New results for higher-order Hadamard-type fractional differential equations on the half-line

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Abstract
The purpose of this paper is to analyze a new kind of Hadamard fractional boundary value problem combining integral boundary condition and multipoint fractional integral boundary condition on an infinite interval. By the help of the Bai-Ge's fixed point theorem, multiplicity results of positive solutions are derived for the Hadamard fractional boundary value problem. In the end, to illustrative the main result, an example is also presented.

2010 Mathematics subject Classification: 34A08; 34B15.
Keywords: Hadamard fractional derivative; Positive solution; Infinite interval.

1 Introduction

Recently, fractional calculus and fractional differential equations have aroused a considerable attention of many scientists because of their capability to modelling real world phenomena in a variety of fields such as physics, applied mathematics, control theory. We refer to excellent books on the subject of fractional calculus and fractional differential equations, see [1, 2, 3, 4, 5]. As a significant topic for the theory of fractional boundary value problems, the existence results of positive solutions have been investigated comprehensively and a variety of results related to fractional boundary value problems has been established with the aid of fixed point theory, monotone iterative method, upper and lower solution technique. See [6, 7, 8, 9, 10, 11, 12, 13] and the references therein.

At the same time, Hadamard fractional differential equations with nonlocal boundary conditions on an unbounded/bounded domain have evolved as an interesting subject of research. This area is open for further development. Especially, Hadamard boundary value problems subject to integral boundary conditions and m-point fractional integral boundary conditions are very popular fields. But as far as we know many results are obtained for the Hadamard fractional boundary value problem using either m-point fractional integral boundary condition or integral boundary condition, see [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26].

In [15], Thiramanus *et al.* investigated the following boundary value problem:

$$\begin{cases} {}^H D^q_{1^+} u(t) + a(t) f(u(t)) = 0, \quad 1 < q \leq 2, \quad t \in (1, +\infty), \\ u(1) = 0, \quad {}^H D^{q-1}_{1^+} u(\infty) = \sum_{i=1}^m \lambda_i {}^H I^{\beta_i}_{1^+} u(\eta), \end{cases}$$

where ${}^H D^q_{1^+}$ denotes the Hadamard-type fractional derivative of order α, $\eta \in (1, +\infty)$; ${}^H I^{\beta_i}_{1^+}$ is the Hadamard-type fractional integral of order $\beta_i > 0$ $(i = 1, 2, ..., m)$.

In [16], Zhang and Lui considered the following problem:

$$\begin{cases} {}^H D^\alpha_{1^+} x(t) + a(t) f(t, x(t)) = 0, \quad n-1 < \alpha \leq n, \quad t \in (1, +\infty), \\ x^{(m)}(1) = 0, \quad {}^H D^{\alpha-1}_{1^+} x(\infty) = \displaystyle\int_1^{+\infty} g(t) x(t) \frac{dt}{t}, \; m = 0, 1, ..., n-2, \end{cases}$$

where $n \in \mathbb{N}$, $n \geq 3$, ${}^H D^\alpha_{1^+}$ is the Hadamard-type fractional derivative of order α. By applying the monotone iterative method, they obtained the minimal and maximal positive solutions of boundary value problem.

Since many authors study Hadamard boundary value problem with integral boundary condition and m-point fractional integral boundary condition, in this work, we are interested in investigating Hadamard boundary value problem which include both integral boundary condition and m-point fractional integral boundary condition on an infinite interval:

$$\begin{cases} {}^HD^{\vartheta}_{1^+}u(t)+p(t)f(t,u(t),{}^HD^{\vartheta-1}_{1^+}u(t))=0, \quad n-1<\vartheta\le n, \quad t\in(1,+\infty),\\ u^{(k)}(1)=0, \quad 0\le k\le n-2, \quad {}^HD^{\vartheta-1}_{1^+}u(\infty)=\displaystyle\int_1^\infty g(t)u(t)\frac{dt}{t}+\sum_{i=1}^m \lambda_i {}^HI^{\beta_i}_{1^+}u(\varsigma), \end{cases} \tag{1.1}$$

where $n\in\mathbb{N}$, $n\ge 3$, ${}^HD^{\vartheta}_{1^+}$ is the Hadamard-type fractional derivative of order ϑ, ${}^HI^{\beta_i}_{1^+}$ is the Hadamard-type fractional integral of order $\beta_i>0$ $(i=1,2,...,m)$, $g\in\mathcal{C}([1,\infty),(0,\infty))$ and $\lambda_i\ge 0$ $(i=1,2,...,m)$, $\varsigma\in(1,+\infty)$ are given constants.

We are interested in the analysis of existence result of positive solutions for Hadamard fractional boundary value problem on an unbounded domain. To the authors knowledge, due to the noncompactness of an infinite interval, research on Hadamard fractional boundary value problems on the half-line has little been discussed up to now. Hence, we attempt to study infinite interval Hadamard fractional differential equation.

Our paper includes new features. Firstly, compared with [15, 16, 26], our nonlinear function f involves Hadamard fractional derivative operator which makes the problem more complex. Secondly, our boundary condition is more general than compared with [15, 16, 26]. Furthermore, the technique is different from [16, 26].

2 Preliminaries

In this part, basic concepts, notations and related lemmas about the Hadamard-type fractional calculus are given for the convenience of the readers.

Now, let us introduce two Banach spaces as below,

$$E=\left\{u\in\mathcal{C}([1,\infty)):\sup_{t\in[1,\infty)}\frac{|u(t)|}{1+(\log t)^{\vartheta-1}}<\infty\right\},$$

$$F=\left\{u\in E, {}^HD^{\vartheta-1}_{1^+}u\in\mathcal{C}([1,\infty)):\sup_{t\in[1,\infty)}|{}^HD^{\vartheta-1}_{1^+}u(t)|<\infty\right\},$$

with norms $||u||_E=\sup_{t\in[1,\infty)}\frac{|u(t)|}{1+(\log t)^{\vartheta-1}}$, and $||u||_F=max\{||u||_E,\sup_{t\in[1,\infty)}|{}^HD^{\vartheta-1}_{1^+}u(t)|\}$, respectively.

Define the cone $P\subset F$ by

$$P=\{u\in F:u(t)\ge 0, t\in[1,\infty)\}.$$

Definition 2.1 ([4]) The Hadamard fractional derivative of fractional order ν for a function $c:[1,\infty)\to\mathbb{R}$ is defined as

$${}^HD^{\nu}_{1^+}c(t)=\frac{1}{\Gamma(n-\nu)}\Big(t\frac{d}{dt}\Big)^n\int_1^t\Big(\log\frac{t}{s}\Big)^{n-\nu-1}c(s)\frac{ds}{s}, \quad n-1<\nu<n,\ n=[\nu]+1,$$

where $[\nu]$ denotes the integer part of the real number ν and $\log(\cdot)=\log_e(\cdot)$.

Definition 2.2 ([4]) The Hadamard fractional integral of order ν for a function $c:[1,\infty)\to\mathbb{R}$ is defined as

$${}^HI^{\nu}_{1^+}c(t)=\frac{1}{\Gamma(\nu)}\int_1^t\big(\log\frac{t}{s}\big)^{\nu-1}c(s)\frac{ds}{s}, \quad \nu>0,$$

provided the integral exists.

Lemma 2.1 ([4]) If $a,\nu,\mu>0$, then

$$({}^HI^{\nu}_a(\log\frac{t}{a})^{\mu-1})(x)=\frac{\Gamma(\mu)}{\Gamma(\mu+\nu)}(\log\frac{x}{a})^{\mu+\nu-1}, \qquad ({}^HD^{\nu}_a(\log\frac{t}{a})^{\mu-1})(x)=\frac{\Gamma(\mu)}{\Gamma(\mu-\nu)}(\log\frac{x}{a})^{\mu-\nu-1}.$$

Lemma 2.2 *Let $k \in \mathcal{C}[1,\infty)$ with $\int_1^\infty k(s)\frac{ds}{s} < \infty$, and $\Upsilon = \Gamma(\vartheta) - \sum_{i=1}^m \frac{\lambda_i \Gamma(\vartheta)}{\Gamma(\vartheta+\beta_i)}(\log\varsigma)^{\vartheta+\beta_i-1}$ with*

$$\Upsilon_1 = \Upsilon - \int_1^\infty g(t)(\log t)^{\vartheta-1}\frac{dt}{t} > 0,$$

then the function $u \in F$ is a solution of the Hadamard-type fractional differential equation

$$^H D_{1^+}^{\vartheta} u(t) + k(t) = 0, \quad n-1 < \vartheta \le n, \quad t \in (1,+\infty), \tag{2.1}$$

with the following boundary conditions

$$u^{(k)}(1) = 0, \quad 0 \le k \le n-2, \quad {}^H D_{1^+}^{\vartheta-1} u(\infty) = \int_1^\infty g(t)u(t)\frac{dt}{t} + \sum_{i=1}^m \lambda_i^H I_{1^+}^{\beta_i} u(\varsigma), \tag{2.2}$$

if and only if u satisfies the integral equation

$$u(t) = \int_1^\infty G(t,s)k(s)\frac{ds}{s}, \qquad t \in [1,\infty) \tag{2.3}$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s), \tag{2.4}$$

and

$$G_1(t,s) = g(t,s) + \sum_{i=1}^m \frac{\lambda_i (\log t)^{\vartheta-1}}{\Upsilon\Gamma(\vartheta+\beta_i)} g_i(\varsigma,s),$$

$$G_2(t,s) = \frac{(\log t)^{\vartheta-1}}{\Upsilon_1}\int_1^\infty G_1(t,s)g(t)\frac{dt}{t},$$

$$g(t,s) = \frac{1}{\Gamma(\vartheta)}\begin{cases}(\log t)^{\vartheta-1} - (\log\frac{t}{s})^{\vartheta-1}, & 1 \le s \le t < \infty,\\ (\log t)^{\vartheta-1}, & 1 \le t \le s < \infty,\end{cases}$$

$$g_i(\varsigma,s) = \begin{cases}(\log\varsigma)^{\vartheta+\beta_i-1} - (\log\frac{\varsigma}{s})^{\vartheta+\beta_i-1}, & 1 \le s \le \varsigma < \infty,\\ (\log\varsigma)^{\vartheta+\beta_i-1}, & 1 \le \varsigma \le s < \infty.\end{cases}$$

Proof. (2.1) can be shown by

$$u(t) = -\frac{1}{\Gamma(\vartheta)}\int_1^t (\log\frac{t}{s})^{\vartheta-1}k(s)\frac{ds}{s} + c_1(\log t)^{\vartheta-1} + c_2(\log t)^{\vartheta-2} + ... + c_n(\log t)^{\vartheta-n},$$

where $c_1, c_2, ..., c_n \in \mathbb{R}$. With the help of the boundary conditions $u^{(k)}(1) = 0$, $k = 0, 1, ..., n-2$, we drive $c_2 = c_3 = ... = c_n = 0$. As a result,

$$u(t) = -\frac{1}{\Gamma(\vartheta)}\int_1^t (\log\frac{t}{s})^{\vartheta-1}k(s)\frac{ds}{s} + c_1(\log t)^{\vartheta-1}. \tag{2.5}$$

Lemma 2.1 results that

$$^H D_{1^+}^{\vartheta-1} u(t) = c_1\Gamma(\vartheta) - \int_1^t k(s)\frac{ds}{s}.$$

Using the second condition of (2.2), we have

$$c_1 = \frac{1}{\Upsilon}(\int_1^\infty g(t)u(t)\frac{dt}{t} + \int_1^\infty k(s)\frac{ds}{s} - \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\vartheta+\beta_i)}\int_1^\varsigma (\log\frac{\varsigma}{s})^{\vartheta+\beta_i-1}k(s)\frac{ds}{s}). \tag{2.6}$$

Substituting (2.6) into (2.5), we get

$$\begin{aligned}
u(t) =&\frac{(\log t)^{\vartheta-1}}{\Upsilon}\int_1^{\infty}k(s)\frac{ds}{s}+\frac{(\log t)^{\vartheta-1}}{\Upsilon}\int_1^{\infty}g(t)u(t)\frac{dt}{t}-\frac{1}{\Gamma(\vartheta)}\int_1^{t}(\log\frac{t}{s})^{\vartheta-1}k(s)\frac{ds}{s}\\
&-\sum_{i=1}^{m}\frac{\lambda_i(\log t)^{\vartheta-1}}{\Upsilon\Gamma(\vartheta+\beta_i)}\int_1^{\varsigma}(\log\frac{\varsigma}{s})^{\vartheta+\beta_i-1}k(s)\frac{ds}{s}\\
=&\frac{(\log t)^{\vartheta-1}}{\Gamma(\vartheta)}\int_1^{\infty}k(s)\frac{ds}{s}+\frac{(\Gamma(\vartheta)-\Upsilon)(\log t)^{\vartheta-1}}{\Upsilon\Gamma(\vartheta)}\int_1^{\infty}k(s)\frac{ds}{s}+\frac{(\log t)^{\vartheta-1}}{\Upsilon}\int_1^{\infty}g(t)u(t)\frac{dt}{t}\\
&-\frac{1}{\Gamma(\vartheta)}\int_1^{t}(\log\frac{t}{s})^{\vartheta-1}k(s)\frac{ds}{s}-\sum_{i=1}^{m}\frac{\lambda_i(\log t)^{\vartheta-1}}{\Upsilon\Gamma(\vartheta+\beta_i)}\int_1^{\varsigma}(\log\frac{\varsigma}{s})^{\vartheta+\beta_i-1}k(s)\frac{ds}{s}\\
=&\frac{(\log t)^{\vartheta-1}}{\Gamma(\vartheta)}\int_1^{\infty}k(s)\frac{ds}{s}+\sum_{i=1}^{m}\frac{\lambda_i(\log t)^{\vartheta-1}}{\Upsilon\Gamma(\vartheta+\beta_i)}\int_1^{\infty}(\log\varsigma)^{\vartheta+\beta_i-1}k(s)\frac{ds}{s}\\
&+\frac{(\log t)^{\vartheta-1}}{\Upsilon}\int_1^{\infty}g(t)u(t)\frac{dt}{t}-\frac{1}{\Gamma(\vartheta)}\int_1^{t}(\log\frac{t}{s})^{\vartheta-1}k(s)\frac{ds}{s}\\
&-\sum_{i=1}^{m}\frac{\lambda_i(\log t)^{\vartheta-1}}{\Upsilon\Gamma(\vartheta+\beta_i)}\int_1^{\varsigma}(\log\frac{\varsigma}{s})^{\vartheta+\beta_i-1}k(s)\frac{ds}{s}\\
=&\int_1^{\infty}g(t,s)k(s)\frac{ds}{s}+\sum_{i=1}^{m}\frac{\lambda_i(\log t)^{\vartheta-1}}{\Upsilon\Gamma(\vartheta+\beta_i)}\int_1^{\infty}g_i(\varsigma,s)k(s)\frac{ds}{s}+\frac{(\log t)^{\vartheta-1}}{\Upsilon}\int_1^{\infty}g(t)u(t)\frac{dt}{t}\\
=&\int_1^{\infty}G_1(t,s)k(s)\frac{ds}{s}+\frac{(\log t)^{\vartheta-1}}{\Upsilon}\int_1^{\infty}g(t)u(t)\frac{dt}{t}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_1^{\infty}g(t)u(t)\frac{dt}{t}=&\int_1^{\infty}g(t)\int_1^{\infty}G_1(t,s)k(s)\frac{ds}{s}\frac{dt}{t}\\
&+\frac{1}{\Upsilon}\int_1^{\infty}g(t)(\log t)^{\vartheta-1}\frac{dt}{t}\int_1^{\infty}g(t)u(t)\frac{dt}{t},
\end{aligned}$$

which provides

$$\int_1^{\infty}g(t)u(t)\frac{dt}{t}=\frac{\Upsilon}{\Upsilon_1}\int_1^{\infty}g(t)\int_1^{\infty}G_1(t,s)k(s)\frac{ds}{s}\frac{dt}{t}.$$

Then,

$$\begin{aligned}
u(t)&=\int_1^{\infty}G_1(t,s)k(s)\frac{ds}{s}+\int_1^{\infty}G_2(t,s)k(s)\frac{ds}{s}\\
&=\int_1^{\infty}G(t,s)k(s)\frac{ds}{s}.
\end{aligned}$$

The proof is completed.

Lemma 2.3 *The Green's function $G(t,s)$, given by (2.4) ensures the following properties:*

(i) $G(t,s)$ is continuous and $G(t,s)\geq 0$ for $(t,s)\in[1,\infty)\times[1,\infty)$;

(ii) $\frac{G(t,s)}{1+(\log t)^{\vartheta-1}}\leq\frac{1}{\Upsilon_1}$ for all $(t,s)\in[1,\infty)\times[1,\infty)$;

(iii) $\min_{\varsigma\leq t\leq k\varsigma}\frac{G(t,s)}{1+(\log t)^{\vartheta-1}}\geq\sum_{i=1}^{m}\frac{\lambda_i(\log\varsigma)^{\vartheta-1}g_i(\varsigma,s)}{\Gamma(\vartheta+\beta_i)(1+(\log\varsigma)^{\vartheta-1})}(\frac{1}{\Upsilon_1}-\frac{1}{\Upsilon})$ for $k>1$ and $s\in[1,\infty)$.

Proof. We can easily see that *(i)* holds. Next, we show that *(ii)* and *(iii)* hold. To prove *(ii)*, for $(t,s)\in[1,\infty)\times[1,\infty)$, we have

$$
\begin{aligned}
\frac{G(t,s)}{1+(\log t)^{\vartheta-1}} &= \frac{G_1(t,s)}{1+(\log t)^{\vartheta-1}} + \frac{G_2(t,s)}{1+(\log t)^{\vartheta-1}} \\
&= \frac{g(t,s)}{1+(\log t)^{\vartheta-1}} + \sum_{i=1}^{m} \frac{\lambda_i (\log t)^{\vartheta-1} g_i(\varsigma,s)}{\Upsilon\Gamma(\vartheta+\beta_i)(1+(\log t)^{\vartheta-1})} \\
&\quad + \frac{(\log t)^{\vartheta-1}}{\Upsilon_1(1+(\log t)^{\vartheta-1})} \int_1^\infty G_1(t,s) g(t) \frac{dt}{t} \\
&\le \frac{1}{\Gamma(\vartheta)} + \sum_{i=1}^{m} \frac{\lambda_i (\log \varsigma)^{\vartheta+\beta_i-1}}{\Upsilon\Gamma(\vartheta+\beta_i)} + \frac{1}{\Upsilon_1\Gamma(\vartheta)} \int_1^\infty (\log t)^{\vartheta-1} g(t) \frac{dt}{t} \\
&\quad + \frac{1}{\Upsilon_1} \sum_{i=1}^{m} \frac{\lambda_i (\log n)^{\vartheta+\beta_i-1}}{\Upsilon\Gamma(\vartheta+\beta_i)} \int_1^\infty (\log t)^{\vartheta-1} g(t) \frac{dt}{t} \\
&= \frac{1}{\Gamma(\vartheta)} + \sum_{i=1}^{m} \frac{\lambda_i (\log \varsigma)^{\vartheta+\beta_i-1}}{\Upsilon\Gamma(\vartheta+\beta_i)} + \frac{1}{\Upsilon_1\Gamma(\vartheta)} \int_1^\infty (\log t)^{\vartheta-1} g(t) \frac{dt}{t} \\
&\quad + \frac{1}{\Upsilon\Upsilon_1\Gamma(\vartheta)} \int_1^\infty (\log t)^{\vartheta-1} g(t) \frac{dt}{t} \big(\Upsilon + \sum_{i=1}^{m} \frac{\lambda_i \Gamma(\vartheta)(\log n)^{\vartheta+\beta_i-1}}{\Gamma(\vartheta+\beta_i)}\big) \\
&= \frac{1}{\Gamma(\vartheta)} + \sum_{i=1}^{m} \frac{\lambda_i \Gamma(\vartheta)(\log \varsigma)^{\vartheta+\beta_i-1}}{\Upsilon\Gamma(\vartheta)\Gamma(\vartheta+\beta_i)} + \frac{1}{\Upsilon_1\Gamma(\vartheta)} \int_1^\infty (\log t)^{\vartheta-1} g(t) \frac{dt}{t} \\
&\quad + \frac{1}{\Upsilon\Upsilon_1} \int_1^\infty (\log t)^{\vartheta-1} g(t) \frac{dt}{t} \\
&= \frac{1}{\Upsilon_1}.
\end{aligned}
$$

To prove (iii), for $(t,s) \in [1,\infty) \times [1,\infty)$ and $k > 1$, we have,

$$
\begin{aligned}
\min_{\varsigma \le t \le k\varsigma} \frac{G(t,s)}{1+(\log t)^{\vartheta-1}} &= \min_{\varsigma \le t \le k\varsigma} \Bigg[\frac{g(t,s)}{1+(\log t)^{\vartheta-1}} + \sum_{i=1}^{m} \frac{\lambda_i (\log t)^{\vartheta-1} g_i(\varsigma,s)}{\Upsilon\Gamma(\vartheta+\beta_i)(1+(\log t)^{\vartheta-1})} \\
&\quad + \frac{(\log t)^{\vartheta-1}}{\Upsilon_1(1+(\log t)^{\vartheta-1})} \int_1^\infty G_1(t,s) g(t) \frac{dt}{t} \Bigg] \\
&\ge \min_{\varsigma \le t \le k\varsigma} \frac{(\log t)^{\vartheta-1}}{\Upsilon_1(1+(\log t)^{\vartheta-1})} \int_1^\infty G_1(t,s) g(t) \frac{dt}{t} \\
&\ge \frac{(\log \varsigma)^{\vartheta-1}}{\Upsilon_1(1+(\log \varsigma)^{\vartheta-1})} \int_1^\infty \sum_{i=1}^{m} \frac{\lambda_i (\log t)^{\vartheta-1}}{\Upsilon\Gamma(\vartheta+\beta_i)} g_i(\varsigma,s) g(t) \frac{dt}{t} \\
&= \sum_{i=1}^{m} \frac{\lambda_i (\log \varsigma)^{\vartheta-1} g_i(\varsigma,s)}{\Gamma(\vartheta+\beta_i)(1+(\log \varsigma)^{\vartheta-1})} \big(\frac{1}{\Upsilon_1} - \frac{1}{\Upsilon}\big).
\end{aligned}
$$

The proof is completed.

Remark 2.1 *By using* (2.3), *we get*

$$
{}^{H}D_{1^+}^{\vartheta-1} u(t) = \int_1^\infty G^*(t,s) k(s) \frac{ds}{s}, \tag{2.7}
$$

where

$$
G^*(t,s) = k(t,s) + \sum_{i=1}^{m} \frac{\lambda_i \Gamma(\vartheta)}{\Upsilon\Gamma(\vartheta+\beta_i)} g_i(\varsigma,s) + \frac{\Gamma(\vartheta)}{\Upsilon_1} \int_1^\infty G_1(t,s) g(t) \frac{dt}{t}, \tag{2.8}
$$

and

$$
k(t,s) = \begin{cases} 0, & 1 \le s \le t < \infty, \\ 1, & 1 \le t \le s < \infty. \end{cases}
$$

Remark 2.2 *It is easy to see that if u is a solution of fractional differential equation* (2.1) – (2.2), *we have, for $s,t\in[1,\infty)$, $G^*(t,s)$ is continuous, $G^*(t,s)\geq 0$ and*

$$
\begin{aligned}
G^*(t,s) &= k(t,s)+\sum_{i=1}^{m}\frac{\lambda_i\Gamma(\vartheta)}{\Upsilon\Gamma(\vartheta+\beta_i)}g_i(\varsigma,s)+\frac{\Gamma(\vartheta)}{\Upsilon_1}\int_1^\infty G_1(t,s)g(t)\frac{dt}{t}\\
&\leq 1+\sum_{i=1}^{m}\frac{\lambda_i\Gamma(\vartheta)(\log\varsigma)^{\vartheta+\beta_i-1}}{\Upsilon\Gamma(\vartheta+\beta_i)}+\frac{1}{\Upsilon_1}\int_1^\infty(\log t)^{\vartheta-1}g(t)\frac{dt}{t}\\
&\quad+\frac{\Gamma(\vartheta)}{\Upsilon_1}\sum_{i=1}^{m}\frac{\lambda_i(\log n)^{\vartheta+\beta_i-1}}{\Upsilon\Gamma(\vartheta+\beta_i)}\int_1^\infty(\log t)^{\vartheta-1}g(t)\frac{dt}{t}\\
&=1+\frac{\Gamma(\vartheta)-\Upsilon}{\Upsilon}+\frac{1}{\Upsilon\Upsilon_1}\int_1^\infty(\log t)^{\vartheta-1}g(t)\frac{dt}{t}\big(\Upsilon+\sum_{i=1}^{m}\frac{\lambda_i\Gamma(\vartheta)(\log n)^{\vartheta+\beta_i-1}}{\Gamma(\vartheta+\beta_i)}\big)\\
&=\frac{\Gamma(\vartheta)}{\Upsilon_1}.
\end{aligned}
$$

In view of Lemma 2.2, we introduce $T:P\to F$, by

$$Tu(t)=\int_1^\infty G(t,s)p(s)f(s,u(s),{}^HD_{1^+}^{\vartheta-1}u(s))\frac{ds}{s}. \tag{2.9}$$

Then, the fixed point of the operator T is the solution of the problem (1.1).

Lemma 2.4 *[17] Let $U\subset F$ be a bounded set. Then U is relatively compact in F if the following conditions hold:*

(i) for any $u(t)\in U$, $\frac{|u(t)|}{1+(\log t)^{\vartheta-1}}$ and ${}^HD_{1^+}^{\vartheta-1}u(t)$ are equicontinuous on any compact interval of $[1,\infty)$.

(ii) for any $\epsilon>0$, there exists a constant $T=T(\epsilon)>1$ such that

$\left|\frac{u(t_1)}{1+(\log t_1)^{\vartheta-1}}-\frac{u(t_2)}{1+(\log t_2)^{\vartheta-1}}\right|<\epsilon$, *and* $|{}^HD_{1^+}^{\vartheta-1}u(t_1)-{}^HD_{1^+}^{\vartheta-1}u(t_2)|<\epsilon$

for any $t_1,t_2\geq T$ and $u\in U$.

As next step, we consider that the assumptions below are satisfied.
(C_1) $f\in\mathcal{C}([1,\infty)\times[0,\infty)\times[0,\infty),[0,\infty))$, $f(t,0,0)\not\equiv 0$ on any subinterval of $[1,\infty)$, when u and v are bounded, $f(t,(1+(\log t)^{\vartheta-1})u,v)$ is bounded on $[1,\infty)$.

(C_2) $p:[1,\infty)\to[0,\infty)$ does not identically vanish on any subinterval of $[1,\infty)$ and

$$0<\int_1^\infty p(s)\frac{ds}{s}<\infty.$$

Lemma 2.5 *Suppose that (C_1) and (C_1) hold. Then $T:P\to P$ is a completely continuous operator.*

Proof. Now, we present the proof with four steps.
(A_1) For any $u\in P$, it is clear that $Tu(t)\geq 0$ for all $t\in[1,\infty)$ i.e., $T:P\to P$. We show that T is continuous. Let $u_n\to u$ as $n\to\infty$ in P, then there exists a constant $k_0>0$ such that $sup_{n\in\mathbb{N}}||u_n||_F\leq k_0$. Let $B_{k_0}=\sup\{f(t,(1+\log t)^{\vartheta-1}u,v)|(t,u,v)\in[1,\infty)\times[0,k_0]^2\}$. By (C_1) and (C_1), We get

$$
\begin{aligned}
\int_1^\infty\frac{G(t,s)}{1+(\log t)^{\vartheta-1}}p(s)f(s,u_n(s),{}^HD_{1^+}^{\vartheta-1}u_n(s))\frac{ds}{s} &\leq\frac{1}{\Upsilon_1}\int_1^\infty p(s)f(s,u_n(s),{}^HD_{1^+}^{\vartheta-1}u_n(s))\frac{ds}{s}\\
&\leq\frac{B_{k_0}}{\Upsilon_1}\int_1^\infty p(s)\frac{ds}{s}<\infty,
\end{aligned}
$$

and

$$\int_1^{\infty} G^*(t,s)p(s)f(s,u_n(s),{}^H D_{1^+}^{\vartheta-1}u_n(s))\frac{ds}{s} \le \frac{\Gamma(\vartheta)}{\Upsilon_1}\int_1^{\infty} p(s)f(s,u_n(s),{}^H D_{1^+}^{\vartheta-1}u_n(s))\frac{ds}{s}$$
$$\le \frac{\Gamma(\vartheta)B_{k_0}}{\Upsilon_1}\int_1^{\infty} p(s)\frac{ds}{s} < \infty.$$

The Lebesgue dominated convergence theorem and continuity of f guarantee that

$$\lim_{n\to\infty}\int_1^{\infty}\frac{G(t,s)}{1+(\log t)^{\vartheta-1}}p(s)f(s,u_n(s),{}^H D_{1^+}^{\vartheta-1}u_n(s))\frac{ds}{s}$$
$$=\int_1^{\infty}\frac{G(t,s)}{1+(\log t)^{\vartheta-1}}p(s)f(s,u(s),{}^H D_{1^+}^{\vartheta-1}u(s))\frac{ds}{s},$$

and

$$\lim_{n\to\infty}\int_1^{\infty} G^*(t,s)p(s)f(s,u_n(s),{}^H D_{1^+}^{\vartheta-1}u_n(s))\frac{ds}{s}$$
$$=\int_1^{\infty} G^*(t,s)p(s)f(s,u(s),{}^H D_{1^+}^{\vartheta-1}u(s))\frac{ds}{s}.$$

Hence,

$$\|Tu_n-Tu\|_E = \sup_{t\in[1,\infty)}\int_1^{\infty}\frac{G(t,s)}{1+(\log t)^{\vartheta-1}}|p(s)f(s,u_n(s),{}^H D_{1^+}^{\vartheta-1}u_n(s))\frac{ds}{s}$$
$$-\,p(s)f(s,u(s),{}^H D_{1^+}^{\vartheta-1}u(s))|\frac{ds}{s} \to 0 \quad (n\to\infty),$$

and

$$\sup_{t\in[1,\infty)}|{}^H D_{1^+}^{\vartheta-1}Tu_n - {}^H D_{1^+}^{\vartheta-1}Tu| \to 0 \quad (n\to\infty),$$

which shows T is continuous.

(A_2) We prove that T is uniformly bounded on P. Let Θ be any bounded subset of cone P, then there exists a constant $k_1>0$ such that $||u||_F \le k_1$ for all $u\in\Theta$. Let $B_{k_1}=\sup\{f(t,(1+\log t)^{\vartheta-1}u,v)|(t,u,v)\in[1,\infty)\times[0,k_1]^2\}$. From Lemma 2.4, Remark 2.2,

$$||Tu||_E = \sup_{t\in[1,\infty)}\int_1^{\infty}\frac{G(t,s)}{1+(\log t)^{\vartheta-1}}|p(s)f(s,u(s),{}^H D_{1^+}^{\vartheta-1}u(s))|\frac{ds}{s}$$
$$\le \frac{1}{\Upsilon_1}\int_1^{\infty}|p(s)f(s,u(s),{}^H D_{1^+}^{\vartheta-1}u(s))|\frac{ds}{s}$$
$$\le \frac{B_{k_1}}{\Upsilon_1}\int_1^{\infty}p(s)\frac{ds}{s} < \infty,$$

and

$$\sup_{t\in[1,\infty)}|{}^H D_{1^+}^{\vartheta-1}Tu(t)| = \sup_{t\in[1,\infty)}\int_1^{\infty}G^*(t,s)|p(s)f(s,u(s),{}^H D_{1^+}^{\vartheta-1}u(s))|\frac{ds}{s}$$
$$\le \frac{\Gamma(\vartheta)}{\Upsilon_1}\int_1^{\infty}|p(s)f(s,u(s),{}^H D_{1^+}^{\vartheta-1}u(s))|\frac{ds}{s}$$
$$\le \frac{\Gamma(\vartheta)B_{k_1}}{\Upsilon_1}\int_1^{\infty}p(s)\frac{ds}{s} < \infty.$$

for all $u\in\Theta$. So $T\Theta$ is uniformly bounded.

(A_3) We show that T and ${}^H D_{1^+}^{\vartheta-1}T$ are equicontinuous on any compact interval of $[1,\infty)$. Let $L \subset [1,\infty)$ be any compact interval. For $\forall t_1, t_2 \in L$, $t_1 < t_2$, and $u \in \Theta$ mentioned above, we have

$$\begin{aligned}
&\left|\frac{Tu(t_2)}{1+(\log t_2)^{\vartheta-1}} - \frac{Tu(t_1)}{1+(\log t_1)^{\vartheta-1}}\right| \\
&= \left|\int_1^\infty \left(\frac{G(t_2,s)}{1+(\log t_2)^{\vartheta-1}} - \frac{G(t_1,s)}{1+(\log t_1)^{\vartheta-1}}\right)p(s)f(s,u(s),{}^H D_{1^+}^{\vartheta-1}u(s))\frac{ds}{s}\right| \\
&\leq B_{k_1}\int_1^{t_2}\left|\frac{G(t_2,s)}{1+(\log t_2)^{\vartheta-1}} - \frac{G(t_1,s)}{1+(\log t_1)^{\vartheta-1}}\right|p(s)\frac{ds}{s} \\
&\quad + B_{k_1}\int_{t_2}^\infty\left|\frac{G(t_2,s)}{1+(\log t_2)^{\vartheta-1}} - \frac{G(t_1,s)}{1+(\log t_1)^{\vartheta-1}}\right|p(s)\frac{ds}{s} \\
&\leq B_{k_1}\int_1^{t_2}\left|\frac{G(t_2,s)}{1+(\log t_2)^{\vartheta-1}} - \frac{G(t_1,s)}{1+(\log t_1)^{\vartheta-1}}\right|p(s)\frac{ds}{s} \\
&\quad + \frac{B_{k_1}}{\Upsilon_1}\int_{t_2}^\infty\left|\frac{(\log t_2)^{\vartheta-1}}{1+(\log t_2)^{\vartheta-1}} - \frac{(\log t_1)^{\vartheta-1}}{1+(\log t_1)^{\vartheta-1}}\right|p(s)\frac{ds}{s}
\end{aligned}$$

Since the functions $\frac{G(t,s)}{1+(\log t)^{\vartheta-1}}$, $\frac{(\log t)^{\vartheta-1}}{1+(\log t)^{\vartheta-1}}$ are uniformly continuous on any compact set $L \times L$ and L, respectively, we obtain:

$$\left|\frac{Tu(t_2)}{1+(\log t_2)^{\vartheta-1}} - \frac{Tu(t_1)}{1+(\log t_1)^{\vartheta-1}}\right| \to 0, \qquad \text{uniformly as } t_1 \to t_2.$$

Hence, it has been ensured that $\frac{Tu(t)}{1+(\log t)^{\vartheta-1}}$ is equicontinuous on L.

Consider that

$${}^H D_{1^+}^{\vartheta-1}Tu(t) = \int_1^\infty G^*(t,s)p(s)f(s,u(s),{}^H D_{1^+}^{\vartheta-1}u(s))\frac{ds}{s}.$$

Here, the Green's function $G^*(t,s) \in C([1,\infty)\times[1,\infty))$ does not depend on t. Then, we can obtain that ${}^H D_{1^+}^{\vartheta-1}Tu(t)$ is equicontinuous on L.

(A_4) We show that T and ${}^H D_{1^+}^{\vartheta-1}T$ are equiconvergent at ∞. For any $u \in \Theta$, by (C_2) we have

$$\begin{aligned}
\lim_{t\to\infty}\left|\frac{Tu(t)}{1+(\log t)^{\vartheta-1}}\right| &= \lim_{t\to\infty}\int_1^\infty \frac{G(t,s)}{1+(\log t)^{\vartheta-1}}|p(s)f(s,u(s),{}^H D_{1^+}^{\vartheta-1}u(s))|\frac{ds}{s} \\
&\leq \frac{B_{k_1}}{\Upsilon_1}\lim_{t\to\infty}\int_1^\infty p(s)\frac{ds}{s} < \infty.
\end{aligned}$$

Thus, $T\Theta$ is equiconvergent at ∞.
From Remark 2.2, for any $u \in \Theta$,

$$\begin{aligned}
\lim_{t\to\infty}\left|{}^H D_{1^+}^{\vartheta-1}Tu(t)\right| &= \lim_{t\to\infty}\int_1^\infty G^*(t,s)|p(s)f(s,u(s),{}^H D_{1^+}^{\vartheta-1}u(s))|\frac{ds}{s} \\
&\leq \frac{B_{k_1}\Gamma(\vartheta)}{\Upsilon_1}\lim_{t\to\infty}\int_1^\infty p(s)\frac{ds}{s} < \infty.
\end{aligned}$$

Then, we prove that ${}^H D_{1^+}^{\vartheta-1}T$ is equiconvergent at ∞. Applying Lemma 2.5, from the above steps, ıt is obvious that $T : P \to P$ is completely continuous. The proof is completed.

Now, let us use the fixed point theorem by Bai and Ge to the operator T so that we have proper existence conditions for the problem (1.1).

Let $p > a > 0$, $N > 0$ be given constants and ζ be a nonnegative continuous concave functional and ψ, β be a nonnegative continuous convex functional on the cone P. Bounded convex sets are given by

$$\begin{aligned}
P(\psi^p, \beta^N) &= \{u \in P : \ \psi(u) < p, \ \beta(u) < N\}, \\
\overline{P}(\psi^p, \beta^N) &= \{u \in P : \ \psi(u) \leq p, \ \beta(u) \leq N\},
\end{aligned}$$

$$P(\psi^p,\beta^N,\zeta_a)=\{u\in P:\ \psi(u)<p,\ \beta(u)<N,\ \zeta(u)>a\},$$
$$\overline{P}(\psi^p,\beta^N,\zeta_a)=\{u\in P:\ \psi(u)\le p,\ \beta(u)\le N,\ \zeta(u)\ge a\}.$$

Here, ψ, β on cone P ensure

(E_1):There exists $M>0$ such that $||u||\le M max\{\psi(u),\beta(u)\}$ *for all* $u\in P$;

(E_2): $\Omega=\{u\in P:\psi(u)<p,\ \beta(u)<N\}\neq\emptyset$, for any $p>0,\ N>0$.

Now, we introduce the below fixed point theorem which we will apply to main result.

Theorem 2.1 *[27] Let $\mathbb{B}$ be a Banach space, $P\subset\mathbb{B}$ be a cone and $p_2\ge d>b>p_1>0$, $N_2\ge N_1>0$ be given. Assume that ψ,β are nonnegative continuous convex functionals on P, such that (E_1) and (E_2) are satisfied, ζ is a nonnegative continuous concave functional on P, such that $\zeta(u)\le\psi(u)$ for all $u\in\overline{P}(\psi^{r_2},\beta^{N_2})$ and let $A:\overline{P}(\psi^{p_2},\beta^{N_2})\to\overline{P}(\psi^{p_2},\beta^{N_2})$ be a completely continuous operator.*

$(B1)$ $\{u\in\overline{P}(\psi^d,\beta^{N_2},\zeta_b):\zeta(u)>b\}\neq\emptyset$, and $\zeta(Au)>b$ for $u\in\overline{P}(\psi^d,\beta^{N_2},\zeta_b)$,

$(B2)$ $\psi(Au)<p_1$, $\beta(Au)<N_1$, for all $u\in\overline{P}(\psi^{p_1},\beta^{N_1})$,

$(B3)$ $\zeta(Au)>b$, for all $u\in\overline{P}(\psi^{p_2},\beta^{N_2},\zeta_b)$ with $\psi(Au)>d$.

Then T has at least three fixed points u_1,u_2,u_3 in $\overline{P}(\psi^{p_2},\beta^{N_2})$ with

$$u_1\in P(\psi^{p_1},\beta^{N_1}),\ u_2\in\{\overline{P}(\psi^{p_2},\beta^{N_2},\zeta_b):\ \zeta(u)>b\},$$
$$u_3\in\overline{P}(\psi^{p_2},\beta^{N_2})\setminus\left(\overline{P}(\psi^{p_2},\beta^{N_2},\zeta_b)\cup\overline{P}(\psi^{p_1},\beta^{N_1})\right).$$

For the readers convenience, let us denote

$$\varrho=\sum_{i=1}^{m}\frac{\lambda_i(\log\varsigma)^{2\vartheta+\beta_i-2}}{\Gamma(\vartheta+\beta_i)(1+(\log\varsigma)^{\vartheta-1})}(\frac{1}{\Upsilon_1}-\frac{1}{\Upsilon})\int_{\varsigma}^{k\varsigma}p(s)\frac{ds}{s},\qquad\Phi=\frac{2[1+(\log\varsigma)^{\vartheta-1}])}{(\log\varsigma)^{\vartheta-1}},$$
$$A_1=\frac{1}{\Upsilon_1}\int_1^{\infty}p(s)\frac{ds}{s},\qquad A_2=\frac{\Gamma(\vartheta)}{\Upsilon_1}\int_1^{\infty}p(s)\frac{ds}{s}.$$

3 Existence theorem

Theorem 3.1 *Assume that there exist constants $\frac{1}{\Phi}p_2>b>p_1>0$, $N_2\ge N_1>0$ such that $\dfrac{p_2}{\Phi\varrho}\le\min\left\{\dfrac{p_2}{A_1},\dfrac{N_2}{A_2}\right\}$ and $\Phi b<\frac{N_2}{\Gamma(\vartheta)}$. Assume*

(a) $f(t,(1+(\log t)^{\vartheta-1})u,v)\le\min\left\{\frac{p_2}{A_1},\frac{N_2}{A_2}\right\}$ for $t\in[1,\infty)$, $u\in[0,p_2]$, $v\in[0,N_2]$.

(b) $f(t,(1+(\log t)^{\vartheta-1})u,v)>\frac{b}{\varrho}$ for $t\in[\varsigma,k\varsigma]$, $u\in[b,\Phi b]$, $v\in[0,N_2]$.

(c) $f(t,(1+(\log t)^{\vartheta-1})u,v)<\min\left\{\frac{p_1}{A_1},\frac{N_1}{A_2}\right\}$ for $t\in[1,\infty)$, $u\in[0,p_1]$, $v\in[0,N_1]$.

(d) $f(t,(1+(\log t)^{\vartheta-1})u,v)\ge\frac{p_2}{\Phi\varrho}$ for $t\in[\varsigma,k\varsigma]$, $u\in[b,p_2]$, $v\in[0,N_2]$.

Then the problem (1.1) has at least three positive solutions u_i $(i=1,2,3)$ with

$$0\le\sup_{t\in[1,\infty)}\frac{u_1(t)}{1+(\log t)^{\vartheta-1}}\le p_1,\quad\sup_{t\in[1,\infty)}{}^{H}D_{1^+}^{\vartheta-1}u_1(t)\le N_1;$$

$$b<\min_{\varsigma\le t\le k\varsigma}\frac{u_2(t)}{1+(\log t)^{\vartheta-1}}\le\sup_{t\in[1,\infty)}\frac{u_2(t)}{1+(\log t)^{\vartheta-1}}\le p_2,\quad\sup_{t\in[1,\infty)}{}^{H}D_{1^+}^{\vartheta-1}u_2(t)\le N_2;$$

$$\sup_{t\in[1,\infty)}\frac{u_3(t)}{1+(\log t)^{\vartheta-1}}\le p_2,\quad\sup_{t\in[1,\infty)}{}^{H}D_{1^+}^{\vartheta-1}u_3(t)\le N_2.$$

Proof. Let P and T be defined as above. Define ψ, β and ζ by

$$\psi(u) = \sup_{t\in[1,\infty)} \frac{|u(t)|}{1+(\log t)^{\vartheta-1}},\ \beta(u) = \sup_{t\in[1,\infty)} \left|{}^H D_{1^+}^{\vartheta-1} u(t)\right|,$$
$$\zeta(u) = \min_{\varsigma\le t\le k\varsigma} \frac{|u(t)|}{1+(\log t)^{\vartheta-1}},$$

for $u \in P$. Obviously, ψ, $\beta : P \to [0,+\infty)$ are nonnegative continuous convex functionals, ζ is nonnegative continuous concave functional with $\zeta(u) \le \psi(u)$ for all $u \in F$. For any $p > 0$ and $N > 0$, let $u^*(t) = \frac{c}{\Gamma(\vartheta)}(\log t)^{\vartheta-1}$, where $0 < c < \min\{p, N\}$. Then $u^*(t) = \frac{c}{\Gamma(\vartheta)}(\log t)^{\vartheta-1} \in P(\psi^p, \beta^N) \neq \emptyset$. Thus, (E_1), (E_2) are satisfied. Then $\psi(u)$, $\beta(u)$, $\zeta(u)$ satisfy the conditions in Theorem 2.1. Now, let us show that the operator T ensures all conditions in Theorem 2.1, which will prove the existence of three fixed points of T. By Lemma 2.6, T is completely continuous. First of all, we show that $T : \overline{P}(\psi^{p_2}, \beta^{N_2}) \to \overline{P}(\psi^{p_2}, \beta^{N_2})$. If $u \in \overline{P}(\psi^{p_2}, \beta^{N_2})$, then $\psi(u) = \sup_{t\in[1,\infty)} \frac{|u(t)|}{1+(\log t)^{\vartheta-1}} \le p_2$, $\beta(u) = \sup_{t\in[1,\infty)} \left|{}^H D_{1^+}^{\vartheta-1} u(t)\right| \le N_2$. Using condition (a),

$$f(t,u,v) = f(t,(1+(\log t)^{\vartheta-1})\frac{u}{(1+(\log t)^{\vartheta-1})}, v) \le \min\left\{\frac{p_2}{A_1}, \frac{N_2}{A_2}\right\}. \tag{3.1}$$

Hence,

$$\begin{aligned}
\psi(Tu) &= \sup_{t\in[1,\infty)} \frac{|Tu(t)|}{1+(\log t)^{\vartheta-1}} \\
&= \sup_{t\in[1,\infty)} \frac{1}{1+(\log t)^{\vartheta-1}} \left| \int_1^\infty G(t,s)p(s)f(s,u(s),{}^H D_{1^+}^{\vartheta-1}u(s))\frac{ds}{s} \right| \\
&\le \frac{p_2}{\Upsilon_1 A_1} \int_1^\infty p(s)\frac{ds}{s} = p_2,
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
\beta(Tu) = \sup_{t\in[1,\infty)} \left|{}^H D_{1^+}^{\vartheta-1} Tu(t)\right| &= \sup_{t\in[1,\infty)} \left| \int_1^\infty G^*(t,s)p(s)f(s,u(s),{}^H D_{1^+}^{\vartheta-1}u(s))\frac{ds}{s} \right| \\
&\le \frac{\Gamma(\vartheta)N_2}{\Upsilon_1 A_2} \int_1^\infty p(s)\frac{ds}{s} = N_2.
\end{aligned} \tag{3.3}$$

Thus, we have $T : \overline{P}(\psi^{p_2}, \beta^{N_2}) \to \overline{P}(\psi^{p_2}, \beta^{N_2})$. With assumption (c) using above case, we can obtain that $T : \overline{P}(\psi^{p_1}, \beta^{N_1}) \to P(\psi^{p_1}, \beta^{N_1})$. Therefore, condition $(B2)$ in Theorem 2.1 is satisfied. As following step, we show that conditon $(B1)$ of Theorem 2.1 holds. Then, choose the function $u(t) = \frac{2b[1+(\log\varsigma)^{\vartheta-1}]}{(\log\varsigma)^{\vartheta-1}}(\log t)^{\vartheta-1}$ for any $t \in [1,\infty)$. It can be easily get

$$\psi(u) = \sup_{t\in[1,\infty)} \frac{1}{1+(\log t)^{\vartheta-1}} \left| \frac{2b[1+(\log\varsigma)^{\vartheta-1}]}{(\log\varsigma)^{\vartheta-1}}(\log t)^{\vartheta-1} \right| \le \frac{2b[1+(\log\varsigma)^{\vartheta-1}]}{(\log\varsigma)^{\vartheta-1}} = \Phi b,$$

$$\beta(u) = \sup_{t\in[1,\infty)} \left| {}^H D_{1^+}^{\vartheta-1} \frac{2b[1+(\log\varsigma)^{\vartheta-1}]}{(\log\varsigma)^{\vartheta-1}}(\log t)^{\vartheta-1} \right| = \frac{2b[1+(\log\varsigma)^{\vartheta-1}]\Gamma(\vartheta)}{(\log\varsigma)^{\vartheta-1}} < N_2,$$

and

$$\zeta(u) = \min_{\varsigma\le t\le k\varsigma} \frac{1}{1+(\log t)^{\vartheta-1}} \left| \frac{2b[1+(\log\varsigma)^{\vartheta-1}]}{(\log\varsigma)^{\vartheta-1}}(\log t)^{\vartheta-1} \right| > b.$$

Then, it can be seen that $u \in \overline{P}(\psi^{\Phi b}, \beta^{N_2}, \zeta_b)$ and $\zeta(u) > b$. So $\{u \in \overline{P}(\psi^{\Phi b}, \beta^{N_2}, \zeta_b) : \zeta(u) > b\} \neq \emptyset$. For any $u \in \overline{P}(\psi^{\Phi b}, \beta^{N_2}, \zeta_b)$, we get

$$b \le \frac{u(t)}{1 + (\log t)^{\vartheta-1}} \le \Phi b \quad for \quad t \in [\varsigma, k\varsigma]$$

and $0 \le^H D_{1^+}^{\vartheta-1} u(t) \le N_2$, $t \in [1, \infty)$. In view of assumption (b), we get

$$\begin{aligned}
\zeta(Tu) &= \min_{\varsigma \le t \le k\varsigma} \frac{|Tu(t)|}{1 + (\log t)^{\vartheta-1}} \\
&\ge \int_1^\infty \min_{\varsigma \le t \le k\varsigma} \Big| \frac{G(t,s)}{1 + (\log t)^{\vartheta-1}} p(s) f(s, u(s), ^H D_{1^+}^{\vartheta-1} u(s)) \frac{ds}{s} \Big| \\
&\ge \sum_{i=1}^m \frac{\lambda_i (\log \varsigma)^{\vartheta-1}}{\Gamma(\vartheta + \beta_i)(1 + (\log \varsigma)^{\vartheta-1})} (\frac{1}{\Upsilon_1} - \frac{1}{\Upsilon}) \int_1^\infty g_i(\varsigma, s) p(s) f(s, u(s), ^H D_{1^+}^{\vartheta-1} u(s)) \frac{ds}{s} \\
&\ge \sum_{i=1}^m \frac{\lambda_i (\log \varsigma)^{\vartheta-1}}{\Gamma(\vartheta + \beta_i)(1 + (\log \varsigma)^{\vartheta-1})} (\frac{1}{\Upsilon_1} - \frac{1}{\Upsilon}) \int_\varsigma^{k\varsigma} g_i(\varsigma, s) p(s) f(s, u(s), ^H D_{1^+}^{\vartheta-1} u(s)) \frac{ds}{s} \\
&\ge \sum_{i=1}^m \frac{\lambda_i (\log \varsigma)^{2\vartheta+\beta_i-2}}{\Gamma(\vartheta + \beta_i)(1 + (\log \varsigma)^{\vartheta-1})} (\frac{1}{\Upsilon_1} - \frac{1}{\Upsilon}) \int_\varsigma^{k\varsigma} p(s) f(s, u(s), ^H D_{1^+}^{\vartheta-1} u(s)) \frac{ds}{s} \\
&> \sum_{i=1}^m \frac{\lambda_i (\log \varsigma)^{2\vartheta+\beta_i-2}}{\Gamma(\vartheta + \beta_i)(1 + (\log \varsigma)^{\vartheta-1})} (\frac{1}{\Upsilon_1} - \frac{1}{\Upsilon}) \frac{b}{\varrho} \int_\varsigma^{k\varsigma} p(s) \frac{ds}{s} \\
&= b.
\end{aligned}$$

So, $\zeta(Tu) > b$ for $u \in \overline{P}(\psi^{\Phi b}, \beta^{N_2}, \zeta_b)$. Finally, we show that the last condition of Theorem 2.1 is satisfied. Assume that $u \in \overline{P}(\psi^{p_2}, \beta^{N_2}, \zeta_b)$ with $\psi(Tu) > \Phi b$. Then, in view of the definition of ζ, we get

$$\begin{aligned}
\min_{\varsigma \le t \le k\varsigma} \frac{|Tu(t)|}{1 + (\log t)^{\vartheta-1}} &\ge \sum_{i=1}^m \frac{\lambda_i (\log \varsigma)^{2\vartheta+\beta_i-2}}{\Gamma(\vartheta + \beta_i)(1 + (\log \varsigma)^{\vartheta-1})} (\frac{1}{\Upsilon_1} - \frac{1}{\Upsilon}) \int_\varsigma^{k\varsigma} p(s) f(s, u(s), ^H D_{1^+}^{\vartheta-1} u(s)) \frac{ds}{s} \\
&\ge \sum_{i=1}^m \frac{\lambda_i (\log \varsigma)^{2\vartheta+\beta_i-2}}{\Gamma(\vartheta + \beta_i)(1 + (\log \varsigma)^{\vartheta-1})} (\frac{1}{\Upsilon_1} - \frac{1}{\Upsilon}) \frac{p_2}{\Phi \varrho} \int_\varsigma^{k\varsigma} p(s) \frac{ds}{s} \\
&\ge \frac{p_2}{\Phi} \ge \frac{\psi(Tu)}{\Phi} > b.
\end{aligned}$$

Hence, $\zeta(Tu) > b$, for all $u \in \overline{P}(\psi^{p_2}, \beta^{N_2}, \zeta_b)$. That is condition $(B3)$ of Theorem 2.1 holds.

Consequently, Theorem 2.1 yields that the operator T has at least three positive solutions u_i, $(i = 1, 2, 3)$ with

$$\begin{aligned}
&u_1 \in P(\psi^{p_1}, \beta^{N_1}),\ u_2 \in \{\overline{P}(\psi^{p_2}, \beta^{N_2}, \zeta_b) :\ \zeta(u) > b\}, \\
&u_3 \in \overline{P}(\psi^{p_2}, \beta^{N_2}) \setminus \big(\overline{P}(\psi^{p_2}, \beta^{N_2}, \zeta_b) \cup \overline{P}(\psi^{p_1}, \beta^{N_1})\big).
\end{aligned}$$

Example 3.1 *Consider the problem*

$$\begin{cases} ^H D_{1^+}^{\frac{5}{2}} u(t) + \frac{1}{t} f(t, u(t), ^H D_{1^+}^{\frac{3}{2}} u(t)) = 0, \qquad t \in (1, +\infty), \\ u(1) = u'(1) = 0, \quad ^H D_{1^+}^{\frac{3}{2}} u(\infty) = \displaystyle\int_1^\infty \frac{10}{195 e^{\log t}} u(t) \frac{dt}{t} + \sum_{i=1}^2 \lambda_i^H I_{1^+}^{\beta_i} u(e^{\frac{1}{4}}), \end{cases} \tag{3.4}$$

where $\vartheta = \frac{5}{2}$, $m = 2$, $n = 3$, $\lambda_1 = \lambda_2 = 28$, $\varsigma = e^{\frac{1}{4}}$, $p(t) = \frac{1}{t}$, $g(t) = \frac{10}{195 e^{\log t}}$,

$$f(t,u,v)=\begin{cases}\dfrac{e^{-\log t}}{1000}+\dfrac{0.15u}{140(1+(\log t)^{\frac{3}{2}})}+\dfrac{v}{10^5}, & u\in[0,140],\\ \dfrac{e^{-\log t}}{1000}+\dfrac{2.85(u-140)+1.5}{10(1+(\log t)^{\frac{3}{2}})}+\dfrac{v}{10^5}, & u\in[140,150],\\ \dfrac{e^{-\log t}}{1000}+\dfrac{0.5(u-150)+10350}{3450(1+(\log t)^{\frac{3}{2}})}+\dfrac{v}{10^5}, & u\in[150,3600],\\ \dfrac{e^{-\log t}}{1000}+\dfrac{7}{2(1+(\log t)^{\frac{3}{2}})}+\dfrac{v}{10^5}, & u\geq 3600,\end{cases}$$

for $t\in[1,\infty), v\in[0,\infty)$. By easy calculation, we get $\varrho\approx 67.8491$ for $k=10^5$ and $A_1\approx 833.3333$, $A_2\approx 1107.4999$, $\Phi=18$. Choosing $p_1=140$, $b=150$, $p_2=3600$, $N_1=2000$, $N_2=4000$ and for $\Phi b=2700$, one gets

$$f(t,(1+(\log t)^{\vartheta-1})u,v)\leq\min\left\{\tfrac{p_2}{A_1},\tfrac{N_2}{A_2}\right\}\approx 3.6117,\ \textit{for } t\in[1,\infty],\ u\in[0,3600],\ v\in[0,4000],$$

$$f(t,(1+(\log t)^{\vartheta-1})u,v)>\frac{b}{\varrho}\approx 2.2107,\ \textit{for } t\in[e^{\frac{1}{4}},10^5e^{\frac{1}{4}}],\ u\in[150,2700],\ v\in[0,4000],$$

$$f(t,(1+(\log t)^{\vartheta-1})u,v)<\min\left\{\tfrac{p_1}{A_1},\tfrac{N_1}{A_2}\right\}\approx 0.168\ \textit{for } t\in[1,\infty],\ u\in[0,140],\ v\in[0,2000],$$

$$f(t,(1+(\log t)^{\vartheta-1})u,v)\geq\tfrac{p_2}{\Phi\varrho}\approx 2.9477\ \textit{for } t\in[e^{\frac{1}{4}},10^5e^{\frac{1}{4}}],\ u\in[150,3600],\ v\in[0,4000].$$

i.e., f holds the conditions of Theorem 3.1. By using Theorem 2.1, the problem (3.4) has at least three positive solutions u_i for $i\in\{1,2,3\}$ with

$$0\leq\sup_{t\in[1,\infty)}\frac{u_1(t)}{1+(\log t)^{\frac{3}{2}}}\leq 140,\quad \sup_{t\in[1,\infty)}{}^HD^{\frac{3}{2}}_{1^+}u_1(t)\leq 2000;$$

$$150<\min_{e^{\frac{1}{4}}\leq t\leq 10^5e^{\frac{1}{4}}}\frac{u_2(t)}{1+(\log t)^{\frac{3}{2}}}\leq\sup_{t\in[1,\infty)}\frac{u_2(t)}{1+(\log t)^{\frac{3}{2}}}\leq 3600,\quad\sup_{t\in[1,\infty)}{}^HD^{\frac{3}{2}}_{1^+}u_2(t)\leq 4000;$$

$$\sup_{t\in[1,\infty)}\frac{u_3(t)}{1+(\log t)^{\frac{3}{2}}}\leq 3600,\quad\sup_{t\in[1,\infty)}{}^HD^{\frac{3}{2}}_{1^+}u_3(t)\leq 4000.$$

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