Rogue waves on the general periodic travelling waves background for an extended modified Korteweg-de Vries equation

Yu Lou[a], Yi Zhang[a,*], Rusuo Ye[a]

[a] *Department of Mathematics, Zhejiang Normal University, Jinhua 321004, PR China*

Abstract

Under consideration in this paper is rogue waves on the general periodic travelling waves background of an integrable extended modified Korteweg-de Vries equation. The general periodic travelling wave solutions are presented in terms of the sub-equation method. By means of the Darboux transformation and the nonlinearization of the Lax pair, we present the first-, second- and third-order rogue waves on the general periodic travelling waves background. Furthermore, the dynamic behaviors of rogue periodic waves are elucidated from the viewpoint of three-dimensional structures.

Keywords: extended modified Korteweg-de Vries equation, Darboux transformation, Jacobian elliptic function, rogue waves, periodic travelling waves

1. Introduction

A special solitary wave, rogue wave, has been extensively investigated in many fields, such as plasmas, oceanic waves, optics and even finance [1–4]. Rogue waves appear from nowhere and disappear without a trace possessing the properties of high peak and being rationally localized [5]. In recent years, rogue waves on the elliptic function background named rogue periodic waves have become more and more popular in nonlinear science. The rogue periodic wave phenomenon is an exact analytic rogue solution of soliton equations on a periodic background. The discovery of rogue periodic waves is of great significance because it discloses novel and absorbing nonlinear phenomena. A new excellent method was proposed to get the rogue periodic waves for the nonlinear Scrödinger equation [6] and modified Korteweg-de Vries (mKdV) equation [7]. Then the rogue periodic waves of the Hirota equation [8, 9], the sine-Gordon equation [10] and the fifth-order Ito equation [11] are obtained. In addition, multi-breather and high-order rogue waves of the nonlinear Scrödinger equation on the elliptic function background have also been presented [12]. As is known to all, Darboux transformation (DT) is an essential method to obtain multi-soliton and localized solutions of soliton equations [13–16], but a few studies of the rogue periodic waves have been done.

It is well known that the mKdV equation is a fundamental completely integrable nonlinear partial differential equation admitted the N-soliton solution [25] and plays a significant role in various physical contexts such as the generation of

*Corresponding author
Email address: zy2836@163.com (Yi Zhang)

supercontinuum in optical fibres [18], propagation of solitons in lattice [19], nonlinear Alfvén waves propagating in plasma [20], physical experiments of ion acoustic solitons in plasmas [21] and fluid mechanics [22]. Therefore, it is necessary to study the extended form of the mKdV equation. In this paper, we investigate an extended modified Korteweg-de Vries (emKdV) equation, which takes the form

$$u_t + \alpha(6u^2u_x + u_{xxx}) + \beta(30u^4u_x + 10u_x^3 + 40uu_xu_{xx} + 10u^2u_{xxx} + u_{xxxxx}) = 0, \tag{1}$$

where $\alpha << 1$ and $\beta << 1$ stand for the third- and fifth-order dispersion coefficients matching with the relevant nonlinear terms, respectively. It is noted that the multi-soliton solutions and rational solutions of Eq. (1) have been obtained [23, 24]. However, the rogue waves on the most general periodic travelling waves background of the emKdV equation (1) have not been reported.

The main aim of the present work is devoted to making a further investigation on rogue periodic waves of Eq. (1). In section 2, we give the Lax pair and the formula of the N-fold DT. In section 3, the most general periodic travelling wave solutions are presented in terms of the sub-equation method. In section 4, we discuss the nonlinearization of the Lax pair. In section 4, we construct the second solution of the Lax pair. In section 5, with the help of the DT method, the first-, second- and third-order rogue periodic waves are generated. The discussion and conclusion are located in the final section.

2. Lax pair and Darboux transformation

To begin with, the Lax pair of Eq. (1) can be given by [25]

$$\Phi_x = U\Phi,\ U = \begin{pmatrix} \lambda & u \\ -u & -\lambda \end{pmatrix}, \tag{2a}$$

$$\Phi_t = V\Phi,\ V = \begin{pmatrix} V_1 & V_2 \\ V_3 & -V_1 \end{pmatrix}, \tag{2b}$$

with

$$\begin{aligned}
V_1 =& -16\beta\lambda^5 - (8\beta u^2 + 4\alpha)\lambda^3 - (6\beta u^4 + 2\alpha u^2 + 4\beta uu_{xx} - 2\beta u_x^2)\lambda, \\
V_2 =& -16\beta u\lambda^4 - 8\beta u_x\lambda^3 - (8\beta u^3 + 4\alpha u + 4\beta u_{xx})\lambda^2 - (12\beta u^2u_x + 2\beta u_{xxx} + 2\alpha u_x)\lambda \\
&- 6\beta u^5 - 2\alpha u^3 - 10\beta u^2u_{xx} - 10\beta uu_x^2 - \beta u_{xxxx} - \alpha u_{xx}, \\
V_3 =& -V_2(-\lambda),
\end{aligned}$$

where Φ is the vector eigenfunction and λ is a spectral parameter. From the zero curvature equation, one can easily obtain Eq. (1). Thereafter, we give the DT of the emKdV equation (1) as follows.

Theorem 1. The unified DT of Eq. (1) reads

$$\Phi[1] = T[1]\Phi,\ T[1] = I - \frac{2\lambda_1}{\lambda + \lambda_1}\frac{\Phi_1\Phi_1^T}{\Phi_1^T\Phi_1}, \tag{3}$$

where $\Phi_1 = (f_1, g_1)^T$ is a special solution for (2) with $\lambda = \lambda_1$. In the following, it is worthy to show that system (2) can be transformed into

$$\Phi[1]_x = U[1]\Phi[1],\ \Phi[1]_t = V[1]\Phi[1], \tag{4}$$

and the transformation between potential functions reads

$$u[1] = u + 4\lambda_1 \frac{f_1 g_1}{f_1^2 + g_1^2}. \tag{5}$$

In what follows, we generalize the aforementioned elementary DT into the N-fold case.

Theorem 2. Assume that $\Phi_l = (f_l, g_l)^T (1 \leq l \leq N)$ be N nonzero solutions of system (2) with $\lambda = \lambda_l (1 \leq l \leq N)$. The N-fold DT of Eq. (1) can be represented as

$$\Phi[N] = T[N]\Phi,\ T[N] = I - XM^{-1}(\lambda I + S)^{-1}X^T, \tag{6}$$

and the transformation between old and new potential functions is

$$u[N] = u - 2\frac{det(M_1)}{det(M)}, \tag{7}$$

where $X = (\Phi_1, \Phi_2, ..., \Phi_N), S = diag(\lambda_1, \lambda_2, ..., \lambda_N)$,

$$M = \begin{pmatrix} \frac{\Phi_1^T\Phi_1}{2\lambda_1} & \frac{\Phi_1^T\Phi_2}{\lambda_2+\lambda_1} & \cdots & \frac{\Phi_1^T\Phi_N}{\lambda_N+\lambda_1} \\ \frac{\Phi_2^T\Phi_1}{\lambda_1+\lambda_2} & \frac{\Phi_2^T\Phi_2}{2\lambda_2} & \cdots & \frac{\Phi_2^T\Phi_N}{\lambda_N+\lambda_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Phi_N^T\Phi_1}{\lambda_1+\lambda_N} & \frac{\Phi_N^T\Phi_2}{\lambda_2+\lambda_N} & \cdots & \frac{\Phi_N^T\Phi_N}{2\lambda_N} \end{pmatrix},\ M_1 = \begin{pmatrix} \frac{\Phi_1^T\Phi_1}{2\lambda_1} & \frac{\Phi_1^T\Phi_2}{\lambda_2+\lambda_1} & \cdots & \frac{\Phi_1^T\Phi_N}{\lambda_N+\lambda_1} & g_1 \\ \frac{\Phi_2^T\Phi_1}{\lambda_1+\lambda_2} & \frac{\Phi_2^T\Phi_2}{2\lambda_2} & \cdots & \frac{\Phi_2^T\Phi_N}{\lambda_N+\lambda_2} & g_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\Phi_N^T\Phi_1}{\lambda_1+\lambda_N} & \frac{\Phi_N^T\Phi_2}{\lambda_2+\lambda_N} & \cdots & \frac{\Phi_N^T\Phi_N}{2\lambda_N} & g_N \\ f_1 & f_2 & \cdots & f_N & 0 \end{pmatrix}. \tag{8}$$

3. Periodic travelling wave solutions

We consider the travelling wave solution of Eq. (1)

$$u(x,t) = u(\xi), \quad \xi = x - vt, \tag{9}$$

where v is the wave speed. Substituting (9) into (1) and integrating once, we get

$$-vu + \alpha(2u^3 + u_{\xi\xi}) + \beta(6u^5 + 10uu_\xi^2 + 10u^2u_{\xi\xi} + u_{\xi\xi\xi\xi}) = \omega, \tag{10}$$

where ω is an integral constant.

Notice that it is laborious to get the exact travelling wave solution. Hence, we will make use of the sub-equation method [26] to obtain the exact travelling wave solution. Firstly, it can be seen that (10) contain the terms $u_{\xi\xi\xi\xi}$, $u_{\xi\xi}$ and u_ξ^2. Assume $u(\xi)$ satisfies the equation

$$u_\xi^2 = P_m(u), \tag{11}$$

where $P_m(u)$ is a polynomial function of degree m, then it solves

$$u_{\xi\xi} = \frac{1}{2}P'_m(u), \tag{12}$$

and

$$u_{\xi\xi\xi\xi} = \frac{1}{2}P'''_m(u)P_m(u) + \frac{1}{4}P''_m(u)P'_m(u). \tag{13}$$

Secondly, inserting (12) and (13) into (10) and then balancing the highest order of derivative term and nonlinear term, we can get $m = 4$. Accordingly, (11) can be expressed as follows

$$u_\xi^2 = P(u), \quad P(u) = a + bu + cu^2 + du^3 + eu^4, \tag{14}$$

where a, b, c ,d and e are constants. Substituting (14) into (10) and comparing the coefficients of like power of u, we have

$$\begin{aligned}
&u^5 : 24\beta(e+1)(e+\frac{1}{4}) = 0,\\
&u^4 : 30\beta(e+\frac{5}{6})d = 0,\\
&u^3 : \frac{1}{2}\beta[15d^5 + 40c(e+1)] + 2\alpha(e+1) = 0,\\
&u^2 : \frac{1}{2}\beta[15cd + 30b(e+1)] + \frac{3}{2}\alpha d = 0,\\
&u^1 : \frac{1}{2}\beta(24ae + obd + 2c^2 + 20a) + \alpha c - v = 0,\\
&u^0 : \frac{1}{2}\beta(6ad + bc) + \frac{1}{2}\alpha b - \omega = 0.
\end{aligned} \tag{15}$$

Solving these equations yields two pairs of solutions

$$a = a,\ b = b,\ c = c,\ d = 0,\ e = -1,\ \omega = \frac{1}{2}b + \frac{1}{2}\beta bc,\ v = \alpha c + \beta(c^2 - 2a), \tag{16}$$

and

$$a = a,\ b = 0,\ c = c,\ d = 0,\ e = -\frac{1}{4},\ \omega = 0,\ v = \beta c^2 + 7\beta a + \alpha c. \tag{17}$$

Thenceforth, we only take the first case (16) into consideration, namely,

$$u_\xi^2 = P(u), \quad P(u) = a + bu + cu^2 - u^4, \tag{18}$$

Besides, it is not difficult to find that the periodic travelling wave solution (9) reveals the following elliptic equations

$$u_{xx} + 2u^3 - cu - \frac{b}{2} = \omega, \tag{19}$$

$$u_{xxx} + 6u^2u_x - cu_x = 0. \tag{20}$$

(I) When the polynomial $P(u)$ admits four real roots ordered as $u_4 \leq u_3 \leq u_2 \leq u_1$ such that $u_1 + u_2 + u_3 + u_4 = 0$, the exact periodic travelling wave solution of the Eq. (1) is given by

$$u = u_4 + \frac{(u_1 - u_4)(u_2 - u_4)}{(u_2 - u_4) + (u_1 - u_2)sn^2(\tau x; \kappa)}, \tag{21}$$

with

$$\begin{cases} 4\tau^2 = (u_1 - u_3)(u_2 - u_4), \\ 4\tau^2\kappa^2 = (u_1 - u_2)(u_3 - u_4), \end{cases} \tag{22}$$

where $\tau > 0$ and $\kappa \in (0, 1)$ is the elliptic modulus. Then, with the aid of the Vieta theorem, we can get

$$\begin{cases} a = -u_1u_2u_3u_4, \\ b = u_1u_2u_3 + u_1u_2u_4 + u_1u_3u_4 + u_2u_3u_4, \\ c = -(u_1u_2 + u_1u_3 + u_1u_4 + u_2u_3 + u_2u_4 + u_3u_4). \end{cases} \tag{23}$$

Remark 1. Since $P(u)$ is even for $b = 0$, one can gain $u_4 = -u_1$ and $u_3 = -u_2$. Therefore, the expressions $a = -u_1^2u_2^2$, $c = u_1^2 + u_2^2$, $u_1 = \tau(1 + \kappa)$ and $u_2 = \tau(1 - \kappa)$ results from (22) and (23). For this reason, the periodic travelling wave solution (21) reduces to

$$u = u_1 dn(u_1\xi; k),\ k = \sqrt{1 - \frac{u_2^2}{u_1^2}}. \tag{24}$$

If $u_1 = 1$ and $u_2 = \sqrt{1 - k^2}$, (24) becomes the normalized dnoidal periodic wave solution

$$u = dn(x - vt; k),\ v = \alpha(2 - k^2) + \beta(k^4 - 6k^2 + 6). \tag{25}$$

(II) When the polynomial $P(u)$ admits two real roots $q \leq p$ and two complex conjugate roots $\mu \pm i\nu$ with $\mu = -\frac{p+q}{2}$, the exact periodic travelling wave solution of Eq. (1) is given by

$$u = p + \frac{(q - p)(1 - cn(\tau x; \kappa))}{1 + \delta + (\delta - 1)cn(\tau x; \kappa)}, \tag{26}$$

with

$$\begin{cases} \delta^2 = \frac{(q-\mu)^2 + \nu^2}{(p-\mu)^2 + \nu^2}, \\ \tau^2 = \sqrt{((p - \mu)^2 + \nu^2)((q - \mu)^2 + \nu^2)}, \\ 2\tau^2\kappa^2 = \tau^2 - (p - \mu)(q - \mu) + \nu^2, \end{cases} \tag{27}$$

where $\delta > 0$, $\tau > 0$. In the same token, using Vieta theorem, we can get

$$\begin{cases} a = -pq(\mu^2 + \nu^2), \\ b = 2pq\mu + (p + q)(\mu^2 + \nu^2), \\ c = -(pq + 2\mu(p + q) + \mu^2 + \nu^2). \end{cases} \tag{28}$$

Remark 2. Since $P(u)$ is even for $b = 0$, one can gain $q = -p$, $\mu = 0$ and $\delta = 1$. Thus, the periodic travelling wave solution (26) reduces to

$$u = pcn(\tau\xi; \kappa),\ \tau = \sqrt{p^2 + \nu^2},\ \kappa = \frac{p}{\sqrt{p^2 + \nu^2}}. \tag{29}$$

If $p = k$ and $\nu = \sqrt{1 - k^2}$, (29) becomes the normalized cnoidal periodic wave solution

$$u = kcn(x - vt; k),\ v = \alpha(2k^2 - 1) + \beta(6k^4 - 6k^2 + 1). \tag{30}$$

4. Nonlinearization of the Lax pair

We take the following Bargmann constraint into consideration

$$u = \phi_1^2 + \phi_2^2 + \varphi_1^2 + \varphi_2^2, \tag{31}$$

where $(\phi_1, \varphi_1)^T$ and $(\phi_2, \varphi_2)^T$ are two nonzero solutions to the Lax pair (2) with $\lambda = \lambda_1$, $\lambda = \lambda_2$ and $\lambda_1 \neq \pm\lambda_2$. Inserting (31) into (2a), we can get the Hamiltonian system of degree two

$$\frac{d\phi_j}{dx} = \frac{\partial H_0}{\partial \varphi_j}, \quad \frac{d\varphi_j}{dx} = -\frac{\partial H_0}{\partial \phi_j}, \quad j = 1, 2, \tag{32}$$

with

$$H_0(\phi_1, \varphi_1, \phi_2, \varphi_2) = \frac{1}{4}(\phi_1^2 + \varphi_1^2 + \phi_2^2 + \varphi_2^2)^2 + \lambda_1\phi_1\varphi_1 + \lambda_2\phi_2\varphi_2. \tag{33}$$

As is shown in Ref. [27], another conserved quantity can be obtained

$$\begin{aligned} H_1(\phi_1, \phi_2, \varphi_1, \varphi_2) = {} & 4(\lambda_1^3\phi_1\varphi_1 + \lambda_2^3\phi_2\varphi_2) - 4(\lambda_1\phi_1\varphi_1 + \lambda_2\phi_2\varphi_2)^2 - (\lambda_1(\phi_1^2 - \varphi_1^2) + \\ & \lambda_2(\phi_2^2 - \varphi_2^2))^2 + 2(\phi_1^2 + \phi_2^2 + \varphi_1^2 + \varphi_2^2)(\lambda_1^2(\phi_1^2 + \varphi_1^2) + \lambda_2^2(\phi_2^2 + \varphi_2^2)). \end{aligned} \tag{34}$$

Because H_1 is independent of x, we define two constants $F_0 = 4H_0$, $F_1 = 4H_1$. Differentiating (31) in x successively, we can derive the first-order differential equation

$$u_x = 2\lambda_1(\phi_1^2 - \varphi_1^2) + 2\lambda_2(\phi_2^2 - \varphi_2^2), \tag{35}$$

the second-order differential equation

$$u_{xx} + 2u^3 = cu - 4\lambda_2^2(\phi_1^2 + \varphi_1^2) - 4\lambda_1^2(\phi_2^2 + \varphi_2^2), \tag{36}$$

the third-order differential equation

$$u_{xxx} + 6u^2u_x - cu_x = -8\lambda_1\lambda_2(\lambda_2(\phi_1^2 - \varphi_1^2) + \lambda_1(\phi_2^2 - \varphi_2^2)), \tag{37}$$

and the fourth-order differential equation

$$u_{xxxx} + 10u^2u_{xx} + 10uu_x^2 + 6u^5 - c(u_{xx} + 2u^3) + 2au = 0, \tag{38}$$

where

$$c = 2F_0 + 4(\lambda_1^2 + \lambda_2^2), \tag{39}$$

$$a = 4F_0(\lambda_1^2 + \lambda_2^2) + 8\lambda_1^2\lambda_2^2 - F_0^2 - F_1, \tag{40}$$

$$F_0 = 4\lambda_1\phi_1\varphi_1 + 4\lambda_2\phi_2\varphi_2 + u^2, \tag{41}$$

$$F_1 = 16(\lambda_1^3\phi_1\varphi_1 + \lambda_2^3\phi_2\varphi_2) - (F_0 - u^2) + 2u(u_{xx} + 2u^3 - 2F_0u) - u_x^2. \tag{42}$$

Let us introduce

$$L = \begin{pmatrix} L_{11}(\lambda) & L_{12}(\lambda) \\ L_{12}(-\lambda) & -L_{11}(-\lambda) \end{pmatrix}, \tag{43}$$

where

$$\begin{aligned} L_{11}(\lambda) &= 1 - \frac{2\lambda_1\phi_1\varphi_1}{\lambda^2-\lambda_1^2} - \frac{2\lambda_2\phi_2\varphi_2}{\lambda^2-\lambda_2^2}, \\ L_{12}(\lambda) &= \frac{\phi_1^2}{\lambda-\lambda_1} + \frac{\varphi_1^2}{\lambda+\lambda_1} + \frac{\phi_2^2}{\lambda-\lambda_2} + \frac{\varphi_2^2}{\lambda+\lambda_2}. \end{aligned}$$

One can check directly the following Lax equation

$$L_x = [U, L]. \tag{44}$$

Computing the determinant of the matrix L yields the following differential constraints

$$\begin{aligned} &(u_{xx} + 2u^3 - 2F_0u)^2 - 16\lambda_1^2\lambda_2^2u - 2u_x(u_{xxx} + 6u^2u_x - cu_x) \\ &- 2(F_0 - u^2)(-a + 8\lambda_1^2\lambda_2^2 + u_x^2 - 2uu_{xx} - 3u^4 + cu^2) \\ &- 4(\lambda_1^2 + \lambda_2^2)\left[u_x^2 + (F_0 - u^2)^2\right] = 0, \end{aligned} \tag{45}$$

and

$$\begin{aligned} &(u_{xxx} + 6u^2u_x - 2F_0u_x)^2 + (F_1 + F_0^2 + u_x^2 - 2uu_{xx} - 3u^4 + 2F_0u^2)^2 \\ &- 4(\lambda_1^2 + \lambda_2^2)(u_{xx} + 2u^3 - 2F_0u)^2 - 16\lambda_1^2\lambda_2^2\left[u_x^2 + (F_0 - u^2)^2\right] \\ &+ 32\lambda_1^2\lambda_2^2u(u_{xx} + 2u^3 - 2F_0u) = 0. \end{aligned} \tag{46}$$

Plugging (20) into the fourth-order differential equation (38) generates

$$u_x^2 - 2u_{xx} - 3u^4 + cu^2 - a = 0. \tag{47}$$

By dint of the (20) and (47), we can rewrite Eqs. (45) and (46) in the equivalent forms

$$(u_{xx} + 2u^3 - 2F_0u)^2 - 4(\lambda_1^2 + \lambda_2^2)\left[u_x^2 + (F_0 - u^2)^2\right] - 16\lambda_1^2\lambda_2^2F_0 = 0, \tag{48}$$

and

$$\begin{aligned} &4(\lambda_1^4 + \lambda_1^2\lambda_2^2 + \lambda_2^4)\left[u_x^2 + (F_0 - u^2)^2\right] + 16\lambda_1^2\lambda_2^2(\lambda_1^+\lambda_2^2)(F_0 - u^2) \\ &+ 16\lambda_1^4\lambda_2^4 - (\lambda_1^2 + \lambda_2^2)(u_{xx} + 2u^3 - 2F_0u)^2 + 8\lambda_1^2\lambda_2^2u(u_{xx} + 2u^3 - 2F_0u) = 0. \end{aligned} \tag{49}$$

Inserting Eqs. (35) and (36) into Eqs. (48) and (49), one can obtain

$$b^2 = 16(\lambda_1^2 + \lambda_2^2)(F_0^2 + a) + 64\lambda_1^2\lambda_2^2F_0, \tag{50}$$

and

$$16(\lambda_1^4 + \lambda_1^2\lambda_2^2 + \lambda_1^4)(F_0^2 + a) + 64\lambda_1^2\lambda_2^2(\lambda_1^2 + \lambda_2^2)F_0 + 64\lambda_1^4\lambda_2^4 - (\lambda_1^2 + \lambda_2^2)b^2 = 0. \tag{51}$$

Solving the above equations (50) and (51), we have

$$a = 4\lambda_1^2\lambda_2^2 - F_0^2 \tag{52}$$

and

$$b^2 = 64\lambda_1^2\lambda_2^2(F_0 + \lambda_1^2 + \lambda_2^2). \tag{53}$$

Theorem 3. The Lax pair (2) with u given by the periodic waves (21) and (26) admits three pairs $\pm\lambda_1$, $\pm\lambda_2$, $\pm\lambda_3$ of eigenvalues with $Re(\lambda) \neq 0$ that corresponds to the periodic eigenfunctions Φ. For the periodic wave (21), the eigenvalues are located at

$$\lambda_1 = \frac{1}{2}(u_1 + u_2), \quad \lambda_2 = \frac{1}{2}(u_1 + u_3), \quad \lambda_3 = \frac{1}{2}(u_2 + u_3). \tag{54}$$

For the periodic wave (26), the eigenvalues are located at

$$\lambda_1 = \frac{1}{4}(p - q) + \frac{i}{2}\nu, \quad \lambda_2 = \frac{1}{4}(p - q) - \frac{i}{2}\nu, \quad \lambda_3 = \frac{1}{2}(p + q). \tag{55}$$

Proof It is noted that the expression (40) and the relation (52) infer

$$F_1 = 4F_0(\lambda_1^2 + \lambda_2^2) + 4\lambda_1^2\lambda_2^2. \tag{56}$$

Then, making use of (40), (52) and (53) results in the cubic equation about F_0

$$4(F_0^2 + a)(2F_0 + c) = b^2. \tag{57}$$

For the periodic wave (21), due to (23), the cubic equation (57) admits three roots in the following forms

$$(1)F_0 = \frac{1}{2}(u_1u_4 + u_2u_3), \quad (2)F_0 = \frac{1}{2}(u_1u_3 + u_2u_4), \quad (3)F_0 = \frac{1}{2}(u_1u_2 + u_3u_4). \tag{58}$$

Because of three possible choices of the two eigenvalues λ_1, λ_2, it is enough to consider one combination of the three eigenvalues

$$\lambda_1 = \frac{1}{2}(u_1 + u_2), \quad \lambda_2 = \frac{1}{2}(u_1 + u_3), \quad \lambda_3 = \frac{1}{2}(u_2 + u_3). \tag{59}$$

For the periodic wave (26), due to (28), the cubic equation (57) admits three roots in the following forms

$$(1)F_0 = \frac{1}{8}(p^2 + 6pq + q^2) + \frac{1}{2}\nu^2, \quad (2_\pm)F_0 = -\frac{1}{4}(p + q)^2 \pm \frac{i}{2}\nu(p - q). \tag{60}$$

Thus, the eigenvalues are in the forms

$$\lambda_1 = \frac{1}{4}(p - q) + \frac{i}{2}\nu, \quad \lambda_2 = \frac{1}{4}(p - q) - \frac{i}{2}\nu, \quad \lambda_3 = \frac{1}{2}(p + q). \tag{61}$$

In what follows, we will give the squared eigenfunctions of the Lax pair (2). By rewriting (35)-(38), we have

$$\begin{cases} \phi_1^2 + \varphi_1^2 + \phi_2^2 + \varphi_2^2 = u, \\ 2\lambda_1(\phi_1^2 - \varphi_1^2) + 2\lambda_2(\phi_2^2 - \varphi_2^2) = u_x, \\ 4\lambda_1^2(\phi_1^2 + \varphi_1^2) + 4\lambda_2^2(\phi_2^2 + \varphi_2^2) = u_{xx} + 2u^3 - 2F_0u, \\ 8\lambda_1^3(\phi_1^2 - \varphi_1^2) + 8\lambda_2^3(\phi_2^2 - \varphi_2^2) = u_{xxx} + 6u^2u_x - 2F_0u_x. \end{cases} \tag{62}$$

By solving (62) with Cramer's rule, one can get

$$\phi_1^2+\varphi_1^2=\frac{u_{xx}+2u^3-2F_0u-4\lambda_2^2u}{4(\lambda_1^2-\lambda_2^2)}, \tag{63}$$

$$\phi_1^2-\varphi_1^2=\frac{u_{xxx}+6u^2u_x-2F_0u-4\lambda_2^2u_x}{8\lambda_1(\lambda_1-\lambda_2^2)}, \tag{64}$$

$$\phi_2^2+\varphi_2^2=\frac{u_{xx}+2u^3-2F_0u-4\lambda_1^2u}{4(\lambda_2^2-\lambda_1^2)}, \tag{65}$$

$$\phi_2^2-\varphi_2^2=\frac{u_{xxx}+6u^2u_x-2F_0u-4\lambda_1^2u_x}{8\lambda_2(\lambda_2-\lambda_1^2)}. \tag{66}$$

Resorting to Eqs. (19), (20) and (39), the above equations can be simplified to the following forms

$$\phi_1^2+\varphi_1^2=\frac{b+8\lambda_1^2u}{8(\lambda_1^2-\lambda_2^2)}, \tag{67}$$

$$\phi_1^2-\varphi_1^2=\frac{\lambda_1u_x}{2(\lambda_1^2-\lambda_2^2)}, \tag{68}$$

$$\phi_2^2+\varphi_2^2=\frac{b+8\lambda_2^2u}{8(\lambda_2^2-\lambda_1^2)}, \tag{69}$$

$$\phi_2^2-\varphi_2^2=\frac{\lambda_2u_x}{2(\lambda_2^2-\lambda_1^2)}. \tag{70}$$

In the same token, we rewrite Eqs. (41) and (42) as a linear system

$$\begin{cases}4\lambda_1\phi_1\varphi_1+4\lambda_2\phi_2\varphi_2=F_0-u^2,\\ 16\lambda_1^3\phi_1\varphi_1+16\lambda_2^3\phi_2\varphi_2=F_1+F_0^2+u_x^2-2uu_x-3u^4+2F_0u^2.\end{cases} \tag{71}$$

Solving the above equations with Carmer's rule, we have

$$\phi_1\varphi_1=\frac{F_1+F_0^2+u_x^2-2uu_{xx}-3u^4+2F_0u^2+4\lambda_2^2u^2-4\lambda_2^2F_0}{16\lambda_1(\lambda_1^2-\lambda_2^2)}, \tag{72}$$

$$\phi_2\varphi_2=\frac{F_1+F_0^2+u_x^2-2uu_{xx}-3u^4+2F_0u^2+4\lambda_1^2u^2-4\lambda_1^2F_0}{16\lambda_2(\lambda_2^2-\lambda_1^2)}. \tag{73}$$

Thanks to Eqs. (39), (47), (52) and (56), we derive

$$\phi_1\varphi_1=\frac{\lambda_1(c-4\lambda_1^2-2u^2)}{8(\lambda_1^2-\lambda_2^2)}, \tag{74}$$

$$\phi_2\varphi_2=\frac{\lambda_2(c-4\lambda_1^2-2u^2)}{8(\lambda_2^2-\lambda_1^2)}. \tag{75}$$

5. Second solution of the Lax pair

For deriving the rogue periodic waves, we construct the second linearly independent solution $\Phi=(\hat{\phi}_1,\hat{\varphi}_1)^T$ of the Lax pair (2) with $\lambda=\lambda_1$.

Let $u(x,t) = u(x - vt)$ be a periodic travelling wave solution of the emKdV equation (1) with the wave speed v, so u satisfies (14)-(20). On the other hand, the eigenfunction $\Phi(x,t) = \Phi(x - vt)$ satisfying the time evolution in the Lax pair (2) should be considered. By virtue of (2b) and Eqs. (14)-(20), it is easy to get

$$\frac{\partial \phi_1}{\partial t} + v\frac{\partial \phi_1}{\partial x} = 0, \quad \frac{\partial \varphi_1}{\partial t} + v\frac{\partial \varphi_1}{\partial x} = 0. \tag{76}$$

Thus, $\phi_1(x,t) = \phi_1(x - vt)$ and $\varphi_1(x,t) = \varphi_1(x - vt)$ satisfy the time evolution in Lax pair (2). In the light of Ref. [28], the second linearly independent solution $\Phi = (\hat{\phi}_1, \hat{\varphi}_1)^T$ can be written by

$$\hat{\phi}_1 = \phi_1\vartheta_1 - \frac{2\varphi_1}{\phi_1^2 + \varphi_1^2}, \quad \hat{\varphi}_1 = \varphi_1\vartheta_1 + \frac{2\phi_1}{\phi_1^2 + \varphi_1^2}, \tag{77}$$

where ϑ_1 is a function of x and t to be determined. Substituting (77) into (2a) yields

$$\frac{\partial \vartheta_1}{\partial x} = -\frac{8\lambda_1\phi_1\varphi_1}{(\phi_1^2 + \varphi_1^2)^2}. \tag{78}$$

Owing to (68) and (72), (78) can be integrated to the form

$$\vartheta_1 = -16(\lambda_1^2 - \lambda_2^2)\left[4\lambda_1^2\int_0^x \frac{c - 4\lambda_1^2 - 2u^2}{(b + 8\lambda_1^2 u)^2}dy + \delta(t)\right], \tag{79}$$

where $\delta(t)$ is an undetermined integral function of t. Then, plugging (77) into (2b) with $\lambda = \lambda_1$ and employing (68), (72) and (78), one can have

$$\vartheta_{1,t} + v\vartheta_{1,x} = -16(\lambda_1^2 - \lambda_2^2)(\alpha + \beta c + 4\beta\lambda_1^2). \tag{80}$$

Hence, ϑ_1 is determined in the following form

$$\vartheta_1 = -16(\lambda_1^2 - \lambda_2^2)\left[4\lambda_1^2\int_0^{x-vt} \frac{c - 4\lambda_1^2 - 2u^2}{(b + 8\lambda_1^2 u)^2}dy + (\alpha + \beta c + 4\beta\lambda_1^2)t\right]. \tag{81}$$

Remark 3. It is noted that the numerators in (67), (68) and (72) for the eigenfunction $(\phi_1, \varphi_1)^T$ can be extended to the other two eigenfunctions $(\phi_2, \varphi_2)^T$ and $(\phi_3, \varphi_3)^T$ by replacing $\lambda_1 \mapsto \lambda_2$ and $\lambda_1 \mapsto \lambda_3$ respectively. On the other hand, the numerical factor $(\lambda_1^2 - \lambda_2^2)$ in ϑ_1 cancels with the denominators in (67), (68) and (72) for the squared eigenfunctions. Thus, the expression for ϑ_2 and ϑ_3 can be obtained with $\lambda_1 \mapsto \lambda_2$ and $\lambda_1 \mapsto \lambda_3$ respectively.

6. Rogue periodic waves

In this section, we derive rogue periodic wave solutions of emKdV equation (1). Inserting (77) and $(f_1, g_1) = (\hat{\phi}_1, \hat{\varphi}_1)$ into the one-fold DT (5), we obtain the first-order rogue periodic wave solution of the emKdV equation (1)

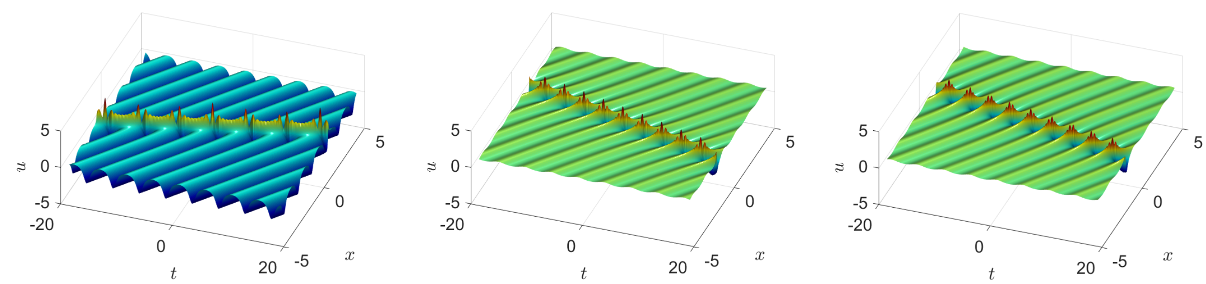
$$u[1] = u + 4\lambda_1\frac{\phi_1\varphi_1\left[(\phi_1^2 + \varphi_1^2)^2\vartheta_1^2 - 4\right] + 2(\phi_1^4 - \varphi_1^4)\vartheta_1}{\left[(\phi_1^2 + \varphi_1^2)^2\vartheta_1^2 + 4\right](\phi_1^2 + \varphi_1^2)}. \tag{82}$$

Similarly, substituting (77) into two- and three-fold DT results in the second- and third-order rogue periodic wave solutions respectively. The exact formulas are lengthy and hence omitted.

(I) Figs. 1-3 show the first-, second- and third-order rogue waves on the periodic travelling wave solution (21) with the parameter values

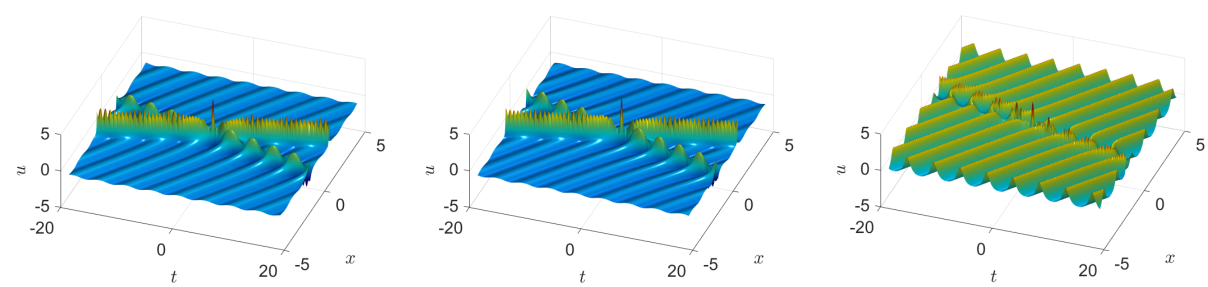
$$u_1 = 2, \quad u_2 = -0.25, \quad u_3 = -0.75, \quad u_4 = -1. \tag{83}$$

By virtue of different choices for the eigenvalue λ_1 given by (54), three new solutions are derived in Figs. 1 which display an algebraic soliton propagating on the periodic travelling wave background (21). The second-order rogue periodic wave solutions with three different choices of an eigenvalue pair (λ_1, λ_2) in (54) are shown in Figs. 2. For the third-order rogue periodic waves in Fig. 3, we only have one choice of $(\lambda_1, \lambda_2, \lambda_3)$ in (54).



(a) (b) (c)

Figure 1: The first-order rogue waves on the periodic travelling wave solution (21) with the parameters $\alpha = 1$, $\beta = 1$, $u_1 = 2$, $u_2 = -0.25$, $u_3 = -0.75$, $u_4 = -1$.



(a) (b) (c)

Figure 2: The second-order rogue waves on the periodic travelling wave solution (21) with the same parameters.

(II) Figs. 4 show the first-, second- and third-order rogue waves on the periodic travelling wave solution (26) with the parameter values

$$p = 1.5, \quad q = -0.5 \quad \mu = -0.5, \quad \nu = 2. \tag{84}$$

Due to the only real eigenvalue λ_3 in (55), we get the first-order rogue wave solution on the periodic travelling wave background (26) in Fig. 4(a). In Fig. 4(b), we choose a complex eigenvalue pair (λ_1, λ_2) in (55) resulting in the second-

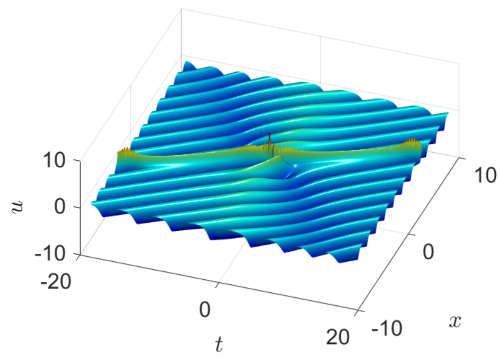
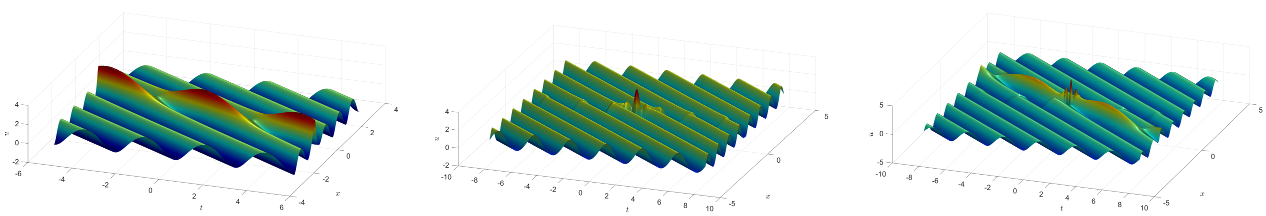


Figure 3: The third-order rogue waves on the periodic travelling wave solution (21) with the same parameters.

order rogue periodic wave solution. For the third-order rogue periodic wave solution in Fig. 4(c), we can only have one choice of $(\lambda_1, \lambda_2, \lambda_3)$ in (55).



(a) (b) (c)

Figure 4: The first-, second- and third-order rogue waves on the periodic travelling wave solution (26) with the parameters $\alpha = 1$, $\beta = 1$, $p = 1.5$, $q = -0.5$, $\mu = -0.5$, $\nu = 2$.

Remark 4. Since the Jacobian elliptic functions $sn^2(\tau x; \kappa)$ and $cn(\tau x; \kappa)$ are periodic in x with the periods $L_1 = \frac{2K(\kappa)}{\tau}$ and $L_2 = \frac{4K(\kappa)}{\tau}$ respectively, the solutions u in (21) and (26) are L_1-period and L_2-period respectively, where $K(\kappa)$ is the complete elliptic integral. Substituting $(f_j, g_j)^T = (\phi_j, \varphi_j)^T$ and $(f_j, g_j)^T = (\hat{\phi}_j, \hat{\varphi}_j)^T (j = 1, 2, 3)$ into the corresponding DTs leads to the solution $\tilde{u}$ and $\hat{u}$ respectively so that $\lim\limits_{|\vartheta_{1,2,3}| \to \infty} \hat{u} = \tilde{u}$ and $\lim\limits_{|\vartheta_{1,2,3}| \to 0} \hat{u} = 2u - \tilde{u}$. The characteristics of rogue periodic waves are summarized in Tables 1 and 2.

7. Conclusion

In this paper, we have studied the rogue waves on the general periodic travelling waves background of emKdV equation which can be seen as a higher-order integrable generalization of the standard mKdV equation. Based on the sub-equation method, the general periodic travelling wave solutions are presented. By means of Darboux transformation

Table 1: Characteristics of rogue waves on the periodic travelling wave background (21)

Solution	$\tilde{u}(0)$	$\hat{u}(0)$	$\tilde{u}(\frac{K(\kappa)}{\tau})$	$\hat{u}(\frac{K(\kappa)}{\tau})$
First-order with λ_1	$-u_2$	$2u_1+u_2$	$-u_1$	$2u_2+u_1$
First-order with λ_2	$-u_3$	$2u_1+u_3$	$-u_4$	$2u_2+u_4$
First-order with λ_3	$-u_4$	$2u_1+u_4$	$-u_3$	$2u_2+u_3$
Second-order with (λ_1,λ_2)	u_4	$2u_1-u_4$	u_3	$2u_2-u_3$
Second-order with (λ_1,λ_3)	u_3	$2u_1-u_3$	u_4	$2u_2-u_4$
Second-order with (λ_2,λ_3)	u_2	$2u_1-u_2$	u_1	$2u_2-u_1$
Third-order with $(\lambda_1,\lambda_2,\lambda_3)$	$-u_1$	$3u_1$	$-u_2$	$3u_2$

Table 2: Characteristics of rogue waves on the periodic travelling wave background (26)

Solution	$\tilde{u}(0)$	$\hat{u}(0)$	$\tilde{u}(\frac{2K(\kappa)}{\tau})$	$\hat{u}(\frac{2K(\kappa)}{\tau})$
First-order with λ_3	$-q$	$2p+q$	$-p$	$2q+p$
Second-order with (λ_1,λ_2)	q	$2p-q$	p	$2q-p$
Third-order with $(\lambda_1,\lambda_2,\lambda_3)$	$-p$	$3p$	$-q$	$3q$

and the nonlinearization of the Lax pair, we obtain the first-, second- and third-order rogue periodic wave solutions. Those solutions cover the known results in the literature. In addition, the dynamic behaviors of rogue periodic waves are discussed on account of three-dimensional figures. It is hoped that this work will be of certain value to investigate the rogue periodic waves in other nonlinear integrable models.

Acknowledge

This work is supported by the National Natural Science Foundation of China (No. 11371326 and No. 11975145).

References

[1] W. M. Moslem, P. K. Shukla, B. Eliasson. Surface plasma rogue waves. EPL. 96 (2011) 25002.

[2] K. Dysthe, H. E. Krogstad, P. Müller. Oceanic rogue waves. Annu. Rev. Fluid Mech. 40 (2008) 287-310.

[3] D. R. Solli, C. Ropers, P. Koonath, et al. Optical rogue waves. Nat. 450 (2007) 1054-1057.

[4] Z. Y. Yan. Vector financial rogue waves. Phys. Lett. A. 375 (2011) 4274-4279.

[5] N. Akhmediev, A. Ankiewicz, M. Taki. Waves that appear from nowhere and disappear without a trace. Phys. Lett. A. 373 (2009) 675-678.

[6] J. B. Chen, D. E. Pelinovsky. Rogue periodic waves of the focusing nonlinear Schrödinger equation. Proc. R. Soc. A. 474 (2018) 20170814.

[7] J. B. Chen, D. E. Pelinovsky. Rogue periodic waves of the modified KdV equation. Nonlinearity. 31 (2018) 1955-1980.

[8] W. Q. Peng, S. F. Tian, X. B. Wang, T. T. Zhang. Characteristics of rogue waves on a periodic background for the Hirota equation. Wave Motion. 93 (2019) 102454.

[9] X. Gao, H. Q. Zhang. Rogue waves for the Hirota equation on the Jacobi elliptic cn-function background. Nonlinear Dyn. 101 (2020) 1159-1168.

[10] R. Li, X. G. Geng. Rogue periodic waves of the sine-Gordon equation. Appl. Math. Lett. 102 (2020) 106147.

[11] H. Q. Zhang, X. Gao, Z. J. Peng, et al. Rogue periodic waves in the fifth-order Ito equation. Appl. Math. Lett. 107 (2020) 106464.

[12] B. F. Feng, L. M. Ling, D. A. Takahashi. Multi-breather and high-order rogue waves for the nonlinear Schrödinger equation on the elliptic function background. Stud. Appl. Math. 144 (2020) 46-101.

[13] B. L. Guo, L. M. Ling, Q. P. Liu, et al. Nonlinear Schrödinger equation: generalized Darboux transformation and rogue wave solutions. Phys. Rev. E. 85 (2012) 026607.

[14] Y. Zhang, R. S. Ye, W. X. Ma. Binary Darboux transformation and soliton solutions for the coupled complex modified Korteweg-de Vries equations. Math. Meth. Appl. Sci. 43 (2019) 613-627.

[15] Q. Y. Zhang, Y. Zhang, R. S. Ye. Exact solutions of nonlocal Fokas-Lenells equation. Appl. Math. Lett. 98 (2019) 336-343.

[16] Y. Zhang, J. W. Yang, K. W. Chow, et al. Solitons, breathers and rogue waves for the coupled Fokas-Lenells system via Darboux transformation. Nonlinear Anal. RWA. 33 (2017) 237-252.

[17] M. Wadati. The modified Korteweg-de Vries equation. J. Phys. Soc. Jpn. 34 (1973) 1289.

[18] H. Leblond, P. Grelu, D. Mihalache. Models for supercontinuum generation beyond the slowly-varying-envelope approximation. Phys. Rev. A. 90 (2014) 053816.

[19] Ono, Hiroaki. Soliton fission in anharmonic lattices with reflectionless inhomogeneity. J. Phys. Soc. Jpn. 61 (1992) 4336.

[20] A. H. Khater, O. H. El-Kalaawy, D. K. Callebaut. Bäcklund transformations and exact solutions for Alfvén solitons in a relativistic electron-positron plasma. Phys. Scr. 58 (1988) 545.

[21] K. E. Lonngren. Ion acoustic soliton experiments in a plasma. Opt. Quantum Electron. 30 (1998) 615.

[22] M. A. Helal. Soliton solution of some nonlinear partial differential equations and its applications in fluid mechanics. Chaos Soliton Fractals. 13 (2002) 1917.

[23] A. M. Wazwaz, G. Q. Xu. An extended modified KdV equation and its Painlevé integrability. Nonlinear Dyn. 86 (2016) 1455.

[24] X. Wang, J. L. Zhang, L. Wang. Conservation laws, periodic and rational solutions for an extended modified Korteweg-de Vries equation. Nonlinear Dyn. 92 (2018) 1507-1516.

[25] M. J. Ablowitz, D. J. Kaup, A. C. Newell, H. Segur. Nonlinear-evolution equations of physical significance. Phys. Rev. Lett. 31 (1973) 125.

[26] L. J. Zhang, C. M. Khalique. Exact solitary wave and periodic wave solutions of the Kaup-Kuper-Schmidt equation. J. Appl. Anal. Comput. 5 (2015) 485-495.

[27] C. W. Cao, Y. T. Wu, X. G. Geng. Relation between the Kadometsev-Petviashvili equation and the confocal involutive system. J. Math. Phys. 40 (1999) 3948-3970.

[28] J. B. Chen, D. E. Pelinovsky. Periodic travelling waves of the modified KdV equation and rogue waves on the periodic background. J. Nonlinear. Sci. 29 (2019) 2797-2843.