Boundary layer study of a nonlinear parabolic equation with a small parameter

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Abstract

This paper is concerned with the initial-boundary value problem for a nonlinear parabolic equation with a small parameter. The existence of a boundary layer as the parameter goes to zero is obtained together with the estimation on the thickness of the boundary layer. The main result extends an earlier work of Frid and Shelukhin (1999).

Keywords: Nonlinear parabolic equation; boundary layer; BL-thickness.
MSC: 35B40; 35K15; 35K65; 76N20.

1 Introduction

In this paper, we study the asymptotic behavior as $\mu \to 0^+$ of solutions of the initial-boundary value problem for the parabolic equation:

$$\begin{cases} \partial_t u = \mu \partial_x \Big((|\partial_x u|^2 + 1)^{\frac{p-2}{2}} \partial_x u \Big), & (x,t) \in Q_T = (0,1) \times (0,T), \\ u(0,t) = u_1(t), \quad u(1,t) = u_2(t), & t \in (0,T), \\ u(x,0) = u_0(x), \quad x \in (0,1), \end{cases} \tag{1.1}$$

where $\mu > 0$, $p > 1$ and $T > 0$ are constants. Such problems appear in non-Newtonian fluids, turbulent flows in porous media, glaciology and other contexts (cf. [1–3]). In the field of viscous fluid, μ is the so-called viscous coefficient describing a nonlinear relation between the internal stress $\sigma(\partial_x u) = \mu(|\partial_x u|^2 + 1)^{\frac{p-2}{2}} \partial_x u$ and the deformation velocity $\partial_x u$ (u is the fluid velocity), and it is noted that the cases $p > 2, p = 2$ and $p < 2$ physically correspond to where the fluid is of dilatant type, Newtonian and pseudo-plastic type, respectively (cf. [1, 3, 4]).

In [5, Section 4], Frid and Shelukhin studied the boundary layer as the shear viscosity $\mu \to 0$ of solutions of initial-boundary value problem for the following Navier-Stokes equations of incompressible flows with cylindrical symmetry:

$$\partial_t v = \mu \partial_x \Big(\partial_x v + \frac{v}{x} \Big), \quad 0 < a < x < b, \quad t > 0,$$

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and

$$\partial_t w = \mu\Big(\partial_x^2 w + \frac{\partial_x w}{x}\Big), \quad 0 < a < x < b, \quad t > 0,$$

where μ is the viscosity coefficient, and v and w represent the angular velocity and axial velocity respectively. They justified the zero viscosity limit as $\mu \to 0^+$ and established the value $O(\mu^\alpha)$ with $\alpha \in (0, 1/2)$ for the boundary layer thickness. Motivated by their work, the main purpose of this paper is to extend their result to problem (1.1) (see the following Theorem 1.2).

It should be pointed out that the theory of boundary layers has been one of the fundamental and important issues in fluid dynamics since the seminal work by Prandtl in 1904 (cf. [6]). There are many papers dedicated to the questions of boundary layers, see for instance [5, 7–11] for the Navier-Stokes equations, [12–14] for the MHD equations, [15–19] for the primitive equations, and [20–22] for some nonlinear evolution equations. Moreover, the boundary layer problem also arises in the theory of hyperbolic systems when parabolic equations with small viscosity are applied as perturbations, see for instance [23–25].

Formally setting $\mu \to 0$ in (1.1), one obtains the following

$$\partial_t v(x,t) = 0, \quad v(x,0) = u_0(x),$$

which implies that $v(x,t) \equiv u_0(x)$.

Now we define the concept of a BL-thickness based on the sprit of [5].

Definition 1.1. *A function $\delta(\mu)$ is called a BL-thickness for problem* (1.1) *with vanishing μ if $\delta(\mu) \downarrow 0$ as $\mu \downarrow 0$, and*

$$\lim_{\mu \to 0^+} \|u - u_0\|_{L^\infty(0,T;L^\infty(\delta(\mu),1-\delta(\mu)))} = 0,$$
$$\inf \lim_{\mu \to 0^+} \|u - u_0\|_{L^\infty(0,T;L^\infty(0,1))} > 0.$$

Remark 1.1. *Clearly, this definition does not determine the BL-thickness uniquely, since any function $\delta_*(\mu)$ satisfying the inequality $\delta_*(\mu) \geq \delta(\mu)$ is also a BL-thickness.*

To make the proof of the existence of a BL-thickness simpler, we shall mainly discuss problem (1.1) with the following initial data:

$$u_0(x) \equiv 0, \quad \forall x \in [0,1]. \tag{1.2}$$

The main result of this paper is as follows.

Theorem 1.2. *Let $p > 1$ and $u_1(t), u_2(t) \in C^1[0,T]$ with $u_1(0) = u_2(0) = 0$. Then*

(a) For any $\mu \in (0,1)$, problem (1.1) *and* (1.2) *admits a unique classical solution $u = u^\mu$ satisfying, for some $C > 0$ independent of μ,*

$$\begin{cases} \displaystyle\sup_{0<t<T} \int_0^1 |\partial_x u(x,t)|^p dx \leq C\mu^{\frac{1-p}{p}}, \\ \displaystyle\sup_{0<t<T} \int_0^1 u^2(x,t) dx \leq C\mu^{\frac{1}{p}}. \end{cases} \tag{1.3}$$

Moreover, if $(u_1, u_2) \not\equiv (0,0)$ in $(0,T)$, then

(b) Any function $\delta(\mu)$ satisfying $\delta(\mu) \downarrow 0$ and $\mu^{1/p}/\delta(\mu) \to 0$ as $\mu \downarrow 0$ is a BL-thickness for problem (1.1) *and* (1.2) *with vanishing μ.*

(c) Any function $\delta(\mu)$ satisfying $\delta(\mu) \downarrow 0$ and $\delta(\mu) = o(\mu^{1/p})$ as $\mu \downarrow 0$ is not a BL-thickness of problem (1.1) *and* (1.2) *with vanishing μ.*

In Section 2, we will prove Theorem 1.2. A summary of the conclusions will be given in Section 3.

2 Proof of Theorem 1.2

Under the assumptions of Theorem 1.2, from Theorem 4.1 of Chapter VI in [26] it follows that for any fixed $\mu > 0$, problem (1.1) and (1.2) admits a unique classical solution $u := u^\mu$. Moreover, $\|u\|_{L^\infty(Q_T)} \le M = \max_{0\le t\le T}(|u_2(t)| + |u_1(t)|)$ by the comparison theorem. Next we will show the following Lemma 2.1, which will paly an important role in the proof of our main results (i.e. Theorem 1.2).

In what follows, we use C to denote a positive generic constant independent of μ.

Lemma 2.1. *Under the assumptions of Theorem 1.2, we have*

$$\begin{cases} \sup\limits_{0<t<T} \displaystyle\int_0^1 \Big(|\partial_x u|^2 + 1\Big)^{\frac{p}{2}} dx + \int_0^T \Big(\|\partial_x u\|^2_{L^\infty(0,1)} + 1\Big)^{\frac{p-1}{2}} dt \le C\mu^{\frac{1-p}{p}}, \\ \sup\limits_{0<t<T} \displaystyle\int_0^1 u^2(x,t)dx + \int_0^T\int_0^1 (\partial_t u)^2 dxdt \le C\mu^{\frac{1}{p}}. \end{cases} \tag{2.1}$$

Proof. For convenience, denote $\Phi(s)$ by $\Phi(s) = (s^2+1)^{(p-2)/2}s$. Multiplying $(1.1)_1$ by $\mu\partial_x(\Phi(\partial_x u))$ and integrating over Q_t, we have

$$\mu \int_0^t\int_0^1 \partial_t u \partial_x(\Phi(\partial_x u)) = \mu^2 \int_0^t\int_0^1 |\partial_x(\Phi(\partial_x u))|^2 dxdt. \tag{2.2}$$

Using integration by parts and noticing $u_0 \equiv 0$, we obtain

$$\begin{aligned} &\mu \int_0^t\int_0^1 \partial_t u \partial_x(\Phi(\partial_x u))\, dxdt \\ &= \frac{\mu}{p} - \frac{\mu}{p}\int_0^1 \Big(|\partial_x u|^2 + 1\Big)^{\frac{p}{2}} dx + \mu \int_0^t \Big(\partial_t u \Phi(\partial_x u)\Big)\Big|_{x=0}^{x=1} dt \\ &\le C\mu - \frac{\mu}{p}\int_0^1 \Big(|\partial_x u|^2 + 1\Big)^{\frac{p}{2}} dx + C\mu \int_0^t \Big(\|\partial_x u\|^2_{L^\infty(0,1)} + 1\Big)^{\frac{p-1}{2}} dt. \end{aligned} \tag{2.3}$$

Now we estimate the final term in the right hand side of (2.3). By the mean value theorem, for every $t \in (0,T)$ there exists some $\xi_t \in (0,1)$ such that

$$\partial_x u(\xi_t, t) = u(1,t) - u(0,t) = u_2(t) - u_1(t),$$

so, for any constant $q > 0$, we obtain

$$\begin{aligned} \Big(|\partial_x u(x,t)|^2 + 1\Big)^{\frac{q}{2}} &= \Big(|\partial_x u(\xi_t,t)|^2 + 1\Big)^{\frac{q}{2}} + \int_{\xi_t}^x \partial_x\Big(|\partial_x u(x,t)|^2 + 1\Big)^{\frac{q}{2}} dx \\ &\le C + C\int_0^1 \Big|\Big(|\partial_x u|^2 + 1\Big)^{\frac{q-2}{2}} \partial_x u \partial_x^2 u\Big| dx \\ &\le C + C\int_0^1 \Big|\Big(|\partial_x u|^2 + 1\Big)^{\frac{p-2}{2}} \partial_x^2 u\Big| \Big(|\partial_x u|^2 + 1\Big)^{\frac{q-p+1}{2}} dx. \end{aligned} \tag{2.4}$$

Noticing $p > 1$ and using the facts:

$$\begin{cases} 1 + (p-2)\dfrac{|\partial_x u|^2}{|\partial_x u|^2 + 1} \in [1, p-1], \quad (p > 2), \\ 1 + (p-2)\dfrac{|\partial_x u|^2}{|\partial_x u|^2 + 1} \in [p-1, 1], \quad (1 < p \le 2), \end{cases}$$

and

$$\partial_x\left(\Phi(\partial_x u)\right)=\Big(|\partial_x u|^2+1\Big)^{(p-2)/2}\left[1+(p-2)\frac{|\partial_x u|^2}{|\partial_x u|^2+1}\right]\partial_x^2 u, \tag{2.5}$$

we have

$$\begin{aligned}\Big(|\partial_x u|^2+1\Big)^{\frac{p-2}{2}}|\partial_x^2 u| &\le C\Big(|\partial_x u|^2+1\Big)^{\frac{p-2}{2}}\left[1+(p-2)\frac{|\partial_x u|^2}{|\partial_x u|^2+1}\right]|\partial_x^2 u|\\ &= C|\partial_x\left(\Phi(\partial_x u)\right)|.\end{aligned}$$

Substituting it into (2.4) yields

$$\Big(|\partial_x u(x,t)|^2+1\Big)^{\frac{q}{2}}\le C+C\int_0^1|\partial_x\left(\Phi(\partial_x u)\right)|\Big(|\partial_x u|^2+1\Big)^{\frac{q-p+1}{2}}dx. \tag{2.6}$$

Taking $q=\frac{3p-2}{2}$ in (2.6) and using Hölder inequality, we obtain

$$\Big(\|\partial_x u\|^2_{L^\infty(0,1)}+1\Big)^{\frac{3p-2}{4}}\le C+C\left(\int_0^1|\partial_x\left(\Phi(\partial_x u)\right)|^2\right)^{\frac{1}{2}}\left(\int_0^1\Big(|\partial_x u|^2+1\Big)^{\frac{p}{2}}dx\right)^{\frac{1}{2}}.$$

Consequently,

$$\begin{aligned}&\Big(\|\partial_x u\|^2_{L^\infty(0,1)}+1\Big)^{\frac{p-1}{2}}\\ &\le C+C\left(\int_0^1\big|\partial_x\big(\Phi(U)\big)\big|^2dx\right)^{\frac{p-1}{3p-2}}\left(\int_0^1\Big(|\partial_x u|^2+1\Big)^{\frac{p}{2}}dx\right)^{\frac{p-1}{3p-2}},\end{aligned}$$

which together with Young inequality give

$$\begin{aligned}&\mu\Big(\|\partial_x u\|^2_{L^\infty(0,1)}+1\Big)^{\frac{p-1}{2}}\\ &\le C\mu+C\mu^{\frac{1}{3p-2}}\left(\mu^2\int_0^1\big|\partial_x\big(\Phi(U)\big)\big|^2dx\right)^{\frac{p-1}{3p-2}}\left(\mu\int_0^1\Big(|\partial_x u|^2+1\Big)^{p/2}dx\right)^{\frac{p-1}{3p-2}}\\ &\le C\mu+C_\epsilon\mu^{\frac{1}{p}}+\epsilon\mu^2\int_0^1\big|\partial_x\big(\Phi(U)\big)\big|^2dx+\mu\int_0^1\Big(|\partial_x u|^2+1\Big)^{\frac{p}{2}}dx,\quad\forall\epsilon>0.\end{aligned}\tag{2.7}$$

Combining (2.7) with (2.2) and (2.3), taking a sufficient small $\epsilon>0$ and using Gronwall inequality, we obtain

$$\mu\int_0^1\Big(|\partial_x u|^2+1\Big)^{\frac{p}{2}}dx+\mu^2\int_0^t\int_0^1|\partial_x\big(\Phi(\partial_x u)\big)|^2dxdt\le C\mu^{1/p},\quad\forall t\in[0,T].$$

It follows from (2.7) and $(1.1)_1$ that

$$\mu\int_0^T\Big(\|\partial_x u\|^2_{L^\infty(0,1)}+1\Big)^{\frac{p-1}{2}}dt\le C\mu^{1/p}, \tag{2.8}$$

and

$$\int_0^T\int_0^1(\partial_t u)^2dxdt=\mu^2\int_0^T\int_0^1|\partial_x\big(\Phi(\partial_x u)\big)|^2dxdt\le C\mu^{1/p}.$$

Multiplying $(1.1)_1$ by u and integrating over Q_t, we obtain

$$\int_0^t\int_0^1 u\partial_t u dxdt = \mu \int_0^t\int_0^1 u\partial_x \left(\Phi(\partial_x u)\right) dxdt.$$

By using integration by parts and (2.8), we have

$$\begin{aligned}\int_0^1 u^2 dx =& -2\mu \int_0^t\int_0^1 \Phi(\partial_x u)\partial_x u dxdt + 2\mu \int_0^t \Phi(\partial_x u)u\Big|_{x=0}^{x=1} dt\\ \leq& C\mu \int_0^t \left(\|\partial_x u\|^2_{L^\infty(0,1)} + 1\right)^{\frac{p-1}{2}} dt \leq C\mu^{\frac{1}{p}},\end{aligned}$$

where we have used $\Phi(\partial_x u)\partial_x u \geq 0$. Thus, (2.1) is justified, and the proof is completed. □

Denote φ_ε for $\varepsilon \in (0,1)$ and ξ_δ for $\delta \in (0,1/2)$ by

$$\varphi_\varepsilon(s) = \sqrt{s^2+\varepsilon^2}, \quad \xi_\delta(x) = \begin{cases} x, & 0 \leq x < \delta, \\ \delta, & \delta \leq x < 1-\delta, \\ 1-x, & 1-\delta \leq x \leq 1. \end{cases}$$

It is easy to check that φ_ε satisfies

$$\begin{cases} |s| \leq |\varphi_\varepsilon(s)| \leq |s| + \varepsilon, \\ \varphi''_\varepsilon(s) \geq 0, \\ \lim\limits_{\varepsilon\to 0} \varphi'_\varepsilon(s) = sgn(s), \quad \forall s \in \mathbb{R}, \end{cases}$$

where

$$sgn(s) = \begin{cases} 1, & s > 0, \\ 0, & s = 0, \\ -1, & s < 0. \end{cases}$$

Now we give the proof of Theorem 1.2 as follows.

Proof. For convenience, we write $U = u_x$. Differentiating (1.1) with respect to x, we have

$$U_t = \mu\big(\Phi(U)\big)_{xx}, \tag{2.9}$$

where $\Phi(s) = (s^2+1)^{(p-2)/2}s$. Multiplying (2.9) by $\varphi'_\varepsilon\left(\Phi(U)\right)\xi_\delta(x)$ and integrating over Q_T, we have

$$\begin{aligned}&\int_0^T\int_0^1 U_t\varphi'_\varepsilon\left(\Phi(U)\right)\xi_\delta(x)dxdt\\ &= \mu \int_0^T\int_0^1 \big(\Phi(U)\big)_{xx}\varphi'_\varepsilon\left(\Phi(U)\right)\xi_\delta(x)dxdt\\ &= -\mu \int_0^T\int_0^1 \big(\Phi(U)\big)_x^2\varphi''_\varepsilon(\Phi(U))\xi_\delta(x)dxdt - \mu\int_0^T\int_0^1 \big(\Phi(U)\big)_x\varphi'_\varepsilon\left(\Phi(U)\right)\xi'_\delta(x)dxdt\\ &=: I_1 + I_2.\end{aligned} \tag{2.10}$$

Noticing $\varphi''_\varepsilon \geq 0$, we obtain

$$I_1 \leq 0. \tag{2.11}$$

Using integration by parts and noticing $|\varphi_\varepsilon(s)| \le |s| + \varepsilon$, we have

$$\begin{aligned} I_2 = & -\mu \int_0^T \int_0^\delta \big(\Phi(U)\big)_x \varphi_\varepsilon' \left(\Phi(U)\right) dxdt \\ & + \mu \int_0^T \int_{1-\delta}^1 \big(\Phi(U)\big)_x \varphi_\varepsilon' \left(\Phi(U)\right) dxdt \\ \le & \, \mu \int_0^T \Big[\varphi_\varepsilon\big(\Phi(U)\big)\Big|_{x=0} + \varphi_\varepsilon\left(\Phi(U)\right)\Big|_{x=1}\Big] dt \\ \le & \, C\mu + \mu \int_0^T \Big(\|\partial_x u\|^2_{L^\infty(0,1)} + 1\Big)^{\frac{p-1}{2}} dt \\ \le & \, C\mu^{1/p}. \end{aligned} \tag{2.12}$$

Combining (2.10) with (2.11) and (2.12), we obtain

$$\int_0^t \int_0^1 U_t \varphi_\varepsilon' \left(\Phi(U)\right) \xi_\delta(x) dxds \le C\mu^{1/p},$$

therefore,

$$\int_0^1 \left(\int_0^{U(x,t)} \varphi_\varepsilon' \left(\Phi(s)\right) ds\right) \xi_\delta(x) dx \le C\mu^{1/p}.$$

Letting $\varepsilon \to 0^+$ and using $\lim_{\varepsilon\to 0} \varphi_\varepsilon'(s) = sgn(s)$, we have

$$\int_0^1 \int_0^{U(x,t)} sgn\left(\Phi(s)\right) \xi_\delta(x) dx \le C\mu^{1/p},$$

consequently,

$$\begin{aligned} \int_0^1 |U(x,t)| \xi_\delta(x) dx &= \int_0^1 \int_0^{U(x,t)} sgn(s) \xi_\delta(x) dx \\ &= \int_0^1 \int_0^{U(x,t)} sgn\left(\Phi(s)\right) \xi_\delta(x) dx \\ &\le C\mu^{1/p}, \end{aligned}$$

namely,

$$\int_0^1 |\partial_x u(x,t)| \xi_\delta(x) dx \le C\mu^{1/p}.$$

This implies that for any $\delta \in (0, 1/2)$,

$$\sup_{0<t<T} \int_\delta^{1-\delta} |\partial_x u(x,t)| dx \le C\frac{\mu^{1/p}}{\delta}. \tag{2.13}$$

On the other hand, we have, by using the embedding $W^{1,1} \hookrightarrow L^\infty$ and (2.13), that for any $\delta \in (0, 1/4)$,

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty(\delta,1-\delta)} \le & C\int_0^1 |u| dx + C\int_\delta^{1-\delta} |\partial_x u| dx \\ \le & C\left(\mu^{\frac{1}{p}} + \frac{\mu^{1/p}}{\delta}\right). \end{aligned}$$

Hence, for any function $\delta(\mu)$ satisfying $\delta(\mu) \downarrow 0$ and $\mu^{1/p}/\delta(\mu) \to 0$ as $\mu \downarrow 0$, we have

$$\lim_{\mu \to 0^+} \|u\|_{L^\infty(0,T;L^\infty(\delta(\mu),1-\delta(\mu)))} = 0.$$

Moreover, $\liminf_{\mu\to 0^+} \|u\|_{L^\infty(0,T;L^\infty(0,1))} > 0$ since $(u_1(t), u_2(t)) \not\equiv (0,0)$ in $(0,T)$. This ends the proof of the conclusion (b).

Finally, we show the conclusion (c). To see this, suppose, on the contrary, that some function $\delta(\mu)$ satisfying $\delta(\mu) \downarrow 0$ and $\delta(\mu) = o(\mu^{1/p})$ as $\mu \downarrow 0$ is a BL-thickness of problem (1.1) and (1.2) with vanishing μ. Then we first obtain from the definition of a BL-thickness that $\lim_{\mu\to 0^+} \|u\|_{L^\infty(0,T;L^\infty(\delta(\mu),1-\delta(\mu))} = 0$. In particular, we have

$$\lim_{\mu \to 0^+} \left[\|u(\delta(\mu),t)\|_{L^\infty(0,T)} + \|u(1-\delta(\mu),t)\|_{L^\infty(0,T)}\right] = 0.$$

On the other hand, by using Hölder inequality and Lemma 2.1, we obtain

$$\begin{aligned}
\int_{1-\delta(\mu)}^1 |u_x|dx + \int_0^{\delta(\mu)} |u_x|dx &\le 2\left(\int_0^1 |u_x|^p dx\right)^{1/p} (\delta(\mu))^{(p-1)/p} \\
&= 2\left(\mu^{\frac{p-1}{p}} \int_0^1 |u_x|^p dx\right)^{1/p} \left(\frac{\delta(\mu)}{\mu^{1/p}}\right)^{(p-1)/p} \\
&\le C\left(\frac{\delta(\mu)}{\mu^{1/p}}\right)^{(p-1)/p} \\
&\to 0 \ \ (\mu \to 0^+),
\end{aligned}$$

which together with Newton-Lebnitz formula give

$$\|u_1\|_{L^\infty(0,T)} \le \|u(\delta(\mu),t)\|_{L^\infty(0,T)} + \sup_{0<t<T} \int_0^{\delta(\mu)} |u_x|dx \to 0 \ \ (\mu \to 0^+),$$

$$\|u_2\|_{L^\infty(0,T)} \le \|u(1-\delta(\mu),t)\|_{L^\infty(0,T)} + \sup_{0<t<T} \int_{1-\delta(\mu)}^1 |u_x|dx \to 0 \ \ (\mu \to 0^+),$$

so $(u_1, u_2) \equiv 0$ on $(0,T)$. This leads to a contradiction, and ends the proof of (c).

Thus, the proof of Theorem 1.2 is completed.

3 Conclusions

In this paper, we study the boundary layer behavior of solutions of problem (1.1) and (1.2) as $\mu \to 0$, and establish the value $O(\mu^\alpha)$ with any $\alpha \in (0, 1/p)$ for the boundary layer thickness (see Theorem 1.2). Thus, we extend the corresponding work by Frid and Shelukhin [5].

Exactly, one can see from the conclusions (b) and (c) in Theorem 1.2 that for any small $\varepsilon > 0$, the function $\delta_1(\mu) = \mu^{1/p-\varepsilon}$ is a BL-thickness, while the function $\delta_2(\mu) = \mu^{1/p+\varepsilon}$ is not. However, we do not know whether the function $\delta(\mu) = \mu^{1/p}$ is a BL-thickness. This will be a problem to be pursued by the authors in the future.

□

Acknowledgments

This work is supported in part by the NSFC (12071058, 11971497), Natural Science Foundation of Guangdong Province (2019B151502041, 2020B1515310004) and the Program for Liaoning Innovative Talents in University (LR2016004).

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