Riemann problem for rate-type materials with non-constant initial conditions

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Abstract

In this paper, using the compatible theory of differential invariants, a class of exact solutions is obtained for nonhomogeneous quasilinear hyperbolic system of partial differential equations (PDEs) describing rate type materials; these solutions exhibit genuine nonlinearity that leads to the formation of discontinuities such as shocks and rarefaction waves. For certain nonconstant and smooth initial data, the solution to the Riemann problem is presented providing a complete characterisation of the solutions.

1 Introduction

It is well known that a large number of physical processes are modelled by systems of quasilinear partial differential equations, but no general methods are available for solving such systems with arbitrary initial or/and boundary conditions ([1], [2]). A variety of mathematical methods for finding exact solutions to such systems have been proposed over the years (see [3]-[6]); however, among several others, the approach based on the use of differential constraints, proposed by Janenko [7] (see also [8] - [9]), has been of considerable interest in recent years (see [10] -[23]). Based on Lie symmetry analysis, an approximate rarefaction wave-type solution to the Riemann problem, with non-classical discontinuous initial data for a system of balance laws describing rate-type materials, was presented in [6]; here the initial data for the variable u are discontinuous whereas the initial data for the variable v are constants. A class of solutions to the partial differential equations, describing rate-type material, was obtained in [6] to solve a generalized Riemann problem through a rarefaction wave. In the present paper, an attempt is made to solve a family of generalized Riemann problems for the system under consideration and to completely characterize solutions that connect the initial data to regions either through shocks or rarefaction waves.

2 Compatibility conditions for Differential Invariants

Consider the following hyperbolic system

$$\frac{\partial v_j}{\partial t} + a_{jk}\frac{\partial v_k}{\partial x} = b_j;\ j,k = 1,2,\cdots,n, \tag{2.1}$$

where the matrices (a_{jk}) and (b_j) may be functions of x, t, and the unknowns v_1, v_2, $\cdots$, v_n. Let $\lambda^{(i)}$ be the real eigenvalues of (a_{jk}) and $R^{(i)}$ the corresponding eigenvectors; here and through out this section, summation from 1 to n over a repeated subscript is automatic unless stated otherwise. The system (2.1) can be written as

$$\frac{\partial v_j}{\partial t} + \lambda^{(i)} \frac{\partial v_j}{\partial x} + \hat{Q}_j^{(i)} = 0, \tag{2.2}$$

where

$$\hat{Q}_j^{(i)} = \left(a_{jk} - \lambda^{(i)} \delta_{kj} \right) \frac{\partial v_k}{\partial x} - b_j, \tag{2.3}$$

with $\delta_{kj} = 0$ for $k \neq j$ and $\delta_{kj} = 1$ for $k = j$. If $\hat{\mathbf{Q}}^{(i)}$ can be determined as functions of x, t and v_1, v_2, $\cdots$, v_n, such that the system (2.3) is consistent, then the system (2.2) can be solved along the characteristic family $\frac{dx}{dt} = \lambda^{(i)}$.

Since the matrix $\left(a_{jk} - \lambda^{(i)} \delta_{kj} \right)$ is of rank $n-1$, it follows from (2.3) that the derivatives $\frac{\partial v_j}{\partial x}$ can be expressed in the form

$$\frac{\partial v_j}{\partial x} = R_j^{(i)} \frac{\partial v_\alpha}{\partial x} + Q_j^{(i)}, \tag{2.4}$$

for some $\alpha \in \{1, 2, \cdots, n\}$ with $R_\alpha^{(i)} = 1$, where $R_j^{(i)}$ is the j^{th} component of eigenvector $\mathbf{R}^{(i)}$, v_α is the α^{th} component of $\mathbf{v}$, and $\mathbf{Q}^{(i)}$ is a particular solution to (2.3) with $Q_\alpha^{(i)} = 0$; thus, the components of $\hat{\mathbf{Q}}^{(i)}$ are determined in terms of the components of $\mathbf{Q}^{(i)}$, given by

$$\hat{Q}_j^{(i)} = \left(a_{jk} - \lambda^{(i)} \delta_{kj} \right) Q_k^{(i)} - b_j. \tag{2.5}$$

The compatibility conditions for determining $Q_j^{(i)}$ are as follows. In view of (2.4), equation (2.1) can be written as

$$\frac{\partial v_j}{\partial t} + a_{jk} \left(R_k^{(i)} \frac{\partial v_\alpha}{\partial x} + Q_k^{(i)} \right) = b_j. \tag{2.6}$$

Following an idea putforth in [3], the method of differential constraints consists of determining $Q_j^{(i)}$, and subsequently $\hat{Q}_j^{(i)}$ from (2.3), subject to the conditions (2.4); thus, equations (2.2) can be solved along a family of characteristics which in turn a gives a class of solutions to the equations (2.1). In order to achieve this objective, we differentiate (2.6) with respect to x to obtain

$$
\begin{aligned}
&\frac{\partial^2 v_j}{\partial x \partial t} + \frac{\partial a_{jk}}{\partial v_\ell}\left(R_\ell^{(i)}\frac{\partial v_\alpha}{\partial x} + Q_\ell^{(i)}\right)\left(R_k^{(i)}\frac{\partial v_\alpha}{\partial x} + Q_k^{(i)}\right) \\
&\qquad + a_{jk}R_k^{(i)}\frac{\partial^2 v_\alpha}{\partial x^2} + \frac{\partial a_{jk}}{\partial x}\left(R_k^{(i)}\frac{\partial v_\alpha}{\partial x} + Q_k^{(i)}\right) \\
&\qquad + a_{jk}\left(\frac{\partial R_k^{(i)}}{\partial x}\frac{\partial v_\alpha}{\partial x} + \frac{\partial Q_k^{(i)}}{\partial x}\right) \\
&\qquad + a_{jk}\left(\frac{\partial R_k^{(i)}}{\partial v_\ell}\frac{\partial v_\alpha}{\partial x} + \frac{\partial Q_k^{(i)}}{\partial v_\ell}\right)\left(R_\ell^{(i)}\frac{\partial v_\alpha}{\partial x} + Q_\ell^{(i)}\right) \\
&\qquad = \frac{\partial b_j}{\partial x} + \frac{\partial b_j}{\partial v_k}\left(R_k^{(i)}\frac{\partial v_\alpha}{\partial x} + Q_k^{(i)}\right). \qquad (2.7)
\end{aligned}
$$

Here and throughout this section, α and i are fixed and these indices are not to be summed. Similarly differentiating the equation (2.4) with respect to t, and replacing $\frac{\partial^2 v_\alpha}{\partial x \partial t}$ from (2.7) when $j = \alpha$, we obtain the following system on using (2.6):

$$
\begin{aligned}
\frac{\partial^2 v_j}{\partial x \partial t} \;=\; & \frac{\partial R_j^{(i)}}{\partial t}\frac{\partial v_\alpha}{\partial x} + \left(\frac{\partial R_j^{(i)}}{\partial v_k}\frac{\partial v_\alpha}{\partial x} + \frac{\partial Q_j^{(i)}}{\partial v_k}\right)\left(b_k - a_{k\ell}R_\ell^{(i)}\frac{\partial v_\alpha}{\partial x} - a_{k\ell}Q_\ell^{(i)}\right) \\
& + \frac{\partial Q_j^{(i)}}{\partial t} + R_j^{(i)}\frac{\partial b_\alpha}{\partial x} + R_j^{(i)}\frac{\partial b_\alpha}{\partial v_k}\left(R_k^{(i)}\frac{\partial v_\alpha}{\partial x} + Q_k^{(i)}\right) \\
& - R_j^{(i)}\frac{\partial a_{\alpha k}}{\partial v_\ell}\left(R_\ell^{(i)}\frac{\partial v_\alpha}{\partial x} + Q_\ell^{(i)}\right)\left(R_k^{(i)}\frac{\partial v_\alpha}{\partial x} + Q_k^{(i)}\right) \\
& - R_j^{(i)}\frac{\partial a_{\alpha k}}{\partial x}\left(R_k^{(i)}\frac{\partial v_\alpha}{\partial x} + Q_k^{(i)}\right) - R_j^{(i)}a_{\alpha k}\left(\frac{\partial R_k^{(i)}}{\partial x}\frac{\partial v_\alpha}{\partial x} + \frac{\partial Q_k^{(i)}}{\partial x}\right) \\
& - R_j^{(i)}a_{\alpha k}\left(\frac{\partial R_k^{(i)}}{\partial v_\ell}\frac{\partial v_\alpha}{\partial x} + \frac{\partial Q_k^{(i)}}{\partial v_\ell}\right)\left(R_\ell^{(i)}\frac{\partial v_\alpha}{\partial x} + Q_\ell^{(i)}\right) \\
& \qquad - R_j^{(i)}a_{\alpha k}R_k^{(i)}\frac{\partial^2 v_\alpha}{\partial x^2}. \qquad (2.8)
\end{aligned}
$$

Equations (2.7) and (2.8) imply that

$$
T_j^{(i)} + S_j^{(i)}\frac{\partial v_\alpha}{\partial x} + \tilde{T}_j^{(i)}\frac{\partial^2 v_\alpha}{\partial x^2} + \tilde{S}_j^{(i)}\left(\frac{\partial v_\alpha}{\partial x}\right)^2 = 0, \qquad (2.9)
$$

where

$$
\begin{aligned}
T_j^{(i)} &= \left(\frac{\partial Q_j^{(i)}}{\partial v_k}\right)\left(b_k - a_{k\ell}Q_\ell^{(i)}\right) + \frac{\partial Q_j^{(i)}}{\partial t} + R_j^{(i)}\frac{\partial b_\alpha}{\partial x} + R_j^{(i)}\frac{\partial b_\alpha}{\partial v_k}Q_k^{(i)} \\
&\quad - R_j^{(i)}\frac{\partial a_{\alpha k}}{\partial v_\ell}Q_\ell^{(i)}Q_k^{(i)} - R_j^{(i)}\frac{\partial a_{\alpha k}}{\partial x}Q_k^{(i)} - R_j^{(i)}a_{\alpha k}\frac{\partial Q_k^{(i)}}{\partial x} \\
&\quad - R_j^{(i)}a_{\alpha k}\frac{\partial Q_k^{(i)}}{\partial v_\ell}Q_\ell^{(i)} + \frac{\partial a_{jk}}{\partial v_\ell}Q_\ell^{(i)}Q_k^{(i)} + \frac{\partial a_{jk}}{\partial x}Q_k^{(i)} + a_{jk}\frac{\partial Q_k^{(i)}}{\partial x} \\
&\quad + a_{jk}\frac{\partial Q_k^{(i)}}{\partial v_\ell}Q_\ell^{(i)} - \frac{\partial b_j}{\partial x} - \frac{\partial b_j}{\partial v_k}Q_k^{(i)}, \\
\tilde{T}_j^{(i)} &= a_{jk}R_k^{(i)} - a_{\alpha k}R_k^{(i)}R_j^{(i)}, \\
\tilde{S}_j^{(i)} &= R_\ell^{(i)}\frac{\partial\left(a_{jk}R_k^{(i)}\right)}{\partial v_\ell} - R_j^{(i)}R_\ell^{(i)}\frac{\partial\left(a_{\alpha k}R_k^{(i)}\right)}{\partial v_\ell} - a_{k\ell}R_\ell^{(i)}\frac{\partial R_j^{(i)}}{\partial v_k},
\end{aligned}
$$

$$
\begin{aligned}
S_j^{(i)} &= \frac{\partial R_j^{(i)}}{\partial t} + \frac{\partial R_j^{(i)}}{\partial v_k}\left(b_k - a_{k\ell}Q_\ell^{(i)}\right) - \frac{\partial Q_j^{(i)}}{\partial v_k}a_{k\ell}R_\ell^{(i)} + R_j^{(i)}\frac{\partial b_\alpha}{\partial v_k}R_k^{(i)} \\
&\quad - R_j^{(i)}\frac{\partial a_{\alpha k}}{\partial v_\ell}\left(R_\ell^{(i)}Q_k^{(i)} + Q_\ell^{(i)}R_k^{(i)}\right) - R_j^{(i)}\frac{\partial a_{\alpha k}}{\partial x}R_k^{(i)} - R_j^{(i)}a_{\alpha k}\frac{\partial R_k^{(i)}}{\partial x} \\
&\quad - R_j^{(i)}a_{\alpha k}\left(\frac{\partial R_k^{(i)}}{\partial v_\ell}Q_\ell^{(i)} + \frac{\partial Q_k^{(i)}}{\partial v_\ell}R_\ell^{(i)}\right) + \frac{\partial a_{jk}}{\partial x}R_k^{(i)} + a_{jk}\frac{\partial R_k^{(i)}}{\partial x} \\
&\quad + \frac{\partial a_{jk}}{\partial v_\ell}\left(R_\ell^{(i)}Q_k^{(i)} + Q_\ell^{(i)}R_k^{(i)}\right) + a_{jk}\left(\frac{\partial R_k^{(i)}}{\partial v_\ell}Q_\ell^{(i)} + \frac{\partial Q_k^{(i)}}{\partial v_\ell}R_\ell^{(i)}\right) \\
&\quad - \frac{\partial b_j}{\partial v_k}R_k^{(i)},
\end{aligned}
$$

for $j = 1$ to n for each $i \in \{1, 2, \cdots, n\}$. Since of, $a_{\alpha k}R_k^{(i)} = \lambda^{(i)}R_\alpha^{(i)} = \lambda^{(i)}$ and $a_{jk}R_k^{(i)} = \lambda^{(i)}R_j^{(i)}$, we have $\tilde{T}_j^{(i)} = 0$ and $\tilde{S}_j^{(i)} = 0$. Thus, equations (2.9) reduce to

$$
T_j^{(i)} + S_j^{(i)}\frac{\partial v_\alpha}{\partial x} = 0, \tag{2.10}
$$

for $j = 1$ to n for each $i \in \{1, 2, \cdots, n\}$. Observe that $T_\alpha^{(i)} \equiv 0$ and $S_\alpha^{(i)} \equiv 0$. In the following section, we use this methodology to a system of conservation laws describing rate-type materials.

3 Solutions to the Cauchy problem:

We consider the following system of balance laws describing rate-type materials ([7]-[9])

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\frac{1}{v}\right) = 0, \quad \frac{\partial v}{\partial t} + \frac{\partial u}{\partial x} = 1 - v, \tag{3.1}$$

where u is the Lagrangian velocity and $1/v$ with $v \neq 0$ denotes the stress in the material that is undergoing loading unloading processes.

The eigenvalues $\lambda^{(i)}$, $i = 1, 2$, representing the characteristic speeds of the system (3.1) and the corresponding right eigenvectors $R^{(i)}$, are given by

$$\lambda^{(1)} = -\frac{1}{v}, \;\; \lambda^{(2)} = \frac{1}{v}, \;\; R^{(1)} = \begin{bmatrix} -v^{-1} \\ 1 \end{bmatrix}, \;\; R^{(2)} = \begin{bmatrix} v^{-1} \\ 1 \end{bmatrix}.$$

As the system (3.1) is strictly hyperbolic and genuinely nonlinear for any smooth initial data:

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \tag{3.2}$$

there exits a unique solution of the Cauchy problem (3.1), (3.2) involving either a rarefaction wave or a shock wave depending on whether $\lambda^{(i)}$ is monotonically increasing or decreasing as (u, v) varies along an integral curve of the vector field $R^{(i)}$.

In view of (2.4), we have $\dfrac{\partial u}{\partial x} = Q_1^{(1)} - \dfrac{1}{v}\dfrac{\partial v}{\partial x}$, and so, equations (3.1) can be written as

$$\frac{du}{dt} = -\frac{Q_1^{(1)}}{v}, \quad \frac{dv}{dt} = 1 - v - Q_1^{(1)}, \tag{3.3}$$

where $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + \lambda^{(1)}\dfrac{\partial}{\partial x}$; and $Q_1^{(1)}$ is a function of x, t, u and v, which is to be determined from the equation (2.10), i.e.,

$$T_1^{(1)} + S_1^{(1)}\frac{\partial v}{\partial x} = 0; \tag{3.4}$$

here $T_1^{(1)}$ and $S_1^{(1)}$ are given by

$$T_1^{(1)} = v\left(\frac{\partial Q_1^{(1)}}{\partial x} + v\frac{\partial Q_1^{(1)}}{\partial t} + Q_1^{(1)}\frac{\partial Q_1^{(1)}}{\partial u} + v\left(1 - v - Q_1^{(1)}\right)\frac{\partial Q_1^{(1)}}{\partial v}\right),$$

$$S_1^{(1)} = 1 - Q_1^{(1)} - 2\frac{\partial Q_1^{(1)}}{\partial u} + 2v\frac{\partial Q_1^{(1)}}{\partial v}.$$

Similarly, when $\dfrac{\partial u}{\partial x} = Q_1^{(2)} + \dfrac{1}{v}\dfrac{\partial v}{\partial x}$, equations (3.1) can be written as

$$\frac{du}{dt} = \frac{Q_1^{(2)}}{v}, \quad \frac{dv}{dt} = 1 - v - Q_1^{(2)}, \tag{3.5}$$

where $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + \lambda^{(2)} \dfrac{\partial}{\partial x}$; and $Q_1^{(2)}$ is a function of x, t, u and v, which is to be determined from the equation (2.10), i.e.,

$$T_1^{(2)} + S_1^{(2)} \frac{\partial v}{\partial x} = 0; \tag{3.6}$$

here $T_1^{(2)}$ and $S_1^{(2)}$ are given by

$$T_1^{(2)} = -v \left(\frac{\partial Q_1^{(2)}}{\partial x} - v \frac{\partial Q_1^{(2)}}{\partial t} + Q_1^{(2)} \frac{\partial Q_1^{(2)}}{\partial u} - v \left(1 - v - Q_1^{(2)}\right) \frac{\partial Q_1^{(2)}}{\partial v} \right),$$

$$S_1^{(2)} = -1 + Q_1^{(2)} - 2 \frac{\partial Q_1^{(2)}}{\partial u} - 2v \frac{\partial Q_1^{(2)}}{\partial v}.$$

Assume that $T_1^{(i)} = 0$ and $S_1^{(i)} = 0$ for $i = 1, 2$. For the case $i = 1$, by solving the equation $S_1^{(1)} = 0$, we get

$$Q_1^{(1)} = 1 + \mathrm{e}^{-u/2} \phi(x, t, \xi), \tag{3.7}$$

where $\xi = u + \log v$ and ϕ being an arbitrary function of x, t and ξ. In view of (3.7), the equation $T_1^{(1)} = 0$ reduces to

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial \xi} - \frac{1}{2}\phi + \mathrm{e}^{(\xi - u)} \left(\frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial \xi} \right) - \frac{1}{2} \mathrm{e}^{-u/2} \phi^2 = 0,$$

which leads to $\phi \equiv 0$, i.e., $Q_1^{(1)} \equiv 1$. Similarly, with the assumption that $T_1^{(2)} = S_1^{(2)} = 0$, we have $Q_1^{(2)} \equiv 1$. Thus, on solving (3.3) and (3.5), equations (3.1) admit the following solutions, which recovers the solution obtained in [23] following a different line of approach:

$$\begin{aligned} v(x,t) &= v_0(\xi) \mathrm{e}^{-t}, \\ u(x,t) &= u_0(\xi) + \delta \left(\frac{\mathrm{e}^t - 1}{v_0(\xi)} \right), \\ x &= \xi + \delta \left(\frac{\mathrm{e}^t - 1}{v_0(\xi)} \right), \quad \delta = \pm 1. \end{aligned} \tag{3.8}$$

Here, $\xi(x,t)$ denotes the unique point on the x-axis, which lies on the characteristic through (x,t) and is given by $(3.8)_3$. For $\delta = \pm 1$, the above equations (3.8) give two solutions of the system (3.1), (3.2), one for each characteristic family; indeed, the above solutions are characterized by the differential constraints:

$$\frac{du_0(x)}{dx} = \frac{\delta}{v_0(x)} \frac{dv_0}{dx} + 1. \tag{3.9}$$

Observe that, for a given x and t, the equations $(3.8)_1$ and $(3.8)_2$ admit unique values for v and u provided there exists a unique ξ satisfying $(3.8)_3$;

in other words, the existence of a unique solution is guaranteed for every x in $(-\infty, \infty)$ and for every $t > 0$ provided that

$$\delta \left(\frac{e^t - 1}{(v_0(\xi))^2} \right) \frac{dv_0}{d\xi} \neq 1. \tag{3.10}$$

4 Shocks and rarefaction waves

There are two distinct families of discontinuous solutions of (3.1), (3.2), referred to as 1-shocks (or back shocks) and 2-shocks (or front shocks). Similarly, there are two families of continuous solutions of (3.1), (3.2), referred to as rarefaction waves corresponding to either characteristic family $\lambda^{(1)}$ or $\lambda^{(2)}$.

Let $x = X(t)$ be a curve representing a discontinuity across which the flow variables u and v are discontinuous and let $\sigma = \frac{dX}{dt}$ be the speed of propagation of the discontinuity. Then R-H conditions for the system (3.1) are

$$\sigma(u_\ell(t) - u_r(t)) = \frac{1}{v_r(t)} - \frac{1}{v_\ell(t)}, \quad \sigma(v_\ell(t) - v_r(t)) = (u_\ell(t) - u_r(t)), \tag{4.1}$$

where $u_\ell(t) = \lim\limits_{x \to X(t)^-} u(x,t)$, $u_r(t) = \lim\limits_{x \to X(t)^+} u(x,t)$, $v_\ell(t) = \lim\limits_{x \to X(t)^-} v(x,t)$, and $v_r(t) = \lim\limits_{x \to X(t)^+} v(x,t)$. Equations (4.1) imply that

$$\sigma = \pm \frac{1}{(v_r v_\ell)^{1/2}}, \quad u_\ell = u_r + \sigma\,(v_\ell - v_r)\,. \tag{4.2}$$

If the admitted discontinuity $x = S_1(t)$ is a consequence of the intersection of characteristics belonging to the family $\frac{dx}{dt} = -\frac{1}{v}$, satisfying

$$-\frac{1}{v_\ell} > \sigma > -\frac{1}{v_r}, \tag{4.3}$$

then the discontinuity $x = S_1(t)$ is called a 1-shock or a back shock; the inequality (4.3) shows that $\sigma < 0$ and therefore, for a 1-shock, we have

$$\sigma = \frac{dS_1}{dt} = -\frac{1}{(v_r v_\ell)^{1/2}}, \quad u_\ell = u_r - \frac{(v_\ell - v_r)}{(v_r v_\ell)^{1/2}}, \tag{4.4}$$

with $v_\ell(t) > v_r(t)$ and $u_\ell(t) < u_r(t)$.

Similarly, if the admitted discontinuity $x = S_2(t)$ is a consequence of the intersection of characteristics belonging to the family $\frac{dx}{dt} = \frac{1}{v}$, satisfying

$$\frac{1}{v_\ell} > \sigma > \frac{1}{v_r}, \tag{4.5}$$

then the discontinuity $x = S_2(t)$ is called a 2-shock or a front shock satisfying $\sigma > 0$ with

$$\sigma = \frac{dS_2}{dt} = \frac{1}{(v_r v_\ell)^{1/2}}, \quad u_\ell = u_r + \frac{(v_\ell - v_r)}{(v_r v_\ell)^{1/2}}. \tag{4.6}$$

$v_\ell < v_r$ and $u_\ell < u_r$.

We now turn to the rarefaction wave solutions of (3.1), (3.2) which are continuous solutions corresponding to the eigen modes $\lambda^{(1)}$ and $\lambda^{(2)}$, referred to as 1-rarefaction wave and 2-rarefaction waves, respectively. Let $u_L = \lim_{x\to 0^-} u_0(x)$, $v_L = \lim_{x\to 0^-} v_0(x)$, $u_R = \lim_{x\to 0^+} u_0(x)$, $v_R = \lim_{x\to 0^+} v_0(x)$, such that the initial step function is expansive with $v_L < v_R$. Let $x = R_1(t)$ and $x = R_2(t)$ be the curves that pass through $(0, 0)$ such that $R_1(t) < R_2(t)$ for all $t > 0$ with $R_1(t)$ and $R_2(t)$ satisfying

$$\frac{dR_1}{dt} = \frac{-1}{v(R_1(t), t)}, \quad \frac{dR_2}{dt} = \frac{-1}{v(R_2(t), t)}. \tag{4.7}$$

In view of (3.8), equations (4.7) lead to $R_1(t) = \frac{(1-\mathrm{e}^t)}{v_L}$, $R_2(t) = \frac{(1-\mathrm{e}^t)}{v_R}$. Since, $v_L < v_R$, we have $v(R_1(t), t) < v(R_2(t), t)$; a continuously varying solution in the region $R_1(t) < x < R_2(t)$, which is continuous across the curves $x = R_1(t)$ and $x = R_2(t)$, referred to as 1-rarefaction wave, can be obtained from (3.8) as follows. Since all the values of u (respectively, v) between u_L and u_R (respectively, v_L and v_R) are taken on characteristics in a fan through origin, where $\xi = 0$, the solution in the fan, bounded by the characteristics $x = R_1(t) = (1 - \mathrm{e}^t)/v_L$ and $x = R_2(t) = (1 - \mathrm{e}^t)/v_R$ is given by

$$\begin{aligned} &v(x,t) = z\mathrm{e}^{-t}, \text{ if } R_1(t) < x < R_2(t), \\ &u(x,t) = \zeta - \left(\frac{\mathrm{e}^t - 1}{z}\right), \text{ if } R_1(t) < x < R_2(t), \\ &x = -\left(\frac{\mathrm{e}^t - 1}{z}\right), \end{aligned} \tag{4.8}$$

where $R_1(t) < x < R_2(t)$, $v_L < z < v_R$, and $u_L < \zeta < u_R$. Here, the characteristics are emanating from the origin and given by $\dfrac{dx}{dt} = -\dfrac{1}{v} = -\dfrac{\mathrm{e}^t}{z}$ whose speeds are varying from $-1/v_L$ to $-1/v_R$. Differentiating the equations (4.8) with respect to x and t and substituting in (3.1) we get

$$\begin{aligned} &\frac{\partial \zeta}{\partial t} + \frac{\mathrm{e}^t}{\mathrm{e}^t - 1} = 0, \\ &\frac{\partial \zeta}{\partial x} - \frac{1}{x} = 0. \end{aligned}$$

Subject to the condition $\zeta = u_L$ when $x = R_1(t)$, the above equations $\zeta = u_L + \log\left(\frac{xv_L}{1-\mathrm{e}^t}\right)$. Thus, the solution for 1- rarefaction wave is given by

$$v(x,t) = \begin{cases} v_0(\xi)\mathrm{e}^{-t} & \text{if} \quad x \leq R_1(t), \\ \left(\frac{\mathrm{e}^{-t}-1}{x}\right) & \text{if} \quad R_1(t) < x < R_2(t), \\ v_0(\xi)\mathrm{e}^{-t} & \text{if} \quad x \geq R_2(t), \end{cases} \tag{4.9}$$

$$u(x,t) = \begin{cases} u_0(\xi) - \left(\frac{\mathrm{e}^t-1}{v_0(\xi)}\right) & \text{if} \quad x \leq R_1(t), \\ u_L + \log\left(\frac{xv_L}{1-\mathrm{e}^t}\right) + x & \text{if} \quad R_1(t) < x < R_2(t), \\ u_0(\xi) - \left(\frac{\mathrm{e}^t-1}{v_0(\xi)}\right) & \text{if} \quad x \geq R_2(t), \end{cases} \tag{4.10}$$

$$x = \xi - \left(\frac{\mathrm{e}^t - 1}{v_0(\xi)}\right), \quad \frac{du_0}{dx} = 1 - \frac{1}{v_0}\frac{dv_0}{dx},$$

with $u_R = u_L + \log\left(\frac{v_L}{v_R}\right)$, $R_1(t) = \frac{(1-\mathrm{e}^t)}{v_L}$, $R_2(t) = \frac{(1-\mathrm{e}^t)}{v_R}$ and $v(R_1(t),t) < v(R_2(t),t)$.

Similarly, let $x = R_3(t)$ and $x = R_4(t)$ be the curves that pass through $(0,0)$ such that $R_3(t) < R_4(t)$ for all $t > 0$

$$\frac{dR_3}{dt} = \frac{1}{v(R_3(t),t)}, \quad \frac{dR_4}{dt} = \frac{1}{v(R_4(t),t)}. \tag{4.11}$$

Then, since $v_L > v_R$ and $R_3(t) < R_4(t)$ for all $t > 0$, we have $v(R_3(t),t) > v(R_4(t),t)$. A continuously varying solution in the region $R_3(t) < x < R_4(t)$, which is continuous across the curves $x = R_3(t)$ and $x = R_4(t)$, referred to as a 2-rarefaction wave, can be obtained in a similar manner, and is given by

$$v(x,t) = \begin{cases} v_0(\xi)\mathrm{e}^{-t} & \text{if} \quad x \leq R_3(t), \\ \left(\frac{1-\mathrm{e}^{-t}}{x}\right) & \text{if} \quad R_3(t) < x < R_4(t), \\ v_0(\xi)\mathrm{e}^{-t} & \text{if} \quad x \geq R_4(t), \end{cases} \tag{4.12}$$

$$u(x,t) = \begin{cases} u_0(\xi) + \left(\frac{\mathrm{e}^t-1}{v_0(\xi)}\right) & \text{if} \quad x \leq R_3(t), \\ u_R - \log\left(\frac{xv_R}{1-\mathrm{e}^t}\right) + x & \text{if} \quad R_3(t) < x < R_4(t), \\ u_0(\xi) + \left(\frac{\mathrm{e}^t-1}{v_0(\xi)}\right) & \text{if} \quad x \geq R_4(t), \end{cases} \tag{4.13}$$

$$x = \xi + \left(\frac{\mathrm{e}^t - 1}{v_0(\xi)}\right), \quad \frac{du_0}{dx} = 1 + \frac{1}{v_0}\frac{dv_0}{dx},$$

with $u_L = u_R - \log\left(\frac{v_R}{v_L}\right)$, $R_3(t) = \frac{(\mathrm{e}^t - 1)}{v_L}$, $R_4(t) = \frac{(\mathrm{e}^t - 1)}{v_R}$ and $v(R_3(t),t) > v(R_4(t),t)$.

Summarizing the above results as:

- Across a 1-shock wave, we have $v_\ell(t) > v_r(t)$ and $u_\ell(t) < u_r(t)$, where $(u_\ell(t), v_\ell(t))$ and $(u_r(t), v_r(t))$ are the limiting values of (u, v) as the discontinuity $x = S_1(t)$ is approached from left and right respectively.
- Across a 2-shock wave, we have $v_\ell(t) < v_r(t)$ and $u_\ell(t) < u_r(t)$, where $(u_\ell(t), v_\ell(t))$ and $(u_r(t), v_r(t))$ are the limiting values of (u, v) as the discontinuity $x = S_2(t)$ is approached from left and right respectively.
- Across a 1-rarefaction wave, we have $v_\ell(t) < v_r(t)$ and $u_\ell(t) > u_r(t)$ where $v_\ell(t) = v(R_1(t), t)$, $u_\ell(t) = u(R_1(t), t)$, $v_r(t) = v(R_2(t), t)$ and $u_r(t) = u(R_2(t), t)$.
- Across a 2-rarefaction wave, we have $v_\ell(t) > v_r(t)$ and $u_\ell(t) > u_r(t)$ where $v_\ell(t) = v(R_3(t), t)$, $u_\ell(t)) = u(R_3(t), t)$, $v_r(t) = v(R_4(t), t)$ and $u_r(t) = u(R_4(t), t)$.

Based on these solutions, we solve a Riemann problem with non-constant and smooth initial data, in the next section.

5 Riemann Problem with non-constant initial state

Consider the initial profile

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)) = \begin{cases} (x + u_L, v_L) & \text{if } \ x < 0, \\ (x + u_R, v_R), & \text{if } \ x \geq 0, \end{cases} \tag{5.1}$$

where u_L, u_R, v_L and v_R are constants.

If 1-wave is a shock wave then

$$v(x,t) = \begin{cases} v_L \mathrm{e}^{-t} & \text{if } \ x \leq S_1(t), \\ \tilde{v}\mathrm{e}^{-t} & \text{if } \ x > S_1(t), \end{cases} \tag{5.2}$$

$$u(x,t) = \begin{cases} u_L + x & \text{if } \ x \leq S_1(t), \\ \tilde{u} + x & \text{if } \ x > S_1(t), \end{cases} \tag{5.3}$$

where $\dfrac{dS_1}{dt} = -\dfrac{\mathrm{e}^t}{\sqrt{(v_L \tilde{v})}}$, which yields on integration that $S_1(t) = \frac{(1-\mathrm{e}^t)}{\sqrt{(v_L\tilde{v})}}$.

In view of $(4.4)_2$ we have $\tilde{u} = u_L - \frac{\tilde{v}-v_L}{\sqrt{(v_L\tilde{v})}}$, $\tilde{v} < v_L$ and $\tilde{u} > u_L$.

Similarly, if 2- wave is a shock wave then

$$v(x,t) = \begin{cases} v_R \mathrm{e}^{-t}, & \text{if } \ x \geq S_2(t), \\ \hat{v}\mathrm{e}^{-t}, & \text{if } \ x < S_2(t) \end{cases} \tag{5.4}$$

$$u(x,t) = \begin{cases} u_R + x, & \text{if } \ x \geq S_2(t), \\ \hat{u} + x, & \text{if } \ x < S_2(t), \end{cases} \tag{5.5}$$

where $\dfrac{dS_2}{dt} = \dfrac{\mathrm{e}^t}{\sqrt{(v_R \hat{v})}}$, which yields on integration that $S_2(t) = \frac{(\mathrm{e}^t-1)}{\sqrt{v_R\hat{v}}}$. In view of $(4.6)_2$ we have $\hat{u} = u_R + \frac{\hat{v}-v_R}{\sqrt{v_R\hat{v}}}$ with $\hat{v} < v_R$ and $\tilde{u} > u_R$.

If 1- wave is a rarefaction wave then

$$v(x,t) = \begin{cases} v_L e^{-t} & \text{if} \quad x \le R_1(t), \\ \left(\frac{e^{-t}-1}{x}\right) & \text{if} \quad R_1(t) < x < R_2(t), \\ \tilde{v} e^{-t} & \text{if} \quad x \ge R_2(t), \end{cases} \tag{5.6}$$

$$u(x,t) = \begin{cases} u_L + x & \text{if} \quad x \le R_1(t), \\ u_L + \log\left(\frac{x v_L}{1-e^t}\right) + x & \text{if} \quad R_1(t) < x < R_2(t), \\ \tilde{u} + x & \text{if} \quad x \ge R_2(t), \end{cases} \tag{5.7}$$

where $\tilde{u} = u_L + \log\left(\frac{v_L}{\tilde{v}}\right)$, $R_1(t) = \frac{(1-e^t)}{v_L}$, $R_2(t) = \frac{(1-e^t)}{\tilde{v}}$ and $\tilde{v} > v_L$.

Similarly, if 2- wave is rarefaction wave then

$$v(x,t) = \begin{cases} \hat{v} e^{-t}, & \text{if} \quad x \le R_3(t), \\ \left(\frac{1-e^{-t}}{x}\right), & \text{if} \quad R_3(t) < x \le R_4(t), \\ v_R e^{-t}, & \text{if} \quad x \ge R_4(t), \end{cases} \tag{5.8}$$

$$u(x,t) = \begin{cases} \hat{u} + x, & \text{if} \quad x \le R_3(t), \\ u_R - \log\left(\frac{x v_R}{e^t-1}\right) + x, & \text{if} \quad R_3(t) < x < R_4(t), \\ u_R + x,, & \text{if} \quad x \ge R_4(t), \end{cases} \tag{5.9}$$

where $\hat{u} = u_R - \log\left(\frac{v_R}{\hat{v}}\right)$, $R_3(t) = \frac{(e^t-1)}{\hat{v}}$, $R_4(t) = \frac{(e^t-1)}{v_R}$ and $\hat{v} > v_R$. Here, $\tilde{v}$ and $\hat{v}$ are arbitrary constants.

Let A and C be the quantities defined by

$$A = u_L - u_R, \quad C = \log(v_R/v_L). \tag{5.10}$$

Then, to continue our development, it is useful to state the following Lemmas:

Lemma 5.1. *Let A and C be defined as in (5.10). If the solution to the Riemann problem for the system (3.1), with initial conditions (5.1), consists of 1- shock wave and 2-shock wave then $A + 2\sinh(|C|/2) < 0$ and $A < 0$.*

Proof. Given that 1-wave is a shock wav, $x = S_1(t)$, implies that $\tilde{u} > u_L$ and $\tilde{v} < v_L$; similarly, if 2-wave is a shock wave, $x = S_2(t)$, then $\hat{u} < u_R$ and $\hat{v} < v_R$. In the region, $S_1(t) < x < S_2(t)$ the solution given in the equations (5.2) and (5.3), through 1-shock, and the solutions (5.4) and (5.5), through 2-shock, should coincide; that is $\tilde{v} = \hat{v} = z$(say), $\tilde{u} = \hat{u}$, i.e., $A = u_L - u_R < 0$ and $f_1(z) = 0$ where

$$f_1(z) = u_L - u_R - \frac{z - v_L}{(v_L z)^{(1/2)}} - \frac{z - v_R}{(v_R z)^{(1/2)}},$$

for $0 < z < \min\{v_L, v_R\}$. Observe that $\lim_{z \to 0} f_1(z) = \infty$ and

$$\frac{df_1}{dz} = -\frac{z + v_L}{2z\sqrt{(v_L z)}} - \frac{z + v_R}{2z\sqrt{(v_R z)}} < 0,$$

implying thereby that f_1 is decreasing.

- Let $v_L < v_R$, i.e., $C > 0$. Since $0 < z < \min\{v_L, v_R\}$, i.e., $0 < z < v_L$, $f_1 = 0$ has a solution if $f_1(v_L) < 0$, where
$$f_1(v_L) = u_L - u_R - \frac{v_L - v_R}{(v_R v_L)^{(1/2)}} = A + 2\sinh(C/2). \tag{5.11}$$
- Similarly, let $v_R < v_L$, i.e., $C < 0$ then $f_1 = 0$ has a solution if $f_1(v_R) < 0$, where
$$f_1(v_R) = u_L - u_R - \frac{v_R - v_L}{(v_R v_L)^{(1/2)}} = A - 2\sinh(C/2). \tag{5.12}$$
Thus, in view of (5.11) and (5.12), if $A < 0$ then the solution exists for $f_1(z) = 0$ only when $A + 2\sinh(|C|/2) < 0$.

This situation is depicted in Figure 1.

Further, when $v_R = v_L$, i.e., $C = 0$, it follows from $f_1(z) = 0$ that $z = v_L\left(\sqrt{\frac{(u_R - u_L)^2}{4} + 1} - \frac{u_R - u_L}{4}\right)^2$ which recovers the result obtained in [6] for $v_L = v_R = v_0$, $u_R = u_r$, $u_L = u_\ell$ and $z = v_m$. □

Lemma 5.2. *Let A and C be defined as in (5.10). If the solution to the Riemann problem for the system (3.1), with initial conditions (5.1), consists of 1- shock wave and 2- rarefaction wave, then either of the following inequalities holds*

(i.) $A < 0$, $C < 0$ *and* $A - 2\sinh(C/2) > 0$.

(ii.) $A > 0$, $C < 0$ *and* $A + C < 0$.

Proof. Let 1-wave be a shock wave and 2-wave be a rarefaction wave. This implies that $\tilde{u} > u_L$, $\tilde{v} < v_L$ and $\hat{v} > v_R$. In view of (5.2), (5.3), (5.8) and (5.9), it follows that the solutions given by (5.2)-(5.3) and (5.8)-(5.9) should coincide in the region $S_1(t) < x < R_3(t)$; this means that $\tilde{v} = \hat{v} = z$(say), i.e., $v_R < z < v_L$ and $f_2(z) = 0$ where
$$f_2(z) = u_L - u_R - \frac{z - v_L}{\sqrt{(v_L z)}} + \log\left(\frac{v_R}{z}\right),$$
for $v_R < z < v_L$. Observe from the equation
$$\frac{df_2}{dz} = -\frac{z + v_L}{2z\sqrt{(v_L z)}} - \frac{1}{z},$$
that f_2 is decreasing. Since, $v_R < v_L$, i.e., $C < 0$, the equation $f_2 = 0$ has a solution only when $f_2(v_R) > 0$ and $f_2(v_L) < 0$, i.e.,
$$f_2(v_L) < 0 \Rightarrow u_L - u_R + \log\left(\frac{v_R}{v_L}\right) < 0, \tag{5.13}$$
$$f_2(v_R) > 0 \Rightarrow u_L - u_R - \frac{v_R - v_L}{\sqrt{(v_L v_R)}} > 0 \tag{5.14}$$

Since $v_R < v_L$, and if $u_L > u_R$ then (5.14) always holds. Thus, a solution for $f_2(z) = 0$ over $[v_R, v_L]$ is possible if (5.13) holds, i.e.,

$$A + C = u_L - u_R + \log\left(\frac{v_R}{v_L}\right) < 0.$$

Hence, $A > 0$, $C < 0$ and $A + C < 0$.

Further, since $v_R < v_L$, and if $u_L < u_R$ then the equation (5.13) always holds. Thus, the number of solutions for $f_2(z) = 0$ over $[v_R, v_L]$ is possible only if (5.14) holds, i.e.,

$$u_L - u_R - \frac{v_R - v_L}{\sqrt{(v_L v_R)}} = A - 2\sin(C/2) > 0.$$

Hence, $A < 0$, $C < 0$ and $A - 2\sinh(C/2) > 0$. This situation is depicted in Figure 2. □

Lemma 5.3. *Let A and C be defined as in (5.10). If the solution to the Riemann problem for the system (3.1), with initial conditions (5.1), consists of 1-rarefaction wave and 2-shock wave then either of the following inequalities holds*

(i.) $A < 0$, $C > 0$ *and* $A + 2\sinh(C/2) > 0$.

(ii.) $A > 0$, $C > 0$ *and* $A - C < 0$.

Proof. Given that 1-wave is a rarefaction wave implies that $\tilde{v} > v_L$; similarly if the 2-wave is a shock wave then $\hat{u} < u_R$ and $\hat{v} < v_R$. In view of the equations (5.4)-(5.7), the solutions given by the equations (5.4)-(5.5) and (5.6)-(5.7) should coincide in the region $R_2(t) < x < S_2(t)$, i.e., $\tilde{v} = \hat{v} = z$(say), $\hat{u} = \tilde{u}$, i.e., $v_L < v_R$, $C < 0$ and $f_3(z) = 0$ where

$$f_3(z) = u_L - u_R + \log\left(\frac{v_L}{z}\right) - \frac{z - v_R}{\sqrt{(v_R z)}},$$

for $v_L < z < v_R$. Observe from the above equation that

$$\frac{df_3}{dz} = -\frac{z + v_R}{2z\sqrt{(v_R z)}} - \frac{1}{z},$$

implying thereby that f_3 is decreasing. Since $C < 0$, the equation $f_3 = 0$ has a solution only when $f_3(v_L) > 0$ and $f_3(v_R) < 0$, i.e.,

$$f_3(v_L) > 0 \Rightarrow u_L - u_R - \frac{v_L - v_R}{\sqrt{(v_L v_R)}} > 0 \tag{5.15}$$

$$f_3(v_R) < 0 \Rightarrow u_L - u_R + \log\left(\frac{v_L}{v_R}\right) < 0, \tag{5.16}$$

If $u_L > u_R$ and and $v_L < v_R$ then $A > 0$, $C < 0$ and the equation (5.15) is always true. Thus, the solution for $f_3(z) = 0$ over $[v_L, v_R]$ exits only when the equation (5.16) is true, i.e.,

$$u_L - u_R + \log\left(\frac{v_L}{v_R}\right) = A - C < 0. \tag{5.17}$$

Thus, $A > 0$, $C > 0$ and $A - C < 0$.

Further, if $v_L < v_R$ and $u_L < u_R$ then $C < 0$, $A > 0$ and the equation (5.16) is always true. Thus, the solution for $f_3(z) = 0$ over $[v_L, v_R]$ exits only when the equation (5.15) is true, i.e.,

$$u_L - u_R - \frac{v_L - v_R}{\sqrt{(v_L v_R)}} = A + 2\sinh(C/2) > 0, \tag{5.18}$$

implying thereby that $A < 0$, $C > 0$ and $A + 2\sinh(C/2) > 0$

This situation is depicted in Figure 3.

□

Lemma 5.4. *Let A and C be defined as in (5.10). If the solution to the Riemann problem for the system (3.1), with initial conditions (5.1), consists of 1-rarefaction wave and 2-rarefaction wave then $A > 0$ and $A - |C| > 0$.*

Proof. Let 1-wave and 2-wave be both rarefaction waves. In view of (5.6)-(5.9), the solutions given in the equations (5.6)-(5.7) and (5.8)-(5.9) should coincide in the region, $R_2(t) < x < R_3(t)$, i.e., $\tilde{v} = \hat{v} = z$(say) and $f_4(z) = 0$ where

$$f_4(z) = u_L - u_R + \log\left(\frac{v_L}{z}\right) + \log\left(\frac{v_R}{z}\right),$$

with $\max\{v_L, v_R\} < z < \infty$. Observe from the above equation that

$$\frac{df_4}{dz} = -\frac{2}{z},$$

showing thereby f_4 is decreasing. Observe that $\lim_{z\to\infty} f_4(z) = -\infty$.

If $v_L < v_R$, $C > 0$, then $f_4 = 0$ has a solution provided $f_4(v_R) > 0$, i.e.,

$$u_L - u_R + \log\left(\frac{v_L}{v_R}\right) = A - C > 0. \tag{5.19}$$

Since $C > 0$ and $A - C > 0$ we have $A > 0$.

Similarly, if $v_R < v_L$ then $f_4 = 0$ has a solution if $f_4(v_L) > 0$, i.e.,

$$u_L - u_R + \log\left(\frac{v_R}{v_L}\right) = A + C > 0. \tag{5.20}$$

Since $C < 0$ and $A + C > 0$ we have $A > 0$.

Also observe that when $v_L > v_R$ (respectively, $v_R > v_L$) and $u_R > u_L$ then equation (5.19) (respectively, equation (5.20)) does not hold. This situation is depicted in Figure 4.

Further, when $v_R = v_L$, i.e., $C = 0$ from the equation $f_4(z) = 0$ yields $z = v_L e^{\frac{u_L - u_R}{2}}$ which on replacing $v_L = v_R = v_0$, $u_R = u_r$, $u_L = u_\ell$ and $z = V_m$ recovers the result in the equation (37) of Ref.[6] .

□

Lemma 5.5. *Let A and C be defined as in (5.10). If the solution to the Riemann problem for the system (3.1), with initial conditions (5.1), consists of only 1-rarefaction wave (respectively, 2-rarefaction wave) then $A > 0$, $C > 0$ and $A - C = 0$ (respectively, $A > 0$, $C < 0$ and $A + C = 0$).*

Proof. Let the solution be given through 1-wave as a rarefaction wave only, then in view of (5.6) – (5.7), we have $\tilde{v} > v_L$, $\tilde{v} = v_R$ and $\tilde{u} = u_R$, i.e., $u_L + \log\left(\frac{v_L}{v_R}\right) = u_R$, which implies that $A - C = 0$. Similarly, when solution is given through 2- rarefaction wave, it can be easily shown that $A + C = 0$. □

Lemma 5.6. *Let A and C be defined as in (5.10). If the solution to the Riemann problem for the system (3.1), with initial conditions (5.1), consists of only 1-shock wave (respectively, 2-shock wave) then $A < 0$, $C < 0$ and $A - 2\sinh(C/2) = 0$ (respectively, $A < 0$, $C > 0$ and $A + 2\sinh(C/2) = 0$).*

Proof. Let solution be given through 1-wave as a shock wave only, then in view of (5.2) – (5.3), we have $\tilde{v} < v_L$, $\tilde{v} = v_R$ and $\tilde{u} = u_R$, i.e., $u_L - \frac{v_R - v_L}{\sqrt{v_L v_R}} = u_R$ which implies $A - 2\sinh(C/2) = 0$. Similarly, when solution is given through 2-wave as a shock wave, it can be proved that $A - C = 0$. □

We next give the following two theorems, which in fact, complete our discussion relating to the complete characterization of the solution of the Riemann problem under consideration.

Theorem 5.1. *Let A and C be defined as in (5.10). Consider the solution to the Riemann problem for the system (3.1), with initial conditions (5.1). Then 1-rarefaction wave (respectively, 1-shock wave) is a solution to the Riemann problem if and only if $A + \max(2\sinh(C/2), C) > 0$ (respectively, $A + \max(2\sinh(C/2), C) < 0$).*

Proof. Observe that if $C > 0$ (respectively; $C < 0$) then $\min(C, 2\sinh(C/2)) = C$ (respectively; $\min(C, 2\sinh(C/2)) = 2\sinh(C/2)$).

Let the 1-wave be a rarefaction wave. Then, from Lemmas 5.3, 5.4 and 5.5, we have

1. $A < 0$, $C > 0$, $A + 2\sinh(C/2) > 0 \Rightarrow A + \max(C, 2\sinh(C/2)) > 0$.

2. $A > 0$, $C < 0$, $A + C > 0 \Rightarrow A + \max(C, 2\sinh(C/2)) > 0$.

3. $A > 0$, $C > 0 \Rightarrow A + C > 0$ and $A + 2\sinh(C/2) > 0$.

Thus, if the 1 -wave is a rarefaction wave then $A + \max(C, 2\sinh(C/2)) > 0$.

Let the 1 -wave be a shock wave then from Lemmas 5.1, 5.2 and 5.6 we have

1. $A < 0$, $C > 0$, $A + 2\sinh(C/2) < 0 \Rightarrow A + \max(C, 2\sinh(C/2)) < 0$.

2. $A > 0$, $C < 0$, $A + C < 0 \Rightarrow A + \max(C, 2\sinh(C/2)) < 0$.

3. $A < 0$, $C < 0 \Rightarrow A + C < 0$ and $A + \max(C, 2\sinh(C/2)) < 0$.

Thus, if the 1 -wave is a shock wave then $A + \max(C, 2\sinh(C/2)) < 0$.

To prove the converse, let $A + \max(2\sinh(C/2), C) > 0$, then the we have one of the following possibilities

- $A > 0$, $C > 0$.
- $A > 0$, $C < 0$, $A + C > 0$
- $A < 0$, $C > 0$, $A + 2\sinh(C/2) > 0$,

which lead us to conclude that the 1-wave cannot be a shock wave as the above inequalities are contradicting the consequences of lemmas 5.1, 5.2 and 5.6 . Hence, the 1-wave is a rarefaction wave.

Now, let $A + \max(2\sinh(C/2), C) < 0$, then the we have one of the following possibilities

- $A < 0$, $C < 0$.
- $A > 0$, $C < 0$, $A + C < 0$.
- $A < 0$, $C > 0$, $A + 2\sinh(C/2) < 0$,

which imply that the 1-wave is not a rarefaction wave as the above inequalities are contradicting the consequences of lemmas 5.3, 5.4 and 5.5. Hence, the 1-wave is a shock wave.

□

Theorem 5.2. *Let A and C be defined as in (5.10). Consider the solution to the Riemann problem for the system (3.1), with initial conditions (5.1). Then, 2-rarefaction wave (respectively, 2-shock wave) is a solution to the Riemann problem if and only if $A - \min(2\sinh(C/2), C) > 0$ (respectively, $A - \min(2\sinh(C/2), C) < 0$).*

Proof. Observe that if $C > 0$ (respectively; $C < 0$) then $\min(C, 2\sinh(C/2)) = C$ (respectively; $\min(C, 2\sinh(C/2)) = 2\sinh(C/2)$).

Let 2 -wave be a rarefaction wave then from Lemmas 5.2, 5.4 and 5.5 we have

1. $A > 0$, $C < 0 \Rightarrow A - C > 0$ and $A - 2\sinh(C/2) > 0$.

2. $A > 0$, $C > 0$, $A - C > 0 \Rightarrow A - \min(C, 2\sinh(C/2)) > 0$.

3. $A < 0$, $C < 0$, $A - 2\sinh(C/2) > 0 \Rightarrow A - \min(C, 2\sinh(C/2)) > 0$.

Thus, if 2 -wave is a rarefaction wave then $A - \min(C, 2\sinh(C/2)) > 0$. However, if the 2 -wave is a shock wave then from Lemmas 5.1, 5.3 and 5.6 we have

1. $A < 0$, $C > 0 \Rightarrow A - C < 0$ and $A - 2\sinh(C/2) < 0$.

2. $A > 0$, $C > 0$, $A - C < 0 \Rightarrow A - \min(C, 2\sinh(C/2) < 0$.

3. $A < 0$, $C < 0$, $A - 2\sinh(C/2) < 0 \Rightarrow A - \min(C, 2\sinh(C/2)) < 0$.

Thus, if 2 -wave is a shock wave then $A - \min(C, 2\sinh(C/2) < 0$.

To prove the converse, let $A - \min(2\sinh(C/2), C) > 0$, then the we have one of the following possibilities

- $A > 0$, $C < 0$.
- $A > 0$, $C > 0$, $A - C > 0$.
- $A < 0$, $C < 0$, $A - 2\sinh(C/2) > 0$.

Assume that 2-wave is a shock wave, then the above inequalities are contradicting the consequences of lemmas 5.1, 5.3 and 5.6. Hence, the 2-wave is a rarefaction wave.

Now, let $A + \max(2\sinh(C/2), C) < 0$; then we have one of the following possibilities

- $A < 0$, $C > 0$.
- $A > 0$, $C < 0$, $A - C < 0$.
- $A < 0$, $C < 0$, $A - 2\sinh(C/2) < 0$.

Assume that 2-wave is a rarefaction wave, then the above possibilities are contradicting lemmas 5.2, 5.4 and 5.5. Hence, the 2-wave is a shock wave. □

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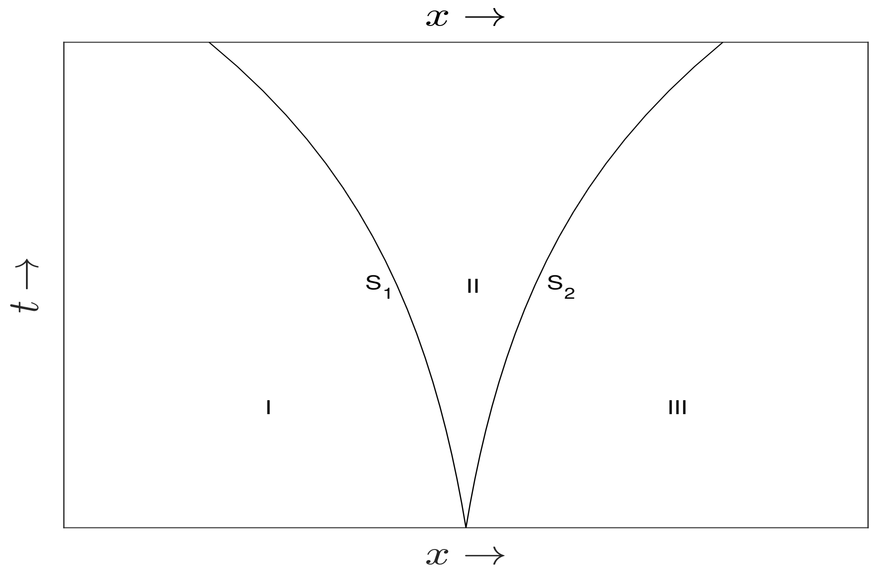

Figure 1: S_1 and S_2 are, respectively, the back-shock and the front shock; regions $x < S_1(t)$, $S_1(t) < x < S_2(t)$, and $x > S_2(t)$ are depicted as I, II and III respectively.

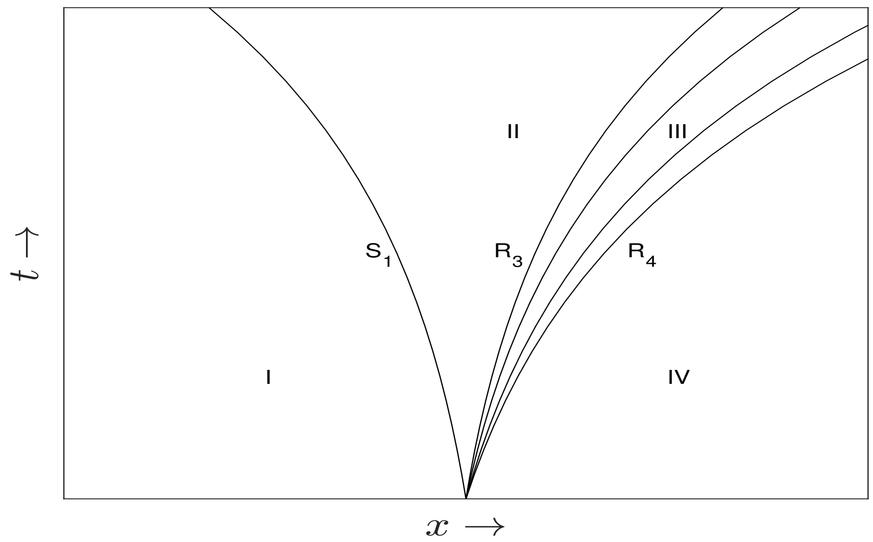

Figure 2: Region behind the back-shock S_1 is depicted as I; region $S_1(t) < x < R_3(t)$ between S_1 and the trail characteristic R_3 of the front rarefaction wave III is depicted as II; region $x > R_4(t)$ ahead of the front rarefaction is depicted as IV.

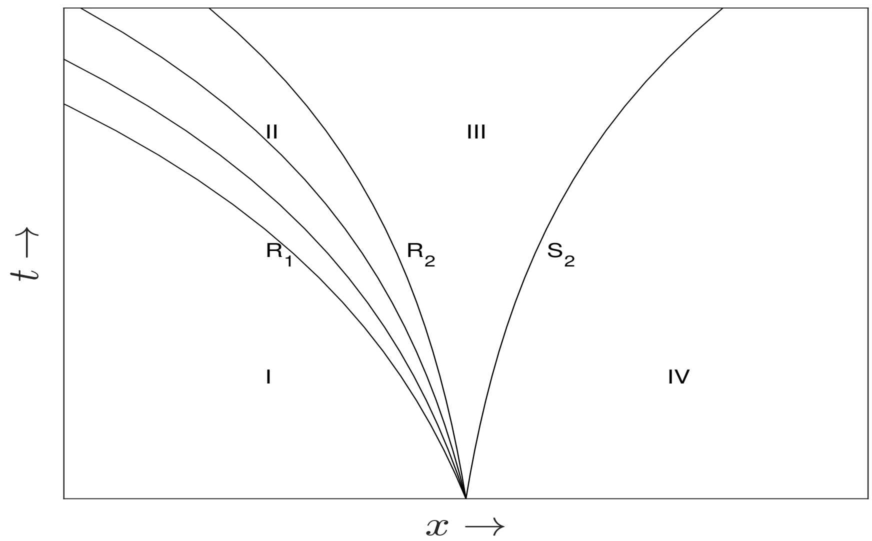

Figure 3: The region $x < R_1(t)$ is depicted as I; back rarefaction region $R_1(t) < x < R_2(t)$ is depicted as II; the region $R_2(t) \leq xS_2(t)$ is depicted as III and the region $x > S_2(t)$ is depicted as IV.

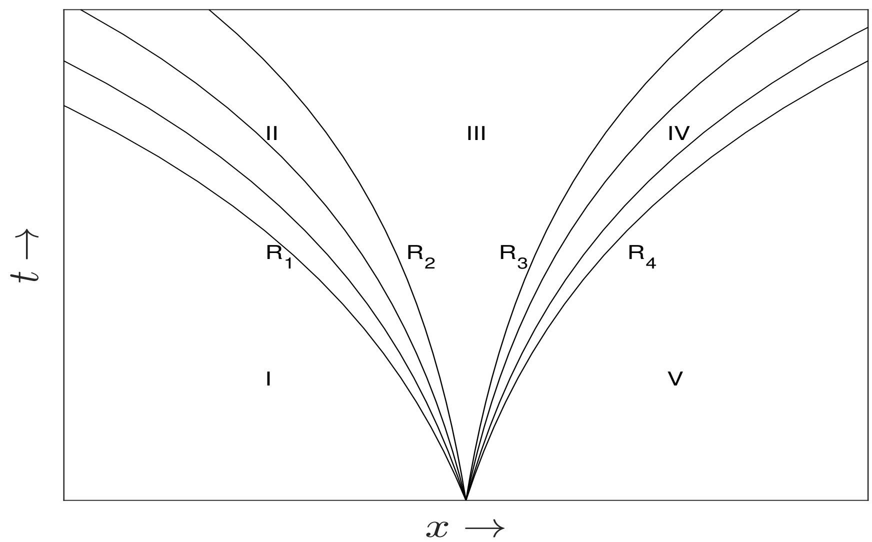

Figure 4: The region $x < R_1(t)$ is depicted as I; II is the back rarefaction wave region; region $R_2(t) \leq x \leq R_3(t)$ between front and back rarefaction is depicted as III; IV is the front rarefaction wave region and region $x > R_4(t)$ is depicted as V.