# The impact of nonlinear perturbation to the dynamics of HIV model

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**Abstract**

In this paper, we developed and studied a stochastic HIV model with nonlinear perturbation. Through a rigorous analysis, we firstly showed that the solution of the stochastic model is positive and global. Then, by employing suitable stochastic Lyapunov functions, we prove that the stochastic model admit a unique ergodic stationary distribution. In addition, sufficient conditions for the extinction of HIV infection are derived. Finally, numerical simulations are employed to confirm our theoretical results.

*Keywords:* HIV model; Nonlinear perturbation; Stochastic Lyapunov functions; Ergodic stationary distribution; Extinction.

## 1. Introduction

HIV stands for human immunodeficiency virus, which attacks the body's immune system by destroying important cells, especially white blood cells named $CD4^+$ T cells [1]. If the person's $CD4^+$ T cell count falls below 200, their immunity is severely compromised, leaving them more susceptible to infections. HIV continues to be a major global public health issue, having claimed almost 33 million lives so far. According to the World Health Organization, there were an estimated 38.0 million people living with HIV at the end of 2019 [2].

It is now well established from a variety of studies [3, 4, 5, 6, 7, 8] that mathematical modelling have been essential tools to study the dynamics of infectious disease. The past thirty years have seen increasingly rapid advances in the field of the pathogen dynamics. In 1999, Perelson and Nelson [3] have proposed different mathematical models for understanding the dynamics of HIV-1 infection in vivo. Wang and Li [4] have studied the global dynamics of HIV infection model with $CD4^+$ T cells. They proved that if the basic reproduction number $R_0 \leq 1$, the HIV infection is cleared from the T-cell population, otherwise the HIV infection persists.

In the real world, the dynamics of infectious diseases is inevitably perturbed by environmental noise [9]. For modeling biological phenomenon, it is appropriate to use the stochastic differential equations due to its realistic approach. Compared to deterministic models, the stochastic models can describe the disease transmission progress exactly and generally result in more valuable conclusions [10]. Generally speaking, there are two ways of stochastic differential equation to reflect the inclusion of random noise in the model approach. The first one is the continuous-time Markov chain model [11, 12, 13], which is derived based on the theory of branching process. The second one is the approach of parameter perturbation, which is widely used in mathematical modelling recently. Researchers have shown an increased interest in stochastic models with parameter perturbations [14, 15, 16, 17, 18]. For instance, Using stochastic Lyapunov method, Han et al. [14] investigated a stochastic AIDS model with the corresponding staged treatment and second-order perturbation. Liu et al. [15] study the dynamical behavior of a higher order stochastically perturbed HIV/AIDS model with differential infectivity and amelioration. Meanwhile some researchers [19, 20, 21, 22] have paid attention to stochastic within-host HIV infection models recently. Wang et al. [19] presented a stochastic HIV infection model with general nonlinear incidence rate and obtained that model has a unique ergodic stationary distribution.

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So far, very little attention has been paid to within-host HIV infection model with nonlinear perturbation. This paper attempts to show that the influence of nonlinear perturbation on dynamical behavior of a within-host HIV infection model. The model consider the interaction of the HIV with two target cells, CD4$^+$ T cells and macrophages. Firstly, by employing a novel combination of Lyapunov functions, we address the existence and uniqueness of the global positive solution. Then, we derive the sufficient conditions for stationary distribution and extinction of HIV infection. Furthermore, numerical simulations are find out with the help of Milstein's higher order method for supporting the theoretical results.

The remaining part of the paper proceeds as follows. In the next section, we derive a new stochastic HIV model with nonlinear perturbation and present some necessary lemmas. In Section 3, we prove that there exists a unique global positive solution of stochastic model (2.2). In Section 4, we get sufficient criteria such that there is an ergodic stationary distribution of stochastic model. Sufficient conditions for the extinction of infected cells and free virus particles are obtained in Section 5. Section 6 illustrates the theoretical results through numerical simulations followed by conclusion in Section 7.

## 2. Models and preliminaries

### *2.1. Ordinary differential equation model*

In [23], Elaiw investigated the global properties of the following nonlinear HIV infection model with CD4$^+$ T cells and macrophages:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \lambda_1 - d_1 x - \beta_1 x v, \\ \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \beta_1 x v - a x_1, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \lambda_2 - d_2 y - \beta_2 y v, \\ \dfrac{\mathrm{d}y_1}{\mathrm{d}t} = \beta_2 y v - \delta y_1, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = p_1 x_1 + p_2 y_1 - c v, \end{cases} \tag{2.1}$$

where $x(t)$ and $x_1(t)$ are the densities of uninfected and infected CD4$^+$ T cells, respectively; $y(t)$ and $y_1(t)$ denote the densities of uninfected and infected macrophages, respectively; $v(t)$ is the density of free virus particles, at time $t > 0$. All parameters and their definitions are summarized in Table 2.1.

Table 2.1: Parameters of model (2.1) and their interpretations.

| Parameters | Biological meaning |
|---|---|
| $\lambda_1$ | Generation rate constant of new CD4$^+$ T cells |
| $\lambda_2$ | Generation rate constant of new macrophages |
| $d_1$ | Death rate constant of CD4$^+$ T cells |
| $d_2$ | Death rate constant of macrophages |
| $\beta_1$ | Transmission rate constant between CD4$^+$ T cells and HIV particles |
| $\beta_2$ | Transmission rate constant between CD4$^+$ T cells and macrophages |
| $a$ | Death rate constant of the infected CD4$^+$ T cells |
| $\delta$ | Death rate constant of the infected macrophages |
| $p_1$ | The rate at which the infected CD4$^+$ T cells produce HIV particles |
| $p_2$ | The rate at which the infected macrophages produce HIV particles |
| $c$ | Clear rate of HIV particles |

There always exists a compact positively invariant set for model (2.1) as follows

$$\Gamma_1 = \left\{ (x, x_1, y, y_1, v) \in \mathbb{R}_+^5 : 0 \le x, x_1 \le L_1, 0 \le y, y_1 \le L_2, 0 \le v \le \frac{p_1 L_1 + p_2 L_2}{c} \right\},$$

with $L_1 = \frac{\lambda_1}{\min\{d_1, a\}}$ and $L_2 = \frac{\lambda_2}{\min\{d_2, \delta\}}$.

In addition, the global behavior of model (2.1) crucially depends on the basic reproduction number given by

$$R_0 = \frac{p_1\beta_1 x_0\delta + p_2\beta_2 y_0 a}{c\delta a},$$

where $x_0 = \frac{\lambda_1}{d_1}$ and $y_0 = \frac{\lambda_2}{d_2}$.

- If $R_0 \le 1$, then $E_0 = (x_0, 0, y_0, 0, 0)$ is globally asymptotically stable (GAS) in $\Gamma_1$.
- If $R_0 > 1$, then $E_1 = (x_0^*, x_1^*, y_0^*, y_1^*, v^*)$ is GAS in $\Gamma_1$.

*2.2. Stochastic differential equation model*

Now we consider random perturbations, assuming the parameters $-d_1$, $-a$, $-d_2$, $-\delta$, $-c$ appearing in model (2.1) are not constants and they always fluctuated by the nonlinear Gaussian white noises:

$$\begin{aligned}
&-d_1 \to -d_1 + (\sigma_{11} + \sigma_{12}x(t))\dot{B}_1(t), &&-a \to -a + (\sigma_{21} + \sigma_{22}x_1(t))\dot{B}_2(t),\\
&-d_2 \to -d_2 + (\sigma_{31} + \sigma_{32}y(t))\dot{B}_3(t), &&-\delta \to -\delta + (\sigma_{41} + \sigma_{42}y_1(t))\dot{B}_4(t),\\
&-c \to -c + (\sigma_{51} + \sigma_{52}v(t))\dot{B}_5(t),
\end{aligned}$$

where $\dot{B}_i(t)$ are independent standard Brownian motions with $B_i(0) = 0$ and $\sigma_{ij}^2 > 0$ denote the intensities of the white noise, for $i = 1, 2, 3, 4, 5$ and $j = 1, 2$. Therefore, model (2.1) with additional nonlinear perturbation can be written as the following stochastic differential equation model:

$$\begin{cases}
\mathrm{d}x(t) = (\lambda_1 - d_1 x - \beta_1 x v)\,\mathrm{d}t + (\sigma_{11} + \sigma_{12}x)x\mathrm{d}B_1(t),\\
\mathrm{d}x_1(t) = (\beta_1 x v - a x_1)\,\mathrm{d}t + (\sigma_{21} + \sigma_{22}x_1)x_1\mathrm{d}B_2(t),\\
\mathrm{d}y(t) = (\lambda_2 - d_2 y - \beta_2 y v)\,\mathrm{d}t + (\sigma_{31} + \sigma_{32}y)y\mathrm{d}B_3(t),\\
\mathrm{d}y_1(t) = (\beta_2 y v - \delta y_1)\,\mathrm{d}t + (\sigma_{41} + \sigma_{42}y_1)y_1\mathrm{d}B_4(t),\\
\mathrm{d}v(t) = (p_1 x_1 + p_2 y_1 - c v)\,\mathrm{d}t + (\sigma_{51} + \sigma_{52}v)v\mathrm{d}B_5(t).
\end{cases} \tag{2.2}$$

The other parameters are the same as in model (2.1).

*2.3. Preliminaries*

Throughout this paper, let $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P}\}$ be the complete probability space with filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the normal conditions. Let $\mathbb{R}^n$ be an $n$-dimensional standard Euclidean space and $\mathbb{R}_+^k = \{(x_1, \ldots, x_k) | x_i > 0, 1 \le i \le k\}$. Denote $a_1 \vee a_2 \vee \cdots \vee a_n = \max\{a_1, a_2, \ldots, a_n\}$ and $a_1 \wedge a_2 \wedge \cdots \wedge a_n = \min\{a_1, a_2, \ldots, a_n\}$.

Considering a d-dimensional stochastic differential equation

$$\mathrm{d}z(t) = f(z(t), t)\mathrm{d}t + g(z(t), t)\mathrm{d}B(t), \quad t \ge t_0, \tag{2.3}$$

with condition $z(t_0) = z_0 \in \mathbb{R}^d$, where $B(t)$ denotes an m-dimensional usual Brownian motion. Define the operator $\mathcal{L}$ related to (2.3) by

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(z,t)\frac{\partial}{\partial z_i} + \frac{1}{2}\sum_{i=1}^{d}\left[g^T(z,t)g(z,t)\right]_{ij}\frac{\partial^2}{\partial z_i \partial z_j}.$$

By operating $\mathcal{L}$ on the function $V$, one gets

$$\mathcal{L}V(z,t) = V_t(z,t) + V_z(z,t)f(z,t) + \frac{1}{2}trace\left[g^T(z,t)V_{zz}(z,t)g(z,t)\right],$$

where $V_t = \frac{\partial V}{\partial t}$, $V_z = \left(\frac{\partial V}{\partial z_1}, \ldots, \frac{\partial V}{\partial z_d}\right)$, $V_{zz} = \left(\frac{\partial^2 V}{\partial z_i \partial z_j}\right)_{d\times d}$.

For the following dynamical investigation of stochastic model (2.2), we shall firstly introduce some important lemmas.

**Lemma 2.1.** *(see [14])For $x \geq 0$, one gets*

$$\frac{x^3}{x^2+1} \geq x - \frac{1}{2}, \quad \frac{x^4}{x^2+1} \geq \frac{3}{4}x^2 - \frac{1}{4}.$$

Next, we shall introduce some results concerning the existence of a stationary distribution. Let $X(t)$ be a regular time-homogeneous Markov process in $\mathbb{R}^n_+$ satisfying the stochastic equation

$$\mathrm{d}X(t) = h(X(t))\mathrm{d}t + \sum_{m=1}^{k} \sigma_m(X(t))\mathrm{d}B_m(t),$$

with the diffusion matrix:

$$\bar{A}(x) = (\bar{a}_{ij}(x)), \quad \bar{a}_{ij}(x) = \sum_{m=1}^{k} \sigma_m^{(i)}(x)\sigma_m^{(j)}(x).$$

For more details see [25, 26, 27].

**Lemma 2.2.** *(see [25]) The Markov process $X(t)$ has a unique ergodic stationary distribution $\mu(\cdot)$. if there exists a bounded domain $U \subset \mathbb{R}^l$ with regular boundary $\Gamma$ such that*

*(i) there is a positive number $M$ such that $\sum_{i,j=1}^{l} a_{ij}(x)\xi_i\xi_j > M|\xi|^2$ for $x \in U$ and $\xi \in \mathbb{R}^d$.*

*(ii) there exists a nonnegative $C^2$-function $V$ such that $\mathcal{L}V$ is negative for any $x \in \mathbb{R}^d \setminus U$.*

*Then*

$$\mathbb{P}\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T h[X^x(t)]\mathrm{d}t = \int_{\mathbb{R}^d} h(x)\mu(\mathrm{d}x)\right\} = 1,$$

*for all $x \in \mathbb{R}^d$, where $h(\cdot)$ is a function integrable with respect to the measure $\mu$.*

Next, consider the following stochastic model with nonlinear perturbation

$$\mathrm{d}X(t) = (\lambda_1 - d_1X(t))\mathrm{d}t + (\sigma_{11} + \sigma_{12}X)X\mathrm{d}B_1(t). \tag{2.4}$$

Making the use of comparison theorem of 1-dimensional stochastic differential equations [28], one has the following lemma.

**Lemma 2.3.** *Let $X(t)$ be the solution of model (2.4) with initial value $X(0) = x(0) > 0$, then $X(t)$ is ergodic and $x(t) \leq X(t)$. System (2.4) has ergodic property and the invariant density is given by*

$$\mu_X = Q_1X^{-2(1+q_1)}(\sigma_{11} + \sigma_{12}X)^{-2(1-q_1)} \exp\left(-\frac{2(\lambda_1 + \sigma_{11}^2 q_1 X)}{\sigma_{11}X(\sigma_{11} + \sigma_{12}X)}\right),$$

*where $q_1 = \frac{2\lambda_1\sigma_{12}+d_1\sigma_{11}}{\sigma_{11}^3}$ and $Q_1$ is a positive constant satisfying $\int_0^\infty \mu_X \mathrm{d}X = 1$.*

*Similarly, for the stochastic model*

$$\mathrm{d}Y(t) = (\lambda_2 - d_2Y(t))\mathrm{d}t + (\sigma_{31} + \sigma_{32}Y)Y\mathrm{d}B_3(t). \tag{2.5}$$

*with the initial value $Y(0) = y(0) > 0$. Then $Y(t)$ is ergodic and $y(t) \leq Y(t)$. System (2.5) has ergodic property and the invariant density is defined by*

$$\mu_Y = Q_2Y^{-2(1+q_2)}(\sigma_{31} + \sigma_{32}Y)^{-2(1-q_2)} \exp\left(-\frac{2(\lambda_2 + \sigma_{31}^2 q_2 Y)}{\sigma_{31}Y(\sigma_{31} + \sigma_{32}Y)}\right),$$

*where $q_2 = \frac{2\lambda_2\sigma_{32}+d_2\sigma_{31}}{\sigma_{31}^3}$ and $Q_2$ is a positive constant satisfying $\int_0^\infty \mu_Y \mathrm{d}Y = 1$.*

The detailed proof of Lemma 2.3 is similar to Theorem 3.1 in literature [24]. We omit it here.

In the following lemma, to prove the extinction theorem in Section 5, we introduce the important exponential martingale inequality.

**Lemma 2.4.** *[29] (exponential martingale inequality) Let $g = (g_1, \dots, g_m) \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R}^{1\times m})$, and let $T$, $\alpha$, $\beta$ be any positive constants. Then*

$$P\left\{\sup_{0\le t\le T}\left[\int_0^t g(s)\mathrm{d}B(s) - \frac{\alpha}{2}\int_0^t |g(s)|^2\mathrm{d}s\right] > \beta\right\} \le e^{-\alpha\beta}.$$

## 3. The existence and uniqueness of global positive solution

Firstly, we give the following fundamental theorem with respect to a unique global positive solution of stochastic model (2.2).

**Theorem 3.1.** *For any initial value $(x(0), x_1(0), y(0), y_1(0), v(0)) \in \mathbb{R}^5_+$, there exists a unique solution $(x(t), x_1(t), y(t), y_1(t), v(t)) \in \mathbb{R}^5_+$ of model (2.2) on $t \ge 0$, and the solution will remain in $\mathbb{R}^3_+$ with probability 1.*

*Proof.* It is noted that the coefficients of model (2.2) are locally Lipschitz continuous, so for any given initial value $(x(0), x_1(0), y(0), y_1(0), v(0)) \in \mathbb{R}^5_+$, there is a unique maximal local solution $(x(t), x_1(t), y(t), y_1(t), v(t))$ on $t \in [0, \tau_e)$, where $\tau_e$ is the explosion time. Let $k_0$ be sufficiently large such that $x(0)$, $x_1(0)$, $y(0)$, $y_1(0)$ and $v(0)$ belong to the interval $[1/k_0, k_0]$. For each integer $k \ge k_0$, define the stopping time

$$\tau_k = \inf\left\{t \in [0, \tau_e) : x(t) \notin (\frac{1}{k}, k), x_1(t) \notin (\frac{1}{k}, k), y(t) \notin (\frac{1}{k}, k), y_1(t) \notin (\frac{1}{k}, k), v(t) \notin (\frac{1}{k}, k)\right\}.$$

Clearly, $\tau_k$ is non-decreasing as $k \to \infty$. We get $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \le \tau_e$ a.s. In order to show local solution $(x(t), x_1(t), y(t), y_1(t), v(t))$ is global, we only need to verify $\tau_\infty = \infty$ a.s..

Define the nonnegative $C^2$-Lyapunov function as follows

$$V_1(x, x_1, y, y_1, v) = (x - \rho_1 - \rho_1 \ln\frac{x}{\rho_1}) + (x_1 - 1 - \ln x_1) + (y - \rho_2 - \rho_2 \ln\frac{y}{\rho_2}) + (y_1 - 1 - \ln y_1) + \rho_3(v - 1 - \ln v),$$

where $\rho_i$ are positive constants which will be determined later, $i = 1, 2, 3$. The nonnegativity of $V_1$ can be obtained by the inequality $x - \rho - \rho \ln\frac{x}{\rho} \ge 0$ for $x, \rho > 0$.

Applying Itô's formula to $V_1$, we have

$$\begin{aligned}
\mathcal{L}V_1 =& \left(1 - \frac{\rho_1}{x}\right)(\lambda_1 - d_1 x - \beta_1 x v) + \frac{\rho_1}{2}(\sigma_{11} + \sigma_{12}x)^2 + \left(1 - \frac{1}{x_1}\right)(\beta_1 x v - a x_1) + \frac{1}{2}(\sigma_{21} + \sigma_{22}x_1)^2\\
&+ \left(1 - \frac{\rho_2}{y}\right)(\lambda_2 - d_2 y - \beta_2 y v) + \frac{\rho_2}{2}(\sigma_{31} + \sigma_{32}y)^2 + \left(1 - \frac{1}{y_1}\right)(\beta_2 y v - \delta y_1) + \frac{1}{2}(\sigma_{41} + \sigma_{42}y_1)^2\\
&+ \rho_3\left(1 - \frac{1}{v}\right)(p_1 x_1 + p_2 y_1 - c v) + \frac{\rho_3}{2}(\sigma_{51} + \sigma_{52}v)^2\\
\le& (\rho_3 p_1 - a)x_1 + (\rho_3 p_2 - \delta)y_1 + (\rho_1\beta_1 + \rho_2\beta_2 - \rho_3 c)v + \lambda_1 + \lambda_2 + a + \delta + \rho_1 d_1 + \rho_2 d_2 + \rho_3 c\\
&+ \frac{\rho_1}{2}(\sigma_{11} + \sigma_{12}x)^2 + \frac{1}{2}(\sigma_{21} + \sigma_{22}x_1)^2 + \frac{\rho_2}{2}(\sigma_{31} + \sigma_{32}y)^2 + \frac{1}{2}(\sigma_{41} + \sigma_{42}y_1)^2 + \frac{\rho_3}{2}(\sigma_{51} + \sigma_{52}v)^2.
\end{aligned}$$

Choose $\rho_1 = \frac{a}{p_1}$, $\rho_2 = \frac{\delta}{p_2}$ and $\rho_3 = \frac{\rho_1\beta_1 + \rho_2\beta_2}{c}$ such that

$$\rho_3 p_1 - a = 0, \quad \rho_3 p_2 - \delta = 0, \quad \rho_1\beta_1 + \rho_2\beta_2 - \rho_3 c = 0.$$

Therefore, one gets

$$\begin{aligned}
\mathcal{L}V_1 \le& \lambda_1 + \lambda_2 + a + \delta + \frac{a d_1}{p_1} + \frac{\delta d_2}{p_2} + \frac{\beta_1 a}{p_1} + \frac{\beta_2 \beta}{p_2} + \frac{a}{2p_1}(\sigma_{11} + \sigma_{12}x)^2 + \frac{1}{2}(\sigma_{21} + \sigma_{22}x_1)^2 + \frac{\delta}{2p_2}(\sigma_{31} + \sigma_{32}y)^2\\
&+ \frac{1}{2}(\sigma_{41} + \sigma_{42}y_1)^2 + \frac{p_2\beta_1 a + p_1\beta_2\delta}{2p_1p_2c}(\sigma_{51} + \sigma_{52}v)^2.
\end{aligned} \tag{3.1}$$

Then define a nonnegative $C^2$-Lyapunov function $V_2$:

$$
\begin{aligned}
V_2 =& \sum_{i=1}^{2} \frac{a_i\left[(x+b_i)^\kappa + (x_1+b_i)^\kappa + (y+b_i)^\kappa + (y_1+b_i)^\kappa + (v+b_i)^\kappa\right]}{\kappa} \\
&+ \sum_{i=1}^{2} a_i b_i^{\kappa-1}\left[\left(\frac{a+p_1}{a}\right)x + \frac{p_1}{a}x_1 + \left(\frac{\delta+p_2}{\delta}\right)y + \frac{p_2}{\delta}y_1\right],
\end{aligned}
$$

where $\kappa \in (0,1)$ is a variable, $a_i$ and $b_i$ will be determined in (3.2), $i = 1, 2$.

Applying the Itô's formula to $V_2$, one has

$$
\begin{aligned}
\mathcal{L}V_2 =& \sum_{i=1}^{2} a_i \Bigg\{ (x+b_i)^{\kappa-1}(\lambda_1 - d_1 x - \beta_1 x v) + (x_1+b_i)^{\kappa-1}(\beta_1 x v - a x_1) + (y+b_i)^{\kappa-1}(\lambda_2 - d_2 y - \beta_2 y v) \\
&+ (y_1+b_i)^{\kappa-1}(\beta_2 y v - \delta y_1) + (v+b_i)^{\kappa-1}(p_1 x_1 + p_2 y_1 - c v) + b_i^{\kappa-1}\Bigg[\left(\frac{a+p_1}{a}\right)\lambda_1 + \left(\frac{\delta+p_2}{\delta}\right)\lambda_2 \\
&- \beta_1 x v - \beta_2 y v - p_1 x_1 - p_2 y_1\Bigg]\Bigg\} - \sum_{i=1}^{2}\frac{a_i(1-\kappa)}{2}\Bigg[\frac{(\sigma_{11}x + \sigma_{12}x^2)^2}{(x+b_i)^{2-\kappa}} + \frac{(\sigma_{21}x_1 + \sigma_{22}x_1^2)^2}{(x_1+b_i)^{2-\kappa}} \\
&+ \frac{(\sigma_{31}y + \sigma_{32}y^2)^2}{(y+b_i)^{2-\kappa}} + \frac{(\sigma_{41}y + \sigma_{42}y_1^2)^2}{(y_1+b_i)^{2-\kappa}} + \frac{(\sigma_{41}y + \sigma_{42}y_1^2)^2}{(v+b_i)^{2-\kappa}}\Bigg] \\
\le& \sum_{i=1}^{2} a_i b_i^{\kappa-1}\left[\frac{(2a+p_1)\lambda_1}{a} + \frac{(2\delta+p_2)\lambda_2}{\delta}\right] - \sum_{i=1}^{2}\frac{a_i(1-\kappa)}{2b_i^{2-\kappa}}\Bigg[\frac{2\sigma_{11}\sigma_{12}x^3 + \sigma_{12}^2x^4}{\left(\frac{x}{b_i}+1\right)^{2-\kappa}} + \frac{2\sigma_{21}\sigma_{22}x_1^3 + \sigma_{22}^2x_1^4}{\left(\frac{x_1}{b_i}+1\right)^{2-\kappa}} \\
&+ \frac{2\sigma_{31}\sigma_{32}y^3 + \sigma_{32}^2y^4}{\left(\frac{y}{b_i}+1\right)^{2-\kappa}} + \frac{2\sigma_{41}\sigma_{12}y_1^3 + \sigma_{42}^2y_1^4}{\left(\frac{y_1}{b_i}+1\right)^{2-\kappa}} + \frac{2\sigma_{51}\sigma_{52}v^3 + \sigma_{52}^2v^4}{\left(\frac{v}{b_i}+1\right)^{2-\kappa}}\Bigg] \\
\le& \sum_{i=1}^{2} a_i b_i^{\kappa-1}\left[\frac{(2a+p_1)\lambda_1}{a} + \frac{(2\delta+p_2)\lambda_2}{\delta}\right] - \frac{a_1(1-\kappa)}{b_1^{2-\kappa}}\Bigg[\frac{\sigma_{11}\sigma_{12}x^3}{\left(\frac{x}{b_1}+1\right)^2} + \frac{\sigma_{21}\sigma_{22}x_1^3}{\left(\frac{x_1}{b_1}+1\right)^2} + \frac{\sigma_{31}\sigma_{32}y^3}{\left(\frac{y}{b_1}+1\right)^2} + \frac{\sigma_{41}\sigma_{42}y_1^3}{\left(\frac{y_1}{b_1}+1\right)^2} \\
&+ \frac{\sigma_{51}\sigma_{52}v^3}{\left(\frac{v}{b_1}+1\right)^2}\Bigg] - \frac{a_2(1-\kappa)}{2b_2^{2-\kappa}}\Bigg[\frac{\sigma_{12}^2x^4}{\left(\frac{x}{b_2}+1\right)^2} + \frac{\sigma_{22}^2x_1^4}{\left(\frac{x_1}{b_2}+1\right)^2} + \frac{\sigma_{32}^2y^4}{\left(\frac{y}{b_2}+1\right)^2} + \frac{\sigma_{42}^2y_1^4}{\left(\frac{y_1}{b_2}+1\right)^2} + \frac{\sigma_{52}^2v^4}{\left(\frac{v}{b_2}+1\right)^2}\Bigg].
\end{aligned}
$$

Making the use of $(x+1)^2 \leq 2(x^2+1)$ and inequalities in Lemma 2.1, we have

$$
\begin{aligned}
\mathcal{L}V_2 \leq & \sum_{i=1}^{2} a_i b_i^{\kappa-1}\left[\frac{(2a+p_1)\lambda_1}{a}+\frac{(2\delta+p_2)\lambda_2}{\delta}\right]-\frac{a_1 b_1^{1+\kappa}(1-\kappa)}{2}\left[\frac{\sigma_{11}\sigma_{12}}{(\frac{x}{b_1})^2+1}\left(\frac{x}{b_1}\right)^3+\frac{\sigma_{21}\sigma_{22}x_1^3}{(\frac{x_1}{b_1})^2+1}\left(\frac{x_1}{b_1}\right)^3\right.\\
& \left.+\frac{\sigma_{31}\sigma_{32}y^3}{(\frac{y}{b_1})^2+1}\left(\frac{y}{b_1}\right)^3+\frac{\sigma_{41}\sigma_{42}y_1^3}{(\frac{y_1}{b_1})^2+1}\left(\frac{y_1}{b_1}\right)^3+\frac{\sigma_{51}\sigma_{52}v^3}{(\frac{v}{b_1})^2+1}\left(\frac{v}{b_1}\right)^3\right]-\frac{a_2 b_2^{2+\kappa}(1-\kappa)}{4}\left[\frac{\sigma_{12}^2}{(\frac{x}{b_2})^2+1}\left(\frac{x}{b_2}\right)^4\right.\\
& \left.+\frac{\sigma_{22}^2}{(\frac{x_1}{b_2})^2+1}\left(\frac{x_1}{b_2}\right)^4+\frac{\sigma_{32}^2}{(\frac{y}{b_2})^2+1}\left(\frac{y}{b_2}\right)^4+\frac{\sigma_{42}^2}{(\frac{y_1}{b_2})^2+1}\left(\frac{y_1}{b_2}\right)^4+\frac{\sigma_{52}^2}{(\frac{v}{b_2})^2+1}\left(\frac{v}{b_2}\right)^4\right]\\
\leq & \sum_{i=1}^{2} a_i b_i^{\kappa-1}\left[\frac{(2a+p_1)\lambda_1}{a}+\frac{(2\delta+p_2)\lambda_2}{\delta}\right]-\frac{a_1 b_1^{1+\kappa}(1-\kappa)}{2}\left[\sigma_{11}\sigma_{12}\left(\frac{x}{b_1}-\frac{1}{2}\right)+\sigma_{21}\sigma_{22}\left(\frac{x_1}{b_1}-\frac{1}{2}\right)\right.\\
& \left.+\sigma_{31}\sigma_{32}\left(\frac{y}{b_1}-\frac{1}{2}\right)+\sigma_{41}\sigma_{42}\left(\frac{y_1}{b_1}-\frac{1}{2}\right)+\sigma_{51}\sigma_{52}\left(\frac{v}{b_1}-\frac{1}{2}\right)\right]-\frac{a_2 b_2^{2+\kappa}(1-\kappa)}{4}\left\{\sigma_{12}^2\left[\frac{3}{4}\left(\frac{x}{b_2}\right)^2-\frac{1}{4}\right]\right.\\
& \left.+\sigma_{22}^2\left[\frac{3}{4}\left(\frac{x_1}{b_2}\right)^2-\frac{1}{4}\right]+\sigma_{32}^2\left[\frac{3}{4}\left(\frac{y}{b_2}\right)^2-\frac{1}{4}\right]+\sigma_{42}^2\left[\frac{3}{4}\left(\frac{y_1}{b_2}\right)^2-\frac{1}{4}\right]+\sigma_{52}^2\left[\frac{3}{4}\left(\frac{v}{b_2}\right)^2-\frac{1}{4}\right]\right\}\\
\leq & a_1 b_1^{\kappa-1}\left[\frac{(2a+p_1)\lambda_1}{a}+\frac{(2\delta+p_2)\lambda_2}{\delta}\right]+\frac{a_1 b_1^{1+\kappa}(1-\kappa)}{4}(\sigma_{11}\sigma_{12}+\sigma_{21}\sigma_{22}+\sigma_{31}\sigma_{32}+\sigma_{41}\sigma_{42}+\sigma_{51}\sigma_{52})\\
& +a_2 b_2^{\kappa-1}\left[\frac{(2a+p_1)\lambda_1}{a}+\frac{(2\delta+p_2)\lambda_2}{\delta}\right]+\frac{a_2 b_2^{2+\kappa}(1-\kappa)}{16}(\sigma_{12}^2+\sigma_{22}^2+\sigma_{32}^2+\sigma_{42}^2+\sigma_{52}^2)\\
& -\frac{a_1 b_1^{\kappa}(1-\kappa)}{2}(\sigma_{11}\sigma_{12}x+\sigma_{21}\sigma_{22}x_1+\sigma_{31}\sigma_{32}y+\sigma_{41}\sigma_{42}y_1+\sigma_{51}\sigma_{52}v)\\
& -\frac{3a_2 b_2^{\kappa}(1-\kappa)}{16}(\sigma_{12}^2x^2+\sigma_{22}^2x_1^2+\sigma_{32}^2y^2+\sigma_{42}^2y_1^2+\sigma_{52}^2v^2),
\end{aligned}
$$

choose

$$
\begin{aligned}
a_1 &= \frac{2}{b_1^{\kappa}(1-\kappa)}, \qquad b_1 = 2\sqrt{\frac{(2a+p_1)\lambda_1\delta+(2\delta+p_2)\lambda_2 a}{a\delta(1-\kappa)(\sigma_{11}\sigma_{12}+\sigma_{21}\sigma_{22}+\sigma_{31}\sigma_{32}+\sigma_{41}\sigma_{42}+\sigma_{51}\sigma_{52})}},\\
a_2 &= \frac{8}{3b_2^{\kappa}(1-\kappa)}, \qquad b_2 = 2\sqrt[3]{\frac{(2a+p_1)\lambda_1\delta+(2\delta+p_2)\lambda_2 a}{a\delta(1-\kappa)(\sigma_{12}^2+\sigma_{22}^2+\sigma_{32}^2+\sigma_{42}^2+\sigma_{52}^2)}}.
\end{aligned} \tag{3.2}
$$

In view of the arbitrariness of $\kappa \in (0,1)$, letting $\kappa \to 0^+$, it leads to

$$
\begin{aligned}
\mathcal{L}V_2 \leq & r-(\sigma_{11}\sigma_{12}x+\sigma_{21}\sigma_{22}x_1+\sigma_{31}\sigma_{32}y+\sigma_{41}\sigma_{42}y_1+\sigma_{51}\sigma_{52}v)\\
& -\frac{1}{2}(\sigma_{12}^2x^2+\sigma_{22}^2x_1^2+\sigma_{32}^2y^2+\sigma_{42}^2y_1^2+\sigma_{52}^2v^2),
\end{aligned} \tag{3.3}
$$

where

$$
\begin{aligned}
r = & 2\sqrt{\left[\frac{(2a+p_1)\lambda_1}{a}+\frac{(2\delta+p_2)\lambda_2}{\delta}\right](\sigma_{11}\sigma_{12}+\sigma_{21}\sigma_{22}+\sigma_{31}\sigma_{32}+\sigma_{41}\sigma_{42}+\sigma_{51}\sigma_{52})}\\
& +2\sqrt[3]{\left[\frac{(2a+p_1)\lambda_1}{a}+\frac{(2\delta+p_2)\lambda_2}{\delta}\right]^2(\sigma_{12}^2+\sigma_{22}^2+\sigma_{32}^2+\sigma_{42}^2+\sigma_{52}^2)}>0.
\end{aligned}
$$

Finally, we define the nonnegative $C^2$-Lyapunov function

$$
V = V_1 + \bar{\rho}V_2,
$$

where $\bar{\rho}=\max\{\frac{a}{p_1},\frac{\delta}{p_2},\frac{p_2\beta_1 a+p_1\beta_2\delta}{p_1p_2c},1\}$. Combining (3.1) and (3.3), one obtains

$$\begin{aligned}\mathcal{L}V\leq&\lambda_1+\lambda_2+a+\delta+\frac{ad_1}{p_1}+\frac{\delta d_2}{p_2}+\frac{\beta_1 a}{p_1}+\frac{\beta_2\beta}{p_2}+\frac{a}{2p_1}\sigma_{11}^2+\frac{\sigma_{21}^2}{2}+\frac{\delta}{2p_2}\sigma_{31}^2+\frac{\sigma_{41}^2}{2}+\frac{p_2\beta_1 a+p_1\beta_2\delta}{2p_1p_2c}\sigma_{51}^2+\bar{\rho}r\\ :=&K \text{ is a positive constant.}\end{aligned}$$

A similar proof of Theorem 3 in literature [30] yields $\tau_\infty=\infty$ a.s., thus $(x(t),x_1(t),y(t),y_1(t),v(t))\in\mathbb{R}_+^5$ a.s. for all $t\geq 0$. This completes the proof. $\square$

## 4. Existence of ergodic stationary distribution

In this section, using the theory of Khasminskii [25], we obtain the sufficient conditions such that model (2.2) has a ergodic stationary distribution.

**Theorem 4.1.** *Let $(x(t),x_1(t),y(t),y_1(t),v(t))$ be a solution of model (2.2) with any initial value $(x(0),x_1(0),y(0),y_1(0),v(0))\in\mathbb{R}_+^5$. If*

$$R_0^s=\frac{p_1\beta_1\lambda_1\bar{\delta}\bar{d}_2+p_2\beta_2\lambda_2\bar{a}\bar{d}_1}{\bar{c}\bar{\delta}\bar{a}\bar{d}_1\bar{d}_2}>1,$$

*then there exists a stationary distribution $\mu(\cdot)$ and the solution $(x(t),x_1(t),y(t),y_1(t),v(t))$ to model (2.2) is ergodic, where $\bar{d}_1=d_1+\frac{\sigma_{11}^2}{2}+2\left(\sigma_{12}\lambda_1\right)^{\frac{2}{3}}+\frac{4\sigma_{11}}{3}\left(\sigma_{12}\lambda_1\right)^{\frac{1}{3}}$, $\bar{a}=a+\frac{\sigma_{21}^2}{2}+2\left(\sigma_{22}\lambda_1\right)^{\frac{2}{3}}+\frac{4\sigma_{21}}{3}\left(\sigma_{22}\lambda_1\right)^{\frac{1}{3}}$, $\bar{d}_2=d_2+\frac{\sigma_{31}^2}{2}+2\left(\sigma_{32}\lambda_2\right)^{\frac{2}{3}}+\frac{4\sigma_{31}}{3}\left(\sigma_{32}\lambda_2\right)^{\frac{1}{3}}$, $\bar{\delta}=\delta+\frac{\sigma_{41}^2}{2}+2\left(\sigma_{42}\lambda_2\right)^{\frac{2}{3}}+\frac{4\sigma_{41}}{3}\left(\sigma_{42}\lambda_2\right)^{\frac{1}{3}}$ and $\bar{c}=c+\frac{\sigma_{51}^2}{2}+2\left(\frac{\sigma_{52}p_1\lambda_1}{a}\right)^{\frac{2}{3}}+\frac{4\sigma_{51}}{3}\left(\frac{\sigma_{52}p_1\lambda_1}{a}\right)^{\frac{1}{3}}+\frac{4p_2\lambda_2}{3\delta}\left(\frac{\sigma_{52}^2 a}{p_1\lambda_1}\right)^{\frac{1}{3}}$.*

*Proof.* In order to prove the existence of stationary distribution and ergodicity of model (2.2), it suffices to verify conditions (i) and (ii) in Lemma 2.2. The diffusion matrix of model (2.2) is

$$\begin{pmatrix}(\sigma_{11}x+\sigma_{12}x^2)^2 & 0 & 0 & 0 & 0\\ 0 & (\sigma_{11}x+\sigma_{12}x^2)^2 & 0 & 0 & 0\\ 0 & 0 & (\sigma_{31}y+\sigma_{32}y^2)^2 & 0 & 0\\ 0 & 0 & 0 & (\sigma_{41}y+\sigma_{42}y_1^2)^2 & 0\\ 0 & 0 & 0 & 0 & (\sigma_{51}v+\sigma_{52}v^2)^2\end{pmatrix},$$

which is a positive definite matrix for any $(x,x_1,y,y_1,v)\in\mathbb{R}_+^5$. Hence, it is obvious that there exists a constant

$$M_0=\min_{(x,x_1,y,y_1,v)\in\bar{U}}\{(\sigma_{11}x+\sigma_{12}x^2)^2,(\sigma_{11}x+\sigma_{12}x^2)^2,(\sigma_{31}y+\sigma_{32}y^2)^2,(\sigma_{41}y+\sigma_{42}y_1^2)^2,(\sigma_{51}v+\sigma_{52}v^2)^2\}>0,$$

with $\bar{U}=\left[\frac{1}{k},k\right]\times\left[\frac{1}{k},k\right]\times\left[\frac{1}{k},k\right]\times\left[\frac{1}{k},k\right]\times\left[\frac{1}{k},k\right]$ and $k>1$ is a sufficiently large integer, such that

$$\begin{aligned}\sum_{i,j=1}^{5}\bar{a}_{ij}(x,x_1,y,y_1,v)\xi_i\xi_j=&(\sigma_{11}x+\sigma_{12}x^2)^2\xi_1^2+(\sigma_{21}x_1+\sigma_{22}x_1^2)^2\xi_2^2+(\sigma_{31}y+\sigma_{32}y^2)^2\xi_3^2\\&+(\sigma_{41}y+\sigma_{42}y_1^2)^2\xi_4^2+(\sigma_{51}v+\sigma_{52}v^2)^2\xi_5^2\geq M_0\|\xi\|^2,\end{aligned}$$

for $(x,x_1,y,y_1,v)\in\bar{U}$ and $\xi=(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)\in\mathbb{R}_+^5$. This implies Lemma 2.2 (i) is satisfied.

To verify (A.2), we only to show that there exist a neighborhood $U\subset\mathbb{R}_+^2$ and a nonnegative $C^2$-function $V$ such that for any $(x,x_1,y,y_1,v)\in\mathbb{R}_+^5\setminus U$, $\mathcal{L}V$ is negative.

Firstly, Define a $C^2$-function $Q_1$ as follows

$$Q_1=-\ln x+\frac{a_3(x+b_3)^\kappa}{\kappa},$$

where $a_3 = \frac{8}{3(1-\kappa)b_3^\kappa}$, $b_3 = 2\left[\frac{\lambda_1}{(1-\kappa)\sigma_{12}^2}\right]^{1/3}$ and $\kappa \in (0,1)$ is a variable. Then applying Itô's formula, we have

$$\begin{aligned}
\mathcal{L}Q_1 =& -\frac{\lambda_1}{x} + \beta_1 v + d_1 + \frac{\sigma_{11}^2}{2} + \sigma_{11}\sigma_{12}x + \frac{\sigma_{12}^2}{2}x^2 + a_3(x+b_3)^{\kappa-1}(\lambda_1 - d_1 x - \beta_1 x v) \\
& - \frac{(1-\kappa)a_3}{2(x+b_3)^{2-\kappa}}(\sigma_{11}x + \sigma_{12}x^2)^2 \\
\le & -\frac{\lambda_1}{x} + \beta_1 v + d_1 + \frac{\sigma_{11}^2}{2} + \sigma_{11}\sigma_{12}x + \frac{\sigma_{12}^2}{2}x^2 + a_3 b_3^{\kappa-1}\lambda_1 - \frac{(1-\kappa)a_3 b_3^{\kappa+2}\sigma_{12}^2\left(\frac{x}{b_3}\right)^4}{4\left[\left(\frac{x}{b_3}\right)^2+1\right]} \\
& - \frac{(1-\kappa)a_3 b_3^{\kappa+1}\sigma_{11}\sigma_{12}\left(\frac{x}{b_3}\right)^3}{2\left[\left(\frac{x}{b_3}\right)^2+1\right]} \\
\le & -\frac{\lambda_1}{x} + \beta_1 v + d_1 + \frac{\sigma_{11}^2}{2} + a_3 b_3^{\kappa-1}\lambda_1 + \frac{(1-\kappa)a_3 b_3^{\kappa+2}\sigma_{12}^2}{16} + \frac{(1-\kappa)a_3 b_3^{\kappa+1}\sigma_{11}\sigma_{12}}{4} \\
& + \left[1 - \frac{(1-\kappa)a_3 b_3^\kappa}{2}\right]\sigma_{11}\sigma_{12}x + \left[\frac{1}{2} - \frac{(1-\kappa)3a_3 b_3^\kappa}{16}\right]\sigma_{12}^2 x^2 \\
= & -\frac{\lambda_1}{x} + \beta_1 v + d_1 + \frac{\sigma_{11}^2}{2} + 2\left(\frac{\sigma_{12}\lambda_1}{1-\kappa}\right)^{\frac{2}{3}} + \frac{4\sigma_{11}}{3}\left(\frac{\sigma_{12}\lambda_1}{1-\kappa}\right)^{\frac{1}{3}}.
\end{aligned}$$

In view of the arbitrariness of the $\kappa$, it follows that if $\kappa \to 0^+$, then

$$\mathcal{L}Q_1 \le -\frac{\lambda_1}{x} + \beta_1 v + \bar{d_1}, \tag{4.1}$$

where $\bar{d_1} = d_1 + \frac{\sigma_{11}^2}{2} + 2(\sigma_{12}\lambda_1)^{\frac{2}{3}} + \frac{4\sigma_{11}}{3}(\sigma_{12}\lambda_1)^{\frac{1}{3}}$.

Then we define $C^2$-functions $Q_2$, $Q_3$ $Q_4$ and $Q_5$ as follows

$$\begin{aligned}
Q_2 &= -\ln x_1 + a_4 b_4^{\kappa-1}x + \frac{a_4(x_1+b_4)^\kappa}{\kappa}, \\
Q_3 &= -\ln y + \frac{a_5(y+b_5)^\kappa}{\kappa}, \\
Q_4 &= -\ln y_1 + a_6 b_6^{\kappa-1}y + \frac{a_6(y_1+b_6)^\kappa}{\kappa}, \\
Q_5 &= -\ln v + a_7 b_7^{\kappa-1}\left[\frac{p_1(x+x_1)}{a} + \frac{p_2(y+y_1)}{\delta}\right] + \frac{a_7(v+b_7)^\kappa}{\kappa},
\end{aligned}$$

where $a_4 = \frac{8}{3(1-\kappa)b_4^\kappa}$, $b_4 = 2\left[\frac{\lambda_1}{(1-\kappa)\sigma_{22}^2}\right]^{\frac{1}{3}}$, $a_5 = \frac{8}{3(1-\kappa)b_5^\kappa}$, $b_5 = 2\left[\frac{\lambda_2}{(1-\kappa)\sigma_{32}^2}\right]^{\frac{1}{3}}$, $a_6 = \frac{8}{3(1-\kappa)b_6^\kappa}$, $b_6 = 2\left[\frac{\lambda_2}{(1-\kappa)\sigma_{42}^2}\right]^{\frac{1}{3}}$, $a_7 = \frac{8}{3(1-\kappa)b_7^\kappa}$, $b_7 = 2\left[\frac{p_1\lambda_1}{(1-\kappa)\sigma_{52}^2 a}\right]^{\frac{1}{3}}$ and $\kappa \in (0,1)$ is a variable.

Applying Itô's formula to $Q_2$, $Q_3$, $Q_4$ and $Q_5$, respectively, one gets

$$\begin{aligned}
\mathcal{L}Q_2 =& -\frac{\beta_1 xv}{x_1} + a + \frac{\sigma_{21}^2}{2} + \sigma_{21}\sigma_{22}x_1 + \frac{\sigma_{22}^2}{2}x_1^2 + a_4 b_4^{\kappa-1}(\lambda_1 - d_1 x - \beta_1 xv)\\
&+ a_4(x_1+b_4)^{\kappa-1}(\beta_1 xv - ax_1) - \frac{(1-\kappa)a_4}{2(x_1+b_4)^{2-\kappa}}(\sigma_{21}x_1 + \sigma_{22}x_1^2)^2\\
\leq& -\frac{\beta_1 xv}{x_1} + a + \frac{\sigma_{21}^2}{2} + a_3 b_3^{\kappa-1}\lambda_1 + \frac{(1-\kappa)a_3 b_3^{\kappa+2}\sigma_{12}^2}{16} + \frac{(1-\kappa)a_3 b_3^{\kappa+1}\sigma_{11}\sigma_{12}}{4}\\
&+ \left[1 - \frac{(1-\kappa)a_3 b_3^{\kappa}}{2}\right]\sigma_{11}\sigma_{12}x + \left[\frac{1}{2} - \frac{(1-\kappa)3a_3 b_3^{\kappa}}{16}\right]\sigma_{12}^2 x^2\\
=& -\frac{\beta_1 xv}{x_1} + a + \frac{\sigma_{21}^2}{2} + 2\left(\frac{\sigma_{22}\lambda_1}{1-\kappa}\right)^{\frac{2}{3}} + \frac{4\sigma_{21}}{3}\left(\frac{\sigma_{22}\lambda_1}{1-\kappa}\right)^{\frac{1}{3}},
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}Q_3 =& -\frac{\lambda_2}{y} + \beta_2 v + d_2 + \frac{\sigma_{31}^2}{2} + \sigma_{31}\sigma_{32}y + \frac{\sigma_{32}^2}{2}y^2 + a_5(x+b_5)^{\kappa-1}(\lambda_2 - d_2 x - \beta_2 yv)\\
&- \frac{(1-\kappa)a_5}{2(y+b_5)^{2-\kappa}}(\sigma_{31}x + \sigma_{32}y^2)^2\\
\leq& -\frac{\lambda_2}{y} + \beta_2 v + d_2 + \frac{\sigma_{31}^2}{2} + a_5 b_5^{\kappa-1}\lambda_1 + \frac{(1-\kappa)a_5 b_5^{\kappa+2}\sigma_{32}^2}{16} + \frac{(1-\kappa)a_5 b_5^{\kappa+1}\sigma_{31}\sigma_{32}}{4}\\
&+ \left[1 - \frac{(1-\kappa)a_5 b_5^{\kappa}}{2}\right]\sigma_{31}\sigma_{32}y + \left[\frac{1}{2} - \frac{(1-\kappa)3a_5 b_5^{\kappa}}{16}\right]\sigma_{32}^2 y^2\\
=& -\frac{\lambda_2}{y} + \beta_2 v + d_2 + \frac{\sigma_{31}^2}{2} + 2\left(\frac{\sigma_{32}\lambda_2}{1-\kappa}\right)^{\frac{2}{3}} + \frac{4\sigma_{31}}{3}\left(\frac{\sigma_{32}\lambda_2}{1-\kappa}\right)^{\frac{1}{3}},
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}Q_4 =& -\frac{\beta_2 yv}{y_1} + \delta + \frac{\sigma_{41}^2}{2} + \sigma_{41}\sigma_{42}y_1 + \frac{\sigma_{42}^2}{2}y_1^2 + a_6 b_6^{\kappa-1}(\lambda_2 - d_2 y - \beta_2 yv)\\
&+ a_6(y_1+b_6)^{\kappa-1}(\beta_2 yv - \delta x_1) - \frac{(1-\kappa)a_6}{2(y_1+b_6)^{2-\kappa}}(\sigma_{41}y_1 + \sigma_{42}y_1^2)^2\\
\leq& -\frac{\beta_2 yv}{y_1} + a + \frac{\sigma_{41}^2}{2} + a_6 b_6^{\kappa-1}\lambda_2 + \frac{(1-\kappa)a_6 b_6^{\kappa+2}\sigma_{42}^2}{16} + \frac{(1-\kappa)a_6 b_6^{\kappa+1}\sigma_{41}\sigma_{42}}{4}\\
&+ \left[1 - \frac{(1-\kappa)a_6 b_6^{\kappa}}{2}\right]\sigma_{41}\sigma_{42}y + \left[\frac{1}{2} - \frac{(1-\kappa)3a_6 b_6^{\kappa}}{16}\right]\sigma_{42}^2 y^2\\
=& -\frac{\beta_2 yv}{y_1} + \delta + \frac{\sigma_{41}^2}{2} + 2\left(\frac{\sigma_{42}\lambda_2}{1-\kappa}\right)^{\frac{2}{3}} + \frac{4\sigma_{41}}{3}\left(\frac{\sigma_{42}\lambda_2}{1-\kappa}\right)^{\frac{1}{3}},
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}Q_5 =& -\frac{p_1 x_1 + p_2 y_1}{v} + c + \frac{\sigma_{51}^2}{2} + \sigma_{51}\sigma_{52}v + \frac{\sigma_{52}^2}{2}v^2 + a_7 b_7^{\kappa-1}\left[\frac{p_1(\lambda_1 - d_1 x - ax_1)}{a} + \frac{p_2(\lambda_2 - d_2 y - \delta y_1)}{\delta}\right]\\
&+ a_7(v+b_7)^{\kappa-1}(p_1 x_1 + p_2 y_1 - cv) - \frac{(1-\kappa)a_7}{2(v+b_7)^{2-\kappa}}(\sigma_{51}v + \sigma_{52}v^2)^2\\
\leq& -\frac{p_1 x_1 + p_2 y_1}{v} + c + \frac{\sigma_{51}^2}{2} + a_7 b_7^{\kappa-1}\left(\frac{p_1\lambda_1}{a} + \frac{p_2\lambda_2}{\delta}\right) + \frac{(1-\kappa)a_7 b_7^{\kappa+2}\sigma_{52}^2}{16} + \frac{(1-\kappa)a_7 b_7^{\kappa+1}\sigma_{51}\sigma_{52}}{4}\\
&+ \left[1 - \frac{(1-\kappa)a_7 b_7^{\kappa}}{2}\right]\sigma_{51}\sigma_{52}v + \left[\frac{1}{2} - \frac{(1-\kappa)3a_7 b_7^{\kappa}}{16}\right]\sigma_{52}^2 v^2\\
=& -\frac{p_1 x_1 + p_2 y_1}{v} + c + \frac{\sigma_{51}^2}{2} + 2\left[\frac{\sigma_{52}p_1\lambda_1}{(1-\kappa)a}\right]^{\frac{2}{3}} + \frac{4\sigma_{51}}{3}\left[\frac{\sigma_{52}p_1\lambda_1}{(1-\kappa)a}\right]^{\frac{1}{3}} + \frac{4p_2\lambda_2}{3\delta}\left[\frac{\sigma_{52}^2 a}{p_1\lambda_1(1-\kappa)^2}\right]^{\frac{1}{3}},
\end{aligned}$$

In view of the arbitrariness of the $\kappa$, it follows that if $\kappa \to 0^+$, then we have

$$\begin{aligned}
\mathcal{L}Q_2 &\leq -\frac{\beta_1 x v}{x_1} + \bar{a},\\
\mathcal{L}Q_3 &\leq -\frac{\lambda_2}{y} + \beta_2 v + \bar{d}_2,\\
\mathcal{L}Q_4 &\leq -\frac{\beta_2 y v}{y_1} + \bar{\delta},\\
\mathcal{L}Q_5 &\leq -\frac{p_1 x_1 + p_2 y_1}{v} + \bar{c},
\end{aligned} \tag{4.2}$$

where $\bar{a} = a + \frac{\sigma_{21}^2}{2} + 2\left(\sigma_{22}\lambda_1\right)^{\frac{2}{3}} + \frac{4\sigma_{21}}{3}\left(\sigma_{22}\lambda_1\right)^{\frac{1}{3}}$, $\bar{d}_2 = d_2 + \frac{\sigma_{31}^2}{2} + 2\left(\sigma_{32}\lambda_2\right)^{\frac{2}{3}} + \frac{4\sigma_{31}}{3}\left(\sigma_{32}\lambda_2\right)^{\frac{1}{3}}$, $\bar{\delta} = \delta + \frac{\sigma_{41}^2}{2} + 2\left(\sigma_{42}\lambda_2\right)^{\frac{2}{3}} + \frac{4\sigma_{41}}{3}\left(\sigma_{42}\lambda_2\right)^{\frac{1}{3}}$ and $\bar{c} = c + \frac{\sigma_{51}^2}{2} + 2\left(\frac{\sigma_{52}p_1\lambda_1}{a}\right)^{\frac{2}{3}} + \frac{4\sigma_{51}}{3}\left(\frac{\sigma_{52}p_1\lambda_1}{a}\right)^{\frac{1}{3}} + \frac{4p_2\lambda_2}{3\delta}\left(\frac{\sigma_{52}^2 a}{p_1\lambda_1}\right)^{\frac{1}{3}}$.

Denote the following $C^2$-function $V_3$:

$$V_3 = c_1 Q_1 + c_2 Q_2 + c_3 Q_3 + c_4 Q_4 + Q_5,$$

and $c_i$ will be determined in (4.3), $i = 1, 2, 3, 4$. Then making use of Itô's formula to $V_3$ and combining $x + y + z \geq 3\sqrt[3]{xyz}$ for $x, y, z \geq 0$, one has

$$\begin{aligned}
\mathcal{L}V_3 =& -\frac{p_1 x_1}{v} - \frac{c_1\lambda_1}{x} - \frac{c_2\beta_1 x v}{x_1} - \frac{p_2 y_1}{v} - \frac{c_3\lambda_2}{y} - \frac{c_4\beta_2 y v}{y_1} + c_1\bar{d}_1 + c_2\bar{a} + c_3\bar{d}_2 + c_4\bar{\delta} + \bar{c} + (c_1\beta_1 + c_3\beta_2)v\\
\leq& -3\sqrt[3]{c_1 c_2 p_1 \beta_1 \lambda_1} - 3\sqrt[3]{c_3 c_4 p_2 \beta_2 \lambda_2} + c_1\bar{d}_1 + c_2\bar{a} + c_3\bar{d}_2 + c_4\bar{\delta} + \bar{c} + (c_1\beta_1 + c_3\beta_2)v.
\end{aligned}$$

Choose

$$c_1 = \frac{p_1\beta_1\lambda_1}{\bar{a}\bar{d}_1^{\,2}}, \quad c_2 = \frac{p_1\beta_1\lambda_1}{\bar{a}^2\bar{d}_1}, \quad c_3 = \frac{p_2\beta_2\lambda_2}{\bar{\delta}\bar{d}_2^{\,2}}, \quad c_4 = \frac{p_2\beta_2\lambda_2}{\bar{\delta}^2\bar{d}_2}. \tag{4.3}$$

Therefore, it leads to

$$\mathcal{L}V_3 \leq -\frac{p_1\beta_1\lambda_1}{\bar{a}\bar{d}_1} - \frac{p_2\beta_2\lambda_2}{\bar{\delta}\bar{d}_2} + \bar{c} + (c_1\beta_1 + c_3\beta_2)v = -\bar{c}(R_0^s - 1) + (c_1\beta_1 + c_3\beta_2)v, \tag{4.4}$$

where $R_0^s = \frac{p_1\beta_1\lambda_1\bar{\delta}\bar{d}_2 + p_2\beta_2\lambda_2\bar{a}\bar{d}_1}{\bar{c}\bar{\delta}\bar{a}\bar{d}_1\bar{d}_2}$.

Next, we define a $C^2$-function $V_4 : \mathbb{R}_+^5 \to \mathbb{R}$ as follows:

$$V_4 = -\ln x - \ln x_1 - \ln y - \ln y_1 + 2V_2 + MV_3,$$

where $M$ is a sufficiently large constant such that

$$-M\bar{c}(R_0^s - 1) + d_1 + \frac{\sigma_{11}^2}{2} + a + \frac{\sigma_{21}^2}{2} + d_2 + \frac{\sigma_{31}^2}{2} + \delta + \frac{\sigma_{41}^2}{2} + 2r \leq -2. \tag{4.5}$$

It is clear that $V_4$ has a minimum value point $(\underline{x}, \underline{x_1}, \underline{y}, \underline{y_1}, \underline{v})$. Therefore, we define the nonnegative $C^2$-function as follows

$$\bar{V} := V_4 - V_4(\underline{x}, \underline{x_1}, \underline{y}, \underline{y_1}, \underline{v}).$$

Then making the use of Itô's formula and combining (3.3), (4.4), one gets

$$\begin{aligned}
\mathcal{L}\bar{V} \leq& -\frac{\lambda_1}{x} - \frac{\beta_1 x v}{x_1} - \frac{\lambda_2}{y} - \frac{\beta_2 y v}{y_1} - M\bar{c}(R_0^s - 1) + d_1 + \frac{\sigma_{11}^2}{2} + a + \frac{\sigma_{21}^2}{2} + d_2 + \frac{\sigma_{31}^2}{2} + \delta + \frac{\sigma_{41}^2}{2} + 2r\\
&+ \left[(1 + Mc_1)\beta_1 + (1 + Mc_3)\beta_2\right] v - \frac{1}{2}(\sigma_{12}^2 x^2 + \sigma_{22}^2 x_1^2 + \sigma_{32}^2 y^2 + \sigma_{42}^2 y_1^2 + \sigma_{52}^2 v^2).
\end{aligned} \tag{4.6}$$

Next, we will construct a bounded set $U \subset \mathbb{R}_+^5$ such that

$$\mathcal{L}\bar{V}(x, x_1, y, y_1, v) \leq -1, \quad \text{for any } (x, x_1, y, y_1, v) \in \mathbb{R}_+^5 \setminus U.$$

Denote

$$U = \left\{ \varepsilon \le x \le \frac{1}{\varepsilon},\ \ \varepsilon^3 \le x_1 \le \frac{1}{\varepsilon^3},\ \ \varepsilon \le y \le \frac{1}{\varepsilon},\ \ \varepsilon^3 \le y_1 \le \frac{1}{\varepsilon^3},\ \ \varepsilon \le v \le \frac{1}{\varepsilon} \right\},$$

where $\varepsilon$ is a sufficiently small positive constant satisfying the following inequalities

$$\begin{aligned}
&\varepsilon \le \frac{1}{(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2},\\
&-\frac{\min\{\lambda_1, \beta_1, \lambda_2, \beta_2\}}{\varepsilon} + \sup_{v\in(0,\infty)} \left\{ [(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]v - \frac{\sigma_{52}^2}{2} v \right\} \le -1,\\
&-\frac{\sigma_{12}^2 \wedge \sigma_{22}^2 \wedge \sigma_{32}^2 \wedge \sigma_{42}^2}{2(\varepsilon^{\theta+1} \vee \varepsilon^{3\theta+3})} + \sup_{v\in(0,\infty)} \left\{ [(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]v - \frac{\sigma_{52}^2}{2} v \right\} \le -1,\\
&-\frac{\sigma_{52}^2}{4\varepsilon^{\theta+1}} + \sup_{v\in(0,\infty)} \left\{ [(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]v - \frac{\sigma_{52}^2}{4} v \right\} \le -1.
\end{aligned} \tag{4.7}$$

For convenience, we divide $\mathbb{R}^5_+ \setminus U$ into ten domains

$$\begin{aligned}
&U_1^c = \left\{ (x, x_1, y, y_1, v) \in \mathbb{R}^5_+ |\ 0 < v < \varepsilon \right\},\ \ U_2^c = \left\{ (x, x_1, y, y_1, v) \in \mathbb{R}^5_+ |\ 0 < x < \varepsilon \right\},\\
&U_3^c = \left\{ (x, x_1, y, y_1, v) \in \mathbb{R}^5_+ |\ 0 < x_1 < \varepsilon^3 \right\},\ \ U_4^c = \left\{ (x, x_1, y, y_1, v) \in \mathbb{R}^5_+ |\ 0 < y < \varepsilon \right\},\\
&U_5^c = \left\{ (x, x_1, y, y_1, v) \in \mathbb{R}^5_+ |\ 0 < y_1 < \varepsilon^3 \right\},\ \ U_6^c = \left\{ (x, x_1, y, y_1, v) \in \mathbb{R}^5_+ |\ v > \frac{1}{\varepsilon} \right\},\\
&U_7^c = \left\{ (x, x_1, y, y_1, v) \in \mathbb{R}^5_+ |\ x > \frac{1}{\varepsilon} \right\},\ \ U_8^c = \left\{ (x, x_1, y, y_1, v) \in \mathbb{R}^5_+ |\ x_1 > \frac{1}{\varepsilon^3} \right\},\\
&U_9^c = \left\{ (x, x_1, y, y_1, v) \in \mathbb{R}^5_+ |\ y > \frac{1}{\varepsilon} \right\},\ \ U_{10}^c = \left\{ (x, x_1, y, y_1, v) \in \mathbb{R}^5_+ |\ y_1 > \frac{1}{\varepsilon^3} \right\}.
\end{aligned}$$

Obviously, $\mathbb{R}^5_+ \setminus U = \bigcup_{i=1}^{10} U_i^c$. Thus we shall verify $\mathcal{L}\bar{V}(x, x_1, y, y_1, v) \le -1$ for any $(x, x_1, y, y_1, v) \in \mathbb{R}^5_+ \setminus U$. Therefore, from (4.5), (4.6) and (4.7), it is not difficult to verify that

$$\mathcal{L}\bar{V}(x, x_1, y, y_1, v) \le -1, \quad \text{for}(x, x_1, y, y_1, v) \in \mathbb{R}^5_+ \setminus U = \bigcup_{i=1}^{10} U_i^c.$$

**Case 1.** If $(x, x_1, y, y_1, v) \in U_1^c$, from (4.5), (4.6) and (4.7), we have

$$\begin{aligned}
\mathcal{L}V \le& -M\bar{c}(R_0^s - 1) + d_1 + \frac{\sigma_{11}^2}{2} + a + \frac{\sigma_{21}^2}{2} + d_2 + \frac{\sigma_{31}^2}{2} + \delta + \frac{\sigma_{41}^2}{2} + 2r\\
&+ [(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]\, v\\
\le& -2 + [(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]\, \varepsilon\\
\le& -1.
\end{aligned}$$

**Case 2.** In domain $U_2^c$,

$$\begin{aligned}
\mathcal{L}V \le& -\frac{\lambda_1}{x} + [(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]\, v - \frac{\sigma_{52}^2}{2} v^2\\
\le& -\frac{\lambda_1}{\varepsilon} + \sup_{v\in(0,\infty)} \left\{ [(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]v - \frac{\sigma_{52}^2}{2} v \right\}\\
\le& -1.
\end{aligned}$$

**Case 3.** In domain $U_3^c$,

$$\begin{aligned}\mathcal{L}V \leq & -\frac{\beta_1 xv}{x_1} + [(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]\, v - \frac{\sigma_{52}^2}{2}v^2 \\ \leq & -\frac{\beta_1}{\varepsilon} + \sup_{v\in(0,\infty)}\left\{[(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]v - \frac{\sigma_{52}^2}{2}v\right\} \\ \leq & -1.\end{aligned}$$

**Case 4.** In domain $U_4^c$,

$$\begin{aligned}\mathcal{L}V \leq & -\frac{\lambda_2}{y} + [(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]\, v - \frac{\sigma_{52}^2}{2}v^2 \\ \leq & -\frac{\lambda_2}{\varepsilon} + \sup_{v\in(0,\infty)}\left\{[(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]v - \frac{\sigma_{52}^2}{2}v\right\} \\ \leq & -1.\end{aligned}$$

**Case 5.** In domain $U_5^c$,

$$\begin{aligned}\mathcal{L}V \leq & -\frac{\beta_2 yv}{y_1} + [(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]\, v - \frac{\sigma_{52}^2}{2}v^2 \\ \leq & -\frac{\beta_2}{\varepsilon} + \sup_{v\in(0,\infty)}\left\{[(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]v - \frac{\sigma_{52}^2}{2}v\right\} \\ \leq & -1.\end{aligned}$$

**Case 6.** In domain $U_6^c$,

$$\begin{aligned}\mathcal{L}V \leq & [(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]\, v - \frac{\sigma_{52}^2}{2}v^2 \\ \leq & -\frac{\sigma_{52}^2}{4}v^2 + \sup_{v\in(0,\infty)}\left\{[(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]v - \frac{\sigma_{52}^2}{4}v\right\} \\ \leq & -\frac{\sigma_{52}^2}{4\varepsilon^{\theta+1}} + \sup_{v\in(0,\infty)}\left\{[(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]v - \frac{\sigma_{52}^2}{4}v\right\} \\ \leq & -1.\end{aligned}$$

**Case 7.** In domain $U_7^c$,

$$\begin{aligned}\mathcal{L}V \leq & -\frac{\sigma_{12}^2}{2}x^2 + [(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]\, v - \frac{\sigma_{52}^2}{2}v^2 \\ \leq & -\frac{\sigma_{12}^2}{2\varepsilon^{\theta+1}} + \sup_{v\in(0,\infty)}\left\{[(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]v - \frac{\sigma_{52}^2}{2}v\right\} \\ \leq & -1.\end{aligned}$$

**Case 8.** In domain $U_8^c$,

$$\begin{aligned}\mathcal{L}V \leq & -\frac{\sigma_{22}^2}{2}x_1^2 + [(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]\, v - \frac{\sigma_{52}^2}{2}v^2 \\ \leq & -\frac{\sigma_{22}^2}{2\varepsilon^{3\theta+3}} + \sup_{v\in(0,\infty)}\left\{[(1+Mc_1)\beta_1 + (1+Mc_3)\beta_2]v - \frac{\sigma_{52}^2}{2}v\right\} \\ \leq & -1.\end{aligned}$$

**Case 9.** In domain $U_9^c$,

$$\begin{aligned}\mathcal{L}V \leq & -\frac{\sigma_{32}^2}{2}y^2+[(1+Mc_1)\beta_1+(1+Mc_3)\beta_2]\,v-\frac{\sigma_{52}^2}{2}v^2\\ \leq & -\frac{\sigma_{32}^2}{2\varepsilon^{\theta+1}}+\sup_{v\in(0,\infty)}\left\{[(1+Mc_1)\beta_1+(1+Mc_3)\beta_2]v-\frac{\sigma_{52}^2}{2}v\right\}\\ \leq & -1.\end{aligned}$$

**Case 10.** In domain $U_{10}^c$,

$$\begin{aligned}\mathcal{L}V \leq & -\frac{\sigma_{42}^2}{2}y_1^2+[(1+Mc_1)\beta_1+(1+Mc_3)\beta_2]\,v-\frac{\sigma_{52}^2}{2}v^2\\ \leq & -\frac{\sigma_{42}^2}{2\varepsilon^{3\theta+3}}+\sup_{v\in(0,\infty)}\left\{[(1+Mc_1)\beta_1+(1+Mc_3)\beta_2]v-\frac{\sigma_{52}^2}{2}v\right\}\\ \leq & -1.\end{aligned}$$

Consequently,

$$\mathcal{L}V(x,x_1,y,y_1,v)\leq -1, \quad \text{for any } (x,x_1,y,y_1,v)\in\mathbb{R}_+^5\setminus U.$$

Therefore, the condition (ii) in Lemma 2.2 is verified. We have verified conditions (i) and (ii) in Lemaa 2.2. Therefore, model (2.2) is ergodic and admits a unique stationary distribution. This completes the proof. □

## 5. Extinction

In this section, we will give sufficient conditions for the extinction of the infected cells and the free virus particles in model (2.2).

**Theorem 5.1.** *Let $(x(t),x_1(t),y(t),y_1(t),v(t))$ be a solution of model (2.2) with any initial value $(x(0),x_1(0),y(0),y_1(0),v(0))\in\mathbb{R}_+^5$. Then*

$$\limsup_{t\to\infty}\frac{1}{t}\ln\left(\frac{p_1}{ac\sqrt{R_0}}x_1(t)+\frac{p_2}{\delta c\sqrt{R_0}}y_1(t)+\frac{1}{c}v(t)\right)\leq\eta, \quad a.s.,$$

*where*

$$\begin{aligned}\eta := & \frac{p_1\beta_1}{a\sqrt{R_0}}\int_0^\infty\left|X-\frac{\lambda_1}{d_1}\right|\mu_X\mathrm{d}X+\frac{p_2\beta_2}{\delta\sqrt{R_0}}\int_0^\infty\left|Y-\frac{\lambda_2}{d_2}\right|\mu_Y\mathrm{d}Y\\ & +\left[(a\wedge\delta\wedge c)\mathbf{I}_{\{\sqrt{R_0}\leq 1\}}+(a\vee\delta\vee c)\mathbf{I}_{\{\sqrt{R_0}>1\}}\right](\sqrt{R_0}-1)-\frac{1}{2\left(\frac{1}{\sigma_{21}^2}+\frac{1}{\sigma_{41}^2}+\frac{1}{\sigma_{51}^2}\right)},\end{aligned}$$

*with $\mathbf{I}_\omega$ denotes the indicator function with respect to set $\omega$. Moreover, if $\eta<0$, it leads to*

$$\lim_{t\to\infty}x_1(t)=0, \quad \lim_{t\to\infty}y_1(t)=0, \quad \lim_{t\to\infty}v(t)=0, \quad a.s.,$$

*which means the infected CD4$^+$ T cells $x_1$, the infected macrophages $y_1$ and the free virus particles $v$ will exponentially go to extinction in a long term.*

*Proof.* Consider

$$\sqrt{R_0}(\omega_1,\omega_2,1)=(\omega_1,\omega_2,1)A,$$

where

$$\omega_1=\frac{p_1}{c\sqrt{R_0}}, \quad \omega_2=\frac{p_2}{c\sqrt{R_0}}, \quad A=\begin{pmatrix}0 & 0 & \frac{\beta_1\lambda_1}{ad_1}\\ 0 & 0 & \frac{\beta_2\lambda_2}{\delta d_2}\\ \frac{p_1}{c} & \frac{p_2}{c} & 0\end{pmatrix}.$$

Define a $C^2$-function $P$: $\mathbb{R}_+^3\to\mathbb{R}_+$ as follows

$$P=\frac{\omega_1}{a}x_1+\frac{\omega_2}{\delta}y_1+\frac{1}{c}v.$$

Applying the Itô's formula to $\ln P$, we have

$$\begin{aligned}\mathrm{d}(\ln P) =&\mathcal{L}\Bigg\{\frac{1}{P}\left[\frac{\omega_1}{a}(\beta_1 xv - ax_1) + \frac{\omega_2}{\delta}(\beta_2 yv - \delta y_1) + \frac{1}{c}(p_1x_1 + p_2y_1 - cv)\right] \\ &- \frac{1}{2P^2}\left(\frac{\omega_1^2}{a^2}(\sigma_{21}x_1 + \sigma_{22}x_1^2)^2 + \frac{\omega_2^2}{\delta^2}(\sigma_{41}y_1 + \sigma_{42}y_1^2)^2 + \frac{1}{c^2}(\sigma_{51}v + \sigma_{52}v^2)^2\right)\Bigg\}\mathrm{d}t \\ &+ \frac{1}{P}\left[\frac{\omega_1}{a}(\sigma_{21}x_1 + \sigma_{22}x_1^2)\mathrm{d}B_2(t) + \frac{\omega_2}{\delta}(\sigma_{41}y_1 + \sigma_{42}y_1^2)\mathrm{d}B_4(t) + \frac{1}{c}(\sigma_{51}v + \sigma_{52}v^2)\mathrm{d}B_5(t)\right],\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}&\frac{1}{P}\left(\frac{\omega_1}{a}(\beta_1 xv - ax_1) + \frac{\omega_2}{\delta}(\beta_2 yv - \delta y_1) + \frac{1}{c}(p_1x_1 + p_2y_1 - cv)\right) \\ =&\frac{v}{P}\left[\frac{\omega_1\beta_1}{a}\left(x - \frac{\lambda_1}{d_1}\right) + \frac{\omega_2\beta_2}{\delta}\left(y - \frac{\lambda_2}{d_2}\right)\right] + \frac{1}{P}\left[\left(\frac{p_1}{c} - \omega_1\right)x_1 + \left(\frac{p_2}{c} - \omega_2\right)x_2 + \left(\frac{\omega_1\beta_1\lambda_1}{ad_1} + \frac{\omega_2\beta_2\lambda_2}{\delta d_2} - 1\right)v\right] \\ \leq&\frac{c\omega_1\beta_1}{a}\left|x - \frac{\lambda_1}{d_1}\right| + \frac{c\omega_2\beta_2}{\delta}\left|y - \frac{\lambda_2}{d_2}\right| + \frac{1}{P}(\omega_1, \omega_2, 1)\left[A(x_1, y_1, v)^T - (x_1, y_1, v)^T\right] \\ \leq&\frac{c\omega_1\beta_1}{a}\left|X - \frac{\lambda_1}{d_1}\right| + \frac{c\omega_2\beta_2}{\delta}\left|Y - \frac{\lambda_2}{d_2}\right| + \frac{1}{P}(\sqrt{R_0} - 1)(\omega_1x_1 + \omega_2y_1 + v) \\ \leq&\frac{c\omega_1\beta_1}{a}\left|X - \frac{\lambda_1}{d_1}\right| + \frac{c\omega_2\beta_2}{\delta}\left|Y - \frac{\lambda_2}{d_2}\right| + \left[(a\wedge\delta\wedge c)\mathbf{I}_{\{\sqrt{R_0}\leq 1\}} + (a\vee\delta\vee c)\mathbf{I}_{\{\sqrt{R_0}>1\}}\right](\sqrt{R_0} - 1).\end{aligned}$$

Integrating (5.1) from 0 to $t$ and dividing by $t$ on both sides, it leads to

$$\begin{aligned}\frac{\ln P(t) - \ln P(0)}{t} \leq&\frac{c\omega_1\beta_1}{at}\int_0^t \left|X(s) - \frac{\lambda_1}{d_1}\right|\mathrm{d}s + \frac{c\omega_2\beta_2}{\delta t}\int_0^t\left|Y(s) - \frac{\lambda_2}{d_2}\right|\mathrm{d}s \\ &+ \left[(a\wedge\delta\wedge c)\mathbf{I}_{\{\sqrt{R_0}\leq 1\}} + (a\vee\delta\vee c)\mathbf{I}_{\{\sqrt{R_0}>1\}}\right](\sqrt{R_0} - 1) + \frac{1}{t}\sum_{i=1}^{3}(M_i(t) - N_i(t)),\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}M_1(t) &= \int_0^t \frac{\omega_1(\sigma_{21}x_1 + \sigma_{22}x_1^2)}{aP(s)}\mathrm{d}B_2(s), &\quad N_1(t) &= \int_0^t \frac{\omega_1^2(\sigma_{21}x_1 + \sigma_{22}x_1^2)^2}{2a^2P^2(s)}\mathrm{d}s \\ M_2(t) &= \int_0^t \frac{\omega_2(\sigma_{41}y_1 + \sigma_{42}y_1^2)}{\delta P(s)}\mathrm{d}B_4(s), &\quad N_2(t) &= \int_0^t \frac{\omega_2^2(\sigma_{41}y_1 + \sigma_{42}y_1^2)^2}{2\delta^2P^2(s)}\mathrm{d}s \\ M_3(t) &= \int_0^t \frac{(\sigma_{51}v + \sigma_{52}v^2)}{cP(s)}\mathrm{d}B_5(s), &\quad N_3(t) &= \int_0^t \frac{(\sigma_{51}v + \sigma_{52}v^2)^2}{2c^2P^2(s)}\mathrm{d}s.\end{aligned}$$

Applying the exponential martingale inequality in Lemma 2.4, we choose $T = n$, $\alpha = \varepsilon$ and $\beta = \frac{2\ln n}{\varepsilon}$ such that

$$\mathbb{P}\left(\sup_{0\leq t\leq n}(M_i(t) - \varepsilon N_i(t)) > \frac{2\ln n}{\varepsilon}\right) \leq \frac{1}{n^2}, \quad i = 1, 2, 3.$$

From Borel-Cantelli lemma [29], one obtains that for almost all $\omega_t \in \Omega$, there exists an integer $k_0 = k_0(\omega_t)$ such that for all $t \in (k-1, k)$, $k \geq k_0$,

$$M_i(t) \leq \varepsilon N_i(t) + \frac{2\ln n}{\epsilon}, \quad i = 1, 2, 3.$$

Since

$$\left(\frac{\omega_1x_1}{a} + \frac{\omega_2y_1}{\delta} + \frac{v}{c}\right)^2 \leq \left(\frac{\sigma_{21}^2\omega_1^2x_1^2}{a^2} + \frac{\sigma_{41}^2\omega_2^2y_1^2}{\delta^2} + \frac{\sigma_{51}^2v^2}{c^2}\right)\left(\frac{1}{\sigma_{21}^2} + \frac{1}{\sigma_{41}^2} + \frac{1}{\sigma_{51}^2}\right),$$

we have

$$
\begin{aligned}
\frac{1}{t}\sum_{i=1}^{3}\big(M_i(t)-N_i(t)\big) \le & -\frac{1-\varepsilon}{t}\sum_{i=1}^{3}N_i(t)+\frac{6\ln n}{\varepsilon t} \\
\le & -\frac{1-\varepsilon}{t}\int_0^t \frac{\frac{\sigma_{21}^2\omega_1^2}{a^2}x_1(s)^2+\frac{\sigma_{41}^2\omega_2^2}{\delta^2}y_1(s)^2+\frac{\sigma_{51}^2}{c^2}v(s)^2}{2\left[\frac{\omega_1}{a}x_1(s)+\frac{\omega_2}{\delta}y_1(s)+\frac{1}{c}v(s)\right]^2}\mathrm{d}s+\frac{6\ln n}{\varepsilon(n-1)} \\
\le & -\frac{(1-\varepsilon)}{2\left(\frac{1}{\sigma_{21}^2}+\frac{1}{\sigma_{41}^2}+\frac{1}{\sigma_{51}^2}\right)}+\frac{6\ln n}{\varepsilon(n-1)}.
\end{aligned}
\tag{5.3}
$$

Making the use of Lemma 2.3 and considering the ergodicity property of $X(t)$ and $Y(t)$, we have

$$
\begin{aligned}
\lim_{t\to\infty}\frac{1}{t}\int_0^t\left|X(s)-\frac{\lambda_1}{d_1}\right|\mathrm{d}s &= \int_0^\infty\left|X-\frac{\lambda_1}{d_1}\right|\mu_X\mathrm{d}X, \\
\lim_{t\to\infty}\frac{1}{t}\int_0^t\left|Y(s)-\frac{\lambda_2}{d_2}\right|\mathrm{d}s &= \int_0^\infty\left|Y-\frac{\lambda_2}{d_2}\right|\mu_Y\mathrm{d}Y.
\end{aligned}
\tag{5.4}
$$

Combining (5.3), (5.4) and taking the superior limit of $t$ on both sides of (5.2), which implies $n\to\infty$. This leads to

$$
\begin{aligned}
\limsup_{t\to\infty}\frac{\ln P(t)}{t} \le & \frac{p_1\beta_1}{a\sqrt{R_0}}\int_0^\infty\left|X-\frac{\lambda_1}{d_1}\right|\mu_X\mathrm{d}X+\frac{p_2\beta_2}{\delta\sqrt{R_0}}\int_0^\infty\left|Y-\frac{\lambda_2}{d_2}\right|\mu_Y\mathrm{d}Y \\
& +\left[(a\wedge\delta\wedge c)\mathbf{I}_{\{\sqrt{R_0}\le 1\}}+(a\vee\delta\vee c)\mathbf{I}_{\{\sqrt{R_0}>1\}}\right](\sqrt{R_0}-1)-\frac{(1-\varepsilon)}{2\left(\frac{1}{\sigma_{21}^2}+\frac{1}{\sigma_{41}^2}+\frac{1}{\sigma_{51}^2}\right)}, \quad a.s.
\end{aligned}
$$

In view of the arbitrariness of $\varepsilon\in(0,1)$, let $\varepsilon\to 0^+$, it leads to

$$
\begin{aligned}
\limsup_{t\to\infty}\frac{\ln P(t)}{t} \le & \frac{p_1\beta_1}{a\sqrt{R_0}}\int_0^\infty\left|X-\frac{\lambda_1}{d_1}\right|\mu_X\mathrm{d}X+\frac{p_2\beta_2}{\delta\sqrt{R_0}}\int_0^\infty\left|Y-\frac{\lambda_2}{d_2}\right|\mu_Y\mathrm{d}Y \\
& +\left[(a\wedge\delta\wedge c)\mathbf{I}_{\{\sqrt{R_0}\le 1\}}+(a\vee\delta\vee c)\mathbf{I}_{\{\sqrt{R_0}>1\}}\right](\sqrt{R_0}-1)-\frac{1}{2\left(\frac{1}{\sigma_{21}^2}+\frac{1}{\sigma_{41}^2}+\frac{1}{\sigma_{51}^2}\right)} \\
:= & \eta, \quad a.s.
\end{aligned}
$$

Furthermore, if $\eta<0$, then $\lim_{t\to\infty}P(t)=0$ a.s., in the other word,

$$
\lim_{t\to\infty}x_1(t)=0, \quad \lim_{t\to\infty}y_1(t)=0, \quad \lim_{t\to\infty}v(t)=0, \quad a.s.
$$

That is to say, the infected CD4$^+$ T cells $x_1$, the infected macrophages $y_1$ and the free virus particles $v$ will exponentially go to extinction in a long term. The proof is completed. □

## 6. Numerical simulations

In this section, to verify the theoretical results obtained in this paper, we present numerical simulations for stochastic model (2.2). Employing Milstein's higher order method [31], we derive the discretization equation as follows,

$$
\begin{cases}
x^{(i+1)} =x^{(i)} + \left(\lambda_1 - d_1 x^{(i)} - \beta_1 x^{(i)} v^{(i)}\right)\Delta t + (\sigma_{11} + \sigma_{12} x^{(i)}) x^{(i)} \sqrt{\Delta t}\xi_1^{(i)} \\
\qquad\qquad + \dfrac{x^{(i)}}{2}\left(\sigma_{11}^2 + 3\sigma_{11}\sigma_{12} x^{(i)} + 2\sigma_{12}^2 (x^{(i)})^2\right)\left(\Delta t (\xi_1^{(i)})^2 - \Delta t\right), \\
x_1^{(i+1)} =x_1^{(i)} + \left(\beta_1 x^{(i)} v^{(i)} - a x_1^{(i)}\right)\Delta t + (\sigma_{21} + \sigma_{22} x_1^{(i)}) x_1^{(i)} \sqrt{\Delta t}\xi_2^{(i)} \\
\qquad\qquad + \dfrac{x_1^{(i)}}{2}\left(\sigma_{21}^2 + 3\sigma_{21}\sigma_{22} x_1^{(i)} + 2\sigma_{22}^2 (x_1^{(i)})^2\right)\left(\Delta t (\xi_2^{(i)})^2 - \Delta t\right), \\
y^{(i+1)} =y^{(i)} + \left(\lambda_2 - d_2 y^{(i)} - \beta_2 y^{(i)} v^{(i)}\right)\Delta t + (\sigma_{31} + \sigma_{32} y^{(i)}) y^{(i)} \sqrt{\Delta t}\xi_3^{(i)} \\
\qquad\qquad + \dfrac{y^{(i)}}{2}\left(\sigma_{31}^2 + 3\sigma_{31}\sigma_{32} y^{(i)} + 2\sigma_{32}^2 (y^{(i)})^2\right)\left(\Delta t (\xi_3^{(i)})^2 - \Delta t\right), \\
y_1^{(i+1)} =y_1^{(i)} + \left(\beta_2 y^{(i)} v^{(i)} - \delta y_1^{(i)}\right)\Delta t + (\sigma_{41} + \sigma_{42} y_1^{(i)}) y_1^{(i)} \sqrt{\Delta t}\xi_4^{(i)} \\
\qquad\qquad + \dfrac{y_1^{(i)}}{2}\left(\sigma_{41}^2 + 3\sigma_{41}\sigma_{42} y_1^{(i)} + 2\sigma_{42}^2 (y_1^{(i)})^2\right)\left(\Delta t (\xi_4^{(i)})^2 - \Delta t\right), \\
v^{(i+1)} =v^{(i)} + \left(p_1 x_1^{(i)} + p_2 y_1^{(i)} - c v^{(i)}\right)\Delta t + (\sigma_{51} + \sigma_{52} v^{(i)}) v^{(i)} \sqrt{\Delta t}\xi_5^{(i)} \\
\qquad\qquad + \dfrac{v^{(i)}}{2}\left(\sigma_{51}^2 + 3\sigma_{51}\sigma_{52} v^{(i)} + 2\sigma_{52}^2 (v^{(i)})^2\right)\left(\Delta t (\xi_5^{(i)})^2 - \Delta t\right),
\end{cases}
$$

where $\Delta t > 0$ is the time increment and $\xi_k$ are $N(0,1)$-distributed eight independent Gaussian random variables, $k = 1, \ldots, 8$.

**Example 5.1.** For the stochastic model (2.2), we choose the parameters value as

$$
\begin{aligned}
&\lambda_1 = 10,\ d_1 = 0.12,\ \beta_1 = 0.015,\ a = 0.2,\ \lambda_2 = 8,\ d_2 = 0.1,\ \beta_2 = 0.01,\ \delta = 0.24, \\
&p_1 = 0.8,\ p_2 = 0.7,\ c = 2.4,\ \sigma_{i1} = 0.01,\ \sigma_{i2} = 0.001, \quad i = 1,2,3,4,5,
\end{aligned}
\tag{6.1}
$$

the initial condition $(x(0), x_1(0), y(0), y_1(0), v(0)) = (2, 0.1, 0.2, 0.1, 0.1)$. Then compute

$$
R_0^s = \frac{p_1\beta_1\lambda_1\bar{\delta}\bar{d}_2 + p_2\beta_2\lambda_2\bar{a}\bar{d}_1}{\bar{c}\bar{\delta}\bar{a}\bar{d}_1\bar{d}_2} = 1.0367 > 1,
$$

Theorem 4.1 claims that there exists an ergodic stationary distribution of stochastic model (2.2). The simulation results can be seen in Fig. 6.1, which clearly supports these results.

**Example 5.2.** In this example, to show the extinction of infected cells and free virus, we let

$$
\begin{aligned}
&d_1 = 0.6, \quad \beta_1 = 0.0015, \quad d_2 = 0.7, \quad \beta_2 = 0.001, \quad \sigma_{11} = \sigma_{31} = \sigma_{51} = 0.2, \\
&\sigma_{12} = \sigma_{32} = \sigma_{52} = 0.02, \quad \sigma_{21} = \sigma_{41} = 0.1, \quad \sigma_{22} = \sigma_{42} = 0.01.
\end{aligned}
$$

The other parameters and initial value are similar to Example 5.1. Then we compute

$$
\begin{aligned}
\eta :=& \frac{p_1\beta_1}{a\sqrt{R_0}} \int_0^\infty \left| X - \frac{\lambda_1}{d_1} \right| \mu_X \mathrm{d}X + \frac{p_2\beta_2}{\delta\sqrt{R_0}} \int_0^\infty \left| Y - \frac{\lambda_2}{d_2} \right| \mu_Y \mathrm{d}Y \\
&+ \left[ (a \wedge \delta \wedge c)\mathbf{I}_{\{\sqrt{R_0} \le 1\}} + (a \vee \delta \vee c)\mathbf{I}_{\{\sqrt{R_0} > 1\}} \right] (\sqrt{R_0} - 1) - \frac{1}{2\left(\frac{1}{\sigma_{21}^2} + \frac{1}{\sigma_{41}^2} + \frac{1}{\sigma_{51}^2}\right)} \\
=& -0.0070 < 0.
\end{aligned}
$$

Therefore, in view of Theorem 5.1, one can obtain that the infected CD4$^+$ T cells $x_1$, the infected macrophages $y_1$ and the free virus particles $v$ will exponentially go to extinction in a long term. The simulation result is displayed in Fig. 6.2.

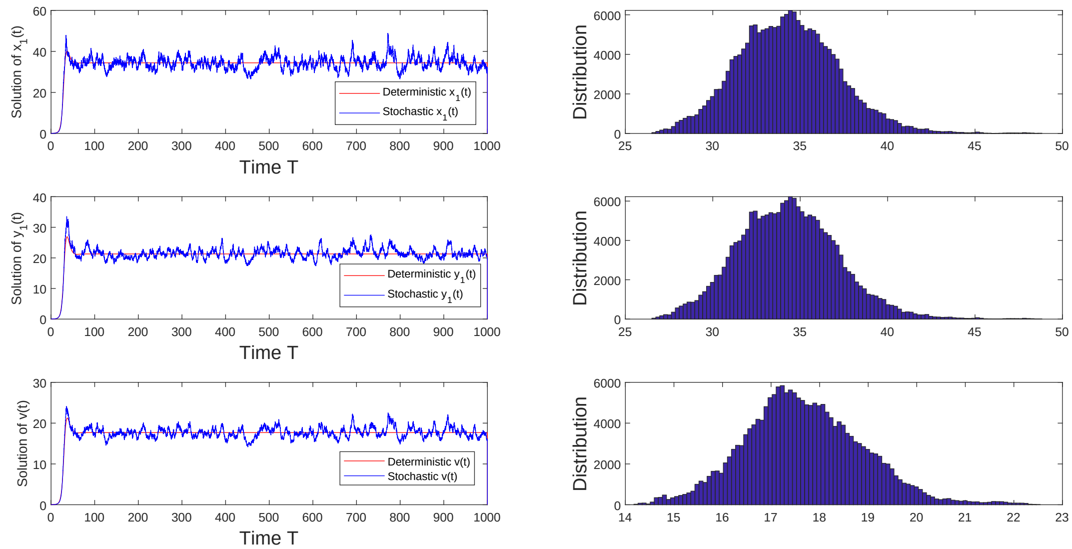


Figure 6.1: The stochastic model (2.2) has ergodic property. The picture on the left and right are the populations size over time and the density functions of $x_1(t)$, $y_1(t)$ and $v(t)$, respectively.

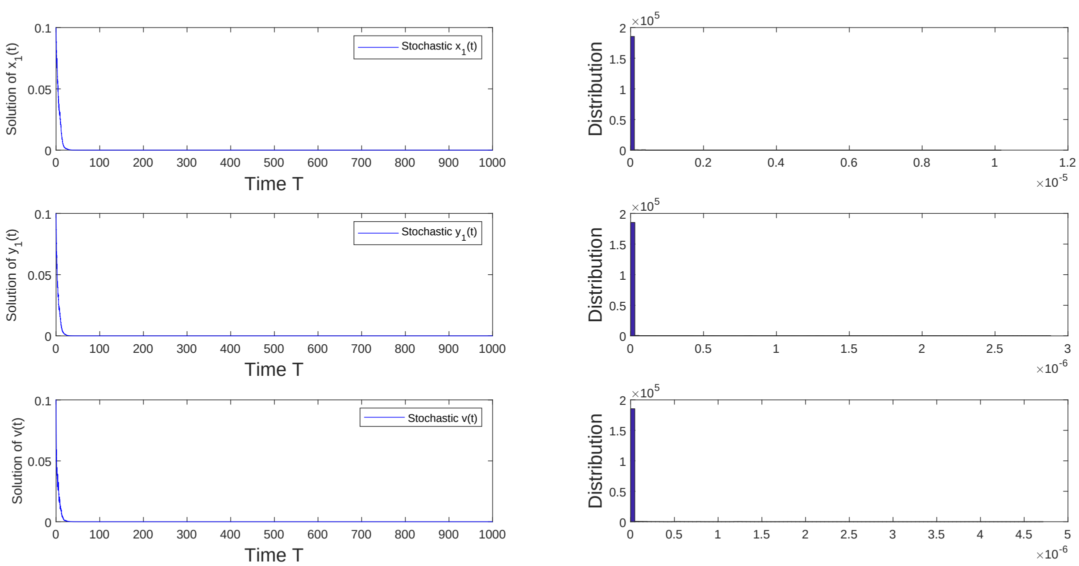


Figure 6.2: The infected cells $x_1$, $y_1$ and free virus particles $v$ of stochastic model (2.2) extinct. The picture on the left and right are the populations size over time and the density functions of $x_1(t)$, $y_1(t)$ and $v(t)$, respectively.

## 7. Conclusion

The purpose of the current study was to determine the effect of nonlinear perturbation on stochastic HIV model, which describes the interaction of the HIV virus with CD4$^+$ T cells and macrophages. The results of this investigation show that the global positive solution of stochastic model (2.2) exists and is unique. According to the ergodic property, we obtain the sufficient conditions for the ergodic stationary distribution of this model. Then, using stochastic comparison theorem and exponential martingales inequality, the sufficient conditions for the extinction of model (2.2) are derived. More precisely, the following conclusions can be drawn from the present study

- Let $(x(t), x_1(t), y(t), y_1(t), v(t))$ be a solution of model (2.2) with any initial value $(x(0), x_1(0), y(0), y_1(0), v(0)) \in \mathbb{R}^5_+$. If

  $$R_0^s = \frac{p_1\beta_1\lambda_1\bar{\delta}\bar{d}_2 + p_2\beta_2\lambda_2\bar{a}\bar{d}_1}{\bar{c}\bar{\delta}\bar{a}\bar{d}_1\bar{d}_2} > 1,$$

  then there exists a stationary distribution $\mu(\cdot)$ and the solution $(x(t), x_1(t), y(t), y_1(t), v(t))$ to model (2.2) is ergodic, where $\bar{d_1} = d_1 + \frac{\sigma_{11}^2}{2} + 2(\sigma_{12}\lambda_1)^{\frac{2}{3}} + \frac{4\sigma_{11}}{3}(\sigma_{12}\lambda_1)^{\frac{1}{3}}$, $\bar{a} = a + \frac{\sigma_{21}^2}{2} + 2(\sigma_{22}\lambda_1)^{\frac{2}{3}} + \frac{4\sigma_{21}}{3}(\sigma_{22}\lambda_1)^{\frac{1}{3}}$, $\bar{d_2} = d_2 + \frac{\sigma_{31}^2}{2} + 2(\sigma_{32}\lambda_2)^{\frac{2}{3}} + \frac{4\sigma_{31}}{3}(\sigma_{32}\lambda_2)^{\frac{1}{3}}$, $\bar{\delta} = \delta + \frac{\sigma_{41}^2}{2} + 2(\sigma_{42}\lambda_2)^{\frac{2}{3}} + \frac{4\sigma_{41}}{3}(\sigma_{42}\lambda_2)^{\frac{1}{3}}$ and $\bar{c} = c + \frac{\sigma_{51}^2}{2} + 2\left(\frac{\sigma_{52}p_1\lambda_1}{a}\right)^{\frac{2}{3}} + \frac{4\sigma_{51}}{3}\left(\frac{\sigma_{52}p_1\lambda_1}{a}\right)^{\frac{1}{3}} + \frac{4p_2\lambda_2}{3\delta}\left(\frac{\sigma_{52}^2 a}{p_1\lambda_1}\right)^{\frac{1}{3}}$.

- Let $(x(t), x_1(t), y(t), y_1(t), v(t))$ be the solution of model (2.2) with any initial value $(x(0), x_1(0), y(0), y_1(0), v(0)) \in \mathbb{R}^5_+$. Then

  $$\limsup_{t\to\infty} \frac{1}{t} \ln\left(\frac{p_1}{ac\sqrt{R_0}}x_1(t) + \frac{p_2}{\delta c\sqrt{R_0}}y_1(t) + \frac{1}{c}v(t)\right) \le \eta, \quad a.s.,$$

  where

  $$\begin{aligned}\eta :=& \frac{p_1\beta_1}{a\sqrt{R_0}}\int_0^\infty \left|X - \frac{\lambda_1}{d_1}\right| \mu_X \mathrm{d}X + \frac{p_2\beta_2}{\delta\sqrt{R_0}}\int_0^\infty \left|Y - \frac{\lambda_2}{d_2}\right| \mu_Y \mathrm{d}Y \\ &+ \left[(a\wedge\delta\wedge c)\mathbf{I}_{\{\sqrt{R_0}\le 1\}} + (a\vee\delta\vee c)\mathbf{I}_{\{\sqrt{R_0}>1\}}\right](\sqrt{R_0}-1) - \frac{1}{2\left(\frac{1}{\sigma_{21}^2} + \frac{1}{\sigma_{41}^2} + \frac{1}{\sigma_{51}^2}\right)}.\end{aligned}$$

  Moreover, if $\eta < 0$, it leads to

  $$\lim_{t\to\infty} x_1(t) = 0, \quad \lim_{t\to\infty} y_1(t) = 0, \quad \lim_{t\to\infty} v(t) = 0, \quad a.s.,$$

  which means the infected CD4$^+$ T cells $x_1$, the infected macrophages $y_1$ and the free virus particles $v$ will exponentially go to extinction in a long term.

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