The Solution of The Linear Delay Differential Equations with Aboodh Transform

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Abstract:

In this paper, we apply Aboodh transform to solve linear delay differential equations. Firstly, The basic properties of the Aboodh Transform are given. Secondly, Existence of the Aboodh transform proved. Then, the two linear delay differential equations are solved by Aboodh transform. This means that Aboodh transform is a powerful tool for solving linear delay differential equations.

Keywords: Delay Differential Equations, Linear Delay Differential Equations, Aboodh Transform, Existence of the Aboodh Transform.

1. Introduction

The Aboodh Transform is integral transform. There are many integral transforms in the literatüre. Some of these transformations are Laplace Transform, Fourier Transform, Sumudu Transform, Elzaki Transform, ZZ Transform [1-7]. These transformations are used to solve for differential equations and integral equations. The most common of these transformations is Laplace transform. The Aboodh transform was first presented by Khalid Aboodh in 2013[9-12]. This transformation has also been applied to the solution of ordinary differential equations and partial differential equations. The purpose of this paper is to solve the linear delay differential equations with the Aboodh transform. The Aboodh transform is obtained from the standard Fourier integral. Based on the mathematical simplicity and basic features of the Aboodh Transform. This transformation facilitates the process of solving ordinary and partial differential equations.

Delay differential equations are used to define many physical phenomena in medicine, engineering, economics, biology, physics, and chemistry. Many methods have been developed to solve these equations[13-15].

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This article was planned as follows: In section second part, the basic properties and existence of the Aboodh transformation are given. In the third part, application was given. In the fourth part, the result is given, respectively.

2. The Aboodh Transform:

The Aboodh transform defined for $t \geq 0$. Let $f(t)$ be an exponential order function in the set A as

$$A = \{f(t): \exists\, M, k_1, k_2 > 0,\ |f(t)| < Me^{-vt}\}.$$

Where, the constant M is finite number and k_1, k_2 are finite or may be infinite numbers. Then the Aboodh transform is,

$$\mathcal{A}[f(t)] = A(v) = \frac{1}{v}\int_0^\infty f(t)e^{-vt}dt \ \ t \geq 0, k_1 \leq v \leq k_2\,. \quad (1)$$

The unique function $f(t)$ in (1) is called the inverse transform of $A(v)$ is indicated by

$$f(t) = \mathcal{A}^{-1}[A(v)]\,.$$

2.1. The Aboodh Transform of Some Required Functions

Theorem 1.

1. $\mathcal{A}(1) = \frac{1}{v^2}$

2. $\mathcal{A}(t^n) = \frac{n!}{v^{n+2}}$

3. $\mathcal{A}(e^{at}) = \frac{1}{v^2 - av}$

4. $\mathcal{A}(sinat) = \frac{a}{v(v^2+a^2)}$

5. $\mathcal{A}(cosat) = \frac{1}{(v^2+a^2)}$

6. $\mathcal{A}(t^n e^{at}) = (-1)^{n+1}\frac{n!}{v(a-v)^{n+1}}$

7. $\mathcal{A}\big(u(x-a)\big) = \frac{e^{-av}}{v^2},\ \ v > 0$

Proof 2. Let $f(t) = t^n$, where $t \geq 0$, then Aboodh transform of this function can be written as

$$\mathcal{A}(t^n) = \frac{1}{v}\int_0^\infty f(t)e^{-vt}dt = \frac{1}{v}\int_0^\infty t^n e^{-vt}dt$$

$$\mathcal{A}(t^n) = \frac{1}{v}\int_0^\infty t^n e^{-vt}dt = \frac{1}{v}\left[\frac{-t^n e^{-vt}}{v}\bigg|_0^\infty + \frac{n}{v}\int_0^\infty t^{n-1}e^{-vt}dt\right]$$

$$\mathcal{A}(t^n) = \frac{1}{v}\left[\frac{n}{v}\mathcal{A}(t^{n-1})\right] = \frac{1}{v}\left[\frac{n!}{v^{n+1}}\right] = \frac{n!}{v^{n+2}} \ .$$

3. Let $f(t) = e^{at}$, where $t \geq 0$, where a is a constant, then Aboodh transform of this function can be written as

$$\mathcal{A}(e^{at}) = \frac{1}{v}\int_0^\infty f(t)e^{-vt}dt = \frac{1}{v}\int_0^\infty e^{at}e^{-vt}dt = \frac{1}{v}\int_0^\infty e^{(a-v)t}dt$$

$$= \frac{1}{v}\int_0^\infty e^{-(v-a)t}dt = \frac{1}{v}\frac{e^{-(v-a)t}}{-(v-a)}\bigg|_0^\infty = \frac{1}{v^2 - av} \ .$$

Others can be proof similarly.

2.2. Existence of the Aboodh Transform

Theorem 2.

If $f(t)$ is piecewise continuous in every finite interval $0 \leq t \leq K$ and of exponential order γ for $t > K$, Then its Aboodh transform $A(f(t))$ exists for all $s > \gamma, v > \gamma$.

Proof:

We have for every positive number K .

$$\frac{1}{v}\int_0^\infty f(t)e^{-vt}dt = \frac{1}{v}\int_0^K f(t)e^{-vt}dt + \frac{1}{v}\int_K^\infty f(t)e^{-vt}dt$$

Since $f(t)$ is piecewise continuous in interval $0 \leq t \leq K$, there is the first integral on the right. Also there is the second integral on the right. So $f(t)$ is of exponential γ for order $t > K$. To see this we have only to observe that in such case:

$$\left|\frac{1}{v}\int_K^\infty f(t)e^{-vt}dt\right| \leq \frac{1}{v}\int_K^\infty |f(t)e^{-vt}|dt \leq \frac{1}{v}\int_0^\infty e^{-vt}|f(t)|dt$$

$$\leq \frac{1}{v}\int_0^\infty e^{-vt}Me^{\gamma t}dt \leq \frac{M}{v}\int_0^\infty e^{-vt}e^{\gamma t}dt \leq \frac{M}{v}\int_0^\infty e^{-(v-\gamma)t}dt$$

$$= \frac{M}{v}\left|\frac{e^{-(v-\gamma)t}}{-(v-\gamma)}\right|_0^\infty = \frac{M}{v}\frac{1}{v-\gamma} = \frac{M}{v(v-\gamma)} \ .$$

Teorem 3.

i. Let $\mathcal{A}\{u(t)\} = \bar{u}(v)$, then

$\mathcal{A}\{u(t+1)\} = \frac{e^v}{v}[\bar{u}(v) - u_0\bar{V}_0(v)], \quad u(0) = u_0$

Proof

i. $\mathcal{A}\{u(t+1)\} = \frac{1}{v}\int_0^\infty e^{-vt}\, u(t+1)dt$

$$= \frac{e^v}{v}\int_1^\infty e^{-v\tau} u(\tau)\, d\tau$$

$$= \frac{e^v}{v}\left[\bar{u}(v) - u(0)\int_0^1 e^{-v\tau} d\tau\right]$$

$$= \frac{e^v}{v}\left[\bar{u}(v) - u_0 \overline{V_0}(v)\right],$$

ii. $\mathcal{A}\{u(t+2)\} = \frac{e^v}{v}\left[\mathcal{A}\{u(t+1)\} - u(1)\overline{V_0}(v)\right]$

$$= \frac{e^{2v}}{v}\left[\bar{u}(v) - u(0)\overline{V_0}(v)\right] - e^v u_1 \overline{V_0}(v)$$

$$= \frac{e^{2v}}{v}\left[\bar{u}(v) - (u_0 + u_1 e^{-v}\overline{V_0}(v)\right] \qquad u(1) = u_1,$$

iii. $\mathcal{A}\{u(t+3)\} = \frac{e^{3v}}{v}\left[\bar{u}(v) - (u_0 + u_1 e^{-v} + u_2 e^{-2v})\overline{V_0}(v)\right],$

Generally

iv. $\mathcal{A}\{u(t+k)\} = \frac{e^{kv}}{v}\left(\bar{u}(v) - \overline{V_0}(v)\sum_{r=0}^{k-1} u_r e^{-rv}\right).$

2.3. Remark

Let $H(t)$ is Heaviside unit step function, then

$$V_n(t) = H(t-n) - H(t-n-1), \;\; n \le t < n+1.$$

The Aboodh Transform of $V_n(t)$ is

$$\overline{V_n}(v) = \mathcal{A}\{V_n(t)\}$$

$$= \frac{1}{v}\int_0^\infty e^{-vt}\{H(t-n) - H(t-n-1)\}dt$$

$$= \frac{1}{v}\int_n^{n+1} e^{-vt} dt$$

$$= \frac{1}{v^2}(1 - e^{-v})e^{-nv}$$

$$= \overline{V_0}(v)\, e^{-nv}.$$

Where

$$\overline{V_0}(v) = \frac{1}{v^2}(1 - e^{-v}).$$

2.4. The Aboodh Transform of Derivatives

Let $A(f(t))$ is Aboodh transform of $f(t)$, then

1. $\mathcal{A}[f'(t)] = vA(v) - \frac{f(0)}{v}$

2.$\mathcal{A}\ [f''(t)] = v^2A(v) - \frac{f'(0)}{v} - f(0)$

Generally

3. $\mathcal{A}[f^n(t)] = v^nA(v) - \sum_{k=0}^{n-1} \frac{f^k(0)}{v^{2-n+k}}$

4. (i) $\mathcal{A}\{tf(t)\} = -\frac{d}{dv}A(v) - \frac{1}{v}A(v)$

(ii) $\mathcal{A}\big(tf'(t)\big) = -\frac{d}{dv}\left[vA(v) - \frac{f(0)}{v}\right] - \frac{1}{v}\left[vA(v) - \frac{f(0)}{v}\right]$

(iii)$\mathcal{A}\{tf''(t)\} = -\frac{d}{dv}\left[v^2A(v) - \frac{f'(0)}{v} - f(0)\right] - \frac{1}{v}\left[v^2A(v) - \frac{f'(0)}{v} - f(0)\right]$

(iv) $\mathcal{A}\{t^2f'(t)\} = v\frac{d^2A(v)}{dv^2} + 2\frac{dA(v)}{dv} - 2\frac{f(0)}{v^3}$

(v) $\mathcal{A}\{t^2f''(t)\} = v^2\frac{d^2A(v)}{dv^2} + 4v\frac{dA(v)}{dv} + 2A(v) - 2\frac{f'(0)}{v^3}$.

Proof

1. $\mathcal{A}[f'(t)] = \frac{1}{v}\int_0^\infty e^{-vt}\, f'(t)dt$

$\mathcal{A}[f'(t)] = \frac{1}{v}\left[e^{-vt}f(t)|_0^\infty - \int_0^\infty -ve^{-vt}\, f(t)dt\right]$

$\mathcal{A}[f'(t)] = vA(v) - \frac{f(0)}{v}$

4. i) $\mathcal{A}[f(t)] = A(v) = \frac{1}{v}\int_0^\infty f(t)e^{-vt}dt \ \ t \geq 0, k_1 \leq v \leq k_2$

$$\frac{d}{dv}A(v) = \frac{d}{dv}\left(\frac{1}{v}\int_0^\infty f(t)e^{-vt}dt\ \right) = -\frac{1}{v^2}\int_0^\infty e^{-vt}f(t)dt - \frac{1}{v}\int_0^\infty te^{-vt}f(t)dt$$

$$-\frac{d}{dv}A(v) - -\frac{1}{v^2}\int_0^\infty f(t)e^{-vt}dt = \frac{1}{v}\int_0^\infty te^{-vt}f(t)dt$$

$$\mathcal{A}[tf(t)] = -\frac{d}{dv}A(v) - \frac{A(v)}{v}\ .$$

Others can be proof similarly.

2.5. Linearity Property of Aboodh Transforms

If $\mathcal{A}\{f(t)\} = A(v)$ and $\mathcal{A}\{g(t)\} = B(v)$ then

$\mathcal{A}\{af(t) + bg(t)\} = a\mathcal{A}\{f(t)\} + b\mathcal{A}\{f(t)\} = aA(v) + bB(v)$,

where a, b are arbitrary constants.

3. Applications

In this section, we will use the Aboodh transformation to find solutions of two the linear delay differential equations.

Application 1:

Consider linear delay differential eqaution

$$u'(t) = u(t-1), \tag{2}$$

with the initial conditions

$$u(0) = 1.$$

Solution:

Taking the Aboodh transform of equation (2), we get

$$\mathcal{A}[u'(t)] = \mathcal{A}[u(t-1)]$$

$$vu(v) - \frac{u(0)}{v} = \frac{e^{-v}}{v}[u(v) - u_0\bar{V}_0(v)]$$

$$vu(v) - \frac{1}{v} = \frac{e^{-v}}{v}\left[u(v) - \frac{1}{v^2}(1 - e^{-v})\right]$$

$$vu(v) - \frac{1}{v} = \frac{e^{-v}}{v}u(v) + \frac{e^{-2v}}{v^3} - \frac{e^{-v}}{v^3}$$

$$u(v)\left[v - \frac{e^{-v}}{v}\right] = \frac{1}{v} + \frac{e^{-v}}{v^3}(e^{-v} - 1)$$

$$u(v) = \frac{v^2 - e^{-v}}{v^2(v^2 - e^{-v})} + \frac{e^{-2v}}{v^2(v^2 - e^{-v})}$$

$$= \frac{1}{v^2} + \frac{e^{-2v}}{v^4\left(1 - \frac{e^{-v}}{v^2}\right)}$$

$$= \frac{1}{v^2} + \frac{e^{-2v}}{v^4}\left(1 + \frac{e^{-v}}{v^2} + \frac{e^{-2v}}{v^4} + \frac{e^{-3v}}{v^6} + \cdots + \frac{e^{-nv}}{v^{2n}} + \cdots\right)$$

$$= \frac{1}{v^2} + \frac{e^{-2v}}{v^4} + \frac{e^{-3v}}{v^6} + \frac{e^{-4v}}{v^8} + \frac{e^{-4v}}{v^{10}} + \cdots + \frac{e^{-nv}}{v^{2n}} + \cdots \qquad .$$

Where $u(v)$ is the Aboodh Transform of function $u(t)$. Now taking the inverse Aboodh Transform, we get

$$\mathcal{A}^{-1}[u(t)] = \mathcal{A}^{-1}\left[\frac{1}{v^2}\right] + \mathcal{A}^{-1}\left[\frac{e^{-2v}}{v^4}\right] + \mathcal{A}^{-1}\left[\frac{e^{-3v}}{v^6}\right] + \mathcal{A}^{-1}\left[\frac{e^{-4v}}{v^8}\right] + \mathcal{A}^{-1}\left[\frac{e^{-4v}}{v^{10}}\right] + \cdots$$

$$+ \mathcal{A}^{-1}\left[\frac{e^{-nv}}{v^{2n}}\right] + \cdots$$

$$u(t) = 1 + \frac{(t-2)^2}{2!} + \frac{(t-3)^4}{4!} + \frac{(t-4)^6}{6!} + \cdots + \frac{(t-n-1)^{2n}}{(2n)!} + \cdots \quad ,t > n$$

where

$$\mathcal{A}^{-1}\left\{\frac{e^{-av}}{v^{2n}}\right\} = \frac{(t-n-1)^{2n}}{\Gamma(2n+1)}\ .$$

Application 2:

Consider linear delay differential eqaution

$$u'(t) - \alpha u(t-1) = \beta, \tag{3}$$

with the initial conditions

$$u(0) = 0\ .$$

Solution:

Taking the Aboodh transform of equation (3), we get

$$\mathcal{A}[u'(t)] - \mathcal{A}[u(t-1)] = \mathcal{A}[\beta]$$

$$vu(v) - \frac{u(0)}{v} - \alpha\frac{e^{-v}}{v}[u(v) - u(0)\bar{u}(v)] = \frac{\beta}{v^2}$$

$$vu(v) - \alpha\frac{e^{-v}}{v}\big(u(v)\big) = \frac{\beta}{v^2}$$

$$u(v)\left(v - \alpha\frac{e^{-v}}{v}\right) = \frac{\beta}{v^2}$$

$$u(v) = \frac{\beta}{v^2}\left(\frac{v}{(v^2 - \alpha e^{-v})}\right)$$

$$u(v) = \frac{\beta}{v^3\left(1-\alpha\frac{e^{-v}}{v^2}\right)}$$

$$u(v) = \frac{\beta}{v^3}\left(1 + \frac{\alpha e^{-v}}{v^2} + \frac{\alpha^2 e^{-2v}}{v^4} + \frac{\alpha^3 e^{-3v}}{v^6} + \cdots + \frac{\alpha^n e^{-nv}}{v^{2n}} + \cdots\right)$$

$$u(v) = \beta\left(\frac{1}{v^3} + \frac{\alpha e^{-v}}{v^5} + \frac{\alpha^2 e^{-2v}}{v^7} + \frac{\alpha^3 e^{-3v}}{v^9} + \cdots + \frac{\alpha^n e^{-nv}}{v^{2n+3}} + \cdots\right).$$

Where $u(v)$ is the Aboodh Transform of function $u(t)$. Now taking the inverse Aboodh Transform, we get

$$\mathcal{A}^{-1}[u(v)] = \beta\left(\mathcal{A}^{-1}\left[\frac{1}{v^3}\right] + \mathcal{A}^{-1}\left[\frac{\alpha e^{-v}}{v^5}\right] + \mathcal{A}^{-1}\left[\frac{\alpha^2 e^{-2v}}{v^7}\right] + \mathcal{A}^{-1}\left[\frac{\alpha^3 e^{-3v}}{v^9}\right] + \cdots + \mathcal{A}^{-1}\left[\frac{\alpha^n e^{-nv}}{v^{2n+3}}\right] + \cdots\right)$$

$$u(t) = \beta\left[t + \frac{\alpha(t-1)^3}{3!} + \frac{\alpha^2(t-2)^5}{5!} + \frac{\alpha^3(t-3)^7}{7!} \ldots + \frac{\alpha^n(t-n)^{2n+1}}{(2n+1)!} + \cdots\right]$$

where

$$\mathcal{A}^{-1}\left\{\frac{e^{-av}}{v^{2n+3}}\right\} = \beta\left[\frac{(t-n)^{2n+1}}{\Gamma(2n+2)}\right].$$

4. Conclusion

The main features of the Aboodh transform are presented in this article. We applied a new integral transform, the Aboodh transform, to solve two the linear delay differential equations. We proved the existence of the Aboodh transformation. Some examples in applications are given to demostrate the effectiveness of Aboodh transform. As a result, the aboodh transform reveals that it is very effective, simple and can be applied to the linear delay differential equations.

Declaration of interest

This work does not have any conflicts of interest and there are no funders to report for this submission.

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