H_∞ State Estimation of Delayed Recurrent Memristive Neural Networks: continuous-time case and discrete-time case

Fangyuan Ma, Xingbao Gao*

School of Mathematics and Information Science, Shaanxi Normal University, Xi'an, 710062, China

Abstract

This paper investigates the problem of H_∞ state estimation of delayed recurrent memristive neural networks (DRMNNs) with both continuous-time and discrete-time cases. By utilizing Lyapunov-Krasovskii functional (LKF) and linear matrix inequalities (LMIs), two criterions are provided to guarantee the asymptotically stable of the estimation error systems with a H_∞ performance. The connection weight parameters of DRMNNs are dealed with logical switching signals, which greatly reduces the computational complexity. The given conditions can be easily checked by solving LMIs, the obtained theoretical results are supported demonstrated by two numerical examples.

Keywords: Recurrent memristive neural network; H_∞ state estimation; Linear matrix inequalities.

1. Introduction

Memristor, the fourth basic circuit elements, which was first proposed by Chua [1]. It is a circuit device that represents the relationship between magnetic flux and charge. Unlike resistance, memristor has the dimension of resistance, the resistance of memristor is determined by the charge flowing through it. Thus, memristor is a kind of nonlinear resistance with memory function because we can by measuring the resistance of the memristor. The resistance value can be changed by controlling current. If the high resistance value is defined as 1 and the low one is defined as 0, the function of data storage can be realized. Due to its small size and low energy

*Corresponding author

Email addresses: `fangyuanma@snnu.edu.cn` (Fangyuan Ma), `xinbaog@snnu.edu.cn` (Xingbao Gao)

[1]This work was supported in part by National Natural Science Foundation of China Grant No. 61273311.

consumption, memristors can store and process information well. Therefore, more and more traditional resistors are replaced by memristors, and memristive neural networks (MNNs) have been widely studied [2]-[5].

It is well known that time delay is inevitable in the implementation and application of neural networks, which often leads to poor performance, such as oscillation, divergence. Thus many good results have been obtained for the MNNs with time delays. For example, Zhao et al. [2] described the memristor in detail and studied the finite-time synchronization of MNNs by using LMIs and finite-time stability theory. In [3], the exponential synchronization of MNNs with non-uniform time delay is discussed by using interval matrix method.

Moreover, it is often necessary to use the state information of neurons to achieve pinning control, system modeling and state feedback control. Then it is very important to estimate the state of neurons by available network outputs in practice. In general, it is difficult and expensive to obtain the complete information of states of all neurons. Few results on state estimation for MNNs are achieved. For example, the exponential state estimation for MNNs has been studied by using multiobjective approach in [6], and Wei et al. [7] concerned the state estimation of MNNs with time-varying delays by constructing LKF and combining Jensen integral inequality.

When investigated the state estimation of neural networks, a H_∞ performance γ is unually considered. The H_∞ state estimator guarantees that the energy-to-energy gain from external disturbances to the estimation error is no more than a prescribed level γ. For the H_∞ state estimation of MNNs, researchers focused on the discrete-time case, because this kinds of models could perform better behaviors in some practical applications, for example, image processing operations, higher brain functions and other fields. Among them, Shen et al. [10] investigated the H_∞ state estimation problem of Markov jump MNNs by virtue of the hidden Markov model approach. The H_∞ state estimation in the discrete-time has been addressed under randomly mixed time-delays and fading measurements in [12]. However, few results on H_∞ state estimation for continuous-time MNNs have been found. Thus, it is necessary to study H_∞ state estimation of continuous-time MNNs.

In addition, non-fragile means that the gain value (controller or observer) does not destroy the stability of closed-loop system, which emphasizes the minimum precision in the controller design. This kind of controller can achieve the goal of cost minimization and adjust the control parameters online, thus non-fragile control can meet the practical requirements, and it has important application value to design non-fragile controllers.

Based on the above consideration, this paper studied the problem of H_∞ state estimation of DRMNNs for continuous and discrete time cases. Since the MNNs are state-dependent system, it is a difficult to overcome the problem of the parameter mismatch. For the continuous-time model, differing from [10]-[12], the connection weight parameters of DRMNNs are dealed with logical switching signals, which reduces the conservatism. The criterion of asymptotic stability and H_∞ performance of the estimation error system are obtained by non-fragile control. For the discrete-time model, our models are more general than [10]-[11], and our theoretical result reduce the control gain and cost effectively, it can be easily checked by solving LMIs.

The rest of this paper is organized as follows. In Section 2, some definitions, and lemmas are given. The H_∞ State Estimation of DRMNNs are studied with both continuous-time and discrete-time cases in Section 3. Numerical simulation on several examples is illustrated in Section 4, and the conclusions are drawn in Section 5.

Notation: $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote the n-dimensional Euclidean space and the set of $n \times n$ real matrices, respectively. $C([a, b], \mathbb{R}^n)$ is the family of continuous function φ from $[a, b]$ to $\mathbb{R}^n$ with $a \leq b$, $\|\cdot\|$ and $\|\cdot\|_2$ represent l_1- and l_2- norms of the vector or matrix, respectively. $\otimes$ stands for the Kronecker protect of two matrices, I_n is the identity matrix of order n, and 0 means zero matrix with proper dimension. '$*$' refers to the ellipsis in symmetric matrices expressions. $l_2[0, \infty)$ denotes the space of square-integrable vector functions defined on $[0, \infty)$.

2. Preliminaries

Before starting discuss, we introduce the following definitions and lemmas.

Definition 2.1. *(Continuous-time) The error system is asymptotically stable and achieves a H_∞ disturbance attenuation level γ, if the following conditions are satisfied:*

(1) The estimation error system is asymptotically stable with the disturbance input $\omega(t) \equiv 0$;

(2) The output $\bar{r}$ under zero initial condition satisfies: $\|\bar{r}\|_2 \leq \gamma\|\omega(t)\|_2$ for any nonzero $\omega(t) \in l_2[0, \infty)$ for all $t > 0$.

Lemma 2.1. *([16]) For any constant matrix $\mathcal{N} \in \mathbb{R}^{n\times n}$, $\mathcal{N} = \mathcal{N}^T$, there exist a scalar $\zeta > 0$, such that*

$$\zeta \int_0^\zeta \chi^T(s)\mathcal{N}\chi(s)ds \geq \Big(\int_0^\zeta \chi(s)ds\Big)^T \mathcal{N}\Big(\int_0^\zeta \chi(s)ds\Big),$$

where vector function $\chi : [0,\zeta] \rightarrow \mathbb{R}^n$.

Lemma 2.2. *(Schur Complement) Given constant matrices Φ_1, Φ_2, Φ_3, where $\Phi_1 = \Phi_1^T$ and $\Phi_2 = \Phi_2^T > 0$, then*

$$\Phi_1 + \Phi_3^T \Phi_2^{-1} \Phi_3 < 0,$$

if and only if

$$\begin{pmatrix} \Phi_1 & \Phi_3^T \\ \Phi_3 & -\Phi_2 \end{pmatrix} < 0.$$

Lemma 2.3. *([17]) For any vectors $x, y \in \mathbb{R}^n$, scalar $\epsilon > 0$, real matrices Φ, Ψ and U of appropriate dimensions with $U^T U \leq I$, then*

$$2x^T \Phi U \Psi y \leq \epsilon^{-1} x^T \Phi \Phi^T x + \epsilon y^T \Psi^T \Psi y.$$

3. Main results

In this section, we will derive the estimators for the continuous and discrete time DRMNN by LKFs.

3.1. The continuous-time for DRMNNs

In this subsection, the state estimator for continuous-time DRMNNs shall be designed, and some sufficient conditions for the asymptotical stability of its error system is derived by using LKF technique.

Consider the following DRMNNs:

$$\begin{cases} \dot{z}(t) = -Cz(t) + A(z(t))f(z(t)) + B(z(t))f(z(t-\tau(t))) + L\omega(t), \\ r(t) = Ez(t), \end{cases} \tag{1}$$

where $z(t) = (z_1(t), \cdots, z_n(t))^T \in \mathbb{R}^n$ is the neuron state vector, $C = \mathrm{diag}\{c_1, \cdots, c_n\}$ with $c_i > 0$ for $i = 1, \cdots, n$, $A(z(t)) = (a_{ij}(z_i(t)))_{n\times n}$ and $B(z(t)) = (b_{ij}(z_i(t)))_{n\times n}$ represent the connection weight matrices, $r(t)$ is the output vector, the given weight $E \in \mathbb{R}^{m\times n}$, $\omega(t) \in l_2[0,\infty)$ is the disturbance input, $\tau(t)$ is the time-varying delay, $0 \leq \tau(t) \leq \tau$ and $\dot{\tau}(t) \leq \mu$ with τ and μ being contants, and the activation function $f(z(t)) = [f_1(z_1(t)), \cdots, f_n(z_n(t))]^T$.

Based on the feature of memristor, one has

$$a_{ij}(z_i(t)) = \begin{cases} \acute{a}_{ij}, & z_i(t) \leq T_i, \\ \grave{a}_{ij}, & z_i(t) > T_i, \end{cases} \quad b_{ij}(z_i(t)) = \begin{cases} \acute{b}_{ij}, & z_i(t) \leq T_i, \\ \grave{b}_{ij}, & z_i(t) > T_i, \end{cases}$$

where the switching jumps $T_i \geq 0$ and $\acute{a}_{ij}, \grave{a}_{ij}, \acute{b}_{ij}, \grave{b}_{ij}$ are constants with $\acute{a}_{ij} \neq \grave{a}_{ij}$ and $\acute{b}_{ij} \neq \grave{b}_{ij}$. The initial value of the system (1) is $z_i(s) = \vartheta_i(s) \in C([-\tau, 0], \mathbb{R}^n)$ for $i = 1, \cdots, n$.

Similar to [15], we set the logical switching signals as follows:

$$\rho_i(t) = \begin{cases} 1, & z_i(t) \leq T_i, \\ 0, & z_i(t) > T_i. \end{cases}$$

Then the system (1) can be rewritten as:

$$\begin{cases} \dot{z}(t) = -Cz(t) + \rho(t)[A^* f(z(t)) + B^* f(z(t-\tau(t)))] \\ \qquad +(1-\rho(t))[A^{**} f(z(t)) + B^{**} f(z(t-\tau(t)))] + L\omega(t), \\ r(t) = Ez(t), \end{cases} \tag{2}$$

where $A^* = (\acute{a}_{ij})_{n\times n}, A^{**} = (\grave{a}_{ij})_{n\times n}$, $B^* = (\acute{b}_{ij})_{n\times n}$ and $B^{**} = (\grave{b}_{ij})_{n\times n}$.

To develop some sufficient conditions for the H_∞ state estimation, we consider the following measurable network output for the system (2):

$$y(t) = Dz(t) + g(z(t)) + \omega(t), \tag{3}$$

where $y(t)$ is the measurement output, $D \in \mathbb{R}^{n\times n}$ is a given matrix, the nonlinear disturbances on the network outputs g satisfies Lipschitz condition, i.e.:

$$|g(z) - g(\tilde{z})| \leq |G(z - \tilde{z})|, \tag{4}$$

with $G \in \mathbb{R}^{n\times n}$ being a given matrix. Then we will consider the following non-fragile observer to estimate the state of (2)

$$\begin{cases} \dot{\tilde{z}}(t) = -C\tilde{z}(t) + \rho(t)[A^*(t) f(\tilde{z}(t)) + B^*(t) f(\tilde{z}(t-\tau(t)))] + (1-\rho(t))[A^{**}(t) f(\tilde{z}(t)) \\ \qquad +B^{**}(t) f(\tilde{z}(t-\tau(t)))] + (H + \Delta H(t))[y(t) - \tilde{y}(t)], \\ \tilde{r}(t) = E\tilde{z}(t), \\ \tilde{y}(t) = D\tilde{z}(t) + g(\tilde{z}(t)), \end{cases} \tag{5}$$

where $\Delta H(t) = WF(t)N$ with $W \in \mathbb{R}^{n\times n^2}$, $N \in \mathbb{R}^{n^2\times n}$ and H is given matrix, $F^T(t)F(t) \leq I$, $\tilde{z}(t)$, $\tilde{r}(t)$ and $\tilde{y}(t)$ are the estimations of $z(t)$, $r(t)$ and $y(t)$, respectively.

Let $e(t) = z(t) - \tilde{z}(t)$ and $\bar{r}(t) = r(t) - \tilde{r}(t)$ be the estimation error and output signal error, respectively. Then

$$\begin{cases} \dot{e}(t) = -Ce(t) + \rho(t)[A^*\varphi(e(t)) + B^*\varphi(e(t-\tau(t)))] + (1-\rho(t))[A^{**}\varphi(e(t)) + B^{**}\varphi(e(t-\tau(t)))] \\ \qquad +\rho(t)[\tilde{A}^*(t) f(\tilde{z}(t)) + \tilde{B}^*(t) f(\tilde{z}(t-\tau(t)))] + L\omega(t) + (1-\rho(t))[\tilde{A}^{**}(t) f(\tilde{z}(t)) \\ \qquad +\tilde{B}^{**}(t) f(\tilde{z}(t-\tau(t)))] + (H + \Delta H(t))[De(t) + \phi(t) + \omega(t)], \\ \bar{r}(t) = Ee(t), \end{cases} \tag{6}$$

where $\tilde{A}^*(t) = A^*(t) - A^*$, $\tilde{A}^{**}(t) = A^{**}(t) - A^{**}$, $\tilde{B}^*(t) = B^*(t) - B^*$, $\tilde{B}^{**}(t) = B^{**}(t) - B^{**}$, $\varphi(e(\cdot)) = f(z(\cdot)) - f(\tilde{z}(\cdot))$ and $\phi(\cdot) = g(z(\cdot)) - g(\tilde{z}(\cdot))$.

For the convenience of the later discussions, we need the following assumptions:

(A_1) The activation function $f_j(\cdot)$ is bounded, and there exist scalars F_j^- and F_j^+, such that:

$$F_j^- \leq \frac{f_j(a) - f_j(b)}{a - b} \leq F_j^+, \ \ |f_j(\cdot)| \leq F_j$$

for all $a, b \in \mathbb{R}^n$, $j = 1, \cdots, n$, $F^- = \mathrm{diag}\{\mathrm{F}_1^-, \cdots, \mathrm{F}_n^-\}$ and $F^+ = \mathrm{diag}\{\mathrm{F}_1^+, \cdots, \mathrm{F}_n^+\}$.

(A_2) There exist positive scalars $\alpha_1, \alpha_2, \beta_1$ and β_2 , such that:

$$|a_{ij}^*(t) - a_{ij}^*| \leq \alpha_1, \ |a_{ij}^{**}(t) - a_{ij}^{**}| \leq \alpha_2, \ |b_{ij}^*(t) - b_{ij}^*| \leq \beta_1, \ |b_{ij}^{**}(t) - b_{ij}^{**}| \leq \beta_2$$

where $\alpha_1, \alpha_2, \beta_1$ and β_2 are constants. for $i, j = 1, \cdots, n$.

Theorem 3.1. *Assume that (A_1)-(A_2) hold. Then the systems (6) will be asymptotically stable if there exist diagonal matrices $\Lambda_1 > 0$, $\Lambda_2 > 0$, and $K > 0$ with*

$$k_i \geq \max\Big\{2\sum_{j=1}^{n} \max\{\alpha_1 + \beta_1, \alpha_2 + \beta_2\}F_j, m_i\Big\}, \tag{7}$$

where $j = 1, \ldots, n$, positive definite matrices Q, Q_1, R, R_1,

$$\begin{pmatrix} M_1 & M_2 \\ * & M_3 \end{pmatrix} \text{ and } \begin{pmatrix} N_1 & N_2 \\ * & N_3 \end{pmatrix}$$

with proper dimension matrices M_s and N_s $(s = 1, 2, 3)$, and positive constants ς, ς_1, such that

$$\Omega = \begin{pmatrix} \Omega_1 & \Omega_2 & \varsigma\Omega_3 & \Omega_4 & \varsigma_1\Omega_5 \\ * & -\varsigma I & 0 & 0 & 0 \\ * & * & -\varsigma I & 0 & 0 \\ * & * & * & -\varsigma_1 I & 0 \\ * & * & * & * & -\varsigma_1 I \end{pmatrix} < 0, \tag{8}$$

where $\Omega_2 = (W/2, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T$, $\Omega_3 = (ND + N, 0, 0, 0, 0, 0, N, 0, 0, 0, N)^T$, $\Omega_4 = (0, 0, 0, 0, 0, 0, JW/2, 0, 0, 0)^T$, $\Omega_5 = (ND, 0, 0, 0, 0, 0, N, 0, 0, 0, N)^T$,

$$\Omega_1 = \begin{pmatrix} \Omega_{11} & HG & 0 & \Omega_{14} & \Omega_{15} & 0 & L + H & 0 & 0 & H \\ * & \Omega_{22} & 0 & 0 & \Omega_{25} & 0 & 0 & 0 & 0 & 0 \\ * & * & \Omega_{33} & 0 & 0 & -M_2 & 0 & 0 & 0 & 0 \\ * & * & * & \Omega_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Omega_{55} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -M_3 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\gamma^2 I_n & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\frac{1}{\tau}Q_1 & 0 & 0 \\ * & * & * & * & * & * & * & 0 & -\frac{1}{\tau}Q_1 & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{pmatrix}, \tag{9}$$

$\Omega_{11} = -C + HD + Q + R + M_1 + N_1 + \tau^2 Q_1 - F^-\Lambda_1 F^+ + \Lambda_1(F^- + F^+) + G^T G + E^T E$, $\Omega_{14} = M_2 + N_2 + \frac{1}{2}(A^* + A^{**})$, $\Omega_{15} = \frac{1}{2}(B^* + B^{**})$, $\Omega_{22} = -(1-\mu)(R + N_1) - F^-\Lambda_2 F^+ + \Lambda_2(F^- + F^+)$, $\Omega_{25} = -(1-\mu)N_2$, $\Omega_{33} = -R - M_1$,$\Omega_{44} = M_3 + N_3 - \Lambda_1$, $\Omega_{55} = -(1-\mu)N_3 - \Lambda_2$. *Moreover, the non-fragile observer gain* $H = N^{-1}X$.

Proof. Consider the following LKF for the system (6):

$$V(t, e(t)) = \sum_{p=1}^{4} V_p(t, e(t)),$$

where

$$\begin{aligned}
V_1(t, e(t)) =& \frac{1}{2} e^T(t)e(t) + \int_0^t e^T(s)K\text{sign}(e(s))ds, \\
V_2(t, e(t)) =& \int_{t-\tau}^{t} \begin{pmatrix} e(s) \\ \varphi(e(s)) \end{pmatrix}^T \begin{pmatrix} M_1 & M_2 \\ * & M_3 \end{pmatrix} \begin{pmatrix} e(s) \\ \varphi(e(s)) \end{pmatrix} ds \\
& + \int_{t-\tau(t)}^{t} \begin{pmatrix} e(s) \\ \varphi(e(s)) \end{pmatrix}^T \begin{pmatrix} N_1 & N_2 \\ * & N_3 \end{pmatrix} \begin{pmatrix} e(s) \\ \varphi(e(s)) \end{pmatrix} ds, \\
V_3(t, e(t)) =& \int_{t-\tau}^{t} e^T(s)Qe(s)ds + \int_{t-\tau(t)}^{t} e^T(s)Re(s)ds, \\
V_4(t, e(t)) =& \tau \int_{-\tau}^{0} \int_{t+\theta}^{t} e^T(s)Q_1 e(s)dsd\theta.
\end{aligned}$$

Then

$$\dot{V}(t, e(t)) = \sum_{p=1}^{4} \dot{V}_p(t, e(t)),$$

and

$$\begin{aligned}
\dot{V}_1(t, e(t)) =& e^T(t)\dot{e}(t) - e^T(t)K\text{sign}(e(t)) \\
=& e^T(t)\Big\{ - Ce(t) + \rho(t)[A^*\varphi(e(t)) + B^*\varphi(e(t - \tau(t)))] \\
& + (1 - \rho(t))[A^{**}\varphi(e(t)) + B^{**}\varphi(e(t - \tau(t)))] \\
& - K\text{sign}(e(t)) + \rho(t)[\tilde{A}^*(t)f(\tilde{z}(t)) + \tilde{B}^*(t)f(\tilde{z}(t - \tau(t)))] \\
& + (1 - \rho(t))[\tilde{A}^{**}(t)f(\tilde{z}(t)) + \tilde{B}^{**}(t)f(\tilde{z}(t - \tau(t)))] + L\omega(t) \\
& + (H + \Delta H(t))[De(t) + \phi(t) + \omega(t)]\Big\} \\
& \leq -e^T(t)(C + HD)e(t) + e^T(t)(A^* + A^{**})\varphi(e(t)) \\
& + e^T(t)(B^* + B^{**})\varphi(e(t - \tau(t))) + e^T(t)(L + H)\omega(t) \\
& + e^T(t)\Delta H(t)De(t) + e^T(t)\Delta H(t)\omega(t) + e^T(t)(H + \Delta H(t))\phi(t)
\end{aligned} \tag{10}$$

where the last step follows by (7) and

$$
\begin{aligned}
& e^T(t)\Big\{\rho(t)[\tilde{A}^*(t)f(\tilde{z}(t)) + \tilde{B}^*(t)f(\tilde{z}(t-\tau(t)))] \\
& \quad + (1-\rho(t))[\tilde{A}^{**}(t)f(\tilde{z}(t)) + \tilde{B}^{**}(t)f(\tilde{z}(t-\tau(t)))]\Big\} \\
\leq & \sum_{i=1}^{n} |e_i(t)|\Big\{\rho(t)[|\tilde{a}^*_{ij}(t)| \cdot |f_j(\tilde{z}_j(t))| + |\tilde{b}^*_{ij}(t)| \cdot |f_j(\tilde{z}_j(t-\tau(t)))|] \\
& \quad + (1-\rho(t))[|\tilde{a}^{**}_{ij}(t)| \cdot |f_j(\tilde{z}_j(t))| + |\tilde{b}^{**}_{ij}(t)| \cdot |f_j(\tilde{z}_j(t-\tau(t)))|]\Big\} \\
\leq & \sum_{i=1}^{n} |e_i(t)|\Big\{[\rho(t)(\alpha_1+\beta_1) + (1-\rho(t))(\alpha_2+\beta_2)] \sum_{j=1}^{n} F_j\Big\} \\
& \leq \sum_{i=1}^{n} k_i|e_i^T(t)| = e^T(t)K\text{sign}(e(t))
\end{aligned}
$$

since $\rho(t) = 0$ or 1.

$$
\begin{aligned}
\dot{V}_2(t,e(t)) \leq & \begin{pmatrix} e(t) \\ \varphi(e(t)) \end{pmatrix}^T \begin{pmatrix} N_1+M_1 & N_2+M_2 \\ * & N_3+M_3 \end{pmatrix} \begin{pmatrix} e(t) \\ \varphi(e(t)) \end{pmatrix} - \begin{pmatrix} e(t-\tau) \\ \varphi(e(t-\tau)) \end{pmatrix}^T \begin{pmatrix} M_1 & M_2 \\ * & M_3 \end{pmatrix} \\
& \times \begin{pmatrix} e(t-\tau) \\ \varphi(e(t-\tau)) \end{pmatrix} - (1-\mu) \begin{pmatrix} e(t-\tau(t)) \\ \varphi(e(t-\tau(t))) \end{pmatrix}^T \begin{pmatrix} N_1 & N_2 \\ * & N_3 \end{pmatrix} \begin{pmatrix} e(t-\tau(t)) \\ \varphi(e(t-\tau(t))) \end{pmatrix},
\end{aligned}
\tag{11}
$$

$$
\dot{V}_3(t,e(t)) \leq e^T(t)(Q+R)e(t) - e^T(t-\tau)Re(t-\tau) - (1-\mu)e^T(t-\tau(t))Re(t-\tau(t)), \tag{12}
$$

$$
\dot{V}_4(t,e(t)) = \tau^2 e^T(t)Q_1e(t) - \tau \int_{t-\tau}^{t} e^T(s)Q_1e(s)ds \tag{13}
$$

According to Lemma 2.1, one has

$$
\begin{aligned}
-\tau \int_{t-\tau}^{t} e^T(s)Q_1e(s)ds = & -\tau \int_{t-\tau}^{t-\tau(t)} e^T(s)Q_1e(s)ds - \tau \int_{t-\tau(t)}^{t} e^T(s)Q_1e(s)ds \\
& \leq -\frac{\tau}{\tau-\tau(t)}\Big(\int_{t-\tau}^{t-\tau(t)} e(s)ds\Big)^T Q_1\Big(\int_{t-\tau}^{t-\tau(t)} e(s)ds\Big) \\
& \quad -\frac{\tau}{\tau(t)}\Big(\int_{t-\tau(t)}^{t} e(s)ds\Big)^T Q_1\Big(\int_{t-\tau(t)}^{t} e(s)ds\Big),
\end{aligned}
\tag{14}
$$

Thus,

$$
\begin{aligned}
\dot{V}_4(t,e(t)) \leq & \tau^2 e^T(t)Q_1e(t) - \frac{\tau}{\tau-\tau(t)}\Big(\int_{t-\tau}^{t-\tau(t)} e(s)ds\Big)^T Q_1\Big(\int_{t-\tau}^{t-\tau(t)} e(s)ds\Big) \\
& -\frac{\tau}{\tau(t)}\Big(\int_{t-\tau(t)}^{t} e(s)ds\Big)^T Q_1\Big(\int_{t-\tau(t)}^{t} e(s)ds\Big).
\end{aligned}
\tag{15}
$$

According to (A_1), one can read that for any positive diagonal matrix Λ_1, Λ_2, the following inequalities are true:

$$\begin{aligned}
&e^T(t)F^-\Lambda_1 F^+ e(t) - e^T(t)\Lambda_1(F^- + F^+)e(t) + \varphi(e^T(t))\Lambda_1\varphi(e(t)) \le 0,\\
&e^T(t-\tau(t))F^-\Lambda_2 F^+ e(t-\tau(t)) - e^T(t-\tau(t))\Lambda_2(F^- + F^+)e(t-\tau(t))\\
&\ + \varphi(e^T(t-\tau(t)))\Lambda_2\varphi(e(t-\tau(t))) \le 0.
\end{aligned} \tag{16}$$

Moreover, (4) future revealed that

$$\phi^T(t)\phi(t) = |g(z(t)) - g(\bar{z}(t))|^2 \le |Ge(t)|^2 = e^T(t)G^TGe(t). \tag{17}$$

Combining (10)-(17) results in

$$\begin{aligned}
\dot{V}(t, e(t)) \le&\eta^T(t)\Theta\eta(t) + e^T(t)\Delta H(t)De(t) + e^T(t)\triangle H(t)\phi(t) + e^T(t)\Delta H(t)\omega(t)\\
&+ 2\dot{e}^T(t)J\Delta H(t)[De(t) + \phi(t) + \omega(t)]\\
=&\eta^T(t)(\Theta + \Omega_2F(t)\Omega_3 + \Omega_2^TF^T(t)\Omega_3^T + \Omega_4F(t)\Omega_5 + \Omega_4^TF^T(t)\Omega_5^T)\eta(t),
\end{aligned} \tag{18}$$

where $\eta^T(t) = (e^T(t),\ e^T(t-\tau(t)),\ e^T(t-\tau),\ \varphi^T(e(t)),\ \varphi^T(e(t-\tau(t))),\ \varphi^T(e(t-\tau)),\ \omega^T(t),$ $\int_{t-\tau}^{t-\tau(t)} e^T(s)ds, \int_{t-\tau(t)}^{t} e^T(s)ds,\ \phi^T(t))$. Then, by using Lemma 2.2 and Lemma 2.3, it is readily checked that there exist a positive constant ς, such that

$$\begin{aligned}
&\Theta + \Omega_2F(t)\Omega_3 + \Omega_2^TF^T(t)\Omega_3^T + \Omega_4F(t)\Omega_5 + \Omega_4^TF^T(t)\Omega_5^T\\
\le&\Theta + \varsigma^{-1}\Omega_2\Omega_2^T + \varsigma\Omega_3\Omega_3^T + \varsigma_1^{-1}\Omega_4\Omega_4^T + \varsigma_1\Omega_5\Omega_5^T < 0,
\end{aligned} \tag{19}$$

if and only if $\Xi < 0$, where

$$\Xi = \begin{pmatrix} \Theta & \Omega_2 & \varsigma\Omega_3 & \Omega_4 & \varsigma_1\Omega_5\\ * & -\varsigma I & 0 & 0 & 0\\ * & * & -\varsigma I & 0 & 0\\ * & * & * & -\varsigma_1 I & 0\\ * & * & * & * & -\varsigma_1 I \end{pmatrix} \tag{20}$$

with

$$\Theta = \begin{pmatrix}
\tilde{\Omega}_{11} & \Omega_{12} & 0 & \Omega_{14} & \Omega_{15} & 0 & L+H & 0 & 0 & H\\
* & \Omega_{22} & 0 & 0 & \Omega_{25} & 0 & 0 & 0 & 0 & 0\\
* & * & \Omega_{33} & 0 & 0 & -M_2 & 0 & 0 & 0 & 0\\
* & * & * & \Omega_{44} & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & \Omega_{55} & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & -M_3 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & -\frac{1}{\tau}Q_1 & 0 & 0\\
* & * & * & * & * & * & * & 0 & -\frac{1}{\tau}Q_1 & 0\\
* & * & * & * & * & * & * & * & * & -I
\end{pmatrix},$$

$$\tilde{\Omega}_{11} = -C + HD + Q + R + M_1 + N_1 + \tau^2Q_1 - F^-\Lambda_1F^+ + \Lambda_1(F^- + F^+) + G^TG.$$

Thus

$$\dot{V}(t, e(t)) + \bar{r}^T(t)\bar{r}(t) - \gamma^2\omega^T(t)\omega(t) \leq \eta^T(t)\Omega\eta(t) < 0, \tag{21}$$

Integrating both sides of (21) yields $\|\bar{r}(t)\|_2 \leq \gamma\|\omega(t)\|_2$, for all nonzero $\omega(t) \in l_2[0, \infty)$, and the H_∞ performance is established under zero conditions.

Following the similar analysis in [19], it can be shown that the estimation error system (6) with $\omega \equiv 0$ is asymptotically stable. This completes the proof. ■

Remark 3.1. *Considering that no literature has been found in H_∞ state estimation systems with the non-fragile control for the continuous-time case, this paper makes up for this gap. Moreover, differing from [11], this paper employs logical switching signals to deal with the connection weight parameters of system (1), which greatly reduces the computation.*

It is worth emphasizing that the continuous-time models may not convenient for some engineering applications, therefore, we will follow with interest of DRMNNs with discrete-time case.

3.2. The discrete-time case

In some cases, we consider the following discrete-time DRMNNs:

$$z(h+1) = Dz(h) + A(z(h))f(z(h)) + B(z(h))f(z(h - \tau(h))) + L\omega(h), \tag{22}$$

where $z(h) = [z_1(h), \cdots, z_n(h)]^T$ is the state vector with n neurons, $D = \mathrm{diag}\{d_1, \cdots, d_n\}$ is the state feedback positive matrix, $f(z(h))$, $\omega(h)$, $A(z(h))$, $B(z(h))$ and L as described in Theorem 3.1, and $\tau(h)$ is the time-varying delay and satisfies $\tau_1 \leq \tau(h) \leq \tau_2$, where τ_1 and τ_2 are positive constants.

The initial condition of neural network (22) is $z(s) = \psi(s)$, for $s \in \mathbb{N}[-\tau_2, 0]$. Similar to [7], the parameters $a_{ij}(z_i(h))$ and $b_{ij}(z_i(h))$ on memristors can be described as follows:

$$a_{ij}(z_i(h)) = \begin{cases} \check{a}_{ij}, & |(z_i(h))| \leq T_i, \\ \hat{a}_{ij}, & |(z_i(h))| > T_i, \end{cases} \quad b_{ij}(z_i(h)) = \begin{cases} \check{b}_{ij}, & |(z_i(h))| \leq T_i, \\ \hat{b}_{ij}, & |(z_i(h))| > T_i, \end{cases}$$

where the switching jumps $T_i > 0$. Furthermore, we define $\bar{a}_{ij} = \max\{\check{a}_{ij}, \hat{a}_{ij}\}$, $\underline{a}_{ij} = \min\{\check{a}_{ij}, \hat{a}_{ij}\}$, $\bar{b}_{ij} = \max\{\check{b}_{ij}, \hat{b}_{ij}\}$, $\underline{b}_{ij} = \min\{\check{b}_{ij}, \hat{b}_{ij}\}$, $\bar{A} = (\bar{a}_{ij})_{n\times n}$, $\underline{A} = (\underline{a}_{ij})_{n\times n}$, $\bar{B} = (\bar{b}_{ij})_{n\times n}$, $\underline{B} = (\underline{b}_{ij})_{n\times n}$, and we have $A(z(h)) \in [\underline{A}, \bar{A}]$, $B(z(h)) \in [\underline{B}, \bar{B}]$.

Then $A(z(h))$ and $B(z(h))$ can be rewritten by

$$A(z(h)) = \breve{A} + \Delta A_{1h}, \quad B(z(h)) = \breve{B} + \Delta B_{1h},$$

where $\breve{A} = (\breve{a}_{ij})_{n\times n}$, $\breve{B} = (\breve{b}_{ij})_{n\times n}$, $\breve{a}_{ij} = \frac{\bar{a}_{ij}+\underline{a}_{ij}}{2}$, $\breve{b}_{ij} = \frac{\bar{b}_{ij}+\underline{b}_{ij}}{2}$, $\Delta A_{1h} = \sum_{i,j=1}^{n} e_i\epsilon_{ij}^a e_j^T$ and $\Delta B_{1h} = \sum_{i,j=1}^{n} e_i\epsilon_{ij}^b e_j^T$, ϵ_{ij}^a, ϵ_{ij}^b are unknown scalars and satisfy $|\epsilon_{ij}^a| \le \delta_{ij}^a$, $|\epsilon_{ij}^b| \le \delta_{ij}^b$ with $\delta_{ij}^a = \frac{\bar{a}_{ij}-\underline{a}_{ij}}{2}$, $\delta_{ij}^b = \frac{\bar{b}_{ij}-\underline{b}_{ij}}{2}$ and $e_i \in \mathbb{R}^n$ is the identity column vector whose hth element is 1.

Then, ΔA_{1h}, ΔB_h can be described as

$$\Delta A_{1h} = N\Gamma^a E^a, \quad \Delta B_{1h} = N\Gamma^b E^b,$$

where $N = [N^1, \cdots, N^n]^T$, $E^a = [E_1^a, \cdots, E_2^a]^T$, $E^b = [E_1^b, \cdots, E_2^b]^T$ are real matrices with $N^i = [e^i, \cdots, e^i]^T$, $E_i^a = [\delta_{i1}^a e_1^T, \cdots, \delta_{in}^a e_n^T]$, $E_i^b = [\delta_{i1}^b e_1^T, \cdots, \delta_{in}^b e_n^T]$, Γ^a and Γ^b are known matrices which are defined by $\Gamma^a = \mathrm{diag}\{\Gamma^{\mathrm{a}_1}, \cdots, \Gamma^{\mathrm{a}_\mathrm{n}}\}$, $\Gamma^b = \mathrm{diag}\{\Gamma^{\mathrm{b}_1}, \cdots, \Gamma^{\mathrm{b}_\mathrm{n}}\}$ with $\Gamma^{ai} = \mathrm{diag}\{\epsilon_{\mathrm{i1}}^{\mathrm{a}}(\delta_{\mathrm{i1}}^{\mathrm{a}})^{-1}, \cdots, \epsilon_{\mathrm{in}}^{\mathrm{a}}(\delta_{\mathrm{in}}^{\mathrm{a}})^{-1}\}$ and $\Gamma^{bi} = \mathrm{diag}\{\epsilon_{\mathrm{i1}}^{\mathrm{b}}(\delta_{\mathrm{i1}}^{\mathrm{b}})^{-1}, \cdots, \epsilon_{\mathrm{in}}^{\mathrm{b}}(\delta_{\mathrm{in}}^{\mathrm{b}})^{-1}\}$, $(\Gamma^a)^T\Gamma^a \le I$, $(\Gamma^b)^T\Gamma^b \le I$.

In this paper, the network output of (22) is of the following form:

$$\begin{cases} y(h) = C_1 z(h) + C_2\omega(h), \\ u(h) = C_3 z(h), \end{cases} \tag{23}$$

where $y(h) \in \mathbb{R}^m$ is the measurement output, $u(h) \in \mathbb{R}^r$ is the output to be estimated, $C_1 \in \mathbb{R}^{m\times n}$, $C_2 \in \mathbb{R}^{m\times l}$, $C_3 \in \mathbb{R}^{r\times n}$ are given matrices.

In order to estimate the neuron state $z(h)$, the full-order state estimator can be proposed as:

$$\begin{cases} \hat{z}(h+1) = D\hat{z}(h) + A(\hat{z}(h))f(\hat{z}(h)) + B(\hat{z}(h))f(\hat{z}(h-\tau(h))) + K[y(h) - \hat{y}(h)], \\ \hat{y}(h) = C_1\hat{z}(h), \\ \hat{u}(h) = C_3\hat{z}(h), \end{cases} \tag{24}$$

where $\hat{z}(h)$ and $\hat{u}(h)$ stand for the estimation of $z(h)$ and $u(h)$ respectively, K is the estimator gain to be determined.

Similar to $A(z(h))$ and $B(z(h))$, $A(\hat{z}(h))$ and $B(\hat{z}(h))$ can be described by $A(\hat{z}(h)) = \breve{A} + \Delta A_{2h}$, $B(\hat{z}(h)) = \breve{B} + \Delta B_{2h}$, where $\Delta A_{2h} = N\bar{\Gamma}^a\bar{E}^a$, $\Delta B_{2h} = N\bar{\Gamma}^b\bar{E}^b$.

The error state $e(h) = z(h) - \hat{z}(h)$, based on (22)-(24), the error dynamical model can be modified as:

$$\begin{cases} e(h+1) = (D - KC_1)e(h) + \breve{A}\varphi(e(h)) + \Delta A_{2h}\varphi(e(h)) + (\Delta A_{1h} - \Delta A_{2h})f(z(h)) + (L \\ \qquad -KC_2)\omega(h) + \breve{B}\varphi(e(h-\tau(h)) + \Delta B_{2h}\varphi(e(h-\tau(h))) + (\Delta B_{1h} - \Delta B_{2h})f(z(h-\tau(h))), \\ \tilde{u}(h) = C_3 e(h), \end{cases} \tag{25}$$

where $\varphi(e(\cdot)) = f(z(\cdot)) - f(\hat{z}(\cdot))$, and $\tilde{u}(h)$ is the estimation error.

Furthermore, let $\eta(h) = [z(h),\ e(h)]^T$, then:

$$\begin{cases} \eta(h+1) = \bar{D}\eta(h) + \bar{A}\xi(\eta(h)) + \bar{B}\xi(\eta(h-\tau(h))) + \bar{K}\phi(\eta(h)) + (\bar{L} - \bar{C}_2)\omega(h), \\ \tilde{u}(h) = \bar{C}_3\eta(h), \end{cases} \tag{26}$$

where

$$\bar{D} = \mathcal{D} + \Delta\mathcal{D}_h, \bar{A} = \mathcal{A} + \Delta\mathcal{A}_h, \bar{B} = \mathcal{B} + \Delta\mathcal{B}_h,$$

$$\mathcal{D} = \mathrm{diag}\{\mathrm{D}, \mathrm{D}\}, \mathcal{A} = \mathrm{diag}\{\breve{\mathrm{A}}, \breve{\mathrm{A}}\}, \mathcal{B} = \mathrm{diag}\{\breve{\mathrm{B}}, \breve{\mathrm{B}}\},$$

$$\Delta\mathcal{D}_h = \begin{pmatrix} 0 & 0 \\ 0 & -KC_1 \end{pmatrix},\ \Delta\mathcal{A}_h = \begin{pmatrix} \Delta A_{1h} & 0 \\ \Delta A_{1h} - \Delta A_{2h} & \Delta A_{2h} \end{pmatrix},\ \Delta\mathcal{B}_h = \begin{pmatrix} \Delta B_{1h} & 0 \\ \Delta B_{1h} - \Delta B_{2h} & \Delta B_{2h} \end{pmatrix},$$

$$\bar{K} = \begin{pmatrix} L & 0 \\ 0 & L - KC_2 \end{pmatrix},\ \bar{C}_2 = \begin{pmatrix} 0 & KC_2 \end{pmatrix}^T,\ \bar{C}_3 = \begin{pmatrix} 0 & C_3 \end{pmatrix}^T,\ \bar{L} = \begin{pmatrix} L & L \end{pmatrix}^T,$$

This part is to tackle the H_∞ state estimator for DRMNNs (22) and look for the gain K such that the following requirements are satisfied:

(1) The augmented system (26) is asymptotically stable with zero disturbance;

(2) Under zero-initial condition, the estimation error $\tilde{u}(h)$ satisfies

$$J = \sum_{h=0}^{\infty} \left[\tilde{u}^T(h)\tilde{u}(h) - \gamma^2\omega^T(h)\omega(h) \right] < 0. \tag{27}$$

Theorem 3.2. *Under assumption (A_1), given positive scalars λ_1 and λ_2, system (26) is asymptotically stable with an H_∞ disturbance attenuation level γ, if there exist positive definite matrices P, K, M, positive diagonal matrices Γ, Γ_1, Λ_1 and Λ_2 such that the following LMI hold:*

$$\Xi = \begin{pmatrix} \tilde{\Xi}_{11} & 0 & \tilde{\Xi}_{13} & \bar{D}^T P\bar{B} + \tau_2(\bar{D} - I)^T M\bar{B} & \tilde{\Xi}_{15} & \tilde{\Xi}_{16} \\ * & \tilde{\Xi}_{22} & 0 & \tilde{\Xi}_{24} & 0 & 0 \\ * & * & \tilde{\Xi}_{33} & \bar{A}^T P\bar{B} + \tau_2\bar{A}^T M\bar{B} & \bar{A}^T P\bar{K} + \tau_2\bar{A}^T M\bar{K} & \tilde{\Xi}_{36} \\ * & * & * & \bar{B}^T P\bar{B} + \tau_2\bar{B}^T M\bar{B} - 2\Gamma_1 & \bar{B}^T P\bar{K} + \tau_2\bar{B}^T M\bar{K} & \tilde{\Xi}_{46} \\ * & * & * & * & \tilde{\Xi}_{55} & \tilde{\Xi}_{56} \\ * & * & * & * & * & \tilde{\Xi}_{66} \end{pmatrix} < 0, \tag{28}$$

where $\tilde{\Xi}_{11} = \bar{D}^T P\bar{D} + \bar{C}_3^T\bar{C}_3 + \tau_2(\bar{D} - I)^T M(\bar{D} - I) - P + K + (\tau_2 - \tau_1)K - 2(I_2 \otimes F^-)\Gamma(I_2 \otimes F^+)$, $\tilde{\Xi}_{13} = \bar{D}^T P\bar{A} + \tau_2(\bar{D} - I)^T M\bar{A} + (I_2 \otimes F^-)\Gamma + \Gamma(I_2 \otimes F^+)$, $\tilde{\Xi}_{15} = \bar{D}^T P\bar{K} + 2\tau_2(\bar{D} - I)^T M\bar{K}$,$\tilde{\Xi}_{16} = \bar{D}^T P(\bar{L} - \bar{C}_2) + \tau_2(\bar{D} - I)^T M(\bar{L} - \bar{C}_2)$, $\tilde{\Xi}_{22} = -K - 2(I_2 \otimes F^-)\Gamma_1(I_2 \otimes F^+)$, $\tilde{\Xi}_{24} = (I_2 \otimes F^-)\Gamma + \Gamma(I_2 \otimes F^+)$, $\tilde{\Xi}_{33} = \bar{A}^T P\bar{A} + \tau_2\bar{A}^T M\bar{A} - 2\Gamma$,$\tilde{\Xi}_{37} = \bar{A}^T P(\bar{L} - \bar{C}_2) + \tau_2\bar{A}^T M(\bar{L} - \bar{C}_2)$, $\tilde{\Xi}_{55} = \bar{K}^T P\bar{K} + \tau_2\bar{K}^T M\bar{K}$, $\tilde{\Xi}_{46} = \bar{B}^T P(\bar{L} - \bar{C}_2) + \tau_2\bar{B}^T M(\bar{L} - \bar{C}_2)$, $\tilde{\Xi}_{56} = \bar{K}^T P(\bar{L} - \bar{C}_2) + \tau_2\bar{K}^T M(\bar{L} - \bar{C}_2)$, $\tilde{\Xi}_{66} = (\bar{L} - \bar{C}_2)^T P(\bar{L} - \bar{C}_2) + \tau_2(\bar{L} - \bar{C}_2)^T M(\bar{L} - \bar{C}_2) - \gamma^2$, Moreover, the state estimator gain matrix can ba designed as $K = P^{-1}X$.

Proof. Consider the following LKF for the system (26):

$$V(\eta(h)) = \sum_{p=1}^{4} V_s(\eta(h)), \tag{29}$$

where

$$V_1(\eta(h)) = \eta^T(h)P\eta(h), \quad V_2(\eta(h)) = \sum_{i=h-\tau(h)}^{h-1} \eta^T(i)K\eta(i),$$

$$V_3(\eta(h)) = \sum_{j=h-\tau_2+1}^{h-\tau_1} \sum_{i=j}^{h-1} \eta^T(i)K\eta(i), \ V_4(\eta(h)) = \sum_{j=-\tau_2}^{-1} \sum_{i=h+j}^{h-1} \varepsilon^T(i)M\varepsilon(i).$$

with $\varepsilon(i) = \eta(i+1) - \eta(i)$.

Let $\Delta V(h) = V(\eta(h+1)) - V(\eta(h))$, then

$$\begin{aligned}
\Delta V_1(h) =& V_1(\eta(h+1)) - V_1(\eta(h)) = \eta^T(h+1)P\eta(h+1) - \eta^T(h)P\eta(h) \\
=&[\bar{D}\eta(h) + \bar{A}\xi(\eta(h)) + \bar{B}\xi(\eta(h-\tau(h))) + \bar{K}\phi(\eta(h)) + (\bar{L} - \bar{C}_2)\omega(h)]^T P[\bar{D}\eta(h) \\
&+ \bar{A}\xi(\eta(h)) + \bar{B}\xi(\eta(h-\tau(h))) + \bar{K}\phi(\eta(h)) + (\bar{L} - \bar{C}_2)\omega(h)] - \eta^T(h)P\eta(h)
\end{aligned} \tag{30}$$

Similarly, we get

$$\begin{aligned}
\Delta V_2(h) =& \sum_{i=h-\tau(h+1)+1}^{h} \eta^T(i)K\eta(i) - \sum_{i=h-\tau(h)}^{h-1} \eta^T(i)K\eta(i) \\
=&\eta^T(h)K\eta(h) - \sum_{i=h-\tau(h+1)+1}^{h-\tau_1} \eta^T(i)K\eta(i) + \sum_{i=h-\tau_1+1}^{h-1} \eta^T(i)K\eta(i) \\
&- \eta^T(h-\tau(h))K\eta(h-\tau(h)) - \sum_{i=h-\tau(h)+1}^{h-1} \eta^T(i)K\eta(i) \\
\leq& \eta^T(h)K\eta(h) - \eta^T(h-\tau(h))K\eta(h-\tau(h)) + \sum_{i=h-\tau_2+1}^{h-\tau_1} \eta^T(i)K\eta(i),
\end{aligned} \tag{31}$$

$$\begin{aligned}
\Delta V_3(h) =& \sum_{j=h+2-\tau_2}^{h+1-\tau_1} \sum_{i=j}^{h} \eta^T(i)K\eta(i) - \sum_{j=h-\tau_2+1}^{h-\tau_1} \sum_{i=j}^{h-1} \eta^T(i)K\eta(i) \\
=& \sum_{j=h+1-\tau_2}^{h-\tau_1} \sum_{i=j+1}^{h} \eta^T(i)K\eta(i) - \sum_{j=h-\tau_2+1}^{h-\tau_1} \sum_{i=j}^{h-1} \eta^T(i)K\eta(i) \\
=& \sum_{j=h+1-\tau_2}^{h-\tau_1} [\eta^T(h)K\eta(h) - \eta^T(j)K\eta(j)], \\
=&(\tau_2 - \tau_1)\eta^T(h)K\eta(h) - \sum_{i=h+1-\tau_2}^{h-\tau_1} \eta^T(i)K\eta(i),
\end{aligned} \tag{32}$$

and

$$\begin{aligned}\Delta V_4(h) &= \sum_{j=-\tau_2}^{-1}[\sum_{i=h+1+j}^{h}\varepsilon^T(i)M\varepsilon(i) - \sum_{i=h+j}^{h-1}\varepsilon^T(i)M\varepsilon(i)] \\ &= \sum_{j=-\tau_2}^{-1}[\varepsilon^T(h)M\varepsilon(h) - \varepsilon^T(h+j)M\varepsilon(h+j)] \\ &=\tau_2\varepsilon^T(h)M\varepsilon(h) - \sum_{i=h-\tau_2}^{h-1}\varepsilon^T(i)M\varepsilon(i).\end{aligned} \tag{33}$$

while

$$\begin{aligned}\varepsilon(h) &= \eta(h+1) - \eta(h) \\ &= (\bar{D} - I)\eta(h) + \bar{A}\xi(\eta(h)) + \bar{B}\xi(\eta(h-\tau(h))) + \bar{K}\phi(\eta(h)) + (\bar{L} - \bar{C}_2)\omega(h),\end{aligned} \tag{34}$$

Thus

$$\begin{aligned}\Delta V_4(h) \leq &\tau_2[(\bar{D} - I)\eta(h) + \bar{A}\xi(\eta(h)) + \bar{B}\xi(\eta(h-\tau(h))) + \bar{K}\phi(\eta(h)) \\ &+ (\bar{L} - \bar{C}_2)\omega(h)]^T M[(\bar{D} - I)\eta(h) + \bar{A}\xi(\eta(h)) + \bar{B}\xi(\eta(h-\tau(h))) \\ &+ \bar{K}\phi(\eta(h)) + (\bar{L} - \bar{C}_2)\omega(h)].\end{aligned} \tag{35}$$

According to (A_1), the following inequalities hold:

$$\begin{cases} 2[(I_2 \otimes F^-)\eta(h) - \xi(\eta(h))]^T\Gamma[(I_2 \otimes F^+)\eta(h) - \xi(\eta(h))] \leq 0, \\ 2[(I_2 \otimes F^-)\eta(h-\tau(h)) - \xi(\eta(h-\tau(h)))]^T\Gamma_1[(I_2 \otimes F^+)\eta(h-\tau(h)) - \xi(\eta(h-\tau(h)))] \leq 0. \end{cases} \tag{36}$$

Then, combining (29)-(36), we can obtain that

$$\Delta V(h) \leq \zeta^T(h)\bar{\Xi}\zeta(h), \tag{37}$$

where

$$\zeta(h) = [\eta^T(h), \eta^T(h-\tau(h)), \xi^T(\eta(h)), \xi^T(\eta(h-\tau(h))), \phi^T(h), \omega^T(h)]^T,$$

$$\bar{\Xi} = \begin{pmatrix} \Xi_{11} & 0 & \tilde{\Xi}_{13} & \bar{D}^TP\bar{B} + \tau_2(\bar{D} - I)^TM\bar{B} & \tilde{\Xi}_{15} & \tilde{\Xi}_{16} \\ * & \tilde{\Xi}_{22} & 0 & \tilde{\Xi}_{24} & 0 & 0 \\ * & * & \tilde{\Xi}_{33} & \bar{A}^TP\bar{B} + \tau_2\bar{A}^TM\bar{B} & \bar{A}^TP\bar{K} + \tau_2\bar{A}^TM\bar{K} & \tilde{\Xi}_{36} \\ * & * & * & \bar{B}^TP\bar{B} + \tau_2\bar{B}^TM\bar{B} - 2\Gamma_1 & \bar{B}^TP\bar{K} + \tau_2\bar{B}^TM\bar{K} & \tilde{\Xi}_{46} \\ * & * & * & * & \tilde{\Xi}_{55} & \tilde{\Xi}_{56} \\ * & * & * & * & * & \Xi_{66} \end{pmatrix}, \tag{38}$$

where $\Xi_{11} = \bar{D}^TP\bar{D} + \tau_2(\bar{D} - I)^TM(\bar{D} - I) - P + K + (\tau_2 - \tau_1)K - 2(I_2 \otimes F^-)\Gamma(I_2 \otimes F^+)$, $\Xi_{66} = (\bar{L} - \bar{C}_2)^TP(\bar{L} - \bar{C}_2) + \tau_2(\bar{L} - \bar{C}_2)^TM(\bar{L} - \bar{C}_2)$.

Considering (23) under zero initial condition, we get that J in (27) is equivalent to

$$\begin{aligned}
J &= \sum_{h=0}^{\infty}\left[\tilde{u}^T(h)\tilde{u}(h) - \gamma^2\omega^T(h)\omega(h)\right] \\
&= \sum_{h=0}^{\infty}\left[\tilde{u}^T(h)\tilde{u}(h) - \gamma^2\omega^T(h)\omega(h) + \Delta V(h)\right] - V(\infty) + V(0) \\
&\le \sum_{h=0}^{\infty}\left[\hat{z}^T(h)C_3^T C_3\hat{z}(h) - \gamma^2\omega^T(h)\omega(h) + \Delta V(h)\right] \\
&\le \sum_{h=0}^{\infty}\zeta^T(h)\Xi\zeta(h) < 0
\end{aligned} \tag{39}$$

by (28).

When $\omega \equiv 0$, following the similar analysis in [11], the asymptotic stability of the system (26) can be obtained. This completes the proof. ■

Remark 3.2. *The state estimation of system (24) with the fixed connection weights is studied in [10] by LKFs and Jensen integral inequality, while Theorem 3.2 ensures the asymptotic stability of the system (24), thus the obtained results are more general and practical.*

4. Numerical examples

Two numerical simulations are given to illustrate the theoretical results.

Example 1. *This example is used to illustrate the obtain results in Theorem 3.1. Consider a two-neuron DRMNNs model as follows:*

$$\begin{cases}
\dot{z}_1(t) = -z_1(t) + a_{11}(z_1(t))f_1(z_1(t)) + 8f_2(z_2(t)) + b_{11}(z_1(t))f_1(z_1(t-1)) + 0.2f_2(z_2(t-1)), \\
\dot{z}_2(t) = -z_2(t) + 0.3f_1(z_1(t)) + a_{22}(z_2(t))f_2(z_2(t)) + 0.5f_1(z_1(t-1)) + b_{22}(z_2(t))f_2(z_2(t-1)),
\end{cases}$$

where the unknown connection weights are assumed to be:

$$a_{11}(z_1(t)) = \begin{cases} -0.90, & z_1(t) \le 0, \\ -0.57, & z_1(t) > 0, \end{cases} \quad a_{22}(z_2(t)) = \begin{cases} -1.40, & z_2(t) \le 0, \\ -2.50, & z_2(t) > 0, \end{cases}$$

$$b_{11}(z_1(t)) = \begin{cases} -0.98, & z_1(t) \le 0, \\ -0.35, & z_1(t) > 0, \end{cases} \quad b_{22}(z_2(t)) = \begin{cases} 0.62, & z_2(t) \le 0, \\ 0.44, & z_2(t) > 0. \end{cases}$$

And the parameters of system (1) and (3) are chosen as: $L = [0.1,\ 0.2]^T$, $E = [0.35,\ 0.3]$,

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ D = \begin{pmatrix} -0.75 & 0.255 \\ 0.135 & 0.28 \end{pmatrix}.$$

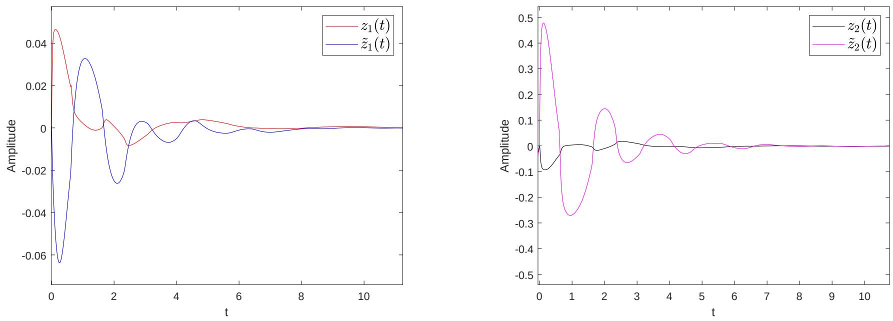

Fig. 1. State trajectories of $z(t)$, $\hat{z}(t)$ in (1) and (5), respectively.

The activation function $f_j(s) = \tanh(0.5s)$*,* $j = 1, 2$*. The time-varying delay is chosen as* $\tau(t) = 2 + sin(\frac{t}{2})$*, the noise input and the nonlinear disturbance are taken by* $\omega(t) = 1/(0.8 + 1.2t)$ *and* $g(t, z(t)) = [0.14cos(2z_1),\ 0.25sin(z_2)]^T$*, respectively. For system (1), the logical switching signals* $\rho_i(t)$ *are defined as*

$$\rho_1(t) = \begin{cases} 1, & z_1(t) \leq 0, \\ 0, & z_1(t) > 0, \end{cases} \quad \rho_2(t) = \begin{cases} 1, & z_2(t) \leq 0, \\ 0, & z_2(t) > 0. \end{cases}$$

Then, it is easy to verify that $\tau = 3$*,* $\mu = 0.5$*,* $F^- = 0$*,* $F^+ = 0.5I_2$*, and*

$$A = \begin{pmatrix} -1.36 & -0.92 \\ -0.67 & -0.752 \end{pmatrix}, \ B = \begin{pmatrix} -0.75 & 0.255 \\ 0.135 & 0.28 \end{pmatrix}.$$

Taking $\varsigma = 0.35$*, we can verify by Matlab toolbox that the LMI is solved and the feasible solution are given below:*

$$K = \begin{pmatrix} 0.5323 & 0.8944 \\ 0.4854 & 0.8718 \end{pmatrix}, \ X = \begin{pmatrix} 0.2757 & 0 \\ 0 & 0.8944 \end{pmatrix}.$$

Thus, the corresponding state estimator gain is found as

$$H = K^{-1}X = \begin{pmatrix} 8.0340 & -26.7387 \\ -4.4731 & 15.9135 \end{pmatrix}.$$

Therefore, from Theorem 3.1, system (2) becomes an asymptotic state estimator of (1) with the given H_∞ *performance index, which is further verified by the simulation results given by Figs. 1-2. Fig. 1 depicts the behaviors of* $z_1(t)$*,* $\hat{z}_1(t)$*, and* $z_2(t)$*,* $\hat{z}_2(t)$*, respectively. Fig. 2 shows that the error states asymptotically converge to zero.*

Example 2. *Consider the DRMNNs (22) in Theorem 3.2 with the following parameters:*

$$a_{11}(z_1(t)) = \begin{cases} -0.90, & z_1(t) \le 0, \\ -0.57, & z_1(t) > 0, \end{cases} \quad a_{12}(z_2(t)) = \begin{cases} -1.40, & z_2(t) \le 0, \\ -2.50, & z_2(t) > 0, \end{cases}$$

$$a_{21}(z_1(t)) = \begin{cases} -0.98, & z_1(t) \le 0, \\ -0.35, & z_1(t) > 0, \end{cases} \quad a_{22}(z_2(t)) = \begin{cases} 0.62, & z_2(t) \le 0, \\ 0.44, & z_2(t) > 0, \end{cases}$$

$$b_{11}(z_1(t)) = \begin{cases} -0.46, & z_1(t) \le 0, \\ -0.76, & z_1(t) > 0, \end{cases} \quad b_{12}(z_2(t)) = \begin{cases} -1.73, & z_2(t) \le 0, \\ -2.14, & z_2(t) > 0, \end{cases}$$

$$b_{21}(z_1(t)) = \begin{cases} -0.28, & z_1(t) \le 0, \\ -0.94, & z_1(t) > 0, \end{cases} \quad b_{22}(z_2(t)) = \begin{cases} 0.32, & z_2(t) \le 0, \\ 0.67, & z_2(t) > 0. \end{cases}$$

and

$$L = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix}, \ C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ C_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, D = C_3 = I_2.$$

Take the activation functions $f_j(s) = \tanh(0.5s)$, $j = 1, 2$, *which satisfy the assumption* (A_1) *with:* $F^- = 0$ *and* $F^+ = 0.5I$. *The discrete time-varying delay is chosen as* $\tau(h) = 10 + 2sin(\frac{h\pi}{2})$, *then, it can be verified that the upper bound and the lower bound of the time varying delays are* $\tau_1 = 8$, $\tau_2 = 12$, *respectively. Choose the disturbance input* $\omega(h) = cos(h - 1)$, *and* $\lambda_1 = 0.5, \lambda_2 = 0.5$, *then by the LMI toolbox, we solve LMI and obtain the matrices P and X as follows:*

$$P = \begin{pmatrix} 0.2847 & 0.1947 \\ 0.3759 & 0.9238 \end{pmatrix}, \ X = \begin{pmatrix} 0.3606 & 0 \\ 0 & 0.4472 \end{pmatrix},$$

Then, according to $K = P^{-1}X$, *the parameter of the desired state estimator are derived:*

$$K = \begin{pmatrix} 1.7550 & -0.4587 \\ -0.7141 & 0.6707 \end{pmatrix}.$$

Under the obtained estimator gain, the simulation results are shown in Figs. 3 and 4. Fig. 3 plots the states $z_i(k)$ *of original system and their estimations* $\hat{z}_i(k)$, $i = 1, 2$, *respectively. Fig. 4 depicts the error states* $e_i(k)$ *between* $z_i(k)$ *and estimated states* $\hat{z}_i(k)$, *which shows that the estimation errors asymptotically converge to zeros.*

5. Conclusion

By constructing appropriate LKFs and LMI strategy, two sufficient conditions had been established to warrant that the estimation error systems is asymptotically stable with a prescribed H_∞ performance. The H_∞ state estimation of DRMNNs with both of continuous-time and discrete-time case are analyzed in this paper. We considered the non-fragile control in H_∞ state estimation

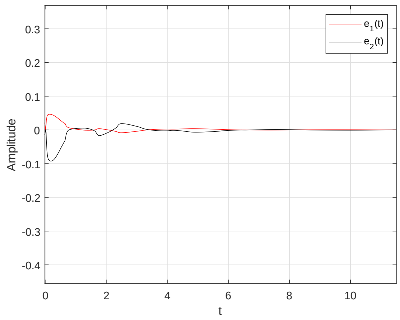

Fig. 2. The behaviors of $\|x(t)\|_1$ of the drive system in Example 2.

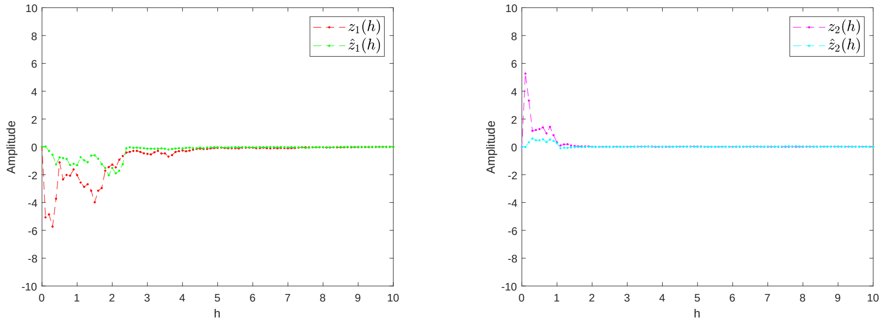

Fig. 3. State trajectories $z_i(t)$ and their estimations $\hat{z}_i(t)$, for $i = 1, 2$.

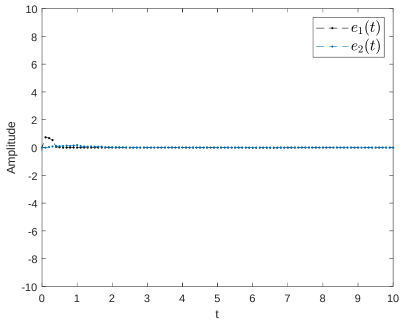

Fig. 4. Estimation error behaviors $e_i(h)$ of the discrete time case, for $i = 1, 2$.

system in continuous-time case, the key skill is to introduce a series of measurable logic switch signals to establish the switching system, which is helpful to reduce the computational cost. The given conditions can be easily examined by solving LMIs, the effectiveness of the obtained results are demonstrated by two numerical examples.

References

[1] Chua L (1971) Memristor-the missing circuit element. IEEE Trans Circuit Theory 18:507–519.

[2] Zhao H, Li L, Peng H, Kurths J, Xiao J, Yang Y (2008) Finite-time robust synchronization of memrisive neural network with perturbation. Neural Process Lett 47:509–533.

[3] Yang X, Cao J, Liang J (2017) Exponential synchronization of memristive neural networks with delays: Interval Matrix Method. IEEE Trans Neural Netw Learn Syst 28:1878–1888.

[4] Zhang R, Zeng D, Park J, Zhong S, Yu Y (2018) Novel discontinuous control for exponential synchronization of memristive recurrent neural networks with heterogeneous time-varying delays. J Frankl Inst 355:2826–2848.

[5] Yang X, Ho D W C (2016) Synchronization of delayed memristive neural networks: Robust Analysis Approach. IEEE Trans Cybern 46:3377–3387.

[6] Li R, Gao X, Cao J. (2019) Exponential State Estimation for Stochastically Disturbed Discrete-Time Memristive Neural Networks: Multiobjective Approach. IEEE Transactions on Neural Networks and Learning Systems, 99:1-10.

[7] Wei H, Li R, Chen C. (2015) State estimation for memristor-based neural networks with time-varying delays. International Journal of Machine Learning and Cybernetics, 6:213-225.

[8] Bao H, Cao J, Kurths J. (2018) State estimation of fractional-order delayed memristive neural networks. Nonlinear Dynamics, 94:1215-1225.

[9] Ji M, He Y, Zhang C, Wu M. (2014) Novel stability criteria for recurrent neural networks with time-varying delay. Neurocomputing, 138:383-391.

[10] Liang S, Jian w, et al. (2020) HMM-based H_∞ state estimation for memristive jumping neural networks subject to fading channel. Neurocomputing, 393:66-75.

[11] Ding S, Wang Z, Wang J, et al. (2016) H_∞ state estimation for memristive neural networks with time-varying delays: The discrete-time case. Neural Networks, 84:47-56.

[12] Liu H, Wang Z, et al. (2020) Delay-Distribution-Dependent H_∞ State Estimation for Discrete-Time Memristive Neural Networks With Mixed Time-Delays and Fading Measurements. IEEE transactions on cybernetics, 50:440-451.

[13] Bao H, Park J H, Cao J. (2019) Non-fragile state estimation for fractional-order delayed memristive BAM neural networks. Neural Networks, 119:190-199.

[14] Yu Y, Dong H, Wang Z, et al. (2016) Design of non-fragile state estimators for discrete time-delayed neural networks with parameter uncertainties. Neurocomputing, 182:18-24.

[15] Ding S, Wang Z, et al. (2017) Stop and Go adaptive strategy for synchronization of delayed memristive recurrent neural networks with unknown synaptic weights. Journal of the Franklin Institute, 354:4989-5010.

[16] Lakshmanan S, Mathiyalagan K, et al. (2014) Delay-dependent H_∞ state estimation of neural networks with mixed time-varying delays. Neurocomputing, 129: 392-400.

[17] Zhang R, Zeng D, Park J, Zhong S, Yu Y (2018) Novel discontinuous control for exponential synchronization of memristive recurrent neural networks with heterogeneous time-varying delays. Journal of the Franklin Institute, 355:2826–2848.

[18] Ding S, Wang Z, Zhang H, (2018) Dissipativity analysis for stochastic memristive neural networks with time-varying delays: a discrete-time case. IEEE Transactions on Neural Networks and Learning Systems, 29:618-630.

[19] Huang H, Huang T, Chen X, (2013) Guaranteed H_∞ performance state estimation of delayed static neural networks. IEEE Transactions on Neural Networks and Learning Systems, 60:371C375.