t) + \int_\Omega u_0 \ln u_0 + 2\int_\Omega |\nabla \sqrt{v_0}|^2 \int_\Omega u_\varepsilon \ln u_\varepsilon(\cdot, t) \right\} \\
&:= I_1 + I_2 + I_3
\end{aligned}
\tag{2.20}
$$

for all $t > 0$ and $\varepsilon \in (0, 1)$. By using Young's inequality, we have

$$I_1 \le \int_0^t \int_\Omega |\nabla v_\varepsilon| |\nabla w_\varepsilon| \le \frac{1}{2} \int_0^t \int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{2} \int_0^t \int_\Omega |\nabla w_\varepsilon|^2$$

and by ([23], Lemma 5.2), we can obtain

$$I_1 \le \frac{1}{2} \int_0^t \int_\Omega |\nabla v_\varepsilon|^2 + C_1 \int_0^t \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + C_1$$

as well as

$$
\begin{aligned}
I_2 = \int_0^t \int_\Omega F_\varepsilon{}'(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla v_\varepsilon &\le \int_0^t \int_\Omega |\nabla u_\varepsilon| |\nabla v_\varepsilon| \\
&\le \frac{1}{4} \int_0^t \int_\Omega |\nabla v_\varepsilon|^4 + \frac{3}{4} \int_0^t \int_\Omega |\nabla u_\varepsilon|^{\frac{4}{3}} \\
&\le \frac{1}{4} \int_0^t \int_\Omega |\nabla v_\varepsilon|^4 + C_2 \int_0^t \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + C_2.
\end{aligned}
$$

Therefore, we have

$$I_1 + I_2 \le C_3 \int_0^t \int_\Omega |\nabla v_\varepsilon|^4 + C_4 \int_0^t \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + C_4$$

and we can obtain $I_1 + I_2 \le C_5$ with some $C_5 > 0$ for all $t > 0$ by ([23] Corollary 5.3). Since $-\xi \ln \xi \le \frac{1}{e}$ for all $\xi > 0$, this shows that

$$
\begin{aligned}
&\frac{1}{2} \sup_{t>0} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + \int_0^\infty \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \int_0^\infty \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 + \frac{1}{2} \int_0^\infty \int_\Omega w_\varepsilon \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \\
&\le C_6 := \int_\Omega u_0 \ln u_0 + 2\int_\Omega |\nabla \sqrt{v_0}|^2 + \frac{|\Omega|}{e} + C_5
\end{aligned}
$$

for all $\varepsilon \in (0, 1)$. Then (2.14), (2.15) and (2.16) are available.
Furthermore, since

$$\frac{1}{2\|v_0\|_{L^\infty(\Omega)}} \int_\Omega |\nabla v_\varepsilon|^2 \le \frac{1}{2} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \le C_6,$$

(2.17) holds and the same reason for (2.19). Moreover, using that $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$ for all $a, b \in \mathbb{R}$, we see that

$$
\begin{aligned}
\int_0^\infty \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 &= \int_0^\infty \int_\Omega v_\varepsilon \sum_{k,l=1}^3 \left| \frac{\partial^2 \ln v_\varepsilon}{\partial x_k \partial x_l} \right|^2 \\
&= \int_0^\infty \int_\Omega v_\varepsilon \sum_{k,l=1}^3 \left| \frac{1}{v_\varepsilon} \frac{\partial^2 v_\varepsilon}{\partial x_k \partial x_l} - \frac{1}{v_\varepsilon^2} \frac{\partial v_\varepsilon}{\partial x_k} \frac{\partial v_\varepsilon}{\partial x_l} \right|^2 \\
&\geq \frac{1}{2} \int_0^\infty \int_\Omega \frac{1}{v_\varepsilon} \cdot \sum_{k,l=1}^3 \left| \frac{\partial^2 v_\varepsilon}{\partial x_k \partial x_l} \right|^2 - \int_0^\infty \int_\Omega \frac{1}{v_\varepsilon^3} \sum_{k,l=1}^3 \left| \frac{\partial v_\varepsilon}{\partial x_k} \cdot \frac{\partial v_\varepsilon}{\partial x_l} \right|^2 \\
&= \frac{1}{2} \int_0^\infty \int_\Omega \frac{|D^2 v_\varepsilon|^2}{v_\varepsilon} - \int_0^\infty \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}.
\end{aligned}
\tag{2.21}
$$

Now by [7, Lemma 3.3] we have

$$
\int_0^\infty \int_\Omega \frac{|\nabla v_\varepsilon|^4}{|v_\varepsilon|^3} \leq (2+\sqrt{3})^2 \int_0^\infty \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2
$$

and

$$
\begin{aligned}
\frac{1}{\|v_0\|_{L^\infty(\Omega)}} \int_0^\infty \int_\Omega |D^2 v_\varepsilon|^2 &\leq \int_0^\infty \int_\Omega \frac{|D^2 v_\varepsilon|^2}{v_\varepsilon} \\
&\leq 2 \int_0^\infty \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 + 2 \int_0^\infty \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \\
&\leq 2C_6 + 2(2+\sqrt{3})^2 \int_0^\infty \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 \\
&\leq \left(2 + 2(2+\sqrt{3})^2 \right) C_6.
\end{aligned}
$$

Therefore, (2.18) is available and the conclusion holds. $\square$

Corollary 2.2 *The weak solution of* (1.5) *obtained from Lemma* 2.1 *has the properties*

$$
\int_0^\infty \int_\Omega \frac{|\nabla u|^2}{u} < \infty,
\tag{2.22}
$$

$$
\int_0^\infty \int_\Omega |\nabla v|^4 < \infty,
\tag{2.23}
$$

$$
\int_0^\infty \int_\Omega |\nabla w|^2 < \infty,
\tag{2.24}
$$

$$
\sup_{t>0} \int_\Omega |\nabla v|^2 < \infty,
\tag{2.25}
$$

$$
\int_0^\infty \int_\Omega |D^2 v|^2 < \infty
\tag{2.26}
$$

and

$$
\int_0^\infty \int_\Omega w |\nabla v|^2 < \infty.
\tag{2.27}
$$

Proof. It can be proved by Corollary 2.1. $\square$

3 A weak stabilization result for u

As a first step on our way to (1.7), let us derive from Corollary 2.1 a provisional statement on convergence of $u_\varepsilon(\cdot, t)$ to $\bar{u}_0$ as $t \to \infty$.

Lemma 3.1 *There exists $C > 0$ such that for all $\varepsilon \in (0,1)$, the solution of (2.2) satisfies*

$$\int_0^\infty \|u_\varepsilon - \overline{u}_0\|_{L^{\frac{3}{2}}(\Omega)}^2 \, dt \le C, \tag{3.1}$$

where $\overline{u}_0 := \frac{1}{|\Omega|} \int_\Omega u_0$. In particular, the weak solution of (1.5) gained from Lemma 2.1 has the property that

$$\int_0^\infty \|u(\cdot, t) - \overline{u}_0\|_{L^{\frac{3}{2}}(\Omega)}^2 \, dt \le C. \tag{3.2}$$

Proof. We apply the Hölder's inequality to (2.14) and recall Lemma 2.2 to obtain

$$
\begin{aligned}
\int_0^\infty \Big(\int_\Omega |\nabla u_\varepsilon| \Big)^2 &= \int_0^\infty \Big(\int_\Omega \frac{|\nabla u_\varepsilon| u_\varepsilon^{\frac{1}{2}}}{u_\varepsilon^{\frac{1}{2}}} \Big)^2 \\
&\le \int_0^\infty \Big(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \Big) \Big(\int_\Omega u_\varepsilon \Big) \\
&\le C_1 \quad \text{for all } \varepsilon \in (0,1)
\end{aligned}
$$

with $C_1 > 0$. By Poincaré's inequality and continuous embedding $W^{1,1}(\Omega) \hookrightarrow L^{\frac{3}{2}}(\Omega)$ in ([20], 8.9), we can yield that

$$\|z\|_{L^{\frac{3}{2}}(\Omega)} \le C_2 \|\nabla z\|_{L^1(\Omega)} \quad \text{for all } z \in W^{1,1}(\Omega) \text{ with } \int_\Omega z = 0.$$

Taking $z = u_\varepsilon(\cdot, t) - \overline{u}_0$ satisfies $\int_\Omega u_\varepsilon(\cdot, t) - \overline{u}_0 = 0$ for all $t > 0$ and $\varepsilon \in (0,1)$, we can obtain that

$$\int_0^\infty \|u_\varepsilon(\cdot, t) - \overline{u}_0\|_{L^{\frac{3}{2}}(\Omega)}^2 \le C_2^2 \int_0^\infty \|\nabla u_\varepsilon(\cdot, t)\|_{L^1(\Omega)}^2 \le C_1 C_2^2.$$

Therefore, (3.1) is established. By Fatou's Lemma, we can see that

$$
\begin{aligned}
\int_0^\infty \|u_\varepsilon(\cdot, t) - \overline{u}_0\|_{L^{\frac{3}{2}}(\Omega)}^2 &= \int_0^\infty \liminf_{\varepsilon \to 0} \|u_\varepsilon(\cdot, t) - \overline{u}_0\|_{L^{\frac{3}{2}}(\Omega)}^2 \\
&\le \liminf_{\varepsilon \to 0} \int_0^\infty \|u_\varepsilon(\cdot, t) - \overline{u}_0\|_{L^{\frac{3}{2}}(\Omega)}^2 \\
&\le C_3,
\end{aligned}
$$

then (3.2) is available. $\square$

Lemma 3.2 *There exists $C > 0$ such that for the solution of (2.2) we have*

$$\int_0^\infty \int_\Omega v_{\varepsilon t}^2 \le C. \tag{3.3}$$

whenever $\varepsilon \in (0,1)$.

Proof. Testing the second PDE in (2.2) by $v_{\varepsilon t}$ we obtain

$$
\begin{aligned}
\int_\Omega v_{\varepsilon t}^2 + \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla v_\varepsilon|^2 &= -\frac{1}{2} \int_\Omega w_\varepsilon (v_\varepsilon^2)_t \\
&= -\frac{1}{2} \frac{d}{dt} \int_\Omega w_\varepsilon v_\varepsilon^2 + \frac{1}{2} \int_\Omega w_{\varepsilon t} v_\varepsilon^2 \quad \text{for all } t > 0.
\end{aligned} \tag{3.4}
$$

It can hold that

$$
\begin{aligned}
\frac{1}{2} \int_0^t \int_\Omega w_{\varepsilon t} v_\varepsilon^2 &\le \frac{1}{2} \int_\Omega w_\varepsilon v_\varepsilon^2(\cdot, t) - \int_0^t \int_\Omega w_\varepsilon v_\varepsilon v_{\varepsilon t} \\
&\le \frac{1}{2} \|v_0\|_{L^\infty(\Omega)} \int_\Omega w_\varepsilon v_\varepsilon(\cdot, t) - \int_0^t \int_\Omega w_\varepsilon v_\varepsilon (\Delta v_\varepsilon - w_\varepsilon v_\varepsilon) \\
&:= I_1 + I_2 \quad \text{for all } t > 0.
\end{aligned} \tag{3.5}
$$

11

By Young's inequality, we obtain that

$$I_2 = -\int_0^t \int_\Omega w_\varepsilon v_\varepsilon \Delta v_\varepsilon + \int_0^t \int_\Omega w_\varepsilon^2 v_\varepsilon^2$$

$$\leq \frac{\sqrt{3}}{2}\|v_0\|_{L^\infty(\Omega)} \int_0^t \int_\Omega |D^2 v_\varepsilon|^2 + \Big(\frac{1}{2} + \|v_0\|_{L^\infty(\Omega)}^2\Big)\int_0^t \int_\Omega w_\varepsilon^2.$$

From Poincaré's inequality, we can yield

$$\|w_\varepsilon - \overline{w}_\varepsilon\|_{L^2(\Omega)} \leq C\|\nabla w_\varepsilon\|_{L^2(\Omega)}$$

and

$$\|w_\varepsilon\|_{L^2(\Omega)}^2 \leq 2\Big(\|\overline{w}_\varepsilon\|_{L^2(\Omega)}^2 + C^2\|\nabla w_\varepsilon\|_{L^2(\Omega)}^2\Big).$$

From $\psi(u_\varepsilon) = w_\varepsilon$ and $\|\overline{w}_\varepsilon\|_{L^2(\Omega)}^2 = \frac{1}{\Omega}\|w_\varepsilon\|_{L^1(\Omega)}^2$, we can obtain that

$$\|w_\varepsilon\|_{L^1(\Omega)} \leq \|w_{\varepsilon 0}\|_{L^1(\Omega)} + \int_\Omega u_\varepsilon \int_0^t e^{\delta(s-t)}ds$$

and

$$\int_0^t \|w_\varepsilon(\cdot,s)\|_{L^1(\Omega)}^2 ds \leq 2\int_0^t \|w_{\varepsilon 0}(\cdot,s)\|_{L^1(\Omega)}^2 ds + 2(|\Omega|\overline{u}_0)^2 \int_0^t \Big(\int_0^t e^{\delta(s-t)}ds\Big)^2 dt.$$

Combining (2.18), (2.16), (3.5) with Lemma 2.4, we can yield that

$$\frac{1}{2}\int_0^t \int_\Omega w_{\varepsilon t} v_\varepsilon^2 \leq C_2 \quad \text{for all } t > 0.$$

Combining (3.5) and integrating (3.4) in time we thus obtain on dropping nonnegative terms that

$$\int_0^t \int_\Omega v_{\varepsilon t}^2 \leq \frac{1}{2}\int_\Omega |\nabla v_0|^2 + \frac{1}{2}\int_\Omega w_0 v_0^2 + \frac{1}{2}\int_0^t \int_\Omega w_{\varepsilon t} v_\varepsilon^2 \leq C$$

holds for all $t > 0$ and each $\varepsilon \in (0,1)$. $\quad\square$

Corollary 3.1 *The function v obtained by Lemma 2.1 satisfies*

$$v \in C^0\Big([0,\infty); L^2(\Omega)\Big). \tag{3.6}$$

Moreover, in (1.7) we may assume without loss of generality that as $\varepsilon = \varepsilon_j \searrow 0$ we have

$$v_\varepsilon \to v \quad in \; L_{loc}^2\Big([0,\infty); L^\infty(\Omega)\Big), \tag{3.7}$$

$$v_\varepsilon(\cdot,t) \to v(\cdot,t) \quad in \; L^\infty(\Omega) \; for \; a.e. \; t > 0 \; and \tag{3.8}$$

$$v_\varepsilon \to v \quad in \; L_{loc}^\infty\Big([0,\infty); L^2(\Omega)\Big). \tag{3.9}$$

Proof. Firstly, we prove $v_\varepsilon \in C^{\frac{1}{2}}([0,T); L^2(\Omega))$. Let $T > 0$. Using the Hölder's inequality and the boundedness of $(v_{\varepsilon t})_{\varepsilon \in (0,1)}$ in $L^2((0,T); L^2(\Omega))$ asserted by Lemma 3.2, for any $t_1 < t_2$, $t_1, t_2 \in [0,T]$ we have

$$\frac{\Big|\|v_\varepsilon(\cdot,t_2)\|_{L^2(\Omega)} - \|v_\varepsilon(\cdot,t_1)\|_{L^2(\Omega)}\Big|}{|t_2 - t_1|^{\frac{1}{2}}} = \frac{\Big|\int_{t_1}^{t_2} \frac{d}{dt}\|v_\varepsilon(\cdot,t)\|_{L^2(\Omega)}dt\Big|}{|t_2 - t_1|^{\frac{1}{2}}}$$

$$= \frac{\Big|\int_{t_1}^{t_2} \frac{\int_\Omega v_\varepsilon(\cdot,t)v_{\varepsilon t}(\cdot,t)dx}{(\int_\Omega v_\varepsilon^2(\cdot,t))dx)^{\frac{1}{2}}}dt\Big|}{|t_2 - t_1|^{\frac{1}{2}}}$$

$$\leq \Big(\int_{t_1}^{t_2} \int_\Omega v_{\varepsilon t}^2\Big)^{\frac{1}{2}}$$

$$\leq C_1.$$

This implies that $(v_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $C^{\frac{1}{2}}([0,T];L^2(\Omega))$. Thus, since $(v_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty((0,T);W^{1,2}(\Omega))$ by (2.17) and Lemma 2.3, and since the embedding $W^{1,2}(\Omega)\hookrightarrow L^2(\Omega)$ is compact, the Arzelà-Ascoli theorem says that $(v_\varepsilon)_{\varepsilon\in(0,1)}$ is relatively compact in $L^\infty((0,T);L^2(\Omega))$, then (3.9) holds which also implies (3.6).

Next, recalling (2.18) we know that $(v_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^2((0,T);W^{2,2}(\Omega))$. Since $W^{2,2}(\Omega)\hookrightarrow\hookrightarrow W^{1,p}(\Omega)$ for each $p<6$, we may combine this with the boundedness of $(v_{\varepsilon t})_{\varepsilon\in(0,1)}$ in $L^2((0,T);L^2(\Omega))$ to obtain from the Aubin-lions Lemma [15] that $(v_\varepsilon)_{\varepsilon\in(0,1)}$ is relatively compact in $L^2((0,T);W^{1,p}(\Omega))$ for any p. In light of the fact that $W^{1,p}(\Omega)\hookrightarrow L^\infty(\Omega)$ for all $p>3$, from this we can deduce (3.7) and (3.8). $\qquad\square$

Throughout the sequel, we fix any sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ such that both (1.7) and the conclusion of Corollary 3.1 hold.

We can now already prove part of the result claimed in Theorem 1.1.

Lemma 3.3 *The second component of the weak solution of* (1.5) *constructed in Lemma* 2.1 *satisfies*

$$v(\cdot,t)\to 0\quad\text{in }L^\infty(\Omega)\text{ as }t\to\infty.$$

Proof. Since v is bounded in $L^\infty(\Omega\times(0,\infty))$ by Lemma 2.4 and $\int_0^\infty\int_\Omega|\nabla v|^4<\infty$ by Corollary 2.1, there exists a sequence of times $t_k\to\infty$ such that $t_k<t_{k+1}\le t_k+1$ for all $k\in\mathbb{N}$ and $(v(\cdot,t_k))_{k\in\mathbb{N}}$ is bounded in $W^{1,4}(\Omega)$ and

$$\int_\Omega v(\cdot,t_k)w(\cdot,t_k)\to 0,\quad k\to\infty. \tag{3.10}$$

Considering in the three-dimensional setting the space $W^{1,4}(\Omega)$ is compactly embedded into $L^\infty(\Omega)$, we may pass to a subsequence, not relabeled for convenience, along which

$$v(\cdot,t_k)\to v_\infty\quad\text{in }L^\infty(\Omega)\text{ with }v_\infty\ge 0.$$

Claim **1** : The limit v_∞ is a constant.
Suppose v_∞ is not a constant, then

$$\lim_{k\to\infty}\frac{1}{|\Omega|}\int_\Omega v(\cdot,t_k)=\frac{1}{|\Omega|}\int_\Omega\lim_{k\to\infty}v(\cdot,t_k)=\frac{1}{|\Omega|}\int_\Omega v_\infty<\|v_\infty\|_{L^\infty(\Omega)},$$

hence there exists $k_0\in\mathbb{N}$ such that

$$\frac{1}{|\Omega|}\int_\Omega v(\cdot,t_{k0})<\|v_\infty\|_{L^\infty(\Omega)}.$$

Set $\overline{v}_\varepsilon(\cdot,t):=e^{t\Delta}v(\cdot,t_{k0})$. It is well known (see for instance [22], Lemma 1.3 (i)) that

$$\left\|\overline{v}(\cdot,t)-\frac{1}{|\Omega|}\int_\Omega v(\cdot,t_{k0})\right\|_{L^\infty(\Omega)}\to 0\quad\text{as }t\to\infty,$$

hence there exist $k_1>k_0$ and $\varepsilon>0$ so small such that

$$\left\|\overline{v}(\cdot,t)-\frac{1}{|\Omega|}\int_\Omega v(\cdot,t_{k0})\right\|_{L^\infty(\Omega)}<\varepsilon$$

and

$$\overline{v}(\cdot,t)\le\|v_\infty\|_{L^\infty(\Omega)}-\varepsilon$$

in $(t_{k1}-t_{k0},\infty)$. Note that $t_{k1}\ge t_{k0}$ as $(t_k)_{k\in\mathbb{N}}$ is increasing. Moreover, let $\underline{v}(\cdot,t):=v_\varepsilon(\cdot,t+t_{k0})$, $t\ge 0$ be the subsolution of the following heat conduction equation

$$\begin{cases}v_t=\Delta v, & x\in\Omega,\,t>0,\\ \dfrac{\partial v}{\partial\nu}=0, & x\in\partial\Omega,\,t>0,\\ v(\cdot,0)=v(\cdot,t_{k0}), & x\in\Omega.\end{cases}$$

From the comparison principle of parabolic equation, we have

$$v(\cdot,t+t_{k0})=\underline{v}(\cdot,t)\le\overline{v}(\cdot,t)\quad\text{for all }t\ge 0$$

13

and

$$\|v_\infty\|_{L^\infty(\Omega)} = \lim_{k\to\infty} \|v(\cdot,t_k)\|_{L^\infty(\Omega)} \leq \lim_{k\to\infty} \|\overline{v}(\cdot,t_k)\|_{L^\infty(\Omega)} \leq \|v_\infty\|_{L^\infty(\Omega)} - \varepsilon.$$

This is a contradiction, hence v_∞ is a constant.

Claim **2**: $v_\infty \equiv 0$.

Suppose that $v_\infty \neq 0$ from Claim **1**, we can obtain $v_\infty \equiv C$ with $C > 0$, thus we may choose $k_2 \in \mathbb{N}$ such that
$v_\varepsilon(\cdot,t_k) \geq \frac{C}{2}$ for all $k \geq k_2$. By (3.10) implies that

$$0 \leq \lim_{k_2 \leq k\to\infty} \int_\Omega w(\cdot,t_k) \leq \lim_{k\to\infty} \frac{2}{C} \int_\Omega v(\cdot,t_k)w(\cdot,t_k) = 0,$$

hence

$$\lim_{k\to\infty} \int_\Omega w(\cdot,t_k) = 0. \tag{3.11}$$

Applying constant variation to the third equation of (2.2), we can yield that

$$w_\varepsilon(\cdot,t) = e^{-\delta t}w_0 + \int_0^t e^{-\delta(t-s)}F_\varepsilon(u_\varepsilon)(\cdot,s)ds.$$

However, by Lemma 2.2 we have for $k \in \mathbb{N}$

$$\begin{aligned}
\int_\Omega w(\cdot,t_k) &= e^{-\delta t_k}\int_\Omega w_0 + \int_0^{t_k}\int_\Omega e^{-\delta(t_k-s)}u(\cdot,s) \\
&\geq \int_0^{t_k} e^{-\delta(t_k-s)}ds \int_\Omega u(\cdot,s) \\
&= \frac{|\Omega|\overline{u}_0}{\delta}(1 - e^{-\delta t_k}) \\
&\geq \frac{|\Omega|\overline{u}_0}{\delta}(1 - e^{-\delta t_1}) \\
&> 0.
\end{aligned} \tag{3.12}$$

Let $k \to \infty$ at both ends of (3.12), it is contradictory to formula (3.11), hence $v_\infty \equiv 0$.

Claim **3**: the statement holds.

From above we know that $\{v(\cdot,t_k)\}$ has a subsequence which convergence to 0 and combining $\|v(\cdot,t)\|_{L^\infty(\Omega)}$ is non-increasing by Lemma 2.3, then the conclusion is valid. □

4   Eventual boundedness and regularity

Combining Lemma 3.3 with (3.8) and Lemma 2.3, we obtain that not only the limit v but also its approximations become conveniently small.

Lemma 4.1 *For any $\delta > 0$ there exist $t_0(\delta) > 0$ and $\varepsilon_0(\delta) \in (0,1)$ such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ fulfilling $\varepsilon < \varepsilon_0(\delta)$, the solution of (2.2) satisfies*

$$v_\varepsilon \leq \delta \quad \text{in } \Omega \times \big(t_0(\delta),\infty\big).$$

Proof. Fixed $\delta > 0$, from Lemma 3.3 we obtain $\tilde{t}_0 > 0$ such that the limit v defined by Lemma 2.1 satisfies $v \leq \frac{\delta}{2}$ in $\Omega \times (\tilde{t}_0,\infty)$. Now (3.8) ensures that we can find some $t_0 \in (\tilde{t}_0,\tilde{t}_0 + 1)$ such that $v_\varepsilon(\cdot,t_0) \to v(\cdot,t_0)$ in $L^\infty(\Omega)$ as $\varepsilon = \varepsilon_j \searrow 0$, so that in particular $v_\varepsilon(\cdot,t_0) \leq \delta$ in Ω whenever $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ is sufficiently small. Since from Lemma 2.3 we conclude that $t \longmapsto \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$ does not increase, we conclude that actually $v_\varepsilon \leq \delta$ in $\Omega \times (t_0,\infty)$ for any such ε, as desired. □

By using the sufficient small property of v_ε and the similar method in [17], the boundedness of u_ε in $L^p(\Omega)$ is obtained for any large p.

Lemma 4.2 *Let $p \in (1, \infty)$. Then there exist $t_1(p) > 0$, $\varepsilon_1(p) \in (0,1)$ and $C(p) > 0$ such that the solution of* (2.2) *has the property*

$$\int_\Omega u_\varepsilon^p(\cdot, t) \leq C(p) \quad \text{for all } t > t_1(p) \tag{4.1}$$

whenever $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$ and $\varepsilon < \varepsilon_1(p)$.

Proof. Given $p \in (1, \infty)$, we can fix $q > 0$ with $q < p - 1$, then let

$$\rho(\delta) := p - 1 - \frac{p}{4} \cdot \frac{4q^2 + (p-1)^2 \cdot (2\delta)}{q(q+1) - pq \cdot 2\delta}, \quad \delta \in (0, \frac{q+1}{2p}),$$

satisfying

$$\rho(0) = p - 1 - \frac{p}{4} \cdot \frac{4q^2}{q(q+1)} = \frac{p - q - 1}{q + 1} > 0.$$

Since $\rho(\delta)$ is continuous in $(0, \frac{q+1}{2p})$, there exists δ so small in $(0, \frac{q+1}{2p})$ such that

$$C_1 := \rho(\delta) > 0. \tag{4.2}$$

We now let

$$\varphi(s) := (2\delta - s)^{-q}, \quad s \in [0, 2\delta)$$

and take $t_0(\delta)$ and $\varepsilon_0(\delta)$ as provided by Lemma 4.1. For the first equation in (2.2) multiplying $pu_\varepsilon^{p-1}\varphi(v_\varepsilon)$ at both ends and then integrating by parts, we can obtain

$$
\begin{aligned}
\frac{d}{dt} \int_\Omega u_\varepsilon^p \varphi(v_\varepsilon) &= p \int_\Omega u_\varepsilon^{p-1} \varphi(v_\varepsilon) \cdot \Big[\Delta u_\varepsilon - \nabla \cdot (u_\varepsilon F_\varepsilon'(u_\varepsilon) \nabla v_\varepsilon) \Big] + \int_\Omega u_\varepsilon^p \varphi'(v_\varepsilon)(\Delta v_\varepsilon - v_\varepsilon w_\varepsilon) \\
&= -p(p-1) \int_\Omega u_\varepsilon^{p-2} \varphi(v_\varepsilon) |\nabla u_\varepsilon|^2 \\
&\quad - \int_\Omega u_\varepsilon^p \Big[\varphi''(v_\varepsilon) - pF_\varepsilon'(u_\varepsilon) \varphi'(v_\varepsilon) \Big] |\nabla v_\varepsilon|^2 \\
&\quad + p \int_\Omega u_\varepsilon^{p-1} \Big[-2\varphi'(v_\varepsilon) + (p-1)F_\varepsilon'(u_\varepsilon)\varphi(v_\varepsilon) \Big] \nabla u_\varepsilon \cdot \nabla v_\varepsilon \\
&\quad - \int_\Omega u_\varepsilon^p w_\varepsilon v_\varepsilon \varphi'(v_\varepsilon) \\
&:= J_1 + J_2 + J_3 + J_4 \quad \text{for all } t > t_0(\delta).
\end{aligned}
\tag{4.3}
$$

Here we note that $J_4 \leq 0$, since $\varphi'(v_\varepsilon) = q(2\delta - v_\varepsilon)^{-q-1} > 0$.
Using that $0 \leq F_\varepsilon'(u_\varepsilon) = \frac{1}{1 + \varepsilon u_\varepsilon} \leq 1$ and $\delta < \frac{q+1}{2p}$, we can yield that

$$J_2 \leq - \int_\Omega u_\varepsilon^p \Big[\varphi''(v_\varepsilon) - p\varphi'(v_\varepsilon) \Big] |\nabla v_\varepsilon|^2$$

and

$$
\begin{aligned}
\varphi''(v_\varepsilon) - p\varphi'(v_\varepsilon) &= (2\delta - v_\varepsilon)^{-q-2}[q(q+1) - pq(2\delta - v_\varepsilon)] \\
&\geq (2\delta - v_\varepsilon)^{-q-2} q(q + 1 - 2p\delta) \\
&> 0 \quad \text{in } \Omega \times \big(t_0(\delta), \infty \big).
\end{aligned}
$$

We may invoke Young's inequality to see that

$$
\begin{aligned}
J_3 &= p \int_\Omega u_\varepsilon^{p-1}[-2\varphi'(v_\varepsilon) + (p-1)F_\varepsilon'(u_\varepsilon)\varphi(v_\varepsilon)] \nabla u_\varepsilon \cdot \nabla v_\varepsilon \cdot \frac{[\varphi''(v_\varepsilon) - pF_\varepsilon'(u_\varepsilon)\varphi'(v_\varepsilon)]^{\frac{1}{2}}}{[\varphi''(v_\varepsilon) - pF_\varepsilon'(u_\varepsilon)\varphi'(v_\varepsilon)]^{\frac{1}{2}}} \\
&\leq \int_\Omega u_\varepsilon^p [\varphi''(v_\varepsilon) - pF_\varepsilon'(u_\varepsilon)\varphi'(v_\varepsilon)] |\nabla v_\varepsilon|^2 + \frac{p^2}{4} \int_\Omega u_\varepsilon^{p-2} \frac{[-2\varphi'(v_\varepsilon) + (p-1)F_\varepsilon'(u_\varepsilon)\varphi(v_\varepsilon)]^2}{\varphi''(v_\varepsilon) - pF_\varepsilon'(u_\varepsilon)\varphi'(v_\varepsilon)} |\nabla u_\varepsilon|^2 \\
&= -J_2 + \frac{p^2}{4} \int_\Omega u_\varepsilon^{p-2} \frac{[-2\varphi'(v_\varepsilon) + (p-1)F_\varepsilon'(u_\varepsilon)\varphi(v_\varepsilon)]^2}{\varphi''(v_\varepsilon) - pF_\varepsilon'(u_\varepsilon)\varphi'(v_\varepsilon)} |\nabla u_\varepsilon|^2.
\end{aligned}
$$

15

To estimate this, we use that $0 \leq F'_\varepsilon \leq 1$ to derive that

$$K(x,t,s) := p(p-1)\varphi(s) - \frac{p^2}{4} \cdot \frac{[-2\varphi'(s) + (p-1)F'_\varepsilon(u_\varepsilon) \cdot \varphi(s)]^2}{\varphi''(s) - p\varphi'(s)},$$

$$(x,t) \in \Omega \times \big(t_0(\delta), \infty\big), s \in [0, 2\delta),$$

satisfies

$$
\begin{aligned}
K(x,t,s) &= p(p-1)\varphi(s) - \frac{p^2}{4} \cdot \frac{4\varphi'^2(s) - 4(p-1)F'_\varepsilon(u_\varepsilon)\varphi(s)\varphi'(s) + (p-1)^2 F'^2_\varepsilon(u_\varepsilon)\varphi^2(s)}{\varphi''(s) - p\varphi'(s)} \\
&\geq p(p-1)\varphi(s) - \frac{p^2}{4} \cdot \frac{4\varphi'^2(s) + (p-1)^2\varphi^2(s)}{\varphi''(s) - p\varphi'(s)} \\
&= p(p-1)\varphi(s) - \frac{p^2}{4} \cdot \frac{4q^2(2\delta-s)^{-2q-2} + (p-1)^2(2\delta-s)^{-2q}}{q(q+1)(2\delta-s)^{-q-2} - pq(2\delta-s)^{-q-1}} \\
&\geq p(2\delta-s)^{-q}\Big\{p-1 - \frac{p}{4} \cdot \frac{4q^2 + (p-1)^2(2\delta-s)}{q(q+1) - pq(2\delta-s)}\Big\} \\
&\geq p(2\delta-s)^{-q}p(\delta) \\
&\geq C_2 := C_1 p(2\delta)^{-q} \quad \text{for all } (x,t,s) \in \Omega \times (t_0(\delta), \infty) \times [0, 2\delta)
\end{aligned}
$$

by definition of ρ. Recalling (4.2), we thus obtain that

$$K(x,t,s) \geq C_2 \quad \text{for all } (x,t,s) \, in \, \Omega \times \big(t_0(\delta), \infty\big) \times [0, 2\delta).$$

In view of $v_\varepsilon \leq \delta$ in $\Omega \times (t_0(\delta), \infty)$ for $\varepsilon < \varepsilon_0(\delta)$ by Lemma 4.1, this entails that

$$
\begin{aligned}
J_3 &\leq -J_2 + \int_\Omega \Big[p(p-1)\varphi(v_\varepsilon) - K(x,t,v_\varepsilon)\Big]|\nabla u_\varepsilon|^2 u_\varepsilon^{p-2} \\
&\leq -J_2 - J_1 - C_2 \int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 \quad \text{for all } t > t_0(\delta).
\end{aligned}
$$

Putting J_1, J_2, J_3 and J_4 into (4.3), we can yield that

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p \varphi(v_\varepsilon) \leq -C_2 \int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 = -\frac{4C_2}{p^2}\int_\Omega |\nabla u_\varepsilon^{\frac{p}{2}}|^2 \quad \text{for all } t > t_0(\delta). \tag{4.4}$$

Since $\varphi(v_\varepsilon) \leq \delta^{-q}$, $\|u_\varepsilon\|_{L^1(\Omega)} \leq C$ for all $t > t_0(\delta)$ and the Gagliardo-Nirenberg inequality in [2], there exist $C_3, C_4 > 0$ such that

$$
\begin{aligned}
\int_\Omega u_\varepsilon^p \varphi(v_\varepsilon) &\leq \delta^{-q}\|u_\varepsilon^{\frac{p}{2}}\|^2_{L^2(\Omega)} \\
&\leq C_3\|\nabla u_\varepsilon^{\frac{p}{2}}\|^{2a}_{L^2(\Omega)} \cdot \|u_\varepsilon^{\frac{p}{2}}\|^{2(1-a)}_{L^{\frac{2}{p}}(\Omega)} + C_3\|u_\varepsilon^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)} \\
&\leq C_4(\|\nabla u_\varepsilon^{\frac{p}{2}}\|^{2a}_{L^2(\Omega)} + 1) \quad \text{for all } t > t_0(\delta).
\end{aligned}
$$

Due to $\frac{3}{2} = a(\frac{3}{2} - 1) + \frac{3p}{2}(1-a)$, we can obtain $a = \frac{3(p-1)}{3p-1} \in (0,1)$. Therefore (4.4) shows that $y_\varepsilon(t) := \int_\Omega u_\varepsilon^p \varphi(v_\varepsilon)(\cdot, t)$, $t > t_0(\delta)$, satisfies

$$y'_\varepsilon(t) \leq -C_5\big(y_\varepsilon(t) - 1\big)^{\frac{1}{a}}_+ \quad \text{for all } t > t_0(\delta)$$

for some $C_5 > 0$, and hence an integration yields

$$y_\varepsilon(t) \leq 1 + \Big\{\frac{C_5(1-a)}{a}(t - t_0(\delta))\Big\}^{-\frac{a}{1-a}} \quad \text{for all } t > t_0(\delta).$$

Since $\varphi(v_\varepsilon) \geq (2\delta)^{-q}$ for all $v_s \in [0, 2\delta)$, we thus conclude that

$$\int_\Omega u_\varepsilon^p(\cdot, t) \leq (2\delta)^q \int_\Omega u_\varepsilon^p \varphi(v_\varepsilon)(\cdot, t) = (2\delta)^q y_\varepsilon(t) \leq (2\delta)^q\Big\{1 + \Big(\frac{C_5(1-a)}{a}\Big)^{-\frac{a}{1-a}}\Big\} \quad \text{for all } t \geq t_0(\delta) + 1,$$

provided that $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$ is sufficiently small. This proves (4.1) upon the choice $t_1(p) := t_0(p) + 1$. $\quad\square$

Lemma 4.3 *There exist $T > 0$ and a subsequence $(\varepsilon_{j_i})_{i \in \mathbb{N}}$ of $(\varepsilon_j)_{j \in \mathbb{N}}$ such that for any $\varepsilon \in (\varepsilon_{j_i})_{i \in \mathbb{N}}$ we have*

$$\|u_\varepsilon(\cdot, t)\|_{C^2(\overline{\Omega})} \leq C \quad \text{for all } t \geq T, \tag{4.5}$$

$$u_\varepsilon \to u \quad \text{and} \quad v_\varepsilon \to v \quad \text{in } C_{loc}^{2,1}\big(\overline{\Omega} \times [0, \infty)\big) \text{ as } \varepsilon = \varepsilon_{j_i} \searrow 0 \tag{4.6}$$

and such that

$$w_\varepsilon \in C^{0,1}\big(\overline{\Omega} \times [T, \infty)\big).$$

Proof. We fix any $p > 6$ and then obtain from Lemma 4.2 some $t_1 = t_1(p) > 0$, $C_1 = C_1(p) > 0$ and $\varepsilon_1(p) \in (0,1)$ such that

$$\|u_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq C_1 \quad \text{for all } t > t_1 \tag{4.7}$$

and any $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$ fulfilling $\varepsilon < \varepsilon_1(p)$. Using the standard regularity estimate in [6] and the variation-of-constants formula for w_ε,

$$w_\varepsilon(\cdot, t) = e^{-\delta t} w_0 + \int_0^t e^{-\delta(t-s)} F_\varepsilon\big(u_\varepsilon(\cdot, s)\big) ds, \quad t \geq 0,$$

we can yield $\|w_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq C_2$, for all $t > t_1$ and $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$. Applying ∇ to both sides of the variation-of-constants formula for v_ε,

$$v_\varepsilon(\cdot, t) = e^{(t-t_1)(\Delta-1)} v_\varepsilon(\cdot, t_1) - \int_{t_1}^t e^{(t-s)(\Delta-1)}(w_\varepsilon - 1) v_\varepsilon(\cdot, s) ds \quad \text{for all } t \geq t_1,$$

recalling that $|F_\varepsilon(u_\varepsilon)| \leq u_\varepsilon$, $v_\varepsilon \leq \|v_0\|_{L^\infty(\Omega)}$ and the estimate of thermal semigroups we therefore obtain $C_3 > 0$ and $C_4 > 0$ such that

$$\|\nabla v_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq \|\nabla e^{(t-t_1)(\Delta-1)} v_\varepsilon(\cdot, t_1)\|_{L^p(\Omega)} + \int_{t_1}^t \|\nabla e^{(t-s)(\Delta-1)}(w_\varepsilon - 1) v_\varepsilon(\cdot, s)\|_{L^p(\Omega)} ds$$

$$\leq C_3 \|v_\varepsilon(\cdot, t_1)\|_{L^\infty(\Omega)} + C_3 \int_{t_1}^t (t-s)^{-\frac{1}{2}} e^{-(t-s)} (\|w_\varepsilon\|_{L^p(\Omega)} + 1) ds$$

$$\leq C_4 \quad \text{for all } t \geq t_2 := t_1 + 1.$$

By Hölder's inequality, this implies that

$$\|u_\varepsilon \nabla v_\varepsilon(\cdot, t)\|_{L^{\frac{p}{2}}(\Omega)} \leq \|u_\varepsilon\|_{L^p(\Omega)} \|\nabla v_\varepsilon\|_{L^p(\Omega)} \leq C_1 C_4 \quad \text{for all } t \geq t_2,$$

the variation-of-constants formula for u_ε in the form

$$u_\varepsilon(\cdot, t) = e^{(t-t_2)\Delta} u_\varepsilon(\cdot, t_2) - \int_{t_2}^t e^{(t-s)\Delta} \nabla \cdot \big(u_\varepsilon F_\varepsilon'(u_\varepsilon) \nabla v_\varepsilon\big)(\cdot, s) ds \quad \text{for all } t \geq t_2, \tag{4.8}$$

along with (4.7) allows us to estimate

$$\|B^\theta u_\varepsilon(\cdot, t)\|_{L^q(\Omega)} \leq C_5 \|u_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq C_5 C_1 := C_6 \quad \text{for all } t \geq t_3 := t_2 + 1 \tag{4.9}$$

and moreover

$$\|B^\theta u_\varepsilon(\cdot, t) - B^\theta u_\varepsilon(\cdot, s)\|_{L^q(\Omega)} \leq C_6 |t-s|^\eta \quad \text{for all } t, s \geq t_3 \text{ such that } |t-s| \leq 1 \tag{4.10}$$

with some $\eta \in (0,1)$, $\theta \in (0,1)$ and $q > 1$ large enough such that $2\theta - \frac{3}{q} > 0$, where B^θ denotes the fractional power of the realization $-\Delta + 1$ in $L^q(\Omega)$ under homogeneous Neumann boundary conditions.

Along with the fact that the domain of definition of B^θ satisfies $D(B^\theta) \hookrightarrow C^\beta(\overline{\Omega})$ for all $\beta \in (0, 2\theta - \frac{3}{q})$, the estimates (4.9) and (4.10) show that $(u_\varepsilon)_{\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}}$ is bounded in both $L^\infty\big(\Omega \times (t_3, \infty)\big)$ and in $C_{loc}^{\beta, \frac{\beta}{2}}\big(\overline{\Omega} \times [t_3, \infty)\big)$ for some $\beta \in (0,1)$ in [5]. Using standard parabolic Schauder estimates applied to the second equation in (2.2) yield boundedness of $(v_\varepsilon)_{\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}}$ in both $L^\infty\big((t_4, \infty); C^{2+\beta}(\overline{\Omega})\big)$ and in $C_{loc}^{2+\beta, 1+\frac{\beta}{2}}\big(\overline{\Omega} \times [t_4, \infty)\big)$ for $t_4 := t_3 + 1$ in [10]. By a similar argument, entails boundedness of $(u_\varepsilon)_{\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}}$ in both $L^\infty\big((t_5, \infty); C^{2+\beta'}(\overline{\Omega})\big)$ and in $C_{loc}^{2+\beta', 1+\frac{\beta'}{2}}\big(\overline{\Omega} \times [t_5, \infty)\big)$ for some $\beta' \in (0,1)$ and $t_5 := t_4 + 1$. The statement about u_ε and v_ε can be completed by applying the Arzelà-Ascoli theorem. Finally, the asserted regularity of w_ε follows from it's constant variation and the third equation of (2.2). $\quad\square$

5 Long time behavior of u

We now aim at improving the rather weak stabilization result for u warranted by Lemma 3.1. As a preparation, we assert that u_t decays at least in some weak sense in the large time limit.

Lemma 5.1 *There exists $C > 0$ such that for all $\varepsilon \in (0, 1)$ the solution of* (2.2) *satisfies*

$$\int_0^\infty \|u_{\varepsilon t}\|^2_{\left(W^{3,2}(\Omega)\right)^*} dt \le C. \tag{5.1}$$

Consequently,

$$\int_0^\infty \|u_t(\cdot, t)\|^2_{\left(W^{3,2}(\Omega)\right)^*} dt \le C. \tag{5.2}$$

Proof. We fix $\psi \in W^{3,2}(\Omega)$ and test the first equation in (2.2) against ψ to obtain

$$
\begin{aligned}
\int_\Omega u_{\varepsilon t} \psi &= \int_\Omega \Delta u_\varepsilon \psi - \int_\Omega \nabla \cdot \left(u_\varepsilon F'_\varepsilon(u_\varepsilon) \nabla v_\varepsilon\right) \psi \\
&= -\int_\Omega \nabla u_\varepsilon \cdot \nabla \psi + \int_\Omega u_\varepsilon F'_\varepsilon(u_\varepsilon) \nabla v_\varepsilon \cdot \nabla \psi \\
&:= I_1 + I_2 \quad \text{for all } t > 0.
\end{aligned}
\tag{5.3}
$$

Here by Hölder's inequality, we can yield that

$$
\begin{aligned}
|I_1| &\le \left(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon}\right)^{\frac{1}{2}} \left(\int_\Omega u_\varepsilon |\nabla \psi|^2\right)^{\frac{1}{2}} \\
&\le \|\nabla \psi\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{2}} \overline{u}_0^{\frac{1}{2}} \left(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon}\right)^{\frac{1}{2}} \\
&:= C_1 \left(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon}\right)^{\frac{1}{2}}.
\end{aligned}
\tag{5.4}
$$

Similarly,

$$
\begin{aligned}
|I_2| &\le \|\nabla \psi\|_{L^\infty(\Omega)} \int_\Omega u_\varepsilon |\nabla v_\varepsilon| \\
&\le C_2 \left\{ \int_\Omega (u_\varepsilon - \overline{u}_0) |\nabla v_\varepsilon| + \overline{u}_0 |\nabla v_\varepsilon| \right\} \\
&\le C_2 \left(\int_\Omega |u_\varepsilon - \overline{u}_0|^{\frac{3}{2}}\right)^{\frac{2}{3}} \left(\int_\Omega |\nabla v_\varepsilon|^3\right)^{\frac{1}{3}} + C_2 \overline{u}_0 \int_\Omega |\nabla v_\varepsilon|.
\end{aligned}
\tag{5.5}
$$

Since $W^{3,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ and hence $\|\nabla z\|_{L^\infty(\Omega)} \le C_3 \|z\|_{W^{3,2}(\Omega)}$ for all $z \in W^{3,2}(\Omega)$ and some $C_1 > 0$, $C_2 > 0$, (5.3), (5.4) and (5.5) show that

$$
\begin{aligned}
\|u_{\varepsilon t}(\cdot, t)\|^2_{\left(W^{3,2}(\Omega)\right)^*} &= \sup_{\psi \in W^{3,2}(\Omega),\, \|\psi\|_{W^{3,2}(\Omega)} \le 1} \left|\int_\Omega u_{\varepsilon t} \psi\right|^2 \\
&\le 2C_1^2 \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + 4C_2^2 \left(\int_\Omega |u_\varepsilon - \overline{u}_0|^{\frac{3}{2}}\right)^{\frac{4}{3}} \left(\int_\Omega |\nabla v_\varepsilon|^3\right)^{\frac{2}{3}} + 4C_2^2 \overline{u}_0^2 \left(\int_\Omega |\nabla v_\varepsilon|\right)^2 \\
&\le 2C_1^2 \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + 2C_2^2 \left(\int_\Omega |u_\varepsilon - \overline{u}_0|^{\frac{3}{2}}\right)^{\frac{4}{3}} \left(\int_\Omega |\nabla v_\varepsilon|^3\right)^{\frac{4}{3}} \\
&\quad + \frac{1}{2} \left(\int_\Omega |u_\varepsilon - \overline{u}_0|^{\frac{3}{2}}\right)^{\frac{4}{3}} + 4C_2^2 \overline{u}_0^2 \left(\int_\Omega |\nabla v_\varepsilon|\right)^2 \\
&\le 2C_1^2 \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + 2C_2^2 |\Omega|^{\frac{1}{3}} \left(\int_\Omega |u_\varepsilon - \overline{u}_0|^{\frac{3}{2}}\right)^{\frac{4}{3}} \int_\Omega |\nabla v_\varepsilon|^4 \\
&\quad + \frac{1}{2} \left(\int_\Omega |u_\varepsilon - \overline{u}_0|^{\frac{3}{2}}\right)^{\frac{4}{3}} + 4C_2^2 \overline{u}_0^2 |\Omega|^{\frac{3}{2}} \left(\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{1}{2}}.
\end{aligned}
$$

18

Hence, in view of (2.14), (2.15) and (3.1) yield (5.1). From lower semicontinuity of the norm in the Hilbert space $L^2\big((0,\infty);W^{3,2}(\Omega)^*\big)$ with respect to weak convergence, we can obtain (5.2). $\square$

Lemma 5.2 *The weak solution of* (1.5) *obtained from Lemma* 2.1 *satisfies*

$$u(\cdot,t) \to \overline{u}_0 \quad in \ L^\infty(\Omega) \ as \ t \to \infty, \tag{5.6}$$

where $\overline{u}_0 = \frac{1}{|\Omega|} \int_\Omega u_0.$

Proof. Let us suppose on the contrary that (5.6) be false. Then we can extract a sequence of times $t_k \to \infty$ such that

$$\inf_{k\in\mathbb{N}} \|u(\cdot,t_k) - \overline{u}_0\|_{L^\infty(\Omega)} > 0, \tag{5.7}$$

where we may assume without loss of generality that $t_k \le t_{k+1} < t_k+1$ and $t_k > T$ for all $k \in \mathbb{N}$ with T as provided by Lemma 4.3. Since then $\{u(\cdot,t_k)\}_{k\in\mathbb{N}}$ is relatively compact in $L^\infty(\Omega)$ according to (4.5) and the Arzelà-Ascoli theorem, we can extract a subsequence denoted by $(t_k)_{k\in\mathbb{N}}$ such that

$$u(\cdot,t_k) \to u_\infty \quad in \ L^\infty(\Omega) \ as \ k \to \infty \tag{5.8}$$

is valid with some non-negative $u_\infty \in L^\infty(\Omega)$. By the Cauchy-Schwarz's inequality, we obtain

$$
\begin{aligned}
\int_{t_k}^{t_{k+1}} \|u_\varepsilon(\cdot,t) - u_\varepsilon(\cdot,t_k)\|_{(W^{3,2}(\Omega))^*}^2 dt &= \int_{t_k}^{t_{k+1}} \Big\| \int_{t_k}^t u_{\varepsilon t}(\cdot,s)ds \Big\|_{(W^{3,2}(\Omega))^*}^2 dt \\
&\le \int_{t_k}^{t_{k+1}} \Big(\int_{t_k}^t \|u_{\varepsilon t}(\cdot,s)\|_{(W^{3,2}(\Omega))^*} ds \Big)^2 dt \\
&\le \int_{t_k}^{t_{k+1}} \Big(\int_{t_k}^t \|u_{\varepsilon t}(\cdot,s)\|_{(W^{3,2}(\Omega))^*}^2 ds \cdot (t - t_k) dt \\
&\le \int_{t_k}^\infty \|u_{\varepsilon t}(\cdot,s)\|_{(W^{3,2}(\Omega))^*}^2 ds \quad \text{for all } \varepsilon \in (0,1)
\end{aligned}
$$

and according to Lemma 5.1, we can yield

$$
\int_{t_k}^{t_{k+1}} \|u(\cdot,t) - u(\cdot,t_k)\|_{(W^{3,2}(\Omega))^*}^2 dt \le \int_{t_k}^\infty \|u_t(\cdot,s)\|_{(W^{3,2}(\Omega))^*}^2 ds
$$
$$\to 0 \quad \text{as } k \to \infty.$$

Due to the fact that $L^\infty(\Omega) \hookrightarrow (W^{3,2}(\Omega))^*$ and (5.8), this ensures that

$$
0 \le \|u(\cdot,t_k) - u_\infty\|_{(W^{3,2}(\Omega))^*} \le C_1 \|u(\cdot,t_k) - u_\infty\|_{L^\infty(\Omega)}
$$
$$\to 0 \quad \text{as } k \to \infty.$$

From the Integral mean value theorem, it yields that

$$
\begin{aligned}
\int_{t_k}^{t_{k+1}} \|u(\cdot,t) - u_\infty\|_{(W^{3,2}(\Omega))^*}^2 dt &= \|u(\cdot,t_k') - u_\infty\|_{(W^{3,2}(\Omega))^*}^2 (t_{k+1} - t_k) \\
&\le \|u(\cdot,t_k') - u_\infty\|_{(W^{3,2}(\Omega))^*}^2 \to 0 \quad \text{as } k \to \infty.
\end{aligned} \tag{5.9}
$$

On the other hand, since also $L^{\frac{3}{2}}(\Omega) \hookrightarrow (W^{3,2}(\Omega))^*$ and Lemma 3.1 asserts that

$$
\int_0^\infty \|u(\cdot,t) - \overline{u}_0\|_{(W^{3,2}(\Omega))^*}^2 dt \le C_2 \int_0^\infty \|u(\cdot,t) - \overline{u}_0\|_{L^{\frac{3}{2}}(\Omega)}^2 dt < \infty
$$

and thus in particular

$$
\int_{t_k}^{t_{k+1}} \|u(\cdot,t) - \overline{u}_0\|_{(w^{3,2}(\Omega))^*}^2 dt \to 0 \quad \text{as } k \to \infty.
$$

Clearly, (5.9) is possible only if $u_\infty = \overline{u}_0$, which contradicts (5.6) and (5.7). $\square$

19

Lemma 5.3 *The weak solution of* (1.5) *obtained from Lemma* 2.1 *satisfies*

$$w(\cdot, t) \to \frac{\overline{u}_0}{\delta} := \overline{w}_0 \quad in \ L^\infty(\Omega) \ for \ t \to \infty. \tag{5.10}$$

Proof. Let $\varepsilon > 0$. According to Lemma 5.2 we may choose $t_1 > 0$ such that

$$\|u(\cdot, t) - \overline{u}_0\|_{L^\infty(\Omega)} < \frac{\varepsilon\delta}{3} \quad \text{for all } t > t_1.$$

Moreover, there exist $t_2, t_3 > 0$ such that

$$e^{-\delta t}\|w_0\|_{L^\infty(\Omega)} < \frac{\varepsilon}{3} \quad \text{for all } t > t_2$$

and

$$\frac{\|u_\varepsilon - \overline{u}_0\|_{L^\infty(\Omega \times [0, t_1])}}{\delta} e^{-\delta(t-t_1)} < \frac{\varepsilon}{3} \quad \text{for all } t > t_3.$$

Let

$$\tilde{w} : \overline{\Omega} \times [0, \infty) \to \mathbb{R}, \quad (x, t) \mapsto w_\varepsilon(x, t) - \frac{\overline{u}_0}{\delta}(1 - e^{-\delta t}),$$

then we have for $t > t_0 := \max\{t_1, t_2, t_3\}$ by the representation formula of w

$$\|\tilde{w}\|_{L^\infty(\Omega)} = \|w_\varepsilon(x, t) - \int_0^t e^{-\delta(t-s)}\overline{u}_0 ds\|_{L^\infty(\Omega)}$$

$$\leq e^{-\delta t}\|w_0\|_{L^\infty(\Omega)} + \int_0^t e^{-\delta(t-s)}\|F_\varepsilon(u_\varepsilon(\cdot, s)) - \overline{u}_0\|_{L^\infty(\Omega)} ds$$

$$< \frac{\varepsilon}{3} + \int_0^{t_1} e^{-\delta(t-s)}\|F_\varepsilon(u_\varepsilon(\cdot, s)) - \overline{u}_0\|_{L^\infty(\Omega)} ds + \int_{t_1}^t e^{-\delta(t-s)}\|u_\varepsilon(\cdot, s) - \overline{u}_0\|_{L^\infty(\Omega)} ds$$

$$< \frac{\varepsilon}{3} + \|u_\varepsilon(\cdot, s) - \overline{u}_0\|_{L^\infty(\Omega \times [0, t_1])} \cdot \frac{e^{-\delta(t-t_1)}}{\delta} + \frac{\varepsilon\delta}{3} \int_{t_1}^t e^{-\delta(t-s)} ds$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$= \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we conclude that

$$\lim_{t \to \infty} \|\tilde{w}\|_{L^\infty(\Omega)} = 0$$

and therefore

$$0 \leq \lim_{t \to \infty} \left\|w(\cdot, t) - \frac{\overline{u}_0}{\delta}\right\|_{L^\infty(\Omega)} \leq \lim_{t \to \infty} \|\tilde{w}(\cdot, t)\|_{L^\infty(\Omega)} + \lim_{t \to \infty} \left\|\frac{\overline{u}_0}{\delta}e^{-\delta t}\right\|_{L^\infty(\Omega)} = 0. \quad \square$$

our main result can now be obtained by simply collecting what we have found so far.

Proof of Theorem 1.1. The statement on eventual boundedness and regularity immediately result from Lemma 4.3 and Lemma 2.1. The convergence results in Theorem 1.1 have already been proved in Lemma 3.3, 5.2 and 5.3. $\square$

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