ON FULL SPARK FRAMES VIA CAUCHY MATRICES

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ABSTRACT. Full spark frames have been widely applied in sparse signal processing, signal reconstruction with erasures and phase retrieval. Since testing whether a given frame is full spark is hard for NP under randomized polynomial-time reductions, hence the deterministic full spark (DFS) frames are particularly significant. However, the degree of freedom of choices of DFS frames is not enough in practical applications because the DFS frames are well known as Vandermonde frames and harmonic frames. In this paper, we focus on the deterministic constructions of full spark frames. We present a new and effective method to construct DFS frames by using Cauchy matrices. We also construct the DFS frames by using Cauchy-Vandermonde matrices. Finally, we show that full spark tight frames can be constructed from generalized Cauchy matrices.

1. INTRODUCTION

Frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [6] in order to study problems in nonharmonic Foutier series, and reintroduced in 1986 by Daubechies, Grossmann and Meyer [5], and popularized from then on. A frame is a generalization of a basis that includes redundancy. The redundancy implies that the frame expansion coefficients of an element may be not unique and the frame expansions are generally more robust to erasures, noises and distortions. Due to this property, frame theory has been a useful tool in many areas such as signal and image processing [3, 17], probability statistics [7], filter bank [12] and coding theory [13].

For a frame $X = \{x_i\}_{i=1}^{M}$, then the linear map

$$X : \mathbb{C}^M \to \mathcal{H}\,,\ X(c) := \sum_{i=1}^{M} c_i x_i,$$

is called the synthesis operator. The matrix representation of the synthesis operator of $X = \{x_i\}_{i=1}^{M}$ is the $N \times M$ matrix with the frame elements as its columns, thus

$$X = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ x_1 & x_2 & \dots & x_M \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}.$$

Note that here and throughout, with a slight abuse of notation, we make no notational distinction between a frame X and its synthesis operator, and we also make no notational distinction between an operator and a matrix in this paper. When we speak of a frame, we always mean the matrix X.

The adjoint of the synthesis operator is the analysis operator which is given by

$$X^* : \mathcal{H} \to \mathbb{C}^M,\ X^*(x) := \{\langle x, x_i\rangle\}_{i=1}^{M},\ \forall x \in \mathcal{H}\,.$$

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The frame operator S of $X = \{x_i\}_{i=1}^M$ is defined as $S = XX^*$. If X is a tight frame with bound a, the frame operator has the form $S = aI_N$, where I_N denotes $N \times N$ identity matrix.

In speech recognition systems, the measurements may be the absolute values of linear combinations of frame coefficients. If we want to reconstruct original signal if and only if the frame has complementary property[4].

In coding theory, a signal vector $x \in \mathcal{H}$ is encoded by a frame $X = \{x_i\}_{i=1}^M$ as frame coefficients $\{\langle x, x_i\rangle\}_{i\in I}$. Then these coefficicents are transmitted to a receiver for decoding to reconstruct the signal x. In a more realistic setting where the channel is not perfect, some coefficients may be erased during the transmission. If we want to perfectly reconstruct signal x from the non-erased coefficients $\{\langle x, x_i\rangle\}_{i\in\Lambda}$ if and only if the $\{x_i\}_{i\in\Lambda}$ is still capable to span the underlying space, i.e. $\{x_i\}_{i\in\Lambda}$ is still a frame for $\mathcal{H}$, where $\Lambda \subset \{1, \cdots, M\}$. In this case, we say that the frame X is robust to erasures.

A frame is said to be robust to k erasures if after randomly removing k vectors the resulting set is still a frame. More precisely, if a frame $X = \{x_i\}_{i=1}^M$ is $M - N$-robust(or maximally robust to erasures [16]), we call it full spark frame for $\mathcal{H}$. Thus a full spark frame remains a basis for $\mathcal{H}$ after removal of any $M - N$ vectors. Hence, full spark frames always have complementary property.

Full spark frame is a useful tool and is widely used in sparse signal processing, data transmission with robustness to erasures, and reconstruction without phase. In [1], the authors prove that testing whether a given frame is full spark is hard for NP under randomized polynomial-time reductions. Hence the DFS frames are particularly significant because they are easily verified to be full spark frames. However, there has not been much progress in constructing DFS frames. In [16], the authors use polynomial transforms to construct real full spark equal norm tight frames. A noteworthy method is Alexeev, Cahill and Mixon's work [1], in which DFS frames are constructed using Vandermonde matrices and discrete Fourier transforms. Currently, Vandermonde frames and harmonic frames are the most commonly used DFS frames. In [2], the authors establish a connection between full spark frames and totally non-singular matrices. This result provides a new way to construct DSF frames. However, in [2], the authors don't provide methods for constructions of totally non-singular matrices. Thus we can't construct full spark frames using totally nonsingular matrices in the current study results.

In this paper, our main goal is devoted to deterministic constructions of full spark frames by using Cauchy matrices, we also answer the question of [2] and provide a method for constructions of totally non-singular matrices.

2. Constructions of full spark frames

The definition of Cauchy matrix is given as follows.

Definition 2.1. [15] *A Cauchy matrix is an $N \times M$ matrix assigned to $N+M$ parameters $a_1, \ldots, a_N, b_1, \ldots, b_M$ as follows:*

$$C = \left(\frac{1}{a_i + b_j}\right),\ i = 1, \ldots, N,\ j = 1, \ldots, M.$$

For generalized Cauchy matrices, additional parameters $u_1, \cdots, u_N$, $v_1, \cdots, v_M$, have to be considered (one of which again superfluous):

$$\widehat{C} = \left(\frac{u_i v_j}{a_i + b_j} \right).$$

If $M = N$ and $a_i + b_j \neq 0$ for all $i,\ j$ in Definition 2.1, there is a formula [9] for the determinant of C,

$$|C| = \frac{\prod_{i,k,i>k}(a_i - a_k)(b_i - b_k)}{\prod_{i,j=1}^{N}(a_i + b_j)}. \tag{2.1}$$

From Definition 2.1, it is easy to find that all submatrices of a Cauchy matrix are also Cauchy matrices. Hence, we can modify the indices in (2.1) to calculate the determinant of the submatrices. These determinants are non-zero precisely when $a_1, \ldots, a_N$ are mutually distinct as well as $b_1, \ldots, b_M$ are mutually distinct and $a_i + b_j \neq 0$ for all $i = 1, \ldots, N$, $j = 1, \ldots, M$, yielding the following result:

Lemma 2.2. *Let C be a Cauchy matrix. All minors of C are nonzero if and only if $a_1, \ldots, a_N$ are mutually distinct as well as $b_1, \ldots, b_M$ are mutually distinct and $a_i + b_j \neq 0$ for all $i = 1, \ldots, N$, $j = 1, \ldots, M$.*

The result of Lemma 2.2 answers the question of [2] and provides a method to construct totally non-singular matrices, where totally non-singular matrix denotes that all minors of it are nonzero.

Next, we give a deterministic construction of full spark frames by using Cauchy matrices. We first give a definition of full spark frames.

Definition 2.3. *A frame $X = \{x_i\}_{i=1}^{M}$ for $\mathcal{H}$ is said to be full spark (or maximally robust) frame if any N of its members make up a basis for $\mathcal{H}$.*

We can see that a frame X is full spark if and only if any $N \times N$ submatrix of X is invertible.

Theorem 2.4. *Let C be a $M \times M$ Cauchy matrix ($M \geq N$), X be any $N \times M$ submatrix of C. Then X is a full spark frame for $\mathcal{H}$ if and only if $a_1, \ldots, a_N$ are mutually distinct as well as $b_1, \ldots, b_M$ are mutually distinct and $a_i + b_j \neq 0$ for all $i = 1, \ldots, N$, $j = 1, \ldots, M$.*

Proof. The proof is directly from Lemma 2.2. □

The result of Theorem 2.4 also holds for generalized Cauchy matrices.

Corollary 2.5. *Let $\widehat{C}$ be a $M \times M$ generalized Cauchy matrix($M \geq N$), X be any $N \times M$ submatrix of C. Then X is a full spark frame for $\mathcal{H}$ if and only if $a_1, \ldots, a_N$ are mutually distinct as well as $b_1, \ldots, b_M$ are mutually distinct and $a_i + b_j \neq 0$ and $u_i \neq 0$, $v_j \neq 0$ for all $i = 1, \ldots, N$, $j = 1, \ldots, M$.*

Let $X = \{x_i\}_{i=1}^{M}$ be a frame for $\mathcal{H}$, there must be an index set $\Lambda = \{i_1, \cdots, i_N\} \subset \{1, \cdots, M\}$ such that $\{x_{i_j}\}_{j=1}^{N}$ is a basis for $\mathcal{H}$. As a convention, let $T = \{x_{i_j}\}_{j \in \Lambda}$ and $V = \{x_{i_j}\}_{j \in \Lambda^c}$. The authors of [2] give a sufficient and necessary condition such that a frame is a full spark frames. We also give a more simple proof for this result as follows.

Lemma 2.6. *$X = \{x_i\}_{i=1}^{M}$ be a frame for $\mathcal{H}$. Then X is a full spark frame if and only if $T^{-1}V$ is a totally non-singular matrix.*

Proof. Let $X = \{x_i\}_{i=1}^{M}$ be a frame for $\mathcal{H}$, from [16, Lemma 2], we know that X is a full spark frame if and only $T^{-1}X$ is a full spark frame. Hence, we only need to prove that $T^{-1}X = [I_N|T^{-1}V]$ is a full spark frame if and only if $T^{-1}V$ is a totally non-singular matrix. Let U be an $N \times N$ submatrix consisting of any k columns of I_N and any $N-k$ columns of $T^{-1}V$. Without loss of generality, suppose that U contains first k columns of I_N , then

$$U = \left[\begin{array}{c|c} I_k & P \\ \hline O_{(N-k)\times k} & Q \end{array}\right], \tag{2.2}$$

where I_k denotes a $k \times k$ identity matrix, $O_{(N-k)\times k}$ denotes a $(N-k) \times k$ zero matrix, P and Q denote the first k rows and last $(N-k)$ rows of any $N \times (N-k)$ submatrix of $T^{-1}V$, respectively. By calculating the determinants on both sides of (2.2), we have

$$|U| = |Q|,$$

which means that U is invertible if and only if Q is invertible. Hence, U is invertible if and only if all minors of $T^{-1}V$ are non-zero. Therefore, $T^{-1}X$ is a full spark frame for $\mathcal{H}$ if and only if all minors of $T^{-1}V$ are non-zero. □

Since an $N \times M$ spark matrix constants at most $N(N-1)$ zero entries, we can construct a deterministic maximally sparse full spark frame by using Cauchy matrix from Lemma 2.2 and Lemma 2.6.

Theorem 2.7. *Let C be a $N \times M$ Cauchy matrix, and let $X = [I_N|C]$. Then X is a full spark frame for $\mathcal{H}$ if and only if $a_1, \dots, a_N$ are mutually distinct as well as $b_1, \dots, b_M$ are mutually distinct and $a_i + b_j \neq 0$ for all $i = 1, \dots, N$, $j = 1, \dots, M$.*

Proof. Let $X = \{x_i\}_{i=1}^{M}$ be a frame for $\mathcal{H}$, and let $T = \{x_{i_j}\}_{j\in\Lambda}$ and $V = \{x_{i_j}\}_{j\in\Lambda^c}$. Without loss of generality, assume that $\{x_{i_j}\}_{j\in\Lambda}$ is a basis for $\mathcal{H}$, thus T is invertible. From [16, Lemma 2], we know that $T^{-1}X = [I_N|T^{-1}V]$ is also a frame for $\mathcal{H}$. From Lemma 2.6, $T^{-1}X$ is a full spark frame if and only if $T^{-1}V$ is a totally non-singular matrix. Let $Y = [I_N|C]$ be a frame for $\mathcal{H}$, C is a $N \times M$ Cauchy matrix. We have that Y is a full spark frame if and only if $a_1, \dots, a_N$ are mutually distinct as well as $b_1, \dots, b_M$ are mutually distinct and $a_i + b_j \neq 0$ for all $i = 1, \dots, N$, $j = 1, \dots, M$. □

Next, we give a simple example showing that it is flexible to construct full spark frames by using Cauchy matrices.

Example 2.8. *Let C be a 3×5 Cauchy matrix given by*

$$C = \left(\frac{1}{a_i + b_j}\right),$$

where $a_1 = -\frac{3}{4}$, $a_2 = \frac{1}{2}$, $a_3 = 1$ and $b_1 = -\frac{3}{4}$, $b_2 = -\frac{2}{3}$, $b_3 = -\frac{1}{3}$, $b_4 = \frac{1}{2}$, $b_5 = 1$. Thus

$$C = \begin{bmatrix} -\frac{2}{3} & -\frac{12}{17} & -\frac{12}{13} & -4 & 4 \\ -4 & -6 & 6 & 1 & \frac{2}{3} \\ 4 & 3 & \frac{3}{2} & \frac{2}{3} & \frac{1}{2} \end{bmatrix}.$$

From Theorem 2.4, C is a full spark frame for $\mathbb{R}^3$. In fact, let U be a 3×3 submatrix of C, we compute the determinant of U with different cardinalities showing in the Fig. 1.

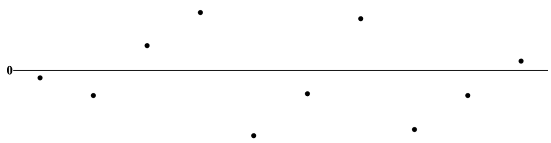

Figure 1. Distribution of determinants of 3×3 submatrices U.

The Fig.1 shows that all 3×3 submatrices of C are non-singular. Hence C is a full spark frame for $\mathbb{R}^3$. Next, we construct maximally sparse full spark frame for $\mathbb{R}^3$ from Cauchy matrix C.

Let $\widehat{X}=[I_3|C]$, from Theorem 2.7, $\widehat{X}$ is a full spark frame with maximally sparse for $\mathbb{R}^3$. In fact, it is easy to see that $\widehat{X}$ constants $3(3-1)=6$ zero entries. And let P be a 3×3 submatrix consisting of any k columns of I_3 and any t columns of C, $k+t=3$. If $k=0$ (or $k=3$), $P=U$ (or $P=I_3$) is invertible. So we only need to compute determinants of P for $1\le k\le 2$. Without loss of generality, assume that P constants first t columns of C. The determinant of P is given in Fig.2. From Fig.1 and Fig.2, we can

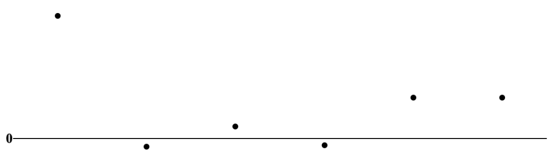

Figure 2. Distribution of determinants of 3×3 submatrices P.

find that all determinants of 3×3 submatrices of $\widehat{X}$ are non-zero. Hence, $\widehat{X}$ is a full spark frame with maximally sparse for $\mathbb{R}^3$.

Since using Vandermonde matrices to construct full spark frames is a very effective method. Next, we consider the constructions of DSF frames from Cauchy-Vandermonde matrices.

A matrix

$$V^T=\begin{bmatrix} \frac{1}{c_1+d_1} & \cdots & \frac{1}{c_1+d_l} & 1 & c_1 & \cdots & c_1^{N-l-1} \\ \frac{1}{c_2+d_1} & \cdots & \frac{1}{c_2+d_l} & 1 & c_2 & \cdots & c_2^{N-l-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{c_N+d_1} & \cdots & \frac{1}{c_N+d_l} & 1 & c_N & \cdots & c_N^{N-l-1} \end{bmatrix}$$

is called a Cauchy-Vandermonde matrix because if $l=0$ it is a Vandermonde matrix and if $l=N$ it is a Cauchy matrix, where $c_i+d_j\neq 0$ for all i,j. We show use the following well-know formula (see [10, 14]) for the determinant of a Cauchy-Vandermonde matrix:

$$|V|=\frac{\left(\prod\limits_{1\le i<j\le N}(c_j-c_i)\right)\left(\prod\limits_{1\le i<j\le l}(d_i-d_j)\right)}{\prod\limits_{1\le i\le N;1\le j\le l}(c_i+d_j)}. \tag{2.3}$$

We now consider the generalized Cauchy-Vandermonde matrix as following:

$$\widehat{V}^T = \begin{bmatrix} \frac{1}{c_1+d_1} & \cdots & \frac{1}{c_1+d_l} & 1 & c_1 & \cdots & c_1^{N-l-1} \\ \frac{1}{c_2+d_1} & \cdots & \frac{1}{c_2+d_l} & 1 & c_2 & \cdots & c_2^{N-l-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{c_M+d_1} & \cdots & \frac{1}{c_M+d_l} & 1 & c_M & \cdots & c_M^{N-l-1} \end{bmatrix} \tag{2.4}$$

Theorem 2.9. *Let $\widehat{V}$ be a defined as in (2.4), where $M \geq N$. Let $\{x_i\}_{i=1}^M$ be the frame consisting of all the columns of $\widehat{V}$. Then $\{x_i\}_{i=1}^M$ is a full spark frames for $\mathcal{H}$ if and only if $c_1, \ldots, c_M$ are mutually distinct as well as $d_1, \ldots, d_l$ are mutually distinct and $c_i + d_j \neq 0$ for all $i = 1, \ldots, N$, $j = 1, \ldots, l$.*

Proof. In fact, we only need to show that any N columns of $\widehat{V}$ are linearly independent. Let A be any $N \times N$ submatrix of $\widehat{V}$,

$$A = \begin{bmatrix} \frac{1}{c_{i_1}+d_{i_1}} & \frac{1}{c_{i_2}+d_{i_1}} & \cdots & \frac{1}{c_{i_{N-1}}+d_{i_1}} & \frac{1}{c_{i_N}+d_{i_1}} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{1}{c_{i_1}+d_{i_l}} & \frac{1}{c_{i_2}+d_{i_l}} & \cdots & \frac{1}{c_{i_{N-1}}+d_{i_l}} & \frac{1}{c_{i_N}+d_{i_l}} \\ 1 & 1 & \cdots & 1 & 1 \\ c_{i_1} & c_{i_2} & \cdots & c_{i_{N-1}} & c_{i_N} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ c_{i_1}^{N-l-1} & c_{i_2}^{N-l-1} & \cdots & c_{i_{N-1}}^{N-l-1} & c_{i_N}^{N-l-1} \end{bmatrix},$$

A is also a Cauchy-Vandermonde matrix, by (2.3), we have

$$|A| = \frac{\left(\prod\limits_{1 \leq n < m \leq N} (c_{i_m} - c_{i_n})\right)\left(\prod\limits_{1 \leq n < m \leq l \leq N} (d_{i_n} - d_{i_m})\right)}{\prod\limits_{1 \leq m \leq N; 1 \leq n \leq l} (c_{i_m} + d_{i_n})}.$$

We can find that $|A| \neq 0$ if and only if $c_{i_1}, \ldots, c_{i_N}$ are mutually distinct as well as $d_{i_1}, \ldots, d_{i_l}$ are mutually distinct and $c_{i_m} + d_{i_n} \neq 0$ for all $m = 1, \ldots, N$, $n = 1, \ldots, l$. Hence the result holds. ☐

As we all know, the tight frames provide simple and stable reconstruction formulas in practical applications. However, the structure of full spark tight frames is more complex than that of full spark frames. Hence it is more difficult to construct full spark tight frame. Finally, we construct real full spark tight frame for $\mathcal{H}$ by using generalized Cauchy matrices. For convenience, let us denote by $D_N^{\varphi} = (\varphi_i)$ the $N \times N$ diagonal matrix. We need the following lemma.

Lemma 2.10. [8] *Let C be an $N \times N$ nonsingular Cauchy matrix. Then there exist diagonal non-singular matrices D_N^v and D_N^w, such that*

$$C^{-1} = D_N^v C^T D_N^w,$$

where

$$v_i = (a_i + b_i) \prod_{k \neq i} \frac{b_i + a_k}{b_i - b_k}, \quad w_i = (a_i + b_i) \prod_{i \neq k} \frac{a_i + b_k}{a_i - a_k}.$$

Note that the inverse of a Cauchy matrix is a generalized Cauchy matrix.

Theorem 2.11. *Let C be a $M \times M$ $(M \geq N)$ real, invertible Cauchy matrix whose inverse admits the factorization $C^{-1} = D_M^v C^T D_M^w$. If $v_i > 0$ and $w_i > 0$ for all $i = 1, \cdots, M$, then there exists a $M \times M$ generalized Cauchy matrix $\widehat{C} = D_M^{\sqrt{w}} C D_M^{\sqrt{v}}$ such that any N rows of $\widehat{C}$ constituting a real, full spark tight frame for $\mathcal{H}$.*

Proof. From [11, Property 2.4], we only need to prove that $\widehat{C}$ is an orthogonal matrix. Let $\widetilde{C}$ be a $M \times M$ generalized Cauchy matrix given by

$$\widetilde{C} = D_M^{\varphi} C D_M^{\psi}.$$

Then

$$\widetilde{C}^{-1} = (D_M^{\psi})^{-1} C^{-1} (D_M^{\varphi})^{-1} = (D_M^{\psi})^{-1} D_M^v C^T D_M^w (D_M^{\varphi})^{-1}.$$

We know that $\widetilde{C}$ is an orthogonal matrix if and only if $\widetilde{C}^{-1} = \widetilde{C}^T$, thus

$$(D_M^{\psi})^{-1} D_M^v C^T D_M^w (D_M^{\varphi})^{-1} = (D_M^{\psi})^T C^T (D_M^{\varphi})^T = D_M^{\psi} C^T D_M^{\varphi}. \tag{2.5}$$

From(2.5), we must have

$$(D_M^{\psi})^2 = D_M^v, \ (D_M^{\varphi})^2 = D_M^w.$$

If $v_i > 0$ and $w_i > 0$ for all $i = 1, \cdots, M$, we have $\psi_i = \sqrt{v_i}$ and $\varphi_i = \sqrt{w_i}$. In this case, $\widehat{C} = \widetilde{C} = D_M^{\sqrt{w}} C D_M^{\sqrt{v}}$ is an orthogonal matrix. And then the result holds. □

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