A REMARK ON ILL-POSEDNESS

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Abstract. In this paper, we construct an example to show that well-posedness and norm inflation are compatible.

1. Introduction and main results

In this paper, we give a remark on the illposedness of the following incompressible Navier-Stokes equations

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u - \nabla p = 0, & \text{in } [0,T) \times \mathbb{R}^n; \\ \nabla \cdot u = 0, & \text{in } [0,T) \times \mathbb{R}^n; \\ u|_{t=0} = u_0, & \text{in } \mathbb{R}^n; \end{cases} \tag{1.1}$$

where $u(t,x)$ and $p(t,x)$ denote the velocity vector field and the pressure of fluid at the point $(t,x) \in [0,T) \times \mathbb{R}^n$ respectively. While u_0 is a given initial velocity vector field. The wellposedness for different initial data spaces have been studied heavily. See Cannone [2], Iwabuchi-Nakamura [7], Koch-Tataru [10], Li-Xiao-Yang [13], Yang-Yang [25]. The solutions of the above Cauchy problem can be obtained via the integral equation:

$$u(t,x) = e^{t\Delta} u_0(x) - B(u,u)(t,x), \tag{1.2}$$

where

$$\begin{cases} B(u,u)(t,x) \equiv \int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla(u \otimes u) ds, \\ \mathbb{P}\nabla(u \otimes u) \equiv \sum\limits_{l} \partial x_l (u_l u) - \sum\limits_{l} \sum\limits_{l'} (-\Delta)^{-1} \partial x_l \partial x_{l'} \nabla(u_l u_{l'}). \end{cases} \tag{1.3}$$

The equation (1.2) can be solved by a fixed-point method whenever the convergence is suitably defined in certain function spaces. For u_0 belongs to some initial space $X^n = (X(\mathbb{R}^n))^n$, denote

$$\begin{cases} u^{(0)}(t,x) = e^{t\Delta} u_0, \\ u^{(\tau+1)}(t,x) = u^{(0)}(t,x) - B(u^{(\tau)}, u^{(\tau)})(t,x), \forall \tau = 0,1,2,\cdots, \end{cases} \tag{1.4}$$

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where $e^{t\Delta}u_0$ belongs to some space $Y^n = (Y([0,T)\times\mathbb{R}^n))^n$.

The above iteration process convergence for $\|u_0\|_{X^n}$ small enough. Such solutions of (1.2) are called mild solutions of (1.1). The notion of such a mild solution was pioneered by Kato-Fujita [9] in 1960s. During the latest decades, many important results about mild solutions to (1.1) have been established. Given $t\in[0,T]$ and $u(t,x)$ belongs to the Banach space X^n. We know $u(t,x)$ belongs to function space $L^\infty([0,T],X^n)$ means

$$(\|u(t,x)\|_{X^n})_{L^\infty([0,T])} < \infty.$$

For initial data $u_0\in X^n$, most often, its solution $u(t,x)$ belong to some solution spaces $Y(X^n)$ which is a subspace of $L^\infty([0,T],X^n)$. See, for example, Cannone [2, 3], Germin-Pavlovic-Staffilani [5], Giga-Miyakawa [6], Kato [8], Lemarié [11, 12], Wu [19, 20, 21, 22] and some author's collaboration work [13, 14, 15, 24].

If a solution space $Y(X^n)$ is not a subspace of $L^\infty([0,T],X^n)$, Bourgain-Pavlović call such solution space have norm inflation phenomenon and the equations are ill-posed in the corresponding initial value spaces. For the end point Triebel-Lizorkin spaces $(\dot F^{-1,q}_\infty)^n(2<q\le\infty)$, Bourgain-Pavlović [1] and Yoneda [26] have shown the norm inflation in the end Triebel-Lizorkin spaces $L^\infty([0,T],(\dot F^{-1,q}_\infty)^n)$. Wang [18] has shown norm inflation in the end point Besov spaces $(\dot B^{-1,q}_\infty)^n(2<q\le\infty)$.

We know $\dot F^{-1,2}_\infty = \mathrm{BMO}^{-1}$. Let $l(Q)$ denote the side length of cube Q. To establish the wellposedness for initial data in $(\mathrm{BMO}^{-1})^n$, Koch-Tataru [10] introduced the following solution space $Y([0,T],(\mathrm{BMO}^{-1})^n)$:

Definition 1.1. $u(t,x)\in Y([0,T],(\mathrm{BMO}^{-1})^n)$ if and only if the following three conditions are satisfied:

$$\nabla u(t,x) = 0 \text{ in } [0,T]\times\mathbb{R}^n, \tag{1.5}$$

$$\sup_{0<t\le T} t^{\frac12}\|u(t,x)\|_\infty < \infty, \tag{1.6}$$

$$\sup_{\text{cube } |Q| \text{ with } l^2(Q)\le T} \{|Q|^{-1}\int_0^{l^2(Q)}\int_Q |u(t,x)|^2 dtdx\}^{\frac12} < \infty. \tag{1.7}$$

For initial data in $(\mathrm{BMO}^{-1})^n$, Koch-Tataru [10] have established wellposedness in solution space $Y([0,T],(\mathrm{BMO}^{-1})^n)$. We find Koch-Tataru's space allow norm inflation in $L^\infty([0,T],(\mathrm{BMO}^{-1})^n)$. In fact, we prove that Koch-Tataru's space $Y([0,T],(\mathrm{BMO}^{-1})^n)$ is not a subspace of $L^\infty([0,T),(\mathrm{BMO}^{-1})^n)$. That is to say, wellposedness and norm inflation are compatible. To simplify the notations, we take $T=1$.

Theorem 1.2. *Koch-Tataru's space $Y([0,1],(\mathrm{BMO}^{-1})^n)$ is not a subspace of $L^\infty([0,1],(\mathrm{BMO}^{-1})^n)$. That is to say, there exists $u(t,x) \in Y([0,1],(\mathrm{BMO}^{-1})^n)$ but $\|u(t,x)\|_{L^\infty([0,1],(\mathrm{BMO}^{-1})^n)} = \infty$.*

Remark 1.3. According to the above Theorem 1.2, norm inflation and well-posedness are not incompatible. Bourgain-Pavlović and Yoneda's illposedness results mean only the norm inflation in $L^\infty([0,T],X^n)$, not real ill-posedness. This supports Chemin and Gallagher's point in [4]: the well-posedness need not the boundedness in $L^\infty([0,T],X^n)$.

The rest of this paper is organized as follows: In section 2, we will present some preliminaries about Meyer wavelets, then we present wavelet characterization for end point Triebel-Lizorkin spaces and Koch-Tataru's solution space. In section 3, we use Meyer wavelets to construct some functions in Koch-Tataru's space and prove Theorems 1.2.

2. Wavelets and function spaces

In this section, we recall first some auxiliary knowledge on wavelets. We indicate that we will use tensorial product real valued orthogonal Meyer wavelets. We refer the reader to [16, 17, 23] for further information. Let Ψ^0 be an even function in $C_0^\infty([-\frac{4\pi}{3},\frac{4\pi}{3}])$ with

$$\begin{cases} 0 \le \Psi^0(\xi) \le 1; \\ \Psi^0(\xi) = 1 \text{ for } |\xi| \le \dfrac{2\pi}{3}. \end{cases}$$

Write

$$\Omega(\xi) = \sqrt{(\Psi^0(\frac{\xi}{2}))^2 - (\Psi^0(\xi))^2}.$$

Then $\Omega(\xi)$ is an even function in $C_0^\infty([-\frac{8\pi}{3},\frac{8\pi}{3}])$. Clearly,

$$\begin{cases} \Omega(\xi) = 0 \text{ for } |\xi| \le \dfrac{2\pi}{3}; \\ \Omega^2(\xi) + \Omega^2(2\xi) = 1 = \Omega^2(\xi) + \Omega^2(2\pi - \xi) \text{ for } \xi \in [\dfrac{2\pi}{3},\dfrac{4\pi}{3}]. \end{cases}$$

Let $\Psi^1(\xi) = \Omega(\xi)e^{-\frac{i\xi}{2}}$. For any $\epsilon = (\epsilon_1,\cdots,\epsilon_n) \in \{0,1\}^n$, define $\Phi^\epsilon(x)$ by $\hat{\Phi}^\epsilon(\xi) = \prod_{i=1}^n \Psi^{\epsilon_i}(\xi_i)$. For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, let $\Phi^\epsilon_{j,k}(x) = 2^{\frac{nj}{2}}\Phi^\epsilon(2^jx - k)$. $\forall \epsilon \in \{0,1\}^n, j \in \mathbb{Z}, k \in \mathbb{Z}^n$ and distribution $f(x)$, denote $f^\epsilon_{j,k} = \langle f, \Phi^\epsilon_{j,k}\rangle$. Furthermore, we put

$$\Lambda_n = \{(\epsilon,j,k), \epsilon \in \{0,1\}^n\backslash\{0\}, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}.$$

Sobolev space $\dot{H}^{\frac{n}{2}-1} = \dot{F}_2^{\frac{n}{2}-1,2}$, Lebesgue space $L^n = \dot{F}_n^{0,2}$, Besov spaces $\dot{B}_p^{\frac{n}{p}-1,p} = \dot{F}_p^{\frac{n}{p}-1,p}$ and $\mathrm{BMO}^{-1} = \dot{F}_\infty^{-1,2}$ are all Triebel-Lizorkin spaces. For an overview of function spaces, we refer to Li-Xiao-Yang [13], Lin-Yang [15],Yang [23] and Yuan-Sickel-Yang [27]. Denote $\mathfrak{D} = \{Q_{j,k} = 2^{-j}k + 2^{-j}[0,1]^n, \forall j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$. We recall then the wavelet characterization of end-point Triebel-Lizorkin spaces $\dot{F}_\infty^{\gamma,q}(\mathbb{R}^n)$ (see [13, 15, 27]).

Lemma 2.1. *Given* $1 \leq q \leq \infty$ *and* $\gamma \in \mathbb{R}$. $f(x) = \sum_{\epsilon,j,k} a_{j,k}^\epsilon \Phi_{j,k}^\epsilon(x) \in \dot{F}_\infty^{\gamma,q}(\mathbb{R}^n) \Leftrightarrow$

$$\sup_{Q\in\mathfrak{D}} \Big\{|Q|^{-1} \sum_{(\epsilon,j,k):Q_{j,k}\subset Q} 2^{jq(\gamma_1+\frac{n}{2}-\frac{n}{q})}|a_{j,k}^\epsilon|^q\Big\}^{\frac{1}{q}} < +\infty. \tag{2.1}$$

At the end of this section, we present one lemma on Koch-Tataru's solution space. Let $u(t,x) = (u_1(t,x), u_2(t,x), \dots, u_n(t,x))^t$. For $i = 1, 2, \cdots, n$, denote $u_i(t,x) = \sum_{(\epsilon,j,k)\in\Lambda_n} a_{j,k}^{i,\epsilon}(t)\Phi_{j,k}^\epsilon(x)$. The following lemma is a direct corollary of the wavelet characterization in Lemma 2.1.

Lemma 2.2. *(i)* $u(t,x)$ *satisfies* (1.7) *if and only if*

$$\sup_{i,j_0\geq 0,k_0\in\mathbb{Z}^n} 2^{nj_0} \int_0^{2^{-2j_0}} \sum_{Q_{j,k}\subset Q_{j_0,k_0}} |a_{j,k}^{i,\epsilon}(t)|^2 dt < \infty. \tag{2.2}$$

(ii) $u(t,x) \in L^\infty([0,1], (\mathrm{BMO}^{-1})^n)$ *if and only if*

$$\sup_{i,0<t\leq 1,k_0\in\mathbb{Z}^n} 2^{nj_0} \sum_{Q_{j,k}\subset Q_{j_0,k_0}} 2^{-2j}|a_{j,k}^{i,\epsilon}(t)|^2 < \infty. \tag{2.3}$$

3. Proof of Theorem 1.2

Now we come to prove Theorem 1.2.

Proof. Denote $\Lambda = \{(e,j,k) \in \Lambda_n, e = (1,\cdots,1), j \geq 0, k \in \mathbb{Z}^n\}$. Take $0 < a < \frac{1}{2}$ and $\frac{n}{2} + 2a - 1 < b < \frac{n}{2}$ and take $u_1(t,x) = \sum_{(e,j,k)\in\Lambda} a_{j,k}^e(t)\Phi_{j,k}^e(x)$ where $a_{j,k}^e(t)$ satisfies

$$a_{j,k}^e(t) = \begin{cases} t^{-a}2^{-bj}, & 1 \leq j \leq -\frac{1}{2}\log_2 t, \quad k \in \mathbb{Z}^n; \\ 0, & j > -\frac{1}{2}\log_2 t \text{ or } t \geq 1; \quad k \in \mathbb{Z}^n. \end{cases}$$

We know, if $0 < b < \frac{n}{2}$ and $b \geq \frac{n}{2}+2a-1$, then $u_1(t,x)$ satisfies the following equation:

$$t^{\frac{1}{2}}\|u_1(t,x)\|_\infty < \infty.$$

The number of k satisfying $Q_{j,k} \subset Q_{j_0,k_0}$ is $2^{n(j-j_0)}$. Hence for $j_0 \geq 0$,

$$\begin{aligned}
& 2^{nj_0} \int_0^{2^{-2j_0}} \sum_{Q_{j,k}\subset Q_{j_0,k_0}} |a^e_{j,k}(t)|^2 dt \\
\leq\ & 2^{nj_0} \int_0^{2^{-2j_0}} t^{-2a} \sum_{j\geq j_0, 1\leq j\leq -\frac{1}{2}\log_2 t} 2^{-2bj} 2^{n(j-j_0)} dt \\
=\ & \int_0^{2^{-2j_0}} t^{-2a} \sum_{\max(j_0,1)\leq j\leq -\frac{1}{2}\log_2 t} 2^{(n-2b)j} dt \\
\leq C\ & \int_0^{2^{-2j_0}} t^{b-\frac{n}{2}-2a} dt.
\end{aligned}$$

If $b > \frac{n}{2} + 2a - 1$, then $u_1(t,x)$ satisfies equation (2.2).

If $a < \frac{1}{2}$ and $\frac{n}{2} + 2a - 1 < b < \frac{n}{2}$, then

$$\begin{aligned}
c_{t,j_0} &= 2^{nj_0} \sum_{Q_{j,k}\subset Q_{j_0,k_0}} 2^{-2j} |a^e_{j,k}(t)|^2 \\
&\leq 2^{nj_0} t^{-2a} \sum_{j\geq j_0, 1\leq j\leq -\frac{1}{2}\log_2 t} 2^{-2j-2bj} 2^{n(j-j_0)} \\
&= t^{-2a} \sum_{\max(j_0,1)\leq j\leq -\frac{1}{2}\log_2 t} 2^{(n-2b-2)j}.
\end{aligned}$$

Take $j_0 = 0, b > \frac{n}{2} - 1$ and $a > 0$, then $c_{t,0} > ct^{-2a}$, hence $\sup\limits_{0<t<1} c_{t,0} = +\infty$. Hence $u_1(t,x)$ does satisfy equation (2.3).

Take $u_2(t,x) = -\frac{1}{\partial_2}\partial_1 u_1(t,x)$. It is easy to see that $u_2(t,x)$ satisfies the same properties as $u_1(t,x)$ does.

For $i = 3, \cdots, n$, take $u_i(t,x) = 0$. By construction, we know that $u(t,x) = (u_1(t,x), u_2(t,x), \cdots, u_n(t,x))^t$ satisfies

$$\nabla u(t,x) = 0 \text{ in } [0,1] \times \mathbb{R}^n.$$

That is to say, $u(t,x)$ satisfies all the conditions in Theorem 1.2. $\square$

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