On the Nonlinear (k, Ψ)-Hilfer Fractional Differential Equations

Kishor D. Kucche [1]
kdkucche@gmail.com

Ashwini D. Mali [2]
maliashwini144@gmail.com

[1,2] Department of Mathematics, Shivaji University, Kolhapur-416 004, Maharashtra, India.

Abstract

In the current paper, we present the most generalized variant of the Hilfer derivative so-called (k, Ψ)-Hilfer fractional derivative operator. The (k, Ψ)-Riemann-Liouville and (k, Ψ)-Caputo fractional derivatives are obtained as special case of (k, Ψ)-Hilfer fractional derivative. We demonstrate a few properties of (k, Ψ)-Riemann-Liouville fractional integral and derivative that expected to build up the calculus of (k, Ψ)-Hilfer fractional derivative operator. We present some significant outcomes about (k, Ψ)-Hilfer fractional derivative operator that require to derive the equivalent fractional integral equation to nonlinear (k, Ψ)-Hilfer fractional differential equation. We prove the existence and uniqueness for the solution of nonlinear (k, Ψ)-Hilfer fractional differential equation. In the conclusion section, we list the various k-fractional derivatives that are specific cases of (k, Ψ)-Hilfer fractional derivative.

Key words: Fractional calculus; (k, Ψ)-Fractional integral; (k, Ψ)-Fractional derivative; (k, Ψ)-Fractional differential equations; Existence and uniqueness; Initial value problem.

2020 Mathematics Subject Classification: 26A33, 33B15, 34A12, 34A08.

1 Introduction

Fractional calculus (FC) is the intensive developing area of mathematical analysis and has extensive applications to real-world problems. Therefore, the field of FC offers enormous features for research. With the aim to preserve different properties of the classical integer-order derivative, different definitions of fractional derivative can be found in the literature [1, 2, 3, 4] which do not coincides in general. In this way, it is expected to define the generalized fractional derivative that consolidates the well-known fractional derivatives as its particular cases. In this sense, we can develop the calculus of all these fractional derivatives under one roof. This work is very well handled by Sousa and Oliveira [4]. They have presented a new definition of the fractional derivative with respect to another function called Ψ-Hilfer fractional derivative. The analysis of nonlinear fractional differential equations (FDEs) involving Ψ-Hilfer fractional derivative concerning various qualitative properties of solutions can be found in the work [5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

In 2007, Díaz and Pariguan [15] introduced k-gamma function $\Gamma_k(z) = \int_0^\infty s^{z-1}\, e^{\frac{-s^k}{k}} ds,\ z \in \mathbb{C},\ \mathrm{Re}(z) > 0,\ k > 0\ (k \in \mathbb{R})$, which is the generalization of the Euler gamma function $\Gamma(\cdot)$, and for $k \to 1$, we obtain $\Gamma_k(z) \to \Gamma(z)$. Several definitions of fractional derivatives and integrals depend on the Euler gamma function. Since the k-gamma function $\Gamma_k(\cdot)$ is the natural generalization of the Euler gamma function $\Gamma(\cdot)$, it is natural to expect the concept of fractional derivatives and integrals with the additional parameter k.

Using the definition of k-gamma function, Mubeen and Habibullah [16] introduced the extended version of the Riemann-Liouville (RL) fractional integral operator called k-RL fractional integral operator. Roused by this new idea, Romero et al. [17] presented a generalized version of RL fractional derivative called k-RL fractional derivative and its properties. In 2015, Dorrego [18] demonstrated that the definition of k-RL fractional derivative defined in [17] is not a left inverse of the corresponding k-RL fractional integral operator. To conquer this trouble, Dorrego [18] presented an alternative definition of k-RL fractional derivative and explored some of its significant properties. Note that the k-Hilfer fractional derivative defined in [19] does not include the k-RL fractional derivative defined in [18]. Therefore, there is a need to give an alternative definition to it. A few researchers have considered the investigation of different types of k-fractional derivatives and analyzed nonlinear FDEs involving it, a few of them are [20, 21, 22, 23, 24, 25, 26, 27].

The work referenced above inspired us to propose a most generalized version of the Hilfer derivative so-called (k,Ψ)-Hilfer fractional derivative. We acquire the (k,Ψ)-RL and (k,Ψ)-Caputo fractional derivatives as a special case of (k,Ψ)-Hilfer fractional derivative. Some properties of (k,Ψ)-RL fractional integral and derivative are demonstrated and used to develop the calculus of (k,Ψ)-Hilfer fractional derivative operator. In the conclusion section, we listed the various fractional derivatives that are specific cases of (k,Ψ)-Hilfer fractional derivative.

Further, we consider the nonlinear FDEs involving (k,Ψ)-Hilfer derivative of the form

$$
\begin{aligned}
{}^{k,H}\mathcal{D}_{a+}^{\eta,\,\nu\,;\,\Psi}y(t) &= f(t,y(t)), \quad t\in(a,b],\ 0<\eta<k,\ 0\le\nu\le 1, && (1.1)\\
{}_{k}\mathfrak{J}_{a+}^{k-\zeta_k\,;\,\Psi}y(a) &= y_a\in\mathbb{R}, \qquad \zeta_k=\eta+\nu\,(k-\eta)\,, && (1.2)
\end{aligned}
$$

where ${}^{k,H}\mathcal{D}_{a+}^{\eta,\,\nu\,;\,\Psi}(\cdot)$ is the (k,Ψ)-Hilfer derivative of order η and type ν, ${}_{k}\mathfrak{J}_{a+}^{k-\zeta_k;\,\Psi}(\cdot)$ is the (k,Ψ)-RL fractional integral of order $k-\zeta_k$ and $f:(a,b]\times\mathbb{R}\to\mathbb{R}$ is an appropriate function specified latter.

We determine the equivalent fractional integral equation to the nonlinear (k,Ψ)-Hilfer FDEs (1.1)-(1.2) and prove the existence and uniqueness for the solution through it. Finally, in the conclusion section, we list the different k-fractional derivatives that are specific cases of (k,Ψ)-Hilfer fractional derivative.

The structure of the paper is as follows: In Section 2, we present some preliminaries about k-RL fractional integral and derivative operators, and Ψ-Hilfer fractional derivative. In Section 3, we define (k,Ψ)-Hilfer fractional derivative operators. Section 4 deals with the properties of (k,Ψ)-RL fractional integral. Properties of (k,Ψ)-RL fractional derivative operators are studied in Section 5. Section 6 deals with calculus of (k,Ψ)-Hilfer fractional derivative. In Section 7, we investigate existence and uniqueness of solution through equivalent fractional integral equation to the nonlinear (k,Ψ)-Hilfer FDEs (1.1)-(1.2). Concluding remarks provided in Section 8.

2 Preliminaries

2.1 k-Riemann-Liouville fractional derivative

Definition 2.1 ([15]) *For $z \in \mathbb{C}$ with $Re(z) > 0$ and $k > 0 (k \in \mathbb{R})$, the k-gamma function $\Gamma_k(\cdot)$ is defined by*

$$\Gamma_k(z) = \int_0^\infty s^{z-1} \, e^{\frac{-s^k}{k}} \, ds.$$

Theorem 2.1 ([15]) *The k-gamma function $\Gamma_k(\cdot)$ satisfies the following properties:*

(i) $\Gamma_k(z+k) = z\,\Gamma_k(z)$
(ii) $\Gamma_k(k) = 1$
(iii) $\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma(\frac{z}{k})$.

Definition 2.2 ([15]) *Let $z, w \in \mathbb{C}$ with $Re(z) > 0$ and $Re(w) > 0$. Then, the k-beta function $B_k(z,w)$ is defined by*

$$B_k(z,w) = \frac{1}{k}\int_0^1 s^{\frac{z}{k}-1} \, (1-s)^{\frac{w}{k}-1} ds.$$

Note that beta function and k-beta function have the following relation

$$B_k(z,w) = \frac{1}{k} \, B\left(\frac{z}{k}, \frac{w}{k}\right).$$

Further, k-beta function and k-gamma function have the following relation

$$B_k(z,w) = \frac{\Gamma_k(z)\,\Gamma_k(w)}{\Gamma_k(z+w)}.$$

Definition 2.3 ([16]) *Let $\mathfrak{h}$ be an integrable function defined on $[a,b]$ and $k > 0$. Then, the k-Riemann-Liouville fractional integral of order $\eta > 0$ ($\eta \in \mathbb{R}$) of the function $\mathfrak{h}$ is given by*

$${}^{k}\mathfrak{I}^{\eta}_{a+}\mathfrak{h}\,(t) = \frac{1}{k\,\Gamma_k\,(\eta)}\int_a^t (t-s)^{\frac{\eta}{k}-1}\,\mathfrak{h}\,(s)\,ds.$$

Definition 2.4 ([20]) *Let $k, \eta \in \mathbb{R}_+ = (0,\infty)$ and $m \in \mathbb{N}$ such that $m = \lceil \frac{\eta}{k} \rceil$ and $\mathfrak{h}$ be an integrable function defined on $[a,b]$. Then, the k-Riemann-Liouville fractional derivative of order η of the function $\mathfrak{h}$ is given by*

$${}^{k,RL}\mathfrak{D}^{\eta}_{a+}\mathfrak{h}\,(t) = \left(k\,\frac{d}{dt}\right)^m \, {}^{k}\mathfrak{I}^{mk-\eta}_{a+}\,\mathfrak{h}\,(t)\,.$$

2.2 Ψ-Hilfer fractional derivative

We review a few definitions, notations and results of Ψ-Hilfer fractional derivative from [4].

Let $\Delta = [a, b]$ $(0 < a < b < \infty)$ be a finite interval and $\Psi : \Delta \to \mathbb{R}$ is an increasing function with $\Psi'(t) \neq 0$, for all $t \in \Delta$. Consider the space $C_{\sigma;\,\Psi}(\Delta, \mathbb{R})$ of weighted functions $\mathfrak{h}$ defined on Δ given by

$$C_{\sigma;\,\Psi}(\Delta, \mathbb{R}) = \left\{ \mathfrak{h} : (a, b] \to \mathbb{R} \;\middle|\; (\Psi\,(\cdot) - \Psi\,(a))^{\sigma}\, \mathfrak{h}\,(\cdot) \in C(\Delta, \mathbb{R}) \right\},\ 0 \le \sigma < 1$$

endowed with the norm

$$\|\mathfrak{h}\|_{C_{\sigma;\,\Psi}(\Delta,\mathbb{R})} = \max_{t\in\Delta} \left| (\Psi\,(t) - \Psi\,(a))^{\sigma}\, \mathfrak{h}\,(t) \right|.$$

Definition 2.5 ([1]) *Let $\mathfrak{h}$ be an integrable function defined on $[a, b]$. Then, the Ψ-Riemann-Liouville fractional integral of order $\eta > 0$ $(\eta \in \mathbb{R})$ of the function $\mathfrak{h}$ is given by*

$$\mathfrak{I}^{\eta\,;\Psi}_{a+}\mathfrak{h}\,(t) = \frac{1}{\Gamma\,(\eta)} \int_a^t \Psi'\,(s)\,(\Psi\,(t) - \Psi\,(s))^{\eta-1}\,\mathfrak{h}\,(s)\,ds.$$

Definition 2.6 ([1]) *Let $m - 1 < \eta \le m$, $\Psi \in C^m[a, b], \Psi'(t) \neq 0, t \in [a, b]$ and $\mathfrak{h} \in C[a, b]$. Then, the Ψ-Riemann-Liouville fractional derivative of a function $\mathfrak{h}$ of order η is defined by*

$${}^{RL}\mathcal{D}^{\eta;\,\Psi}_{a+}\mathfrak{h}(t) = \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^m \mathfrak{I}^{m-\eta;\,\Psi}_{a+}\mathfrak{h}(t).$$

Definition 2.7 ([28]) *Let $m - 1 < \eta \le m, \Psi \in C^m[a, b], \Psi'(t) \neq 0, t \in [a, b]$ and $\mathfrak{h} \in C^m[a, b]$. Then, the Ψ-Caputo fractional derivative of a function $\mathfrak{h}$ of order η is defined by*

$${}^{C}\mathcal{D}^{\eta;\,\Psi}_{a+}\mathfrak{h}(t) = \mathfrak{I}^{m-\eta;\,\Psi}_{a+} \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^m \mathfrak{h}(t).$$

Definition 2.8 ([4]) *Let $m - 1 < \eta \le m$, $\nu \in [0, 1]$, $\Psi \in C^m[a, b], \Psi'(t) \neq 0, t \in [a, b]$and $\mathfrak{h} \in C^m[a, b]$. The Ψ-Hilfer fractional derivative of a function $\mathfrak{h}$ of order η and type ν is defined by*

$${}^{H}\mathcal{D}^{\eta,\,\nu;\,\Psi}_{a+}\mathfrak{h}(t) = \mathfrak{I}^{\nu(m-\eta);\,\Psi}_{a+} \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^m \mathfrak{I}^{(1-\nu)(m-\eta);\,\Psi}_{a+}\mathfrak{h}(t). \tag{2.1}$$

3 (k, Ψ)-Hilfer fractional derivative

Motivated by the definitions of k-RL derivative [16] and Ψ-Hilfer derivative [4], in this section we define the most generalized version of Hilfer derivative namely (k, Ψ)-Hilfer derivative.

To define the (k, Ψ)-Hilfer fractional derivative operator, we first introduce the (k, Ψ)-RL fractional integral defined in [29].

Definition 3.1 ([29]) *Let $\mathfrak{h} \in L^1[a, b]$ and $k > 0\,(k \in \mathbb{R})$. Then, the (k, Ψ)-Riemann-Liouville fractional integral of order $\eta > 0$ $(\eta \in \mathbb{R})$ of the function $\mathfrak{h}$ is given by*

$${}_{k}\mathfrak{I}^{\eta\,;\Psi}_{a+}\mathfrak{h}\,(t) = \frac{1}{k\,\Gamma_k\,(\eta)} \int_a^t \Psi'\,(s)\,(\Psi\,(t) - \Psi\,(s))^{\frac{\eta}{k}-1}\,\mathfrak{h}\,(s)\,ds. \tag{3.1}$$

Definition 3.2 *Let $\eta, k \in \mathbb{R}_+ = (0,\infty))$, $\nu \in [0,1]$, $\Psi \in C^m[a,b], \Psi'(t) \neq 0, t \in [a,b]$ and $\mathfrak{h} \in C^m[a,b]$. Then, the (k,Ψ)-Hilfer fractional derivative of a function $\mathfrak{h}$ of order η and type ν is defined by*

$$ {}^{k,H}\mathcal{D}^{\eta,\nu;\Psi}_{a+}\mathfrak{h}(t) = {}_k\mathfrak{I}^{\nu(mk-\eta);\Psi}_{a+}\left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right)^m {}_k\mathfrak{I}^{(1-\nu)(mk-\eta);\Psi}_{a+}\mathfrak{h}(t),\ m = \left\lceil \frac{\eta}{k} \right\rceil. \tag{3.2} $$

- For $\Psi(t) = t$ and $\nu = 0$, (k,Ψ)-Hilfer fractional derivative reduces to (k,Ψ)-RL fractional derivative operator

$$ {}^{k,RL}\mathcal{D}^{\eta;\Psi}_{a+}\mathfrak{h}(t) = \left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right)^m {}_k\mathfrak{I}^{mk-\eta;\Psi}_{a+}\mathfrak{h}(t). \tag{3.3} $$

  If we take $\Psi(t) = t$ in equation (3.3), we obatin the definition of k-RL fractional derivative [20].

- For $\Psi(t) = t$ and $\nu = 1$, (k,Ψ)-Hilfer fractional derivative reduces to (k,Ψ)-Caputo fractional derivative operator

$$ {}^{k,C}\mathcal{D}^{\eta;\Psi}_{a+}\mathfrak{h}(t) = {}_k\mathfrak{I}^{mk-\eta;\Psi}_{a+}\left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right)^m \mathfrak{h}(t). \tag{3.4} $$

  If we take $\Psi(t) = t$ in equation (3.4), we obatin the definition of k-Caputo fractional derivative

$$ {}^{k,C}\mathcal{D}^{\eta}_{a+}\mathfrak{h}(t) = {}_k\mathfrak{I}^{(mk-\eta)}_{a+}\left(k\frac{d}{dt}\right)^m \mathfrak{h}(t). \tag{3.5} $$

- For $\zeta_k = \eta + \nu(mk-\eta)$, we have $\nu(mk-\eta) = \zeta_k - \eta$ and $(1-\nu)(mk-\eta) = mk - \zeta_k$, and hence (k,Ψ)-Hilfer fractional derivative can be defined in the form of (k,Ψ)-RL fractional derivative as follows

$$ \begin{aligned} {}^{k,H}\mathcal{D}^{\eta,\nu;\Psi}_{a+}\mathfrak{h}(t) &= {}_k\mathfrak{I}^{\zeta_k-\eta;\Psi}_{a+}\left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right)^m {}_k\mathfrak{I}^{mk-\zeta_k;\Psi}_{a+}\mathfrak{h}(t) && (3.6)\\ &= {}_k\mathfrak{I}^{\zeta_k-\eta;\Psi}_{a+}\ {}^{k,RL}\mathcal{D}^{\zeta_k;\Psi}_{a+}\mathfrak{h}(t). && (3.7) \end{aligned} $$

  Note that for $\nu \in [0,1]$ and $m-1 < \frac{\eta}{k} < m$, we have $m - 1 < \frac{\zeta_k}{k} \leq m$.

Remark 3.1

1. *For $k = 1$, (k,Ψ)-Hilfer fractional derivative reduces to Ψ-Hilfer fractional derivative [4]. For $\Psi(t) = t$ and $k = 1$, (k,Ψ)-Hilfer fractional derivative reduces to Hilfer fractional derivative [30].*
2. *For $\Psi(t) = t$, (k,Ψ)-Hilfer fractional derivative reduces to k-Hilfer fractional derivative*

$$ {}^{k,H}\mathcal{D}^{\eta,\nu}_{a+}\mathfrak{h}(t) = {}_k\mathfrak{I}^{\nu(mk-\eta)}_{a+}\left(k\frac{d}{dt}\right)^m {}_k\mathfrak{I}^{(1-\nu)(mk-\eta)}_{a+}\mathfrak{h}(t). \tag{3.8} $$

3. *It is observed that the k-Hilfer fractional derivative defined in [19] does not includes the k-RL fractional derivative. But the formula which we have defined in* (3.8) *includes the k-RL fractional derivative as a particular case of it for $\nu = 0$.*

4 Properties of (k, Ψ)-Riemann-Liouville fractional integral

In this section, we prove few properties of (k, Ψ)-RL fractional integral that are needed to investigate the properties of (k, Ψ)-Hilfer fractional derivative.

Theorem 4.1 *Let $\mu_i, k \in \mathbb{R}_+ = (0, \infty)(i = 1, 2)$. Then,*

$$
{}^k\mathfrak{I}_{a+}^{\mu_1;\,\Psi}\ {}^k\mathfrak{I}_{a+}^{\mu_2;\,\Psi} = {}^k\mathfrak{I}_{a+}^{\mu_1+\mu_2;\,\Psi}.
$$

Proof: The proof of the theorem one can obtain easily using the definition of (k, Ψ)-RL fractional integral, Dirichlet's formula, the substitution $\Psi(s) = \Psi(a) + z\,(\Psi(t) - \Psi(a))$ and the properties of k-gamma function given in the Theorem 2.1. Thus, we omit the details. □

Theorem 4.2 *Let $\sigma, k \in \mathbb{R}$ with $0 \leq \sigma < k$, $d \in (a, b)$, $g \in C_{\frac{\sigma}{k};\,\Psi}[a, d]$ and $g \in C[d, b]$. Then, $g \in C_{\frac{\sigma}{k};\,\Psi}[a, b]$.*

Proof: Since $g \in C_{\frac{\sigma}{k};\,\Psi}[a, d]$, we have $(\Psi\,(t) - \Psi\,(a))^{\frac{\sigma}{k}}\, g(t)$ is continuous on $[a, d]$. Further, $g \in C[d, b]$, $\Psi \in C[a, b]$ and $\frac{\sigma}{k} > 0$, the function $(\Psi\,(t) - \Psi\,(a))^{\frac{\sigma}{k}}\, g(t)$ is continuous on $[d, b]$. From above discussion, it follows that $(\Psi\,(t) - \Psi\,(a))^{\frac{\sigma}{k}}\, g(t)$ is continuous on $[a, b]$. This implies $g \in C_{\frac{\sigma}{k};\,\Psi}[a, b]$. □

Theorem 4.3 *Let $\mu, k \in \mathbb{R}_+ = (0, \infty)$ and let $\xi \in \mathbb{R}$ such that $\frac{\xi}{k} > -1$. Then,*

$$
{}^k\mathfrak{I}_{a+}^{\mu;\,\Psi}\,(\Psi(t) - \Psi(a))^{\frac{\xi}{k}} = \frac{\Gamma_k(\xi + k)}{\Gamma_k(\xi + k + \mu)}\,(\Psi(t) - \Psi(a))^{\frac{\xi+\mu}{k}}.
$$

Proof: One can obtain the proof easily, using the substitution $\Psi(s) = \Psi(a) + z\,(\Psi(t) - \Psi(a))$ and the definition of k-beta function given in the Definition 2.2. □

Remark 4.4 *Taking $k = 1$ and $\Psi(t) = t$ in the Theorem 4.3, we obtain the following result*

$$
\mathfrak{I}_{a+}^{\mu}\,(t - a)^{\xi} = \frac{\Gamma(\xi)}{\Gamma(\xi + \mu)}\,(t - a)^{\xi+\mu}.
$$

which is proved in [31].

Theorem 4.5 *Let $\mu, \sigma, k \in \mathbb{R}_+ = (0, \infty)$ with $\sigma \leq \mu < k$. Then, the (k, Ψ)-Riemann-Liouville fractional integral operator ${}^k\mathfrak{I}_{a+}^{\mu;\,\Psi}$ is bounded from $C_{\frac{\sigma}{k};\,\Psi}[a, b]$ to $C[a, b]$ and for any $\mathfrak{h} \in C_{\frac{\sigma}{k};\,\Psi}[a, b]$,*

$$
\left\| {}^k\mathfrak{I}_{a+}^{\mu;\,\Psi}\,\mathfrak{h} \right\|_{C[a,b]} \leq \|\mathfrak{h}\|_{C_{\frac{\sigma}{k};\,\Psi}[a,b]}\ \frac{\Gamma_k(k - \sigma)}{\Gamma_k(k - \sigma + \mu)}\,(\Psi\,(b) - \Psi\,(a))^{\frac{\mu-\sigma}{k}}.
$$

Proof: Let any $\mathfrak{h} \in C_{\frac{\sigma}{k};\Psi}[a,b]$ and $t_1, t_2 \in [a,b]$ with $t_2 > t_1$. Then, using the definition of (k,Ψ)-RL fractional integral and the Theorem 4.3, we obtain

$$\begin{aligned}
&\left| {}^{k}\mathfrak{J}_{a+}^{\mu;\Psi}\,\mathfrak{h}(t_2) - {}^{k}\mathfrak{J}_{a+}^{\mu;\Psi}\,\mathfrak{h}(t_1) \right| \\
&= \left| \frac{1}{k\,\Gamma_k(\mu)} \int_a^{t_2} \Psi'(s)\left(\Psi(t_2) - \Psi(s)\right)^{\frac{\mu}{k}-1} \mathfrak{h}(s)\, ds \right. \\
&\quad \left. - \frac{1}{k\,\Gamma_k(\mu)} \int_a^{t_1} \Psi'(s)\left(\Psi(t_1) - \Psi(s)\right)^{\frac{\mu}{k}-1} \mathfrak{h}(s)\, ds \right| \\
&\le \left| \frac{1}{k\,\Gamma_k(\mu)} \int_a^{t_2} \Psi'(s)\left(\Psi(t_2) - \Psi(s)\right)^{\frac{\mu}{k}-1} \left(\Psi(s) - \Psi(a)\right)^{\frac{-\sigma}{k}} \left| \left(\Psi(s) - \Psi(a)\right)^{\frac{\sigma}{k}} \mathfrak{h}(s) \right| ds \right. \\
&\quad \left. - \frac{1}{k\,\Gamma_k(\mu)} \int_a^{t_1} \Psi'(s)\left(\Psi(t_1) - \Psi(s)\right)^{\frac{\mu}{k}-1} \left(\Psi(s) - \Psi(a)\right)^{\frac{-\sigma}{k}} \left| \left(\Psi(s) - \Psi(a)\right)^{\frac{\sigma}{k}} \mathfrak{h}(s) \right| ds \right| \\
&\le \|\mathfrak{h}\|_{C_{\frac{\sigma}{k};\Psi}[a,b]} \left| {}^{k}\mathfrak{J}_{a+}^{\mu;\Psi} \left(\Psi(t_2) - \Psi(a)\right)^{\frac{-\sigma}{k}} - {}^{k}\mathfrak{J}_{a+}^{\mu;\Psi} \left(\Psi(t_1) - \Psi(a)\right)^{\frac{-\sigma}{k}} \right| \\
&\le \|\mathfrak{h}\|_{C_{\frac{\sigma}{k};\Psi}[a,b]} \, \frac{\Gamma_k(k-\sigma)}{\Gamma_k(k-\sigma+\mu)} \left| \left(\Psi(t_2) - \Psi(a)\right)^{\frac{\mu-\sigma}{k}} - \left(\Psi(t_1) - \Psi(a)\right)^{\frac{\mu-\sigma}{k}} \right|.
\end{aligned}$$

Since $\sigma \le \mu < k$, using the continuity of Ψ, we have

$$\left| {}^{k}\mathfrak{J}_{a+}^{\mu;\Psi}\,\mathfrak{h}(t_2) - {}^{k}\mathfrak{J}_{a+}^{\mu;\Psi}\,\mathfrak{h}(t_1) \right| \to 0 \text{ as } |t_2 - t_1| \to 0.$$

This implies ${}^{k}\mathfrak{J}_{a+}^{\mu;\Psi}\mathfrak{h} \in C[a,b]$. Following similar type of steps as above, one can verify that

$$\left\| {}^{k}\mathfrak{J}_{a+}^{\mu;\Psi}\,\mathfrak{h} \right\|_{C[a,b]} \le \|\mathfrak{h}\|_{C_{\frac{\sigma}{k};\Psi}[a,b]} \, \frac{\Gamma_k(k-\sigma)}{\Gamma_k(k-\sigma+\mu)} \left(\Psi(b) - \Psi(a)\right)^{\frac{\mu-\sigma}{k}}.$$

$\square$

Theorem 4.6 *Let $\mu, \sigma, k \in \mathbb{R}_+ = (0,\infty)$ with $\sigma < k$. Then, the (k,Ψ)-Riemann-Liouville fractional integral operator ${}^{k}\mathfrak{J}_{a+}^{\mu;\Psi}$ is bounded from $C_{\frac{\sigma}{k};\Psi}[a,b]$ to $C_{\frac{\sigma}{k};\Psi}[a,b]$ and for any $\mathfrak{h} \in C_{\frac{\sigma}{k};\Psi}[a,b]$,*

$$\left\| {}^{k}\mathfrak{J}_{a+}^{\mu;\Psi}\,\mathfrak{h} \right\|_{C_{\frac{\sigma}{k};\Psi}[a,b]} \le \|\mathfrak{h}\|_{C_{\frac{\sigma}{k};\Psi}[a,b]} \, \frac{\Gamma_k(k-\sigma)}{\Gamma_k(k-\sigma+\mu)} \left(\Psi(b) - \Psi(a)\right)^{\frac{\mu}{k}}.$$

Proof: Let any $\mathfrak{h} \in C_{\frac{\sigma}{k};\Psi}[a,b]$ and $t_1, t_2 \in [a,b]$ with $t_2 > t_1$. Then, using the definition of (k,Ψ)-RL fractional integral operator and the Theorem 4.3, we have

$$\begin{aligned}
&\left| \left(\Psi(t_2) - \Psi(a)\right)^{\frac{\sigma}{k}} \, {}^{k}\mathfrak{J}_{a+}^{\mu;\Psi}\,\mathfrak{h}(t_2) - \left(\Psi(t_1) - \Psi(a)\right)^{\frac{\sigma}{k}} \, {}^{k}\mathfrak{J}_{a+}^{\mu;\Psi}\,\mathfrak{h}(t_1) \right| \\
&= \left| \frac{\left(\Psi(t_2) - \Psi(a)\right)^{\frac{\sigma}{k}}}{k\,\Gamma_k(\mu)} \int_a^{t_2} \Psi'(s)\left(\Psi(t_2) - \Psi(s)\right)^{\frac{\mu}{k}-1} \mathfrak{h}(s)\, ds \right. \\
&\quad \left. - \frac{\left(\Psi(t_1) - \Psi(a)\right)^{\frac{\sigma}{k}}}{k\,\Gamma_k(\mu)} \int_a^{t_1} \Psi'(s)\left(\Psi(t_1) - \Psi(s)\right)^{\frac{\mu}{k}-1} \mathfrak{h}(s)\, ds \right|
\end{aligned}$$

$$\leq \left| \left(\Psi\left(t_2\right)-\Psi\left(a\right)\right)^{\frac{\sigma}{k}}\ {}^{k}\mathfrak{I}^{\mu;\,\Psi}_{a+}\left(\Psi\left(t_2\right)-\Psi\left(a\right)\right)^{\frac{-\sigma}{k}}-\left(\Psi\left(t_1\right)-\Psi\left(a\right)\right)^{\frac{\sigma}{k}}\ {}^{k}\mathfrak{I}^{\mu;\,\Psi}_{a+}\left(\Psi\left(t_1\right)-\Psi\left(a\right)\right)^{\frac{-\sigma}{k}}\right| \times \|\mathfrak{h}\|_{C_{\frac{\sigma}{k};\,\Psi}[a,b]}$$

$$\leq \|\mathfrak{h}\|_{C_{\frac{\sigma}{k};\,\Psi}[a,b]}\ \frac{\Gamma_k(k-\sigma)}{\Gamma_k(k-\sigma+\mu)}\left|\left(\Psi\left(t_2\right)-\Psi\left(a\right)\right)^{\frac{\mu}{k}}-\left(\Psi\left(t_1\right)-\Psi\left(a\right)\right)^{\frac{\mu}{k}}\right|.$$

Using the continuity of Ψ, we have

$$\left|\left(\Psi\left(t_2\right)-\Psi\left(a\right)\right)^{\frac{\sigma}{k}}\ {}^{k}\mathfrak{I}^{\mu;\,\Psi}_{a+}\mathfrak{h}(t_2)-\left(\Psi\left(t_1\right)-\Psi\left(a\right)\right)^{\frac{\sigma}{k}}\ {}^{k}\mathfrak{I}^{\mu;\,\Psi}_{a+}\mathfrak{h}(t_1)\right| \to 0 \text{ as } |t_2-t_1|\to 0.$$

This proves, for any $\mu>0$ and $\mathfrak{h}\in C_{\frac{\sigma}{k};\,\Psi}[a,b]$, we have ${}^{k}\mathfrak{I}^{\mu;\,\Psi}_{a+}\mathfrak{h}\in C_{\frac{\sigma}{k};\,\Psi}[a,b]$. Further, following the similar types of steps as above, one can easily check that

$$\left\|{}^{k}\mathfrak{I}^{\mu;\,\Psi}_{a+}\,\mathfrak{h}\right\|_{C_{\frac{\sigma}{k};\,\Psi}[a,b]}\leq \|\mathfrak{h}\|_{C_{\frac{\sigma}{k};\,\Psi}[a,b]}\ \frac{\Gamma_k(k-\sigma)}{\Gamma_k(k-\sigma+\mu)}\left(\Psi\left(b\right)-\Psi\left(a\right)\right)^{\frac{\mu}{k}}.$$

□

Theorem 4.7 *Let $\mu\geq 0\,(\mu\in\mathbb{R})$, ${}^{k}\mathfrak{I}^{\mu;\,\Psi}_{a+}$ maps $C[a,b]$ into $C[a,b]$.*

Proof: Proof can be completed following similar types of steps as in the proof of Theorem 4.5 and Theorem 4.6. □

Theorem 4.8 *Let $\mu,\sigma,k\in\mathbb{R}_+=(0,\infty)$ with $\sigma<\mu<k$ and $\mathfrak{h}\in C_{\frac{\sigma}{k};\,\Psi}[a,b]$. Then,*

$$ {}^{k}\mathfrak{I}^{\mu;\,\Psi}_{a+}\mathfrak{h}(a)=\lim_{t\to a+}\ {}^{k}\mathfrak{I}^{\mu;\,\Psi}_{a+}\mathfrak{h}(t)=\,0. \tag{4.1}$$

Proof: Let any $\mathfrak{h}\in C_{\frac{\sigma}{k};\,\Psi}[a,b]$. Then, by Theorem 4.5, ${}^{k}\mathfrak{I}^{\mu;\,\Psi}_{a+}\mathfrak{h}\in C[a,b]$. Further,

$$\begin{aligned}\left|{}^{k}\mathfrak{I}^{\mu;\,\Psi}_{a+}\,\mathfrak{h}(t)\right| &= \left|\frac{1}{k\,\Gamma_k\left(\mu\right)}\int_a^t \Psi'\left(s\right)\left(\Psi\left(t\right)-\Psi\left(s\right)\right)^{\frac{\mu}{k}-1}\mathfrak{h}\left(s\right)ds\right|\\ &\leq \frac{1}{k\,\Gamma_k\left(\mu\right)}\int_a^t \Psi'\left(s\right)\left(\Psi\left(t_2\right)-\Psi\left(s\right)\right)^{\frac{\mu}{k}-1}\left(\Psi\left(s\right)-\Psi\left(a\right)\right)^{\frac{-\sigma}{k}}\left|\left(\Psi\left(s\right)-\Psi\left(a\right)\right)^{\frac{\sigma}{k}}\mathfrak{h}\left(s\right)\right|ds\\ &\leq \|\mathfrak{h}\|_{C_{\frac{\sigma}{k};\,\Psi}[a,b]}\ {}^{k}\mathfrak{I}^{\mu;\,\Psi}_{a+}\left(\Psi\left(t\right)-\Psi\left(a\right)\right)^{\frac{-\sigma}{k}}\\ &\leq \|\mathfrak{h}\|_{C_{\frac{\sigma}{k};\,\Psi}[a,b]}\ \frac{\Gamma_k(k-\sigma)}{\Gamma_k(k-\sigma+\mu)}\left(\Psi\left(t\right)-\Psi\left(a\right)\right)^{\frac{\mu-\sigma}{k}}.\end{aligned} \tag{4.2}$$

Since $\sigma<\mu$, by continuity of Ψ, from inequality (4.2), we obtain

$$\lim_{t\to a+}\left|{}^{k}\mathfrak{I}^{\mu;\,\Psi}_{a+}\,\mathfrak{h}(t)\right|=0.$$

This gives the desired equation (4.1). □

Theorem 4.9 *Let μ,$k\in\mathbb{R}_+=(0,\infty)$. Then, ${}^{k}\mathfrak{I}^{\mu;\,\Psi}_{a+}\mathfrak{h}(t)=\,k^{\frac{-\mu}{k}}\,\mathfrak{I}^{\frac{\mu}{k};\,\Psi}_{a+}\mathfrak{h}(t),\ \mathfrak{h}\in C[a,b]$.*

Proof: Using the Theorem 2.1 (iii), we obtain

$$\begin{aligned}
{}_k\mathcal{J}_{a+}^{\mu;\Psi}\mathfrak{h}(t) &= \frac{1}{k\,\Gamma_k(\mu)}\int_a^t \Psi'(s)\,(\Psi(t)-\Psi(s))^{\frac{\mu}{k}-1}\,\mathfrak{h}(s)\,ds \\
&= \frac{1}{k\,k^{\frac{\mu}{k}-1}\Gamma\left(\frac{\mu}{k}\right)}\int_a^t \Psi'(s)\,(\Psi(t)-\Psi(s))^{\frac{\mu}{k}-1}\,\mathfrak{h}(s)\,ds \\
&= \frac{1}{k^{\frac{\mu}{k}}\Gamma\left(\frac{\mu}{k}\right)}\int_a^t \Psi'(s)\,(\Psi(t)-\Psi(s))^{\frac{\mu}{k}-1}\,\mathfrak{h}(s)\,ds \\
&= k^{\frac{-\mu}{k}}\,\mathcal{J}_{a+}^{\frac{\mu}{k};\Psi}\mathfrak{h}(t).
\end{aligned}$$

□

Lemma 4.10 *Let $m \in \mathbb{N}$, $k > 0\,(k \in \mathbb{R})$ and $\mathfrak{h} \in C[a,b]$. Then,*

(a) $\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^m \mathcal{J}_{a+}^{m;\Psi}\mathfrak{h}(t) = \mathfrak{h}(t)$

(b) $\left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right)^m {}_k\mathcal{J}_{a+}^{mk;\Psi}\mathfrak{h}(t) = \mathfrak{h}(t).$

Proof: (a) Using the Lemma 2.4 [1], for $m-1 < \mu \leq m \in \mathbb{N}$ and $\mathfrak{h} \in C[a,b]$, we have

$$\mathfrak{h}(t) = {}^{RL}\mathcal{D}_{a+}^{\mu;\Psi}\,\mathcal{J}_{a+}^{\mu;\Psi}\mathfrak{h}(t).$$

In particular for $\mu = m$, we have

$$\begin{aligned}
\mathfrak{h}(t) &= {}^{RL}\mathcal{D}_{a+}^{m;\Psi}\,\mathcal{J}_{a+}^{m;\Psi}\mathfrak{h}(t) \\
&= \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^m \mathcal{J}_{a+}^{m-m;\Psi}\,\mathcal{J}_{a+}^{m;\Psi}\mathfrak{h}(t) \\
&= \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^m \mathcal{J}_{a+}^{m;\Psi}\mathfrak{h}(t).
\end{aligned}$$

(b) Using the definition for (k,Ψ)-RL fractional integral operator, Theorem 2.1(iii) and (a), we obtain

$$\begin{aligned}
\left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right)^m {}_k\mathcal{J}_{a+}^{mk;\Psi}\mathfrak{h}(t) &= k^m\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^m \frac{1}{k\,\Gamma_k(mk)}\int_a^t \Psi'(s)\,(\Psi(t)-\Psi(s))^{\frac{mk}{k}-1}\,\mathfrak{h}(s)\,ds \\
&= k^m\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^m \frac{1}{k\,k^{m-1}\Gamma(m)}\int_a^t \Psi'(s)\,(\Psi(t)-\Psi(s))^{m-1}\,\mathfrak{h}(s)\,ds \\
&= \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^m \mathcal{J}_{a+}^{m;\Psi}\mathfrak{h}(t) \\
&= \mathfrak{h}(t).
\end{aligned}$$

□

5 Properties of (k,Ψ)-Riemann-Liouville fractional derivative

In this section, we prove few properties of (k,Ψ)-RL fractional derivative that are needed to investigate the properties of (k,Ψ)-Hilfer fractional derivative.

Theorem 5.1 *Let $\mu, k \in \mathbb{R}_+ = (0,\infty)$. Then,* ${}^{k,RL}\mathcal{D}^{\mu;\,\Psi}_{a+}\mathfrak{h}(t) = k^{\frac{\mu}{k}}\ {}^{RL}\mathcal{D}^{\frac{\mu}{k};\,\Psi}_{a+}\mathfrak{h}(t),\ \mathfrak{h} \in C[a,b]$.

Proof: Using the Theorem 4.9, we obtain

$$\begin{aligned}
{}^{k,RL}\mathcal{D}^{\mu\,;\Psi}_{a+}\mathfrak{h}\,(t) &= \left(\frac{k}{\Psi'\,(t)}\frac{d}{dt}\right)^{m}\ {}_{k}\mathcal{J}^{mk-\mu\,;\Psi}_{a+}\mathfrak{h}\,(t)\\
&= k^{m}\left(\frac{1}{\Psi'\,(t)}\frac{d}{dt}\right)^{m}\ k^{-\left(\frac{mk-\mu}{k}\right)}\,\mathcal{J}^{\frac{mk-\mu}{k}\,;\,\Psi}_{a+}\mathfrak{h}(t)\\
&= k^{\frac{\mu}{k}}\left(\frac{1}{\Psi'\,(t)}\frac{d}{dt}\right)^{m}\ \mathcal{J}^{m-\frac{\mu}{k}\,;\,\Psi}_{a+}\mathfrak{h}(t)\\
&= k^{\frac{\mu}{k}}\ {}^{RL}\mathcal{D}^{\frac{\mu}{k}\,;\Psi}_{a+}\mathfrak{h}(t).
\end{aligned}$$

□

Theorem 5.2 *Let $\mu, k \in \mathbb{R}_+ = (0,\infty)$ and let $\xi \in \mathbb{R}$ such that $\frac{\xi}{k} > -1$. Then,*

$$ {}^{k,RL}\mathcal{D}^{\mu;\,\Psi}_{a+}\left(\Psi(t)-\Psi(a)\right)^{\frac{\xi}{k}} = \frac{\Gamma_k(\xi+k)}{\Gamma_k(\xi+k-\mu)}\left(\Psi(t)-\Psi(a)\right)^{\frac{\xi-\mu}{k}}. $$

Proof: Proof follows by using the definition of (k,Ψ)-RL fractional derivative and the Theorem 4.3. □

Theorem 5.3 *Let $\eta, k \in \mathbb{R}_+ = (0,\infty)$ with $\eta < k$, $\nu \in [0,1]$ and $\zeta_k = \eta + \nu\,(k-\eta)$. Then, for $\mathfrak{h} \in C^{\zeta_k}_{1-\frac{\zeta_k}{k};\Psi}[a,b]$,*

$$ {}^{k,RL}\mathcal{D}^{\zeta_k;\,\Psi}_{a+}\ {}_{k}\mathcal{J}^{\eta;\,\Psi}_{a+}\mathfrak{h}(t) = {}^{k,RL}\mathcal{D}^{\nu(k-\eta);\,\Psi}_{a+}\mathfrak{h}(t). $$

Proof: Since $\eta < k$, we have $m = \lceil \frac{\eta}{k} \rceil = 1$, hence $\left\lceil \frac{\zeta_k}{k} \right\rceil = 1$. Thus, using the definition of (k,Ψ)-RL fractional derivative and the semigroup property of (k,Ψ)-RL fractional integral, we have

$$\begin{aligned}
{}^{k,RL}\mathcal{D}^{\zeta_k;\,\Psi}_{a+}\ {}_{k}\mathcal{J}^{\eta;\,\Psi}_{a+}\mathfrak{h}(t) &= \left(\frac{k}{\Psi'\,(t)}\frac{d}{dt}\right)\ {}_{k}\mathcal{J}^{k-\zeta_k\,;\Psi}_{a+}\ {}_{k}\mathcal{J}^{\eta\,;\Psi}_{a+}\mathfrak{h}(t)\\
&= \left(\frac{k}{\Psi'\,(t)}\frac{d}{dt}\right)\ {}_{k}\mathcal{J}^{k-\zeta_k+\eta\,;\Psi}_{a+}\mathfrak{h}(t)\\
&= \left(\frac{k}{\Psi'\,(t)}\frac{d}{dt}\right)\ {}_{k}\mathcal{J}^{k-\nu(k-\eta)\,;\Psi}_{a+}\mathfrak{h}(t)\\
&= {}^{k,RL}\mathcal{D}^{\nu(k-\eta);\,\Psi}_{a+}\mathfrak{h}(t).
\end{aligned}$$

□

Theorem 5.4 *Let $\mu, k \in \mathbb{R}_+ = (0,\infty)$, $m = \lceil \frac{\mu}{k} \rceil$. Then, for $h \in C_{m-\frac{\varsigma_k}{k};\Psi}[a,b]$, we have*

$$^{k,RL}\mathfrak{D}_{a+}^{\mu;\,\Psi}\, {}^{k}\mathfrak{I}_{a+}^{\mu;\,\Psi}\mathfrak{h}(t) = \mathfrak{h}(t).$$

Proof: Using the Theorem 4.1 and Theorem 4.10 (b), we have

$$\begin{aligned} ^{k,RL}\mathfrak{D}_{a+}^{\mu;\,\Psi}\, {}^{k}\mathfrak{I}_{a+}^{\mu;\,\Psi}\mathfrak{h}(t) &= \left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right)^{m} {}^{k}\mathfrak{I}_{a+}^{mk-\mu;\,\Psi}\, {}^{k}\mathfrak{I}_{a+}^{\mu;\,\Psi}\mathfrak{h}(t) \\ &= \left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right)^{m} {}^{k}\mathfrak{I}_{a+}^{mk;\,\Psi}\,\mathfrak{h}(t) \\ &= \mathfrak{h}(t). \end{aligned}$$

□

Theorem 5.5 *Let $\mu, k, \sigma \in \mathbb{R}_+ = (0,\infty)$ with $\sigma < k$ and $m = \lceil \frac{\mu}{k} \rceil$. Assume that $\mathfrak{h} \in C_{\frac{\sigma}{k};\Psi}[a,b]$ and ${}^{k}\mathfrak{I}_{a+}^{mk-\mu;\,\Psi}\mathfrak{h} \in C^{m}_{\frac{\sigma}{k};\Psi}[a,b]$. Then,*

$${}^{k}\mathfrak{I}_{a+}^{\mu;\,\Psi}\left({}^{k,RL}\mathfrak{D}_{a+}^{\mu;\,\Psi}\mathfrak{h}(t)\right) = \mathfrak{h}(t) - \sum_{j=1}^{m}\frac{(\Psi(t)-\Psi(a))^{\frac{\mu}{k}-j}}{\Gamma_k(\mu - jk + k)}\left[\left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right)^{m-j} {}^{k}\mathfrak{I}_{a+}^{mk-\mu;\,\Psi}\,\mathfrak{h}(t)\right]_{t=a}.$$

Proof: Using the Theorem 4.10 (b) for $m = 1$, we obtain

$$\left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right) {}^{k}\mathfrak{I}_{a+}^{k;\,\Psi}\mathfrak{h}(t) = \mathfrak{h}(t). \tag{5.1}$$

Using the equation (5.1) with $\mathfrak{h}(t)$ replaced by ${}^{k}\mathfrak{I}_{a+}^{\mu;\,\Psi}\, {}^{k,RL}\mathfrak{D}_{a+}^{\mu;\,\Psi}\mathfrak{h}(t)$, we obtain

$${}^{k}\mathfrak{I}_{a+}^{\mu;\,\Psi}\left({}^{k,RL}\mathfrak{D}_{a+}^{\mu;\,\Psi}\mathfrak{h}(t)\right) = \left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right) {}^{k}\mathfrak{I}_{a+}^{k;\,\Psi}\left[{}^{k}\mathfrak{I}_{a+}^{\mu;\,\Psi}\left({}^{k,RL}\mathfrak{D}_{a+}^{\mu;\,\Psi}\mathfrak{h}(t)\right)\right]. \tag{5.2}$$

Using the relation (5.2), the semigroup property of (k,Ψ)-RL fractional integrals and the Theorem 2.1 (iii), we obtain

$$\begin{aligned} {}^{k}\mathfrak{I}_{a+}^{\mu;\,\Psi}\left({}^{k,RL}\mathfrak{D}_{a+}^{\mu;\,\Psi}\mathfrak{h}(t)\right) &= \left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right)\left[{}^{k}\mathfrak{I}_{a+}^{k+\mu;\,\Psi}\left({}^{k,RL}\mathfrak{D}_{a+}^{\mu;\,\Psi}\mathfrak{h}(t)\right)\right] \\ &= \frac{k}{\Psi'(t)}\frac{d}{dt}\left[\frac{1}{k\,\Gamma_k(k+\mu)}\int_a^t \Psi'(s)\left(\Psi(t)-\Psi(s)\right)^{\frac{k+\mu}{k}-1}\, {}^{k,RL}\mathfrak{D}_{a+}^{\mu;\,\Psi}\mathfrak{h}(s)\,ds\right] \\ &= \frac{1}{\Psi'(t)}\frac{d}{dt}\left[\frac{1}{k^{\frac{\mu}{k}}\,\Gamma\left(\frac{\mu}{k}+1\right)}\int_a^t \Psi'(s)\left(\Psi(t)-\Psi(s)\right)^{\frac{\mu}{k}}\, {}^{k,RL}\mathfrak{D}_{a+}^{\mu;\,\Psi}\mathfrak{h}(s)\,ds\right]. \end{aligned} \tag{5.3}$$

Now, consider using the definition of (k,Ψ)-RL fractional derivative and integration by parts, we obtain

$$\frac{1}{k^{\frac{\mu}{k}}\,\Gamma\left(\frac{\mu}{k}+1\right)}\int_a^t \Psi'(s)\left(\Psi(t)-\Psi(s)\right)^{\frac{\mu}{k}}\, {}^{k,RL}\mathfrak{D}_{a+}^{\mu;\,\Psi}\mathfrak{h}(s)\,ds$$

$$
\begin{aligned}
&= \frac{1}{k^{\frac{\mu}{k}}\,\Gamma\left(\frac{\mu}{k}+1\right)} \int_a^t \Psi'(s)\left(\Psi(t)-\Psi(s)\right)^{\frac{\mu}{k}} \left(\frac{k}{\Psi'(s)}\frac{d}{ds}\right)^{m} {}_k\mathfrak{I}_{a+}^{mk-\mu;\,\Psi}\mathfrak{h}(s)\,ds \\
&= \frac{k}{k^{\frac{\mu}{k}}\,\Gamma\left(\frac{\mu}{k}+1\right)} \int_a^t \left(\Psi(t)-\Psi(s)\right)^{\frac{\mu}{k}} \frac{d}{ds}\left[\left(\frac{k}{\Psi'(s)}\frac{d}{ds}\right)^{m-1} {}_k\mathfrak{I}_{a+}^{mk-\mu;\,\Psi}\mathfrak{h}(s)\right] ds \\
&= \frac{-k^{1-\frac{\mu}{k}}}{\Gamma\left(\frac{\mu}{k}+1\right)} \left(\Psi(t)-\Psi(a)\right)^{\frac{\mu}{k}} \left[\left(\frac{k}{\Psi'(s)}\frac{d}{ds}\right)^{m-1} {}_k\mathfrak{I}_{a+}^{mk-\mu;\,\Psi}\mathfrak{h}(s)\right]_{s=a} \\
&\quad + \frac{k^{2-\frac{\mu}{k}}}{\Gamma\left(\frac{\mu}{k}\right)} \int_a^t \left(\Psi(t)-\Psi(s)\right)^{\frac{\mu}{k}-1} \frac{d}{ds}\left[\left(\frac{k}{\Psi'(s)}\frac{d}{ds}\right)^{m-2} {}_k\mathfrak{I}_{a+}^{mk-\mu;\,\Psi}\mathfrak{h}(s)\right] ds.
\end{aligned}
$$

Repeating the process of integration by parts at n^{th} step, we obtain

$$
\begin{aligned}
&\frac{1}{k^{\frac{\mu}{k}}\,\Gamma\left(\frac{\mu}{k}+1\right)} \int_a^t \Psi'(s)\left(\Psi(t)-\Psi(s)\right)^{\frac{\mu}{k}} {}^{k,RL}\mathcal{D}_{a+}^{\mu;\,\Psi}\mathfrak{h}(s)\,ds \\
&= \sum_{j=1}^{m} \frac{-k^{j-\frac{\mu}{k}}}{\Gamma\left(\frac{\mu}{k}-j+2\right)} \left(\Psi(t)-\Psi(a)\right)^{\frac{\mu}{k}-j+1} \left[\left(\frac{k}{\Psi'(s)}\frac{d}{ds}\right)^{m-j} {}_k\mathfrak{I}_{a+}^{mk-\mu;\,\Psi}\mathfrak{h}(s)\right]_{s=a} \\
&\quad + \frac{k^{m-\frac{\mu}{k}}}{\Gamma\left(\frac{\mu}{k}-(m-1)\right)} \int_a^t \Psi'(s)\left(\Psi(t)-\Psi(s)\right)^{\frac{\mu}{k}-m} \left(\frac{k}{\Psi'(s)}\frac{d}{ds}\right)^{m-m} {}_k\mathfrak{I}_{a+}^{mk-\mu;\,\Psi}\mathfrak{h}(s)\,ds.
\end{aligned}
$$

Using the definition of (k,Ψ)-RL fractional integral, its semigroup property and the Theorem 2.1, we obtain

$$
\begin{aligned}
&\frac{1}{k^{\frac{\mu}{k}}\,\Gamma\left(\frac{\mu}{k}+1\right)} \int_a^t \Psi'(s)\left(\Psi(t)-\Psi(s)\right)^{\frac{\mu}{k}} {}^{k,RL}\mathcal{D}_{a+}^{\mu;\,\Psi}\mathfrak{h}(s)\,ds \\
&= \sum_{j=1}^{m} \frac{-k^{j-\frac{\mu}{k}}}{\Gamma\left(\frac{\mu}{k}-j+2\right)} \left(\Psi(t)-\Psi(a)\right)^{\frac{\mu}{k}-j+1} \left[\left(\frac{k}{\Psi'(s)}\frac{d}{ds}\right)^{m-j} {}_k\mathfrak{I}_{a+}^{mk-\mu;\,\Psi}\mathfrak{h}(s)\right]_{s=a} \\
&\quad + \frac{k}{k\,\Gamma_k(\mu+k-mk)} \int_a^t \Psi'(s)\left(\Psi(t)-\Psi(s)\right)^{\frac{\mu+k-mk}{k}-1} {}_k\mathfrak{I}_{a+}^{mk-\mu;\,\Psi}\mathfrak{h}(s)\,ds \\
&= \sum_{j=1}^{m} \frac{-k^{j-\frac{\mu}{k}}}{\Gamma\left(\frac{\mu}{k}-j+2\right)} \left(\Psi(t)-\Psi(a)\right)^{\frac{\mu}{k}-j+1} \left[\left(\frac{k}{\Psi'(s)}\frac{d}{ds}\right)^{m-j} {}_k\mathfrak{I}_{a+}^{mk-\mu;\,\Psi}\mathfrak{h}(s)\right]_{s=a} \\
&\quad + k\; {}_k\mathfrak{I}_{a+}^{\mu+k-mk;\,\Psi}\, {}_k\mathfrak{I}_{a+}^{mk-\mu;\,\Psi}\mathfrak{h}(t) \\
&= \sum_{j=1}^{m} \frac{-k^{j-\frac{\mu}{k}}}{\Gamma\left(\frac{\mu}{k}-j+2\right)} \left(\Psi(t)-\Psi(a)\right)^{\frac{\mu}{k}-j+1} \left[\left(\frac{k}{\Psi'(s)}\frac{d}{ds}\right)^{m-j} {}_k\mathfrak{I}_{a+}^{mk-\mu;\,\Psi}\mathfrak{h}(s)\right]_{s=a} \\
&\quad + k\; {}_k\mathfrak{I}_{a+}^{k;\,\Psi}\mathfrak{h}(t) \\
&= \sum_{j=1}^{m} \frac{-k^{j-\frac{\mu}{k}}}{\Gamma\left(\frac{\mu}{k}-j+2\right)} \left(\Psi(t)-\Psi(a)\right)^{\frac{\mu}{k}-j+1} \left[\left(\frac{k}{\Psi'(s)}\frac{d}{ds}\right)^{m-j} {}_k\mathfrak{I}_{a+}^{mk-\mu;\,\Psi}\mathfrak{h}(s)\right]_{s=a} \\
&\quad + \int_a^t \Psi'(s)\mathfrak{h}(s)ds. \qquad (5.4)
\end{aligned}
$$

Using the equation (5.4) in the equation (5.3) and the Theorem 2.1 (iii), we obtain

${}_k\mathfrak{I}_{a+}^{\mu;\,\Psi}\, {}^{k,RL}\mathcal{D}_{a+}^{\mu;\,\Psi}\mathfrak{h}(t)$

$$= \frac{1}{\Psi'(t)} \frac{d}{dt} \left\{ \int_a^t \Psi'(s)\mathfrak{h}(s) ds \right.$$

$$\left. - \sum_{j=1}^{m} \frac{k^{j-\frac{\mu}{k}}}{\Gamma\left(\frac{\mu}{k} - j + 2\right)} \left(\Psi(t) - \Psi(a)\right)^{\frac{\mu}{k}-j+1} \left[\left(\frac{k}{\Psi'(s)} \frac{d}{ds} \right)^{m-j} {}_k\mathfrak{I}_{a+}^{mk-\mu;\,\Psi} \mathfrak{h}(s) \right]_{s=a} \right\}$$

$$= h(t) - \sum_{j=1}^{m} \frac{k^{j-\frac{\mu}{k}}}{\Gamma\left(\frac{\mu}{k} - j + 1\right)} \left(\Psi(t) - \Psi(a)\right)^{\frac{\mu}{k}-j} \left[\left(\frac{k}{\Psi'(s)} \frac{d}{ds} \right)^{m-j} {}_k\mathfrak{I}_{a+}^{mk-\mu;\,\Psi} \mathfrak{h}(s) \right]_{s=a}$$

$$= \mathfrak{h}(t) - \sum_{j=1}^{m} \frac{k^{j-\frac{\mu}{k}}}{k^{1-\left(\frac{\mu}{k}-j+1\right)} \Gamma_k(\mu - jk + k)} \left(\Psi(t) - \Psi(a)\right)^{\frac{\mu}{k}-j} \left[\left(\frac{k}{\Psi'(s)} \frac{d}{ds} \right)^{m-j} {}_k\mathfrak{I}_{a+}^{mk-\mu;\,\Psi} \mathfrak{h}(s) \right]_{s=a}$$

$$= \mathfrak{h}(t) - \sum_{j=1}^{m} \frac{\left(\Psi(t) - \Psi(a)\right)^{\frac{\mu}{k}-j}}{\Gamma_k(\mu - jk + k)} \left[\left(\frac{k}{\Psi'(t)} \frac{d}{dt} \right)^{m-j} {}_k\mathfrak{I}_{a+}^{mk-\mu;\,\Psi} \mathfrak{h}(t) \right]_{t=a}.$$

□

Remark 5.6

1. *For $\Psi(t) = t$, the above result reduces to*

$$ {}_k\mathfrak{I}_{a+}^{\mu}\, {}^{k,RL}\mathfrak{D}_{a+}^{\mu}\mathfrak{h}(t) = \mathfrak{h}(t) - \sum_{j=1}^{m} \frac{(t-a)^{\frac{\mu}{k}-j}}{\Gamma_k(\mu - jk + k)} \left[\left(k\frac{d}{dt} \right)^{m-j} {}_k\mathfrak{I}_{a+}^{mk-\mu} \mathfrak{h}(t) \right]_{t=a},$$

   which improves the result of [18].

2. *For $\Psi(t) = t$ and $k = 1$, the result obtained in the above theorem reduces to the result obtained in [1, 31],*

$$\mathfrak{I}_{a+}^{\mu}\, {}^{RL}\mathfrak{D}_{a+}^{\mu}\mathfrak{h}(t) = \mathfrak{h}(t) - \sum_{j=1}^{m} \frac{(t-a)^{\mu-j}}{\Gamma(\mu - j + 1)} \left[\left(\frac{d}{dt} \right)^{m-j} \mathfrak{I}_{a+}^{m-\mu} \mathfrak{h}(t) \right]_{t=a}.$$

6 Calculus of (k, Ψ)-Hilfer fractional derivative

Theorem 6.1 *Let $\eta, k \in \mathbb{R}_+ = (0, \infty)$ and $\nu \in [0,1]$. Then, ${}^{k,H}\mathfrak{D}_{a+}^{\eta,\nu;\,\Psi}\mathfrak{h}(t) = k^{\frac{\eta}{k}}\, {}^{H}\mathfrak{D}_{a+}^{\frac{\eta}{k},\nu;\,\Psi}\mathfrak{h}(t)$, $\mathfrak{h} \in C^m[a,b]$.*

Proof: Using the Theorem 4.9, we have

$$\begin{aligned}
{}^{k,H}\mathfrak{D}_{a+}^{\eta,\nu;\,\Psi}\mathfrak{h}(t) &= {}_k\mathfrak{I}_{a+}^{\nu(mk-\eta);\,\Psi} \left(\frac{k}{\Psi'(t)} \frac{d}{dt} \right)^m {}_k\mathfrak{I}_{a+}^{(1-\nu)(mk-\eta);\,\Psi}\mathfrak{h}(t) \\
&= k^{-\left(\frac{\nu(mk-\eta)}{k}\right)} \mathfrak{I}_{a+}^{\frac{\nu(mk-\eta)}{k};\,\Psi}\, k^m \left(\frac{1}{\Psi'(t)} \frac{d}{dt} \right)^m k^{-\left(\frac{(1-\nu)(mk-\eta)}{k}\right)} \mathfrak{I}_{a+}^{\frac{(1-\nu)(mk-\eta)}{k};\,\Psi}\mathfrak{h}(t) \\
&= k^{\frac{\eta}{k}}\, \mathfrak{I}_{a+}^{\nu\left(m-\frac{\eta}{k}\right);\,\Psi} \left(\frac{1}{\Psi'(t)} \frac{d}{dt} \right)^m \mathfrak{I}_{a+}^{(1-\nu)\left(m-\frac{\eta}{k}\right);\,\Psi}\mathfrak{h}(t)
\end{aligned}$$

$$= k^{\frac{\eta}{k}}\, {}^{H}\mathcal{D}_{a+}^{\frac{\eta}{k},\nu;\Psi}\mathfrak{h}(t).$$

□

Theorem 6.2 *Let $\eta, k \in \mathbb{R}_+ = (0,\infty)$ and let $\xi \in \mathbb{R}$ such that $\frac{\xi}{k} > -1$. Then,*

$$ {}^{k,H}\mathcal{D}_{a+}^{\eta,\nu;\Psi}\left(\Psi(t)-\Psi(a)\right)^{\frac{\xi}{k}} = \frac{\Gamma_k(\xi+k)}{\Gamma_k(\xi+k-\eta)}\left(\Psi(t)-\Psi(a)\right)^{\frac{\xi-\eta}{k}}.$$

Proof: Proof follows by using the definition of (k,Ψ)-Hilfer fractional derivative and an application of Theorem 4.3 and Theorem 5.2. □

Theorem 6.3 *Let $\eta, k \in \mathbb{R}_+ = (0,\infty)$ with $\eta < k$, $\nu \in [0,1]$ and $\zeta_k = \eta + \nu\,(k-\eta)$. Then,*

$$ {}^{k}\mathcal{J}_{a+}^{\zeta_k;\Psi}\, {}^{k,RL}\mathcal{D}_{a+}^{\zeta_k;\Psi}\mathfrak{h}(t) = {}^{k}\mathcal{J}_{a+}^{\eta;\Psi}\, {}^{k,H}\mathcal{D}_{a+}^{\eta,\nu;\Psi}\mathfrak{h}(t), \quad \mathfrak{h} \in C_{1-\frac{\zeta_k}{k};\Psi}^{\zeta_k}[a,b].$$

Proof: Using the semigroup property of (k,Ψ)-RL fractional integral and the definition of (k,Ψ)-Hilfer fractional derivative, we have

$$\begin{aligned} {}^{k}\mathcal{J}_{a+}^{\zeta_k;\Psi}\, {}^{k,RL}\mathcal{D}_{a+}^{\zeta_k;\Psi}\mathfrak{h}(t) &= {}^{k}\mathcal{J}_{a+}^{\eta+\nu(k-\eta);\Psi}\, {}^{k,RL}\mathcal{D}_{a+}^{\zeta_k;\Psi}\mathfrak{h}(t)\\ &= {}^{k}\mathcal{J}_{a+}^{\eta;\Psi}\, {}^{k}\mathcal{J}_{a+}^{\nu(k-\eta);\Psi}\, {}^{k,RL}\mathcal{D}_{a+}^{\zeta_k;\Psi}\mathfrak{h}(t)\\ &= {}^{k}\mathcal{J}_{a+}^{\eta;\Psi}\, {}^{k}\mathcal{J}_{a+}^{\zeta_k-\eta;\Psi}\, {}^{k,RL}\mathcal{D}_{a+}^{\zeta_k;\Psi}\mathfrak{h}(t)\\ &= {}^{k}\mathcal{J}_{a+}^{\eta;\Psi}\, {}^{k,H}\mathcal{D}_{a+}^{\eta,\nu;\Psi}\mathfrak{h}(t). \end{aligned}$$

□

Theorem 6.4 *Let $\mathfrak{h} \in L^1[a,b]$. Assume that ${}^{k,RL}\mathcal{D}_{a+}^{\nu(k-\eta);\Psi}\mathfrak{h}$ exists and it lies in $L^1[a,b]$. Then,*

$$ {}^{k,H}\mathcal{D}_{a+}^{\eta,\nu;\Psi}\, {}^{k}\mathcal{J}_{a+}^{\eta;\Psi}\mathfrak{h}(t) = {}^{k}\mathcal{J}_{a+}^{\nu(k-\eta);\Psi}\, {}^{k,RL}\mathcal{D}_{a+}^{\nu(k-\eta);\Psi}\mathfrak{h}(t).$$

Proof: Using the definition of (k,Ψ)-Hilfer fractional derivative and an application of Theorem 5.3, we obtain

$$\begin{aligned} {}^{k,H}\mathcal{D}_{a+}^{\eta,\nu;\Psi}\, {}^{k}\mathcal{J}_{a+}^{\eta;\Psi}\mathfrak{h}(t) &= {}^{k}\mathcal{J}_{a+}^{\nu(k-\eta);\Psi}\, {}^{k,RL}\mathcal{D}_{a+}^{\zeta_k;\Psi}\, {}^{k}\mathcal{J}_{a+}^{\eta;\Psi}\mathfrak{h}(t)\\ &= {}^{k}\mathcal{J}_{a+}^{\nu(k-\eta);\Psi}\, {}^{k,RL}\mathcal{D}_{a+}^{\nu(k-\eta);\Psi}\mathfrak{h}(t). \end{aligned}$$

□

7 (k,Ψ)-Hilfer FDEs

In this section, we investigate the existence and uniqueness of solution for the (k,Ψ)-Hilfer FDEs (1.1)-(1.2).

First we define the weighted space as follows. Let $[a,b]$ $(0<a<b<\infty)$ be a finite interval and $\Psi\in C^1([a,b],\mathbb{R})$ be an increasing function such that $\Psi'(t)\neq 0$, for all $t\in[a,b]$. Let $0<\eta<k\,(\eta,k\in\mathbb{R})$, $\nu\in[0,1]$ and $\zeta_k=\eta+\nu\,(k-\eta)$. On the line of [4], we define the following weighted space.

$$C_{1-\frac{\zeta_k}{k};\,\Psi}[a,b]=\left\{\mathfrak{h}\big|\mathfrak{h}:(a,b]\to\mathbb{R},\mathfrak{h}(a+)\text{ exists and }(\Psi(\cdot)-\Psi(a))^{1-\frac{\zeta_k}{k}}\mathfrak{h}(\cdot)\in C[a,b]\right\},\tag{7.1}$$

$0<\frac{\zeta_k}{k}\leq 1$ endowed with the norm

$$\|\mathfrak{h}\|_{C_{1-\frac{\zeta_k}{k};\Psi}[a,b]}=\max_{t\in[a,b]}\left|(\Psi(t)-\Psi(a))^{1-\frac{\zeta_k}{k}}\mathfrak{h}(t)\right|.$$

Further, we consider the weighted space

$$C^{\zeta_k}_{1-\frac{\zeta_k}{k};\,\Psi}[a,b]=\left\{\mathfrak{h}\in C_{1-\frac{\zeta_k}{k};\,\Psi}[a,b]:\ {}^{k,RL}\mathcal{D}^{\zeta_k;\,\Psi}_{a+}\mathfrak{h}\in C_{1-\frac{\zeta_k}{k};\,\Psi}[a,b]\right\}\tag{7.2}$$

and for $m\in\mathbb{N}$, we consider the weighted space

$$C^{m}_{1-\frac{\zeta_k}{k};\,\Psi}[a,b]=\left\{\mathfrak{h}:\left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right)^{m-1}\mathfrak{h}(t)\in C[a,b]\ \text{ and }\ \left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right)^{m}\mathfrak{h}(t)\in C_{1-\frac{\zeta_k}{k};\,\Psi}[a,b]\right\}.\tag{7.3}$$

7.1 Equivalent fractional integral equation

Theorem 7.1 *Let $\eta,k\in\mathbb{R}$ with $\eta<k,\nu\in[0,1]$ and $\zeta_k=\eta+\nu\,(k-\eta)$. Assume that $f:(a,b]\times\mathbb{R}\to\mathbb{R}$ be a function such that $f(\cdot,y(\cdot))\in C_{1-\frac{\zeta_k}{k};\Psi}[a,b]$ for each $y\in C_{1-\frac{\zeta_k}{k};\Psi}[a,b]$. Then, $y\in C^{\zeta_k}_{1-\frac{\zeta_k}{k};\Psi}[a,b]$ satisfies (k,Ψ)-Hilfer FDEs (1.1)-(1.2) if and only if y satisfies the following fractional integral equation*

$$y(t)=\frac{(\Psi(t)-\Psi(a))^{\frac{\zeta_k}{k}-1}}{\Gamma_k(\zeta_k)}y_a+\frac{1}{k\,\Gamma_k(\eta)}\int_a^t\Psi'(s)\,(\Psi(t)-\Psi(s))^{\frac{\eta}{k}-1}f(s,y(s))\,ds,\ t\in(a,b].\tag{7.4}$$

Proof: Assume that $y\in C^{\zeta_k}_{1-\frac{\zeta_k}{k};\Psi}[a,b]$ is a solution of (k,Ψ)-Hilfer FDEs (1.1)-(1.2). We prove that y satisfy the fractional integral equation (7.4). Since $y\in C^{\zeta_k}_{1-\frac{\zeta_k}{k};\Psi}[a,b]$, we have $y\in C_{1-\frac{\zeta_k}{k};\Psi}[a,b]$ and

$$\left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right){}^{k}\mathfrak{J}^{k-\zeta_k;\,\Psi}_{a+}y={}^{k,RL}\mathcal{D}^{\zeta_k\,;\,\Psi}_{a+}y\in C_{1-\frac{\zeta_k}{k};\Psi}[a,b].\tag{7.5}$$

Further, by applying Theorem 4.5 with $\sigma=\mu=k-\zeta_k$, we obtain

$${}^{k}\mathfrak{J}^{k-\zeta_k;\,\Psi}_{a+}y\in C[a,b].\tag{7.6}$$

Using the equations (7.5) and (7.6), and the definition of weighted space given in equation (7.3), we have ${}^{k}\mathcal{J}_{a+}^{k-\zeta_k;\,\Psi}y \in C^1_{1-\frac{\zeta_k}{k};\Psi}[a,b]$. Since $y \in C_{1-\frac{\zeta_k}{k};\Psi}[a,b]$ and ${}^{k}\mathcal{J}_{a+}^{k-\zeta_k;\,\Psi}y \in C^1_{1-\frac{\zeta_k}{k};\Psi}[a,b]$ by applying the Theorem 5.5 with $\sigma = k-\zeta_k, \mu = \zeta_k$ and $m = \lceil \frac{\zeta_k}{k} \rceil = 1$, we obtain

$$\begin{aligned}{}^{k}\mathcal{J}_{a+}^{\zeta_k;\,\Psi}\ {}^{k,RL}\mathcal{D}_{a+}^{\zeta_k;\,\Psi}y(t) &= y(t) - \frac{(\Psi(t)-\Psi(a))^{\frac{\zeta_k}{k}-1}}{\Gamma_k(\zeta_k)}\left[{}^{k}\mathcal{J}_{a+}^{k-\zeta_k;\,\Psi}y(t)\right]_{t=a} \\ &= y(t) - \frac{(\Psi(t)-\Psi(a))^{\frac{\zeta_k}{k}-1}}{\Gamma_k(\zeta_k)}y_a. \end{aligned} \tag{7.7}$$

Since $y \in C^{\zeta_k}_{1-\frac{\zeta_k}{k};\Psi}[a,b]$, by Theorem 6.3 and the equation (1.1), we obtain

$$\begin{aligned}{}^{k}\mathcal{J}_{a+}^{\zeta_k;\,\Psi}\,{}^{k,RL}\mathcal{D}_{a+}^{\zeta_k;\,\Psi}y(t) &= {}^{k}\mathcal{J}_{a+}^{\eta;\,\Psi}\,{}^{k,H}\mathcal{D}_{a+}^{\eta,\nu;\,\Psi}y(t) \\ &= {}^{k}\mathcal{J}_{a+}^{\eta;\,\Psi}\,f(t,y(t)). \end{aligned} \tag{7.8}$$

From equations (7.7) and (7.8), we have

$$\begin{aligned} y(t) &= \frac{(\Psi(t)-\Psi(a))^{\frac{\zeta_k}{k}-1}}{\Gamma_k(\zeta_k)}y_a + {}^{k}\mathcal{J}_{a+}^{\eta;\,\Psi}\,f(t,y(t)) \\ &= \frac{(\Psi(t)-\Psi(a))^{\frac{\zeta_k}{k}-1}}{\Gamma_k(\zeta_k)}y_a + \frac{1}{k\,\Gamma_k(\eta)}\int_a^t \Psi'(s)\,(\Psi(t)-\Psi(s))^{\frac{\eta}{k}-1}\,f(s,y(s))\,ds,\ t\in(a,b], \end{aligned}$$

which is desired fractional integral equation (7.4).

Conversely, suppose that $y \in C^{\zeta_k}_{1-\frac{\zeta_k}{k};\Psi}[a,b]$ satisfy equation (7.4). Then,

$$y(t) = \frac{(\Psi(t)-\Psi(a))^{\frac{\zeta_k}{k}-1}}{\Gamma_k(\zeta_k)}y_a + {}^{k}\mathcal{J}_{a+}^{\eta;\,\Psi}\,f(t,y(t)),\ t\in(a,b].$$

Operating ${}^{k,RL}\mathcal{D}_{a+}^{\zeta_k;\,\Psi}$ on both sides of above equation, we obtain

$${}^{k,RL}\mathcal{D}_{a+}^{\zeta_k;\,\Psi}y(t) = \frac{y_a}{\Gamma_k(\zeta_k)}\,{}^{k,RL}\mathcal{D}_{a+}^{\zeta_k;\,\Psi}\,(\Psi(t)-\Psi(a))^{\frac{\zeta_k}{k}-1} + {}^{k,RL}\mathcal{D}_{a+}^{\zeta_k;\,\Psi}\,{}^{k}\mathcal{J}_{a+}^{\eta;\,\Psi}\,f(t,y(t)).$$

Using the Theorem 5.1 and Theorem 5.3, the above equation reduces to

$$\begin{aligned}{}^{k,RL}\mathcal{D}_{a+}^{\zeta_k;\,\Psi}y(t) &= \frac{y_a}{\Gamma_k(\zeta_k)}\,k^{\frac{\zeta_k}{k}}\ {}^{RL}\mathcal{D}_{a+}^{\frac{\zeta_k}{k};\,\Psi}\,(\Psi(t)-\Psi(a))^{\frac{\zeta_k}{k}-1} + {}^{k,RL}\mathcal{D}_{a+}^{\nu(k-\eta);\,\Psi}\,f(t,y(t)) \\ &= {}^{k,RL}\mathcal{D}_{a+}^{\nu(k-\eta);\,\Psi}\,f(t,y(t)). \end{aligned} \tag{7.9}$$

Since $y \in C^{\zeta_k}_{1-\frac{\zeta_k}{k};\Psi}[a,b]$, we get ${}^{k,RL}\mathcal{D}_{a+}^{\zeta_k;\,\Psi}y \in C_{1-\frac{\zeta_k}{k};\Psi}[a,b]$. Therefore, from (7.9) it follows that

$${}^{k,RL}\mathcal{D}_{a+}^{\nu(k-\eta);\,\Psi}\,f(\cdot,y(\cdot)) \in C_{1-\frac{\zeta_k}{k};\Psi}[a,b]. \tag{7.10}$$

Since $\eta < k$, $\nu \in [0,1]$ and $0 < 1-\frac{\eta}{k} < 1$, we have $\frac{\nu(k-\eta)}{k} = \nu\left(1-\frac{\eta}{k}\right) < 1$. Therefore,

$$\left\lceil \nu\left(1-\frac{\eta}{k}\right)\right\rceil = 1. \tag{7.11}$$

In this case the definition of (k,Ψ)-RL derivative reduces to

$$ {}^{k,RL}\mathcal{D}_{a+}^{\nu(k-\eta);\,\Psi} f(t,y(t)) = \left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right) {}^{k}\mathcal{J}_{a+}^{k-\nu(k-\eta);\,\Psi} f(t,y(t)). \tag{7.12} $$

Using equations (7.10) and (7.12), we have

$$ \left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right) {}^{k}\mathcal{J}_{a+}^{k-\nu(k-\eta);\,\Psi} f(\cdot,y(\cdot)) \in C_{1-\frac{\zeta_k}{k};\Psi}[a,b]. \tag{7.13} $$

Since $\zeta_k = \eta + \nu\,(k-\eta) > \nu\,(k-\eta)$, we have $k-\zeta_k < k-\nu\,(k-\eta)$. Since $f(\cdot,y(\cdot)) \in C_{1-\frac{\zeta_k}{k};\Psi}[a,b]$, applying Theorem 4.5 with $\sigma = k-\zeta_k$, $\mu = k-\nu\,(k-\eta)$, we obtain

$$ {}^{k}\mathcal{J}_{a+}^{k-\nu(k-\eta);\,\Psi} f(\cdot,y(\cdot)) \in C[a,b]. \tag{7.14} $$

Using the definition of weighted space given in the equation (7.3), from equations (7.13) and (7.14), it follows that

$$ {}^{k}\mathcal{J}_{a+}^{k-\nu(k-\eta);\,\Psi} f(\cdot,y(\cdot)) \in C^{1}_{1-\frac{\zeta_k}{k};\Psi}[a,b]. $$

By applying ${}^{k}\mathcal{J}_{a+}^{\nu(k-\eta);\,\Psi}$ on both sided of equation (7.9) and using Theorem 5.5 with $\sigma = k-\zeta_k, \mu = \nu(k-\eta)$ and $m=1$, we obtain

$$ \begin{aligned} {}^{k}\mathcal{J}_{a+}^{\nu(k-\eta);\,\Psi}\, {}^{k,RL}\mathcal{D}_{a+}^{\zeta_k;\,\Psi} y(t) &= {}^{k}\mathcal{J}_{a+}^{\nu(k-\eta);\,\Psi}\, {}^{k,RL}\mathcal{D}_{a+}^{\nu(k-\eta);\,\Psi} f(t,y(t)) \\ &= f(t,y(t)) - \frac{(\Psi\,(t)-\Psi\,(a))^{\frac{\nu(k-\eta)}{k}-1}}{\Gamma_k(\nu(k-\eta))} \left[{}^{k}\mathcal{J}_{a+}^{k-\nu(k-\eta);\,\Psi} f(t,y(t))\right]_{t=a}. \end{aligned} \tag{7.15} $$

Using the Theorem 4.8 with $\sigma = k-\zeta_k$ and $\mu = k-\nu(k-\eta)$, we obtain

$$ \left[{}^{k}\mathcal{J}_{a+}^{k-\nu(k-\eta);\,\Psi} f(t,y(t))\right]_{t=a} = 0. \tag{7.16} $$

Using the definition of (k,Ψ)-Hilfer fractional derivative and equation (7.16), from equation (7.15), we have

$$ {}^{k,H}\mathcal{D}_{a+}^{\eta,\,\nu;\,\Psi} y(t) = f(t,y(t)),\ t \in (a,b]. $$

Hence, equation (1.1) is verified. Now, it remains to verify initial condition (1.2). Taking ${}^{k}\mathcal{J}_{a+}^{k-\zeta_k;\,\Psi}$ on both sides of equation (7.4), using semigroup property for (k,Ψ)-RL fractional integral and Theorem 4.3, we have

$$ \begin{aligned} {}^{k}\mathcal{J}_{a+}^{k-\zeta_k;\,\Psi} y(t) &= \frac{y_a}{\Gamma_k(\zeta_k)}\, {}^{k}\mathcal{J}_{a+}^{k-\zeta_k;\,\Psi} (\Psi\,(t)-\Psi\,(a))^{\frac{\zeta_k}{k}-1} + {}^{k}\mathcal{J}_{a+}^{k-\zeta_k+\eta;\,\Psi} f(t,y(t)) \\ &= \frac{y_a}{\Gamma_k(k)} + {}^{k}\mathcal{J}_{a+}^{k-\nu(k-\eta);\,\Psi} f(t,y(t)). \end{aligned} $$

Using the fact $\Gamma_k(k) = 1$, from above equation, we obtain

$$ \left[{}^{k}\mathcal{J}_{a+}^{k-\zeta_k;\,\Psi} y(t)\right]_{t=a} = y_a + \left[{}^{k}\mathcal{J}_{a+}^{k-\nu(k-\eta);\,\Psi} f(t,y(t))\right]_{t=a}. $$

Using the equation (7.16) in above equation, we obtain

$$\left[{}^{k}\mathfrak{J}_{a+}^{k-\zeta_k;\,\Psi} y(t)\right]_{t=a} = y_a.$$

This proves the initial condition (1.2) is verified. □

Remark 7.2

1. *For $k = 1$, the above theorem include the result of [32].*
2. *For $\Psi(t) = t$ and $k = 1$, the Theorem 7.1 includes the result of [3].*

7.2 Existence and uniqueness of solution

Theorem 7.3 *Let $f : (a,b] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(\cdot, y(\cdot)) \in C^{\nu(k-\eta)}_{1-\frac{\zeta_k}{k};\Psi}[a,b]$, for any $y \in C_{1-\frac{\zeta_k}{k};\Psi}[a,b]$. Further, f satisfies Lipschitz condition in second argument as*

$$|f(t,x) - f(t,y)| \leq L\,|x-y|\,, \ \textit{for all}\ t \in (a,b], \tag{7.17}$$

where $L > 0$ and $x, y \in \mathbb{R}$. Then, there exists a unique solution $y \in C^{\zeta_k}_{1-\frac{\zeta_k}{k};\Psi}[a,b]$ for the (k,Ψ)-Hilfer FDEs (1.1)-(1.2).

Proof: We prove the existence and uniqueness of solution for the (k,Ψ)-Hilfer FDEs (1.1)-(1.2) through its equivalent fractional integral equation (7.4). Consider the operator $\mathcal{A}$ defined by

$$\mathcal{A}y(t) = y_0(t) + {}^{k}\mathfrak{J}_{a+}^{\eta;\,\Psi} f(t, y(t)),\ t \in (a,b],$$

where

$$y_0(t) = \frac{y_a}{\Gamma_k(\zeta_k)} \left(\Psi\left(t\right) - \Psi\left(a\right)\right)^{\frac{\zeta_k}{k}-1}.$$

Then, the fractional integral equation (7.4) can be written as an operator equation

$$y(t) = \mathcal{A}y(t),\ t \in (a,b].$$

Firstly, it is proved that the (k,Ψ)-Hilfer FDEs (1.1)-(1.2) has unique solution in the weighted space $C_{1-\frac{\zeta_k}{k};\Psi}[a,b]$. We prove that the operator $\mathcal{A}$ is contraction on the spaces which depends on the subinterval that obtained from partitioning the interval $[a,b]$.

Let any $c_1, c_2 \in [a,b]$ with $c_1 < c_2$. Then, $C_{1-\frac{\zeta_k}{k};\Psi}[c_1,c_2]$ is complete normed linear space with norm

$$\|y\|_{C_{1-\frac{\zeta_k}{k};\Psi}[c_1,c_2]} = \max_{t\in[c_1,c_2]} \left|\left(\Psi\left(t\right) - \Psi\left(c_1\right)\right)^{1-\frac{\zeta_k}{k}} y\left(t\right)\right|.$$

Since Ψ is continuous on $[a,b]$ and $\frac{\eta}{k} > 0$ it is possible to select $t_1 \in (a,b]$ such that

$$w_1 = \frac{L\,\Gamma_k(\zeta_k)}{\Gamma_k(\zeta_k+\eta)} \left(\Psi\left(t_1\right) - \Psi\left(a\right)\right)^{\frac{\eta}{k}} < 1, \tag{7.18}$$

where, $L > 0$ is the Lipschitz constant of the function $f : (a, b] \times \mathbb{R} \to \mathbb{R}$. Since for any $t \in [a, t_1]$,

$$(\Psi(t) - \Psi(a))^{1-\frac{\zeta_k}{k}} y_0(t) = \frac{y_a}{\Gamma_k(\zeta_k)} \in C[a, t_1],$$

we have

$$y_0 \in C_{1-\frac{\zeta_k}{k};\Psi}[a, t_1]. \tag{7.19}$$

Since $f(\cdot, y(\cdot)) \in C^{\nu(k-\eta)}_{1-\frac{\zeta_k}{k};\Psi}[a, b]$, for any $y \in C_{1-\frac{\zeta_k}{k};\Psi}[a, b]$, we have $f(\cdot, y(\cdot)) \in C^{\nu(k-\eta)}_{1-\frac{\zeta_k}{k};\Psi}[a, t_1]$, for any $y \in C_{1-\frac{\zeta_k}{k};\Psi}[a, t_1]$. Therefore, by Theorem 4.6, we have

$${}^{k}\mathfrak{I}^{\eta;\Psi}_{a+} f(\cdot, y(\cdot)) \in C_{1-\frac{\zeta_k}{k};\Psi}[a, t_1]. \tag{7.20}$$

From equations (7.19) and (7.20), it follows that $\mathcal{A}y \in C_{1-\frac{\zeta_k}{k};\Psi}[a, t_1]$, and hence $\mathcal{A}$ is mapping from $C_{1-\frac{\zeta_k}{k};\Psi}[a, t_1]$ into itself.

Let any $y_1, y_2 \in C_{1-\frac{\zeta_k}{k};\Psi}[a, t_1]$. By using definition of an operator $\mathcal{A}$, Theorem 4.3 and Lipschitz condition on f, we obtain

$$\begin{aligned}
&\|\mathcal{A}y_1 - \mathcal{A}y_2\|_{C_{1-\frac{\zeta_k}{k};\Psi}[a,t_1]} \\
&= \max_{t\in[a,t_1]} \left| \frac{(\Psi(t) - \Psi(a))^{1-\frac{\zeta_k}{k}}}{k\,\Gamma_k(\eta)} \int_a^t \Psi'(s)\,(\Psi(t) - \Psi(s))^{\frac{\eta}{k}-1} \left[f(s, y_1(s)) - f(s, y_2(s))\right] ds \right| \\
&\le \max_{t\in[a,t_1]} \frac{L\,(\Psi(t) - \Psi(a))^{1-\frac{\zeta_k}{k}}}{k\,\Gamma_k(\eta)} \int_a^t \Psi'(s)\,(\Psi(t) - \Psi(s))^{\frac{\eta}{k}-1} (\Psi(s) - \Psi(a))^{\frac{\zeta_k}{k}-1} \times \\
&\qquad \left| (\Psi(s) - \Psi(a))^{1-\frac{\zeta_k}{k}} (y_1(s) - y_2(s)) \right| ds \\
&\le \|y_1 - y_2\|_{C_{1-\frac{\zeta_k}{k};\Psi}[a,t_1]} L\,(\Psi(t) - \Psi(a))^{1-\frac{\zeta_k}{k}}\ {}^{k}\mathfrak{I}^{\eta;\Psi}_{a+} (\Psi(t) - \Psi(a))^{\frac{\zeta_k}{k}-1} \\
&\le \|y_1 - y_2\|_{C_{1-\frac{\zeta_k}{k};\Psi}[a,t_1]} L\,(\Psi(t) - \Psi(a))^{1-\frac{\zeta_k}{k}}\ \frac{\Gamma_k(\zeta_k)}{\Gamma_k(\eta + \zeta_k)} (\Psi(t) - \Psi(a))^{\frac{\zeta_k+\eta}{k}-1} \\
&\le \frac{L\,\Gamma_k(\zeta_k)}{\Gamma_k(\eta + \zeta_k)} (\Psi(t_1) - \Psi(a))^{\frac{\eta}{k}} \|y_1 - y_2\|_{C_{1-\frac{\zeta_k}{k};\Psi}[a,t_1]}.
\end{aligned}$$

Using the inequality (7.18), we have

$$\|\mathcal{A}y_1 - \mathcal{A}y_2\|_{C_{1-\frac{\zeta_k}{k};\Psi}[a,t_1]} \le w_1 \|y_1 - y_2\|_{C_{1-\frac{\zeta_k}{k};\Psi}[a,t_1]}.$$

Since $0 < w_1 < 1$, the operator $\mathcal{A}$ is contraction on $(a, t_1]$. By Banach fixed point theorem, there exists a unique solution $y_0^* \in C_{1-\frac{\zeta_k}{k};\Psi}[a, t_1]$ to the fractional integral equation (7.4).

If $t_1 \neq b$, then we consider the interval $[t_1, b]$. Let $y \in C[t_1, b]$ is the solution of the fractional integral equation

$$y(t) = \mathcal{A}y(t) := y_{01}(t) + \frac{1}{k\,\Gamma_k(\eta)} \int_{t_1}^t \Psi'(s)\,(\Psi(t) - \Psi(s))^{\frac{\eta}{k}-1} f(s, y(s))ds,\ t \in [t_1, b], \tag{7.21}$$

where

$$y_{01}(t)=\frac{y_a}{\Gamma_k(\zeta_k)}\left(\Psi\left(t\right)-\Psi\left(a\right)\right)^{\frac{\zeta_k}{k}-1}+\frac{1}{k\,\Gamma_k\left(\eta\right)}\int_a^{t_1}\Psi'\left(s\right)\left(\Psi\left(t\right)-\Psi\left(s\right)\right)^{\frac{\eta}{k}-1}f(s,y(s))ds. \tag{7.22}$$

Note that y_{01} is known function as it is uniquely defined on $(a,t_1]$. Again, by continuity of Ψ, it is possible to select $t_2\in(t_1,b]$ such that

$$w_2=\frac{L\,\left(\Psi\left(t_2\right)-\Psi\left(t_1\right)\right)^{\frac{\eta}{k}}}{\eta\,\Gamma_k(\eta)}<1. \tag{7.23}$$

Since $f\left(\cdot,y(\cdot)\right)\in C[t_1,t_2]$, for any $y\in C[t_1,t_2]$. Therefore, by Theorem 4.7, we have

$$^{k}\mathfrak{J}^{\eta;\,\Psi}_{t_1+}f\left(\cdot,y(\cdot)\right)\in C[t_1,t_2]. \tag{7.24}$$

Using the equation (7.22) and condition (7.24), it follows that $\mathcal{A}$ maps $C[t_1,t_2]$ into itself. Let any $y_1,y_2\in C[t_1,t_2]$. Then, using the Lipschitz condition on the function f, we have

$$\begin{aligned}\left|\mathcal{A}y_1(t)-\mathcal{A}y_2(t)\right| &= \left|{}^{k}\mathfrak{J}^{\eta;\,\Psi}_{t_1+}\left[f\left(t,y_1(t)\right)-f\left(t,y_2(t)\right)\right]\right|\\ &\leq L\,{}^{k}\mathfrak{J}^{\eta;\,\Psi}_{t_1+}\left|y_1(t)-y_2(t)\right|\\ &\leq L\,\left\|y_1-y_2\right\|_{C[t_1,t_2]}\;{}^{k}\mathfrak{J}^{\eta;\,\Psi}_{a+}(1)\\ &= \frac{L}{\eta\,\Gamma_k(\eta)}\left(\Psi\left(t_2\right)-\Psi\left(t_1\right)\right)^{\frac{\eta}{k}}\;\left\|y_1-y_2\right\|_{C[t_1,t_2]}.\end{aligned} \tag{7.25}$$

Using (7.23) in the inequality (7.25), we have

$$\left\|\mathcal{A}y_1-\mathcal{A}y_2\right\|_{C[t_1,t_2]}\leq w_2\left\|y_1-y_2\right\|_{C[t_1,t_2]}.$$

Since $0<w_2<1$, the operator $\mathcal{A}$ is contraction on $[t_1,t_2]$.

Therefore, by Banach fixed point theorem, there exists a unique solution $y_1^*\in C[t_1,t_2]$ to the fractional integral equation (7.4). Note that at point t_1, we have two different solutions y_0^* and y_1^*. But due to unique solution, we must have $y_0^*(t_1)=y_1^*(t_1)$. Define $y^*:(a,t_2]\to\mathbb{R}$ by

$$y^*(t)=\begin{cases}y_0^*(t), & t\in(a,t_1]\\ y_1^*(t), & t\in(t_1,t_2].\end{cases} \tag{7.26}$$

Then, by Theorem 4.2, we have $y^*\in C_{1-\frac{\zeta_k}{k};\Psi}[a,t_2]$. Hence, y^* is the unique solution of integral equation (7.4) in $C_{1-\frac{\zeta_k}{k};\Psi}[a,t_2]$. If $t_2\neq b$ then we repeat the above procedure as necessary times, say $N-2$ times to obtain unique solution $y_k^*\in C[t_k,t_{k+1}], k=2,3,\cdots,N-1$, where $a=t_0<t_1<t_2<\cdots t_N=b$ such that

$$w_{k+1}=\frac{L}{\eta\,\Gamma_k(\eta)}\left(\Psi\left(t_{k+1}\right)-\Psi\left(t_k\right)\right)^{\frac{\eta}{k}}<1.$$

Proceeding in the similar way as discussed above, we obtain a unique solution $y^*\in C_{1-\frac{\zeta_k}{k};\Psi}[a,b]$ to the fractional integral equation (7.4), given by

$$y^*(t)=y_k^*(t), t\in(t_k,t_{k+1}], k=0,1,2,\cdots,N-1.$$

Now,we prove that the unique solution $y^* \in C_{1-\frac{\zeta_k}{k};\Psi}[a,b]$ lies in $C^{\zeta_k}_{1-\frac{\zeta_k}{k};\Psi}[a,b]$. From equation (7.4), we have

$$y^*(t) = \frac{(\Psi(t) - \Psi(a))^{\frac{\zeta_k}{k}-1}}{\Gamma_k(\zeta_k)} y_a + {}^{k}\mathcal{I}^{\eta;\,\Psi}_{a+} f(t, y^*(t)),\ t \in (a,b]. \tag{7.27}$$

Applying ${}^{k,\,RL}\mathcal{D}^{\zeta_k;\,\Psi}_{a+}$ on both sides of equation (7.27) and using the Theorem 5.3, we obtain

$$\begin{aligned} {}^{k,\,RL}\mathcal{D}^{\zeta_k;\,\Psi}_{a+} y^*(t) &= \frac{y_a}{\Gamma_k(\zeta_k)}\, {}^{k,\,RL}\mathcal{D}^{\zeta_k;\,\Psi}_{a+} \left(\Psi(t) - \Psi(a)\right)^{\frac{\zeta_k}{k}-1} + {}^{k,\,RL}\mathcal{D}^{\zeta_k;\,\Psi}_{a+}\, {}^{k}\mathcal{I}^{\eta;\,\Psi}_{a+} f(t, y^*(t)) \\ &= {}^{k,\,RL}\mathcal{D}^{\nu(k-\eta);\,\Psi}_{a+} f(t, y^*(t)). \end{aligned} \tag{7.28}$$

By assumption $f(\cdot, y^*(\cdot)) \in C^{\nu(k-\eta)}_{1-\frac{\zeta_k}{k};\Psi}[a,b]$ and hence, we get

$${}^{k,\,RL}\mathcal{D}^{\nu(k-\eta);\,\Psi}_{a+} f(\cdot, y^*(\cdot)) \in C_{1-\frac{\zeta_k}{k};\Psi}[a,b]. \tag{7.29}$$

From equation (7.28) and condition (7.29), it follows that

$${}^{k,\,RL}\mathcal{D}^{\zeta_k;\,\Psi}_{a+} y^*(t) \in C_{1-\frac{\zeta_k}{k};\Psi}[a,b].$$

This implies $y^* \in C^{\zeta_k}_{1-\frac{\zeta_k}{k};\Psi}[a,b]$. By Theorem 7.1, y^* is the unique solution of the (k,Ψ)-Hilfer FDEs (1.1)-(1.2). □

8 Conclusion

- We introduced the most generalized variant of the Hilfer derivative namely (k,Ψ)-Hilfer fractional derivative and proved few its properties. Many other properties of (k,Ψ)-Hilfer fractional derivative are open for investigation.
- For $k = 1$, (k,Ψ)-Hilfer fractional derivative reduces to Ψ-Hilfer fractional derivative [4]. For $\Psi(t) = t$ and $k = 1$, (k,Ψ)-Hilfer fractional derivative reduces to Hilfer fractional derivative [30].
- It is observed that the k-Hilfer fractional derivative defined in [19] does not includes the k-RL fractional derivative. But the formula which we have defined in (3.8) includes the k-RL fractional derivative as a particular case of it for $\nu = 0$.
- For $\Psi(t) = t$ and $\nu = 0$, (k,Ψ)-Hilfer fractional derivative reduces to (k,Ψ)-RL fractional derivative operator. Few properties of (k,Ψ)-RL fractional integral and derivative are obtained.
- For $\Psi(t) = t$ and $\nu = 1$, (k,Ψ)-Hilfer fractional derivative reduces to (k,Ψ)-Caputo fractional derivative operator. The properties of (k,Ψ)-Caputo fractional derivative are open for investigation.
- For different function Ψ and the different values of the parameter ν, the (k,Ψ)-Hilfer fractional derivative ${}^{k,H}\mathcal{D}^{\eta,\,\nu\,;\,\Psi}_{a+}$ produces distinct types of fractional derivative operators recorded in the Table 1.

${}_{a}^{k,H}\mathcal{D}_{t}^{\eta,\nu;\Psi}$		Special Cases		
$\Psi(t)$	ν	$k>0$		$k=1$
$\Psi(t)$	0	(k,Ψ)-RL Derivative	${}_{a}^{k,RL}\mathcal{D}_{t}^{\eta;\Psi}$	Ψ-RL Derivative
$\Psi(t)$	1	(k,Ψ)-Caputo Derivative	${}_{a}^{k,C}\mathcal{D}_{t}^{\eta;\Psi}$	Ψ-Caputo Derivative
t	0	k-RL Derivative	${}_{a}^{k,RL}\mathcal{D}_{t}^{\eta}$	RL Derivative
t	1	k-Caputo Derivative	${}_{a}^{k,C}\mathcal{D}_{t}^{\eta}$	Caputo Derivative
t	ν	k-Hilfer Derivative	${}_{a}^{k,H}\mathcal{D}_{t}^{\eta,\nu}$	Hilfer Derivative
t^{ρ}	0	k-Katugampola Derivative	${}_{a}^{k}\mathcal{D}_{t}^{\eta,\rho}$	Katugampola Derivative
t^{ρ}	1	k-Caputo-Katugampola Derivative	${}_{a}^{k,C}\mathcal{D}_{t}^{\eta,\rho}$	Caputo-Katugampola Derivative
t^{ρ}	ν	k-Hilfer-Katugampola Derivative	${}_{a}^{k,HK}\mathcal{D}_{t}^{\eta,\nu}$	Hilfer-Katugampola Derivative
$\log t$	0	k-Hadmard Derivative	${}_{a}^{k,H}\mathcal{D}_{t}^{\eta}$	Hadmard Derivative
$\log t$	1	k-Caputo-Hadmard Derivative	${}_{a}^{k,CH}\mathcal{D}_{t}^{\eta}$	Caputo-Hadmard Derivative
$\log t$	ν	k-Hilfer-Hadmard Derivative	${}_{a}^{k,HH}\mathcal{D}_{t}^{\eta,\nu}$	Hilfer-Hadmard Derivative

Table 1: List of particular cases of (k,Ψ)-Hilfer fractional derivatives

- We have treated the existence and uniqueness of solution for nonlinear (k,Ψ)-Hilfer FDEs. Many other qualitative properties of solution for nonlinear (k,Ψ)-Hilfer FDEs are open for investigation

Acknowledgment

The first author acknowledges the Science and Engineering Research Board (SERB), New Delhi, India for the Research Grant (Ref: File no. EEQ/2018/000407).

Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Credit author statement

All authors contributed equally to this work.

References

[1] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, North–Holland Mathematics Studies, Elsevier, Amsterdam, Vol. 207, 2006.

[2] I. Podlubny, Fractional differential equations, Academic Press, New York, 1999.

[3] K. M. Furati, M. D. Kassim, N. E. Tatar, Existence and uniqueness for a problem involving Hilfer fractional derivative, Comp. Math. Appl., 64(2012), 1616–1626.

[4] J. Vanterler da C. Sousa, E. Capelas de Oliveira, On the Ψ-Hilfer fractional derivative, Commun. Nonlinear Sci. Numer. Simul., 60(2018), 72–91.

[5] K. D. Kucche, A. D. Mali, J. Vanterler da C. Sousa, On the nonlinear Ψ-Hilfer fractional differential equations, Computational and Applied Mathematics, 38(2)(2019): 73.

[6] K. D. Kucche, A. D. Mali, Initial time difference quasilinearization method for fractional differential equations involving generalized Hilfer fractional derivative, Computational and Applied Mathematics, 39(1)(2020): 31.

[7] A. D. Mali, K. D. Kucche, Nonlocal boundary value problem for generalized Hilfer implicit fractional differential equations, Mathematical Methods in the Applied Sciences, 43(15)(2020), 8608–8631.

[8] K. D. Kucche, A. D. Mali, On the nonlinear Ψ-Hilfer hybrid fractional differential equations, (2020), arXiv preprint: arXiv:2008.06306.

[9] A. D. Mali, K. D. Kucche, On the boundary value problems of Ψ-Hilfer fractional differential equations, (2020), arXiv preprint: arXiv:2009.09175.

[10] J. P. Kharade, K. D. Kucche, On the impulsive implicit Ψ-Hilfer fractional differential equations with delay, Mathematical Methods in the Applied Sciences, 43(4)(2019), 1938–1952.

[11] M. S. Abdo, S. K. Panchal, H. A. Wahash, Ulam-Hyers-Mittag-Leffler stability for a Ψ-Hilfer problem with fractional order and infinite delay, Results in Applied Mathematics, 7(2020): 100115, doi:10.1016/j.rinam.2020.100115.

[12] M. Ahmad, A. Zada, X. Wang, Existence, uniqueness and stability of implicit switched coupled fractional differential equations of Ψ-Hilfer type, International Journal of Nonlinear Sciences and Numerical Simulation, 21(3-4)(2020), 327–337.

[13] J. Vanterler da C. Sousa, J. T. Machado, E. Capelas de Oliveira, The Ψ-Hilfer fractional calculus of variable order and its applications, Computational and Applied Mathematics, 39(4)(2020): 296.

[14] H. Afshari, E. Karapnar, A discussion on the existence of positive solutions of the boundary value problems via Ψ-Hilfer fractional derivative on b-metric spaces, Advances in Difference Equations, 2020 (2020): 616, doi:10.1186/s13662-020-03076-z.

[15] R. Díaz, E. Pariguan, On hypergeometric functions and Pochhammer k-symbol, Divulgaciones Matemticas, 15(2)(2007), 179–192.

[16] S. Mubeen, G. M. Habibullah, k-Fractional integrals and application, Int. J. Contemp. Math. Sci, 7(2)(2012), 89–94.

[17] L. G. Romero, L. L. Luque, G. A. Dorrego, R. A. Cerutti, On the k-Riemann-Liouville fractional derivative, Int. J. Contemp. Math. Sciences, 8(1)(2013), 41–51.

[18] G. A. Dorrego, An alternative definition for the k-Riemann-Liouville fractional derivative, Applied Mathematical Sciences, 9(10)(2015), 481–491.

[19] G. A. Dorrego, R. A. Cerutti, The k-fractional Hilfer derivative, International Journal of Mathematical Analysis, 7(11)(2013), 543–550.

[20] G. A. Dorrego, Generalized Riemann-Liouville fractional operators associated with a generalization of the Prabhakar integral operator, Progr. Fract. Differ. Appl, 2(2)(2016), 131–140.

[21] M. K. Azam, G. Farid, M. A. Rehman, Study of generalized type k-fractional derivatives, Advances in Difference Equations, 2017 (2017): 249, doi:10.1186/s13662-017-1311-2.

[22] S. K. Panchal, P. V. Dole, A. D. Khandagale, k-Hilfer-Prabhakar fractional derivatives and its applications, Indian Journal of Mathematics, 59(3)(2017), 367–383.

[23] P. I. Pucheta, On the k-Caputo-Fabrizio fractional derivative and its applications, Int. J. Math. And Appl., 6(1-A)(2018), 179–185.

[24] G. Farid, N. Latif, M. Anwar, A. Imran, M. Ozair, M. Nawaz, On applications of Caputo k-fractional derivatives, Advances in Difference Equations, 2019 (2019): 439, doi:10.1186/s13662-019-2369-9.

[25] O. S. Iyiola, Solving k-fractional Hilfer differential equations via combined fractional integral transform methods, Journal of Advances in Mathematics and Computer Science, 4(10)(2014), 1427–1436.

[26] E. Mittal, S. Joshi, Note on a k-generalised fractional derivative, Discrete & Continuous Dynamical Systems-S, 13(3)(2020), 797–804.

[27] G. Farid, A. Javed, A. U. Rehman, On Hadamard inequalities for n-times differentiable functions which are relative convex via Caputo k-fractional derivatives, Nonlinear Analysis Forum, 22(2)(2017), 17–28.

[28] R. Almeida, A Caputo fractional derivative of a function with respect to another function, Commun. Nonlinear Sci. Numer. Simul., 44 (2017), 460–481.

[29] Y. C. Kwun, G. Farid, W. Nazeer, S. Ullah, S. M. Kang, Generalized Riemann-Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard inequalities, IEEE access, 6(2018), 64946–64953.

[30] R. Hilfer, Applications of fractional calculus in Physics, World Scientific, Singapore, 2000.

[31] K. Diethelm, The analysis of fractional differential equations, Lecture Notes in Mathematics, Springer–Verlag Berlin Heidelberg, 2010.

[32] J. Vanterler da C. Sousa , E. Capelas de Oliveira, A Gronwall inequality and the Cauchy-type problem by means of ψ-Hilfer operator, Differ. Equ. Appl, 11(1)(2019), 87–106.