

p-th mean pseudo almost automorphic solutions of class *r* under the light of measure theory

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Abstract. The objective in this work is to present a new concept of *p*-th mean pseudo almost automorphic by use of the measure theory. We use the (μ, ν) -ergodic process to define the spaces of (μ, ν) -pseudo almost automorphic process of class *r* in the *p*-th sense. To do this, firstly we show some interesting results regarding the completeness and composition theorems. Secondly we study the existence, uniqueness of the *p*-th mean (μ, ν) -pseudo almost automorphic solution of class *r* for the stochastic evolution equation.

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1. Introduction

In this work, we study some properties of the *p*-th mean (μ, ν) -pseudo almost automorphic process using the measure theory and we use those results to study the following stochastic evolution equations in a Hilbert space *H*,

$$dx(t) = [Ax(t) + L(x_t) + f(t)]dt + g(t)dW(t), \text{ for } t \in \mathbb{R} \quad (1.1)$$

where $A : D(A) \subset H$ is the infinitesimal generator of a C_0 -semigroup $(T(t))_{t \geq 0}$ on *H* such that

$$\|T(t)\| \leq Me^{-\omega t}, \text{ for } t \geq 0,$$

for some $M, \omega > 0$, $f : \mathbb{R} \rightarrow L^p(\Omega, H)$ and $g : \mathbb{R} \rightarrow L^p(\Omega, H)$ are appropriate functions specified later, and $W(t)$ is a two-sided standard Brownian motion with values in *H*.

$\mathcal{C} = C([-r, 0], L^p(\Omega, H))$ denotes the space of continuous functions from $[-r, 0]$ to $L^p(\Omega, H)$ endowed with the uniform topology norm. For every $t \geq 0$, x_t denotes the history function of \mathcal{C} defined by $x_t(\theta) = x(t + \theta)$ for $-r \leq \theta \leq 0$.

We assume $(H, \|\cdot\|)$ is a real separable Hilbert space and $L^p(\Omega, H)$ is the space of all *H*-valued random variables *x* such that

$$\mathbb{E}\|x\|^p = \int_{\Omega} \|x\|^p dP < \infty.$$

The concept of almost automorphy is a generalization of the classical periodicity. It was introduced in literature by Bochner. This work is an extension of [11] whose authors had studied equation (1.1) in the deterministic case. Some recent contributions concerning *p*-th mean pseudo

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almost automorphic for abstract differential equations similar to equation (1.1) have been made. For example [12, 25] the authors studied equation (1.1) without operator L . They showed that equation has unique p -th mean μ -pseudo almost periodic and μ -pseudo almost automorphic solutions on \mathbb{R} when f, g are p -th mean pseudo almost periodic or p -th mean pseudo almost automorphic functions.

This work is organized as follows, in section 2, we give the spectral decomposition of the phase space, in section 3, we study p -th mean (μ, ν) -ergodic process of class r , in section 4, we study p -th mean (μ, ν) -pseudo almost automorphic process and we discuss the existence and uniqueness of p -th mean (μ, ν) -pseudo almost automorphic solution of class r , the last section is devoted to application.

2. Spectral decomposition

To equation (1.1), associate the following initial value problem

$$\begin{cases} du_t = [Au_t + Lu_t + f(t)]dt + g(t)dW(t) \text{ for } t \geq 0 \\ u_0 = \varphi \in C = C([-r, 0], L^p(\Omega, H)), \end{cases} \quad (2.1)$$

where $f : \mathbb{R}^+ \rightarrow L^p(\Omega, H)$ and $g : \mathbb{R}^+ \rightarrow L^p(\Omega, H)$ are two stochastic processes continuous.

Definition 2.1. We say that a continuous function $u : [-r, +\infty[\rightarrow L^p(\Omega, H)$ is an integral solution of equation, if the following conditions hold :

- (1) $\int_0^t u(s)ds \in D(A)$ for $t \geq 0$,
- (2) $u(t) + A \int_0^t u(s)ds + \int_0^t (L(u_s) + f(s))ds + \int_0^t g(s)dW(s)$ for $t \geq 0$,
- (3) $u_0 = \varphi$.

If $\overline{D(A)} = L^p(\Omega, H)$, the integral solution coincide with the know mild solutions. One can see that if $u(t)$ is an integral solution of equation (2.1), then $u(t) \in \overline{D(A)}$ for all $t \geq 0$, in particular $\varphi(0) \in \overline{D(A)}$. Let us introduce the part A_0 of the operator A which defined by

$$\begin{cases} D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\} \\ A_0x = Ax \text{ for } x \in D(A_0). \end{cases}$$

We make the following assumption.

H₀ A satisfies the Hille-Yosida condition.

Proposition 2.2. A_0 generates a strongly continuous semigroup $(T_0(t))_{t \geq 0}$ on $\overline{D(A)}$. The phase C_0 of equation (2.1) is defined by

$$C_0 = \{\varphi \in C : \varphi(0) \in \overline{D(A)}\}.$$

For each $t \geq 0$, we define the linear operator $\mathcal{U}(t)$ on C_0 by

$$\mathcal{U}(t) = v_t(., \varphi),$$

where $v(., \varphi)$ is the solution of the following homogeneous equation

$$\begin{cases} \frac{d}{dt}v(t) = Av(t) + L(v_t) \text{ for } t \geq 0 \\ v_0 = \varphi \in C \end{cases}$$

Proposition 2.3. $(\mathcal{U}(t))_{t \geq 0}$ is strongly continuous semigroup of linear operators on C_0 . Moreover $(\mathcal{U}(\sqcup))_{t \geq 0}$ satisfies for $t \geq 0$ and $\theta \in [-r, 0]$ the following translation property

$$(\mathcal{U}(t))_{\geq 0} = \begin{cases} (\mathcal{U}(t + \theta)\varphi)(0) \text{ for } t + \theta \geq 0. \\ \varphi(t + \theta) \text{ for } t + \theta \leq 0. \end{cases}$$

Proposition 2.4. [23] Let $\mathcal{A}_{\mathcal{U}}$ defined on C_0 by

$$\begin{cases} D(\mathcal{A}_{\mathcal{U}}) = \{\varphi \in C^1([-r, 0]; X), \varphi(0) \in (D(A), \varphi(0)' \in \overline{D(A)}) \text{ and } \varphi(0)' = A\varphi(0) + L(\varphi)\} \\ \mathcal{A}_{\mathcal{U}}\varphi = \varphi' \in D(\mathcal{A}_{\mathcal{U}}) \end{cases}$$

Then $\mathcal{A}_{\mathcal{U}}$ is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_{t \geq 0}$ on C_0 . Let $\langle X_0 \rangle$ be the space defined by

$$\langle X_0 \rangle = \{X_0 c : c \in X\},$$

where the function $X_0 c$ is defined by

$$(X_0 c)(\theta) = \begin{cases} 0 & \text{if } \theta \in [-r, 0[\\ c & \text{if } \theta = 0. \end{cases}$$

The space $C_0 \oplus \langle X_0 \rangle$ equipped with the norm $|\phi + X_0 c|_{\mathcal{C}} = |\phi|_{\mathcal{C}} + |c|$ for $(\phi, c) \in C_0 \times X$ is a Banach space. Consider the extension $\widetilde{\mathcal{A}}_{\mathcal{U}}$ defined on $C_{\alpha} \oplus \langle X_0 \rangle$ by

$$\begin{cases} D(\widetilde{\mathcal{A}}_{\mathcal{U}}) = \{\varphi \in C^1([-r, 0], X) : \varphi(0) \in D(A) \text{ and } \varphi(0)' \in \overline{D(A)}\} \\ \widetilde{\mathcal{A}}_{\mathcal{U}}\varphi = X_0(A\varphi(0) + L(\varphi) - \varphi(0)'). \end{cases}$$

Lemma 2.5. [24] Assume that (\mathbf{H}_0) holds. Then, $\widetilde{\mathcal{A}}_{\mathcal{U}}$ satisfies the Hile-Yosida condition on $C_0 \oplus \langle X_0 \rangle$ there exist $\widetilde{M} \geq 0$, $\widetilde{\omega} \in \mathbb{R}$ such that $]\widetilde{\omega}, +\infty[\subset \rho(\widetilde{\mathcal{A}}_{\mathcal{U}})$ and

$$|(\lambda I - \widetilde{\mathcal{A}}_{\mathcal{U}})^{-n}| \leq \frac{\widetilde{M}}{(\lambda - \widetilde{\omega})^n} \text{ for } n \in \mathbb{N} \text{ and } \lambda > \widetilde{\omega}.$$

Moreover, the part of $\widetilde{\mathcal{A}}_{\mathcal{U}}$ on $D(\widetilde{\mathcal{A}}_{\mathcal{U}}) = C_0$ is exactly the operator $\widetilde{\mathcal{A}}_{\mathcal{U}}$.

Definition 2.6. We say a semigroup, $(\mathcal{U}(t))_{t \geq 0}$ is hyperbolic if

$$\sigma(\mathcal{A}_{\mathcal{U}}) \cap i\mathbb{R} = \emptyset.$$

For the sequel, we make the following assumption :

(H₁) $(T(t))_{t \geq 0}$ is compact on $\overline{D(A)}$ for $t > 0$.

Proposition 2.7. Assume that (\mathbf{H}_0) and (\mathbf{H}_1) hold. Then the semigroup $(\mathcal{U}(t))_{t \geq 0}$ is compact for $t > r$.

We get the following result on the spectral decomposition of the phase space C_0 .

Proposition 2.8. Assume that (\mathbf{H}_0) and (\mathbf{H}_1) hold. If the semigroup $\mathcal{U}(t)_{t \geq 0}$ is hyperbolic, then the space C_0 is decomposed as a direct sum

$$C_0 = S \oplus U$$

of two $\mathcal{U}(t)$ invariant closed subspaces S and U such that the restriction of $(\mathcal{U}(t))_{t \geq 0}$ on U is a group and there exist positive constants \overline{M} and ω such that

$$|\mathcal{U}(t)\varphi| \leq \overline{M}e^{-\omega t}|\varphi| \text{ for } t \geq 0 \text{ and } \varphi \in S,$$

$$|\mathcal{U}(t)\varphi| \leq \overline{M}e^{-\omega t}|\varphi| \text{ for } t \leq 0 \text{ and } \varphi \in U,$$

where S and U are called respectively the stable and unstable space, Π^s and Π^u denote respectively the projection operator on S and U .

3. (μ, ν) -ergodic process in p -th mean sense of class r

Let \mathcal{N} the Lebesgue σ -field of \mathbb{R} and by \mathcal{M} the set of all positive measures μ on \mathcal{N} satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < \infty$ for all $a, b \in \mathbb{R}$ ($a \leq b$). Let $p \geq 2$. $L^p(\Omega, H)$ is a Hilbert space with the following norm

$$\|x\|_{L^p} = \left(\int_{\Omega} \|x\|^p dP \right)^{\frac{1}{p}}$$

Definition 3.1. [20] Let $x : \mathbb{R} \rightarrow L^p(\Omega, H)$ be a stochastic process.

(1) x said to be stochastically bounded in p -th mean sense, if there exists $M > 0$ such that

$$\mathbb{E}\|x(t)\|^p \leq M \text{ for all } t \in \mathbb{R}.$$

(2) x said to be stochastically continuous in p -th mean sense if

$$\lim_{t \rightarrow s} \mathbb{E}\|x(t) - x(s)\|^p \leq M \text{ for all } t, s \in \mathbb{R}.$$

Let $BC(\mathbb{R}, L^p(\Omega, H))$ denote the space of all the stochastically bounded continuous processes.

Remark 3.2. [20] $(BC(\mathbb{R}, L^p(\Omega, H)), \|\cdot\|_{\infty})$ is a Banach space, where

$$\|x\|_{\infty} = \sup_{t \in \mathbb{R}} (\mathbb{E}(\|x(t)\|^p))^{\frac{1}{p}}$$

Definition 3.3. Let $\mu, \nu \in \mathcal{M}$. A stochastic process f is said to be (μ, ν) -ergodic in p -th ($p \geq 2$) mean sense, if $f \in BC(\mathbb{R}, L^p(\Omega, H))$ and satisfies

$$\lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \mathbb{E}\|f(t)\|^p d\mu(t) = 0.$$

We denote by $\mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu)$, the space of all such process.

Proposition 3.4. Let $\mu, \nu \in \mathcal{M}$. Then $\mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu)$ is a Banach space with the supremum norm $\|\cdot\|_{\infty}$.

Definition 3.5. Let $\mu, \nu \in \mathcal{M}$. A stochastic process f is said to be p -th mean (μ, ν) -ergodic of class r if $f \in BC(\mathbb{R}, L^p(\Omega, H))$ and satisfies

$$\lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \mathbb{E}\|f(\theta)\|^p d\mu(t) = 0.$$

We denote by $\mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$, the space of all such process. For $\mu \in \mathcal{M}$ and $a \in \mathbb{R}$, we denote μ_a the positive measure on $(\mathbb{R}, \mathcal{N})$ defined by

$$\mu_a(A) = \mu([a + b : b \in A]) \text{ for } A \in \mathcal{N} \quad (3.1)$$

From $\mu, \nu \in \mathcal{M}$, we formulate the following hypotheses.

(H₂) Let $\mu, \nu \in \mathcal{M}$ be such that

$$\limsup_{\tau \rightarrow +\infty} \frac{\mu([- \tau, \tau])}{\nu([- \tau, \tau])} = \delta < \infty.$$

(H₃) For all a, b and $c \in \mathbb{R}$ such that $0 \leq a < b < c$, there exist δ_0 and $\alpha_0 > 0$ such that

$$|\delta| \geq \delta_0 \Rightarrow \mu(a + \delta, b + \delta) \geq \alpha_0 \mu(\delta, c + \delta).$$

(H₄) For all $\tau \in \mathbb{R}$ there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a + \tau : a \in A\}) \leq \beta\mu(A) \text{ when } A \in \mathcal{N} \text{ and satisfies } A \cap I = \emptyset.$$

Proposition 3.6. Assume that (\mathbf{H}_2) holds. Then $\mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ is a Banach space with the norm $\|\cdot\|_\infty$

Proof. We can see that $\mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ is a vector subspace of $BC(\mathbb{R}, L^p(\Omega, H))$. To complete the proof is enough to prove that (\mathbf{H}_2) holds. Then $\mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ is closed in $BC(\mathbb{R}, L^p(\Omega, H))$. Let $(f_n)_n$ be a sequence in $\mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ such that $\lim_{n \rightarrow +\infty} f_n = f$ uniformly in $BC(\mathbb{R}, L^p(\Omega, H))$. From $\nu(\mathbb{R}) = +\infty$, it follows that $\nu([- \tau, \tau]) > 0$ for τ sufficiently large. Let $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\|f_n - f\|_\infty < \varepsilon$. Let $n \geq n_0$, then we have

$$\begin{aligned} \frac{1}{\nu([- \tau, \tau])} \int_{- \tau}^{+ \tau} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f(t)\|^p \right) d\mu(t) &\leq \frac{2^{p-1}}{\nu([- \tau, \tau])} \int_{- \tau}^{+ \tau} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f_n(t) - f(t)\|^p \right) d\mu(t) \\ &\quad + \frac{2^{p-1}}{\nu([- \tau, \tau])} \int_{- \tau}^{+ \tau} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f_n(t)\|^p \right) d\mu(t) \\ &\leq \frac{2^{p-1}}{\nu([- \tau, \tau])} \int_{- \tau}^{+ \tau} \left(\sup_{t \in \mathbb{R}} \mathbb{E} \|f_n(t) - f(t)\|^p \right) d\mu(t) \\ &\quad + \frac{2^{p-1}}{\nu([- \tau, \tau])} \int_{- \tau}^{+ \tau} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f_n(t)\|^p \right) d\mu(t) \\ &\leq 2^{p-1} \|f_n - f\|_\infty^p \frac{\mu([- \tau, \tau])}{\nu([- \tau, \tau])} + \frac{2^{p-1}}{\nu([- \tau, \tau])} \int_{- \tau}^{+ \tau} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f_n(t)\|^p \right) d\mu(t). \end{aligned}$$

We deduce that

$$\limsup_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{- \tau}^{+ \tau} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f(t)\|^p \right) d\mu(t) \leq 2^{p-1} \delta \varepsilon \text{ for any } \varepsilon > 0.$$

■

Next result is a characterisation of p -th mean (μ, ν) -ergodic processes of class r .

Theorem 3.7. Assume that (\mathbf{H}_2) holds and let $\mu, \nu \in \mathcal{M}$ and I be a bounded interval (eventually $I = \emptyset$). Assume that $f \in BC(\mathbb{R}, L^p(\Omega, H))$. The following assertions are equivalent

i) $f \in \mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$

ii) $\lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau] \setminus I)} \int_{[- \tau, \tau] \setminus I} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p \right) d\mu(t) = 0.$

iii) For any $\varepsilon > 0$, $\lim_{\tau \rightarrow +\infty} \frac{\mu(\{t \in [- \tau, \tau] \setminus I : \mathbb{E} \|f(\theta)\|^p > \varepsilon\})}{\nu([- \tau, \tau] \setminus I)} = 0$

Proof. The proof uses the same arguments of the proof of Theorem 2.22 in [28].

i) \Leftrightarrow ii). Denote By $A = \mu(I)$ and $B = \int_I \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p \right) d\mu(t).$

Since the interval I is bounded and the process f is stochastically bounded continuous. Then A, B and C are finite.

For $\tau > 0$, such that $I \subset [- \tau, \tau]$ and $\nu([- \tau, \tau] \setminus I) > 0$, we have

$$\begin{aligned}
\frac{1}{\nu([- \tau, \tau]) \setminus I} \int_{[- \tau, \tau] \setminus I} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p \right) d\mu(t) &= \frac{1}{\nu([- \tau, \tau]) - A} \left[\int_{[- \tau, \tau]} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p \right) d\mu(t) - B \right] \\
&= \frac{\nu([- \tau, \tau])}{\nu([- \tau, \tau]) - A} \left[\frac{1}{\nu([- \tau, \tau])} \int_{[- \tau, \tau]} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p \right) d\mu(t) \right. \\
&\quad \left. - \frac{B}{\nu([- \tau, \tau])} \right]
\end{aligned}$$

From above equalities and the fact $\nu(\mathbb{R}) = +\infty$, we deduce *ii*) is equivalent to

$$\lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{[- \tau, \tau]} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p \right) d\mu(t) = 0,$$

that *i*). *iii*) \Rightarrow *ii*) Denote by A_τ^ε and B_τ^ε the following sets

$$A_\tau^\varepsilon = \left\{ t \in [- \tau, \tau] \setminus I : \sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p > \varepsilon \right\} \text{ and } B_\tau^\varepsilon = \left\{ t \in [- \tau, \tau] \setminus I : \sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p \leq \varepsilon \right\}.$$

Assume that *ii*) holds, that is

$$\lim_{\tau \rightarrow +\infty} \frac{\mu(A_\tau^\varepsilon)}{\nu([- \tau, \tau]) \setminus I} = 0. \quad (3.2)$$

From the equality

$$\int_{[- \tau, \tau] \setminus I} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p \right) d\mu(t) = \int_{A_\tau^\varepsilon} \left(\sup_{\theta \in [t-r, t]} \|f(\theta)\|^p \right) d\mu(t) + \int_{B_\tau^\varepsilon} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p \right) d\mu(t)$$

we deduce that for τ sufficient large

$$\frac{1}{\nu([- \tau, \tau] \setminus I)} \int_{[- \tau, \tau] \setminus I} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p \right) d\mu(t) \leq \|f\|_\infty \times \frac{\mu(A_\tau^\varepsilon)}{\nu([- \tau, \tau] \setminus I)} + \varepsilon \frac{\mu(B_\tau^\varepsilon)}{\nu([- \tau, \tau] \setminus I)}$$

Since $\mu(\mathbb{R}) = \nu(\mathbb{R}) = \infty$ and by using **(H₂)** then for all $\varepsilon > 0$ we have

$$\frac{1}{\nu([- \tau, \tau] \setminus I)} \int_{[- \tau, \tau] \setminus I} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p \right) d\mu(t) \leq \delta \varepsilon$$

Consequently *ii*) holds.

ii) \Rightarrow *iii*)

$$\begin{aligned}
\int_{[- \tau, \tau] \setminus I} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p \right) d\mu(t) &\geq \int_{A_\tau^\varepsilon} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p \right) d\mu(t) \\
\frac{1}{\nu([- \tau, \tau] \setminus I)} \int_{[- \tau, \tau] \setminus I} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p \right) d\mu(t) &\geq \varepsilon \frac{\mu(A_\tau^\varepsilon)}{\nu([- \tau, \tau] \setminus I)} \\
\frac{1}{\varepsilon \nu([- \tau, \tau] \setminus I)} \int_{[- \tau, \tau] \setminus I} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p \right) d\mu(t) &\geq \frac{\mu(A_\tau^\varepsilon)}{\nu([- \tau, \tau] \setminus I)},
\end{aligned}$$

for τ sufficiently large, we obtain equation (3.2), that is *iii*). ■

Definition 3.8. Let $\mu, \nu \in \mathcal{M}$. A function $f : \mathbb{R} \times L^p(\Omega, H) \rightarrow L^p(\Omega, H)$ is said to be (μ, ν) -ergodic in p -th mean sense in $t \in \mathbb{R}$ uniformly with the respect to $x \in \mathcal{K}$, if $f \in BC(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H))$ and satisfies

$$\lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \mathbb{E} \|f(t, x)\|^p d\mu(t) = 0,$$

where $\mathcal{K} \subset L^p(\Omega, H)$ is compact.

We denote $\mathcal{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu)$ the set of all such functions.

Definition 3.9. Let $\mu, \nu \in \mathcal{M}$. A function $f : \mathbb{R} \times L^p(\Omega, H) \rightarrow L^p(\Omega, H)$ is said to be p -th mean (μ, ν) -ergodic of class r in $t \in \mathbb{R}$ uniformly with the respect to $x \in \mathcal{K}$, if $f \in BC(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H))$ and satisfies

$$\lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta, x)\|^p d\mu(t) = 0,$$

where $\mathcal{K} \subset L^p(\Omega, H)$ is compact.

We denote $\mathcal{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$ the set of all such functions.

Definition 3.10. Let $\mu_1, \mu_2 \in \mathcal{M}$. We say that μ_1 is equivalent to μ_2 , denoting this as $\mu_1 \sim \mu_2$ if there exist constants α and $\beta > 0$ and a bounded interval I (eventually $I = \emptyset$) such that $\alpha\mu_1(A) \leq \mu_2(A) \leq \beta\mu_1(A)$, when $A \in \mathcal{N}$ satisfies $A \cap I = \emptyset$.

Remark 3.11. The relation \sim is an equivalence relation on \mathcal{M} .

Theorem 3.12. Let $\mu_1, \nu_1, \mu_2, \nu_2 \in \mathcal{M}$. If $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then $\mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu_1, \nu_1, r) = \mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu_2, \nu_2, r)$.

Proof. Since $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, there exists some constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and a bounded interval I (eventually $I = \emptyset$) such that $\alpha_1\mu_1(A) \leq \mu_2(A) \leq \beta_1\mu_1(A)$ and $\alpha_2\nu_1(A) \leq \nu_2(A) \leq \beta_2\nu_1(A)$ for each $A \in \mathcal{N}$ satisfies $A \cap I = \emptyset$, i.e

$$\frac{1}{\beta_2\nu_1(A)} \leq \frac{1}{\nu_2(A)} \leq \frac{1}{\alpha_2\nu_1(A)}$$

Since $\mu_1 \sim \mu_2$ and \mathcal{N} is the Lebesgue σ -field for τ sufficiently large,

$$\begin{aligned} \frac{\alpha_1\mu_1\left(\{t \in [-\tau, \tau] \setminus I : \sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p > \varepsilon\}\right)}{\beta_2\mu_2([-\tau, \tau] \setminus I)} &\leq \frac{\mu_2\left(\{t \in [-\tau, \tau] \setminus I : \sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p > \varepsilon\}\right)}{\nu_2([-\tau, \tau] \setminus I)} \\ &\leq \frac{\beta_1\mu_1\left(\{t \in [-\tau, \tau] \setminus I : \sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p > \varepsilon\}\right)}{\alpha_2\nu_1([-\tau, \tau] \setminus I)}. \end{aligned}$$

By using Theorem 3.7, we deduce that $\mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu_1, \nu_1, r) = \mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu_2, \nu_2, r)$.

Let $\mu, \nu \in \mathcal{M}$, we denote by

$$cl(\mu, \nu) = \left\{ \bar{\omega}_1, \bar{\omega}_2 \in \mathcal{M} : \mu_1 \sim \mu_2, \nu_1 \sim \nu_2 \right\}$$

Lemma 3.13. [14] Let $\mu \in \mathcal{M}$ satisfy (H_4) . Then the measures μ and μ_τ are equivalent for all $\tau \in \mathbb{R}$.

Lemma 3.14. [14] (H_4) implies

$$\text{for all } \sigma > 0, \quad \limsup_{\tau \rightarrow +\infty} \frac{\mu([- \tau - \sigma, \tau + \sigma])}{\nu([- \tau, \tau])} < \infty.$$

Theorem 3.15. Assume that (H_4) holds. Then $\mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ is invariant by translation.

Proof. The proof is inspired by Theorem 3.5 in [13].

Let $f \in \mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ and $a \in \mathbb{R}$. Since $\nu(\mathbb{R}) = +\infty$, there exists $a_0 > 0$ such that $\nu([- \tau - |a|, \tau + |a|]) > 0$ for $|a| > a_0$. Denote

$$M_a(\tau) = \frac{1}{\nu_a([- \tau, \tau])} \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p \right) d\mu_a(t) \quad \forall \tau > 0 \text{ and } a \in \mathbb{R},$$

where ν_a is the positive measure define by equation (3.1) By using Lemma (3.13), it follows that ν and ν_a are equivalent, μ and μ_a are equivalent and by Theorem (3.12), we have $\mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu_a, \nu_a, r) = \mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$, therefore $f \in \mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu_a, \nu_a, r)$ that is $\lim_{t \rightarrow +\infty} M_a(\tau) = 0$ for all $a \in \mathbb{R}$.

For all $A \in \mathcal{N}$, we denote χ_A the characteristic function of A . By using definition of the μ_a , we obtain that

$$\int_{[-\tau, \tau]} \chi_A(t) d\mu_a(t) = \int_{[-\tau, \tau]} \chi_A(t) d\mu_a(t+a) = \int_{[-\tau+a, \tau+a]} \chi_A(t) d\mu_a(t).$$

Since $t \mapsto \sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p$ is the pointwise limit of an increasing sequence of function see([19, Theorem 1.17, p.15]), we deduce that

$$\int_{[-\tau, \tau]} \sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta)\|^p d\mu_a(t) = \int_{[-\tau+a, \tau+a]} \sup_{\theta \in [t-a-r, t-a]} \mathbb{E} \|f(\theta)\|^p d\mu(t).$$

We denote by $a^+ = \max(a, 0)$ and $a^- = \max(-a, 0)$. Then we have $|a| + a = 2a^+$, $|a| - a = 2a^-$ and $[-\tau + a - |a|, \tau + a + |a|] = [-\tau - 2a^-, \tau + 2a^+]$. Therefore we obtain

$$M_a(\tau + |a|) = \frac{1}{\nu([- \tau - 2a^-, \tau + 2a^+])} \int_{[-\tau-2a^-, \tau+2a^+]} \sup_{\theta \in [t-a-r, t-a]} \mathbb{E} \|f(\theta)\|^p d\mu(t) \quad (3.3)$$

From (3.3) and the following inequality

$$\frac{1}{\nu([- \tau, \tau])} \int_{[-\tau, \tau]} \sup_{\theta \in [t-a-r, t-a]} \mathbb{E} \|f(\theta)\|^p d\mu(t) \leq \frac{1}{\nu([- \tau, \tau])} \int_{[-\tau-2a^-, \tau+2a^+]} \sup_{\theta \in [t-a-r, t-a]} \mathbb{E} \|f(\theta)\|^p d\mu(t),$$

we obtain

$$\frac{1}{\nu([- \tau, \tau])} \int_{[-\tau, \tau]} \sup_{\theta \in [t-a-r, t-a]} \mathbb{E} \|f(\theta)\|^p d\mu(t) \leq \frac{\nu([- \tau - 2a^-, \tau + 2a^+])}{\nu([- \tau, \tau])} \times M_a(\tau + |a|).$$

This implies

$$\frac{1}{\nu([- \tau, \tau])} \int_{[-\tau, \tau]} \sup_{\theta \in [t-a-r, t-a]} \mathbb{E} \|f(\theta)\|^p d\mu(t) \leq \frac{\nu([- \tau - 2|a|, \tau + 2|a|])}{\nu([- \tau, \tau])} \times M_a(\tau + |a|). \quad (3.4)$$

From equation (3.3) and equation (3.4) and using Lemma 3.14, we deduce that

$$\frac{1}{\nu([- \tau, \tau])} \int_{[-\tau, \tau]} \sup_{\theta \in [t-a-r, t-a]} \mathbb{E} \|f(\theta)\|^p d\mu(t) = 0,$$

which equivalent to

$$\frac{1}{\nu([- \tau, \tau])} \int_{[- \tau, \tau]} \sup_{\theta \in [t-r, t]} \mathbb{E} \|f(\theta - a)\|^p d\mu(t) = 0,$$

that is $f_a \in \mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$. We have proved that $f \in \mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ then $f_{-a} \in \mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ for all $a \in \mathbb{R}$, that is $\mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ invariant by translation. ■

Proposition 3.16. *The space $PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ is invariant by translation, that is for all $a \in \mathbb{R}$ and $f \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$, $f_a \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$.*

4. p -th mean (μ, ν) -pseudo almost automorphic processes

In this section, we define p -th mean (μ, ν) -pseudo almost automorphic and their properties.

Definition 4.1. [4] *A continuous function stochastic process $f : \mathbb{R} \rightarrow L^p(\Omega, H)$ is said to be almost automorphic process in the p -th mean sense if for every sequence of real numbers $(s_m)_{m \in \mathbb{N}}$, there exists a subsequence $(s_n)_{n \in \mathbb{N}}$ and a stochastic process $g : \mathbb{R} \rightarrow L^p(\Omega, H)$ such that*

$$\lim_{n \rightarrow \infty} \mathbb{E} \|f(t + s_n) - g(t)\|^p = 0$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n \rightarrow \infty} \mathbb{E} \|g(t - s_n) - f(t)\|^p = 0$$

for each $t \in \mathbb{R}$.

We denote the space of all such stochastic processes by $AA(\mathbb{R}, L^p(\Omega, H))$

Lemma 4.2. [4] *The space $AA(\mathbb{R}, L^p(\Omega, H))$ of p -th mean almost automorphic stochastic processes equipped with the norm $\|\cdot\|_\infty$ is a Banach space.*

Definition 4.3. [4] *A continuous function stochastic process $f : \mathbb{R} \times L^p(\Omega, H) \rightarrow L^p(\Omega, H)$, $(t, x) \mapsto f(t, x)$ is said to be almost automorphic process in the p -th mean sense in $t \in \mathbb{R}$ uniformly with respect to $x \in K$, if for every sequence of real numbers $(s_m)_{m \in \mathbb{N}}$, there exists a subsequence $(s_n)_{n \in \mathbb{N}}$ and a stochastic process $g : \mathbb{R} \times L^p(\Omega, H) \rightarrow L^p(\Omega, H)$ such that*

$$\lim_{n \rightarrow \infty} \mathbb{E} \|f(t + s_n, x) - g(t, x)\|^p = 0$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n \rightarrow \infty} \mathbb{E} \|g(t - s_n, x) - f(t, x)\|^p = 0$$

for each $t \in \mathbb{R}$, where $K \subset L^p(\Omega, H)$ is compact.

We denote the space of all such stochastic processes by $AA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H))$.

Lemma 4.4. [4] *If x and y are two automorphic processes in p -th mean sense, then*

- (1) $x + y$ is almost automorphic in p -th mean sense;
- (2) for every scalar λ , λx is almost automorphic in p -th mean sense;
- (3) there exists a constant $M > 0$ such that

$$\sup_{t \in \mathbb{R}} \mathbb{E} \|x(t)\|^p \leq M,$$

that is, x is bounded in $L^p(\Omega, H)$.

We now introduce some new spaces used in the sequel.

Definition 4.5. Let $\mu, \nu \in \mathcal{M}$. A bounded continuous stochastic process $f : \mathbb{R} \rightarrow L^p(\Omega, H)$ is said to be (μ, ν) -pseudo almost automorphic in p -th mean sense, if it can be decomposed as follows

$$f = g + \phi,$$

where $g \in AA(\mathbb{R}, L^p(\Omega, H))$ and $\phi \in \mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu)$.

We denote the space of all such stochastic processes by $PAA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu)$.

Definition 4.6. Let $\mu, \nu \in \mathcal{M}$. A bounded continuous stochastic process $f : \mathbb{R} \times L^p(\Omega, H) \rightarrow L^p(\Omega, H)$ is said to be (μ, ν) -pseudo almost automorphic in p -th mean sense, if it can be decomposed as follows

$$f = g + \phi,$$

where $g \in AA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H))$ and $\phi \in \mathcal{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu)$.

Proposition 4.7. [28] Assume that (H_3) holds. Then the decomposition of (μ, ν) -pseudo almost automorphic function in the form $f = g + \phi$, where $g \in AA(\mathbb{R}, X)$ and $\phi \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$ is unique.

Remark 4.8. Let $X = L^p(\Omega, H)$. Then the Proposition 4.7 always holds.

Proposition 4.9. [11] Assume that (H_3) holds. Then the decomposition of (μ, ν) -pseudo almost automorphic function of class r in the form $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA(\mathbb{R}, X)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}, X, \mu, \nu, r)$ is unique.

We denote the space of all such stochastic processes by $PAA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu)$.

Definition 4.10. Let $\mu, \nu \in \mathcal{M}$. A bounded continuous stochastic process $f : \mathbb{R} \rightarrow L^p(\Omega, H)$ is said to be (μ, ν) -pseudo almost automorphic of class r in p -th mean sense, if it can be decomposed as follows

$$f = g + \phi,$$

where $g \in AA(\mathbb{R}, L^p(\Omega, H))$ and $\phi \in \mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$.

We denote by $PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ the space of all such stochastic processes.

Proposition 4.11. Assume that (H_2) holds. Let $\mu, \nu \in \mathcal{M}$. The space $PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ endowed with the uniform topology norm is a Banach space.

Proof. This Proposition is the consequence of Lemma 4.2 and Proposition 3.6 ■

Definition 4.12. Let $\mu, \nu \in \mathcal{M}$. A bounded continuous stochastic process $f : \mathbb{R} \times L^p(\Omega, H) \rightarrow L^p(\Omega, H)$ is said to be (μ, ν) -pseudo almost automorphic of class r in p -th mean sense, if it can be decomposed as follows

$$f = g + \phi,$$

where $g \in AA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H))$ and $\phi \in \mathcal{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$.

We denote the space of all such stochastic processes by $PAA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$.

Proposition 4.13. Let μ_1, μ_2, ν_1 and $\nu_2 \in \mathcal{M}$ if $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then $PAA(\mathbb{R}, L^p(\Omega, H), \mu_1, \nu_1, r) = PAA(\mathbb{R}, L^p(\Omega, H), \mu_2, \nu_2, r)$.

This Proposition is just a consequence of Theorem 3.12.

Theorem 4.14. Assume that (H_3) holds. Let $\mu, \nu \in \mathcal{M}$ and $\phi \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ then the function $t \rightarrow \phi_t$ belongs to $PAA((C[-r, 0], L^p(\Omega, H), \mu, \nu, r))$.

Proof. Assume that $\phi = g + h$, where $g \in AA(\mathbb{R}, L^p(\Omega, H))$ and $h \in \mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$. Then we can see that $\phi_t = g_t + h_t$ and g_t is p -th mean almost automorphic process. Let us denote

$$M_a = \frac{1}{\nu_a([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \mathbb{E} \|h(\theta)\|^p d\mu_a(t),$$

where μ_a and ν_a are the positive measures defined by equation (3.1). By using Lemma 3.13 it follows that μ and μ_a are equivalent, ν and ν_a are equivalent by using theorem 3.12 $\mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r) = \mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu_a, \nu_a, r)$ therefore $f \in \mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu_a, \nu_a, r)$ that is $\lim_{\tau \rightarrow \infty} M_a(\tau) = 0$ for all $a \in \mathbb{R}$.

On the other hand for $\tau > 0$, we have

$$\begin{aligned} & \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left(\sup_{\theta \in [-r, 0]} \mathbb{E} \|h(\theta + \xi)\|^p \right) d\mu(t) \leq \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-2r, t]} \mathbb{E} \|h(\theta)\|^p \right) d\mu(t) \\ & \leq \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-2r, t-r]} \mathbb{E} \|h(\theta)\|^p + \sup_{\theta \in [t-r, t]} \mathbb{E} \|h(\theta)\|^p \right) d\mu(t) \\ & \leq \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-2r, t-r]} \mathbb{E} \|h(\theta)\|^p \right) d\mu(t) + \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \mathbb{E} \|h(\theta)\|^p d\mu(t) \\ & \leq \frac{1}{\nu([- \tau, \tau])} \int_{-\tau-r}^{\tau-r} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|h(\theta)\|^p \right) d\mu(t) + \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \mathbb{E} \|h(\theta)\|^p d\mu(t) \\ & \leq \frac{1}{\nu([- \tau, \tau])} \int_{-\tau-r}^{\tau-r} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|h(\theta)\|^p \right) d\mu(t+r) + \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \mathbb{E} \|h(\theta)\|^p d\mu(t) \\ & \leq \frac{\mu([- \tau-r, \tau+r])}{\nu([- \tau, \tau])} \left(\frac{1}{\mu([- \tau-r, \tau+r])} \int_{-\tau-r}^{\tau-r} \left(\sup_{\theta \in [t-r, t]} \mathbb{E} \|h(\theta)\|^p \right) d\mu(t+r) \right) + \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \mathbb{E} \|h(\theta)\|^p d\mu(t). \end{aligned}$$

Consequently

$$\begin{aligned} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left(\sup_{\theta \in [-r, 0]} \mathbb{E} \|h(\theta + \xi)\|^p \right) d\mu(t) & \leq \frac{\mu([- \tau-r, \tau+r])}{\nu([- \tau, \tau])} \times M_{\delta}(\tau+r) \\ & + \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \mathbb{E} \|h(\theta)\|^p d\mu(t), \end{aligned}$$

which shows using Lemma 3.13 and Lemma 3.14 that ϕ_t belongs to $PAA(C[-r, 0], \mu, \nu, r)$. Thus we obtain the desired result. ■

Next, we study the composition of (μ, ν) -pseudo almost automorphic process in p -th mean sense.

Theorem 4.15. [5] Let $f : \mathbb{R} \times L^p(\Omega, H) \rightarrow L^p(\Omega, H)$, $(t, x) \mapsto f(t, x)$ be almost automorphic in p -th sense in $t \in \mathbb{R}$, for each $x \in L^p(\Omega, H)$ and assume that f satisfies the Lipschitz condition in the following sense

$$\mathbb{E} \|f(t, x) - f(t, y)\|^p \leq L \|x - y\|^p \quad \forall x, y \in L^p(\Omega, H),$$

where L is positive number. Then $t \mapsto f(t, x(t)) \in AA(\mathbb{R}, L^p(\Omega, H))$ for any $x \in AA(\mathbb{R}, L^p(\Omega, H))$.

Theorem 4.16. Let (H_2) holds and $\mu, \nu \in \mathcal{M}$ satisfy (H_4) . Suppose that $f \in PAA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$ satisfies the Lipschitz condition in the second variable that is, there exists a positive number L such that for any $x, y \in L^p(\Omega, H)$,

$$\mathbb{E} \|f(t, x) - f(t, y)\|^p \leq L \|x - y\|^p, \quad t \in \mathbb{R}.$$

Then $t \mapsto f(t, x(t)) \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ for any $x \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$.

Proof. Since $x \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$, then we can decompose $x = x_1 + x_2$, where $x_1 \in AA(\mathbb{R}, L^p(\Omega, H))$ and $x_2 \in \mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$. Otherwise, since $f \in PAA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$

then $f = f_1 + f_2$, where $f_1 \in AA(\mathbb{R} \times L^p(\Omega, H))$ and $f_2 \in \mathcal{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$. Then the function f can be decomposed as follows

$$\begin{aligned} f(t, x(t)) &= f_1(t, x_1(t)) + [f(t, x(t)) - f(t, x_1(t))] + [f(t, x_1(t)) - f_1(t, x_1(t))] \\ &= f_1(t, x_1(t)) + [f(t, x(t)) - f(t, x_1(t))] + f_2(t, x_1(t)). \end{aligned}$$

Using Theorem 4.15, we have $t \mapsto f_1(t, x_1) \in AA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H))$. It remains to show that the both functions $t \mapsto [f(t, x_1(t)) - f_1(t, x_1(t))]$ and $t \mapsto +f_2(t, x_1(t))$ belong to $\mathcal{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$.

We have

$$\begin{aligned} \mathbb{E}\|f(t, x(t)) - f(t, x_1(t))\|^p &\leq L\|x(t) - x_1(t)\|^p \\ \sup_{\theta \in [t-r, t]} \mathbb{E}\|f(\theta, x(\theta)) - f(\theta, x_1(\theta))\|^p &\leq L \sup_{\theta \in [t-r, t]} \|x(\theta) - x_1(\theta)\|^p. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{\nu([- \tau, \tau])} \int_{[- \tau, \tau]} \sup_{\theta \in [t-r, t]} \mathbb{E}\|f(\theta, x(\theta)) - f(\theta, x_1(\theta))\|^p d\mu(t) &\leq \frac{L}{\nu([- \tau, \tau])} \int_{[- \tau, \tau]} \sup_{\theta \in [t-r, t]} \mathbb{E}\|x(\theta) - x_1(\theta)\|^p d\mu(t) \\ &\leq \frac{L}{\nu([- \tau, \tau])} \int_{[- \tau, \tau]} \sup_{\theta \in [t-r, t]} \mathbb{E}\|x_2(\theta)\|^p d\mu(t) \end{aligned}$$

Since $x_2 \in \mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ then

$$\lim_{\tau \rightarrow +\infty} \frac{L}{\nu([- \tau, \tau])} \int_{[- \tau, \tau]} \sup_{\theta \in [t-r, t]} \mathbb{E}\|x_2(\theta)\|^p d\mu(t) = 0.$$

We deduce that

$$\limsup_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{[- \tau, \tau]} \sup_{\theta \in [t-r, t]} \mathbb{E}\|f(\theta, x(\theta)) - f(\theta, x_1(\theta))\|^p d\mu(t) = 0,$$

therefore $[f(t, x(t)) - f(t, x_1(t))] \in \mathcal{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$. Now to complete the proof it is enough to prove that $t \mapsto f_2(t, x_1(t)) \in \mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$

In fact for each $t \in \mathbb{R}$, we have

$$\begin{aligned} \|f_2(t, x) - f_2(t, y)\|^p &= \|f(t, x) - f_1(t, x) - f_1(t, y) + f(t, y)\|^p \\ &\leq 2^{p-1}\|f(t, x) - f(t, y)\|^p + 2^{p-1}\|f_1(t, x) - f_1(t, y)\|^p. \end{aligned}$$

By using the Lipschitz condition, we have

$$\begin{aligned} \mathbb{E}\|f_2(t, x) - f_2(t, y)\|^p &\leq 2^{p-1}\mathbb{E}\|f(t, x) - f(t, y)\|^p + 2^{p-1}\mathbb{E}\|f_1(t, x) - f_1(t, y)\|^p \\ &\leq 2^p\|x - y\|^p \end{aligned}$$

Since $K = \overline{\{x_1(t) : t \in \mathbb{R}\}}$ is compact. Then for $\varepsilon > 0$, there exists a finite number x_1, \dots, x_m such that

$$K \subset \bigcup_{i=1}^m B\left(x_i, \frac{\varepsilon}{2^{2p-1}L}\right),$$

where $B\left(x_i, \frac{\varepsilon}{2^{2p-1}L}\right) = \{x \in K, \|x_i - x\|^p \leq \frac{\varepsilon}{2^{2p-1}L}\}$. Its implies that

$$K \subset \bigcup_{i=1}^m \left\{x \in K, \forall t \in \mathbb{R}, \|f_2(t, x) - f_2(t, x_i)\|^p \leq \frac{\varepsilon}{2^{p-1}}\right\}$$

Let $t \in \mathbb{R}$ and $x \in K$, there exists $i_0 \in \{1, \dots, m\}$ such that

$$\mathbb{E}\|f_2(t, x) - f_2(t, x_{i_0})\|^p \leq \frac{\varepsilon}{2^{p-1}},$$

therefore

$$\begin{aligned} \mathbb{E}\|f_2(t, x_1(t))\|^p &\leq 2^{p-1}\|f_2(t, x_1(t)) - f_2(t, x_{i_0}(t))\|^p + 2^{p-1}\mathbb{E}\|f_2(t, x_{i_0}(t))\|^p \\ &\leq \varepsilon + 2^{p-1}\mathbb{E}\|f_2(t, x_{i_0}(t))\|^p \\ &\leq \varepsilon + 2^{p-1}\sum_{i=1}^m \mathbb{E}\|f_2(t, x_{i_0}(t))\|^p. \end{aligned}$$

It follows that

$$\frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \mathbb{E}\|f_2(\theta, x_1(\theta))\|^p d\mu(t) \leq \left(\frac{\varepsilon \mu([- \tau, \tau])}{\nu([- \tau, \tau])} + \sum_{i=1}^m \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \mathbb{E}\|f_2(\theta, x_i(\theta))\|^p d\mu(t) \right).$$

By the fact $\forall i \in \{1, \dots, m\}$, $\lim_{\tau \rightarrow +\infty} \sum_{i=1}^m \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \mathbb{E}\|f_2(\theta, x_i(\theta))\|^p d\mu(t) = 0$, we deduce that

$$\limsup_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \mathbb{E}\|f_2(\theta, x_1(\theta))\|^p d\mu(t) \leq \varepsilon \delta.$$

Therefore $t \mapsto f_2(t, x_1(t)) \in \mathcal{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$. ■

Proposition 4.17. *Assume that (H_4) holds. Then the space $PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ is invariant by translation, that is $f \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ implies $f_\alpha \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ for all $\alpha \in \mathbb{R}$.*

Lemma 4.18. [6] *Let $G : [0, T] \times \Omega \rightarrow \mathcal{L}(L^p(\Omega, H))$ be an \mathcal{F}_t -adapted measurable stochastic process satisfying*

$$\int_0^T \mathbb{E}\|G(t)\|^2 < \infty$$

almost surely, where $\mathcal{L}(L^p(\Omega, H))$ denote the space of all linear operators from $L^p(\Omega, H)$ to itself. Then for any $p \geq 1$, there exists a constant $C_p > 0$ such that

$$\mathbb{E} \sup_{0 \leq s \leq T} \left\| \int_0^s G(s) dW(s) \right\|^p \leq C_p \mathbb{E} \left(\int_0^T \|G(s)\|^2 ds \right)^{p/2}, T > 0.$$

We make the following assumption

(H₅) g is a stochastically bounded process in p -th mean sense.

Proposition 4.19. *Assume that (H_0) , (H_1) and (H_5) hold and the semigroup $(U(t))_{t \geq 0}$ is hyperbolic. If f is bounded on \mathbb{R} , then there exists a unique bounded solution u of equation (1.1) on \mathbb{R} , given by*

$$\begin{aligned} u_t &= \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 f(s)) ds + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 f(s)) ds \\ &+ \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) dW(s) + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 g(s)) dW(s) \end{aligned}$$

where $\tilde{B}_\lambda = \lambda(\lambda I - \mathcal{A}_U)^{-1}$ for $\lambda > \tilde{\omega}$, Π^s and Π^u are projections of C_0 onto the stable and unstable subspaces respectively.

Proof. Let

$$u_t = v(t) + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) dW(s) + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 g(s)) dW(s),$$

where

$$v(t) = \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 f(s)) ds + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 f(s)) ds$$

Let us first prove that u_t exists. The existence of $v(t)$ have proved by [23]. Now we show that the limit $\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 f(s)) ds$ exists.

For each $t \in \mathbb{R}$ and by Lemma 4.18, we have

$$\begin{aligned} \mathbb{E} \left\| \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) dW(s) \right\|^p &\leq C_p \mathbb{E} \left(\int_{-\infty}^t \|\mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 g(s))\|^2 ds \right)^{p/2} \\ &\leq C_p (\overline{M} \widetilde{M}) |\Pi^s|^p \mathbb{E} \left(\int_{-\infty}^t e^{-2\omega(t-s)} \|g(s)\|^2 ds \right)^{p/2} \\ &\leq C_p (\overline{M} \widetilde{M}) |\Pi^s|^p \sum_{n=1}^{+\infty} \mathbb{E} \left(\int_{t-n}^{t-n+1} e^{-2\omega(t-s)} \|g(s)\|^2 ds \right)^{p/2} \\ &\leq C_p (\overline{M} \widetilde{M}) |\pi^s|^p \sum_{n=1}^{+\infty} \mathbb{E} \left(\int_{t-n}^{t-n+1} e^{-2(\frac{p-2}{p})\omega(t-s)} \right. \\ &\quad \left. \times e^{-\frac{4}{p}\omega(t-s)} \|g(s)\|^2 ds \right)^{p/2}. \end{aligned}$$

By using Hölder inequality, we obtain

$$\begin{aligned} \mathbb{E} \left\| \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) dW(s) \right\|^p &\leq C_p (\overline{M} \widetilde{M}) |\Pi^s|^p \sum_{n=1}^{+\infty} \left[\left(\int_{t-n}^{t-n+1} (e^{-2(\frac{p-2}{p})\omega(t-s)})^{\frac{p}{p-2}} ds \right)^{\frac{p}{p-2}} \right]^{p/2} \\ &\quad \times \mathbb{E} \left[\left(\int_{t-n}^{t-n+1} (e^{-\frac{4}{p}\omega(t-s)} \|g(s)\|^2)^{\frac{p}{2}} ds \right)^{\frac{2}{p}} \right]^{p/2} \\ &\leq C_p (\overline{M} \widetilde{M}) |\Pi^s|^p \sum_{n=1}^{+\infty} \left(\int_{t-n}^{t-n+1} e^{-2\omega(t-s)} ds \right)^{\frac{p-2}{2}} \\ &\quad \times \left(\int_{t-n}^{t-n+1} e^{-2\omega(t-s)} \mathbb{E} \|g(s)\|^p ds \right). \end{aligned}$$

Since g stochastic bounded process in p -th mean sense, then there exists, $M > 0$ such that $\mathbb{E} \|g(s)\|^p \leq M$.

It follows that

$$\begin{aligned}
\mathbb{E} \left\| \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) dW(s) \right\|^p &\leq C_p M(\overline{M} \widetilde{M}) |\Pi^s|^p \sum_{n=1}^{+\infty} \left(\int_{t-n}^{t-n+1} e^{-2\omega(t-s)} ds \right)^{p/2} \\
&\leq C_p M(\overline{M} \widetilde{M}) |\Pi^s|^p \sum_{n=1}^{+\infty} \frac{1}{(2\omega)^{p/2}} (e^{2\omega} - 1)^{p/2} \times e^{-\omega p n} \\
&\leq C_p M(\overline{M} \widetilde{M}) |\Pi^s|^p \frac{1}{(2\omega)^{p/2}} (e^{2\omega} - 1)^{p/2} \times \sum_{n=1}^{+\infty} e^{-\omega p n}.
\end{aligned}$$

Since the serie $\sum_{n=1}^{+\infty} e^{-\omega p n} = 1 - \frac{1}{1 - e^{-\omega p}} = \frac{e^{-\omega p}}{1 - e^{-\omega p}} < \infty$.

It follows that

$$\mathbb{E} \left\| \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) dW(s) \right\|^p \leq \gamma, \quad (4.1)$$

where

$$\gamma = \frac{C_p M(\overline{M} \widetilde{M}) |\Pi^s|^p}{(2\omega)^{p/2}} (e^{2\omega} - 1)^{p/2} \times \frac{e^{-\omega p}}{1 - e^{-\omega p}}.$$

Set

$$F(n, s, t) = \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 f(s)) \text{ for } n \in \mathbb{N} \text{ for } s \leq t.$$

For n is sufficiently large and $\sigma \leq t$, we have

$$\begin{aligned}
\mathbb{E} \left\| \int_{-\infty}^\sigma \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) dW(s) \right\|^p &\leq C_p (\overline{M} \widetilde{M}) |\Pi^s|^p \sum_{n=1}^{+\infty} \left(\int_{\sigma-n}^{\sigma-n+1} e^{-2\omega(t-s)} ds \right)^{\frac{p-2}{2}} \\
&\quad \times \left(\int_{\sigma-n}^{\sigma-n+1} e^{-2\omega(t-s)} \mathbb{E} \|g(s)\|^p ds \right) \\
&\leq C_p M(\overline{M} \widetilde{M}) |\Pi^s|^p \sum_{n=1}^{+\infty} \left(\int_{\sigma-n}^{\sigma-n+1} e^{-2\omega(t-s)} ds \right)^{p/2} \\
&\leq C_p M(\overline{M} \widetilde{M}) |\Pi^s|^p \frac{1}{(2\omega)^{p/2}} (e^{2\omega} - 1)^{p/2} e^{-\omega p(t-\sigma)} \times \sum_{n=1}^{+\infty} e^{-\omega p n} \\
&\leq \gamma e^{-\omega p(t-\sigma)}.
\end{aligned}$$

It follow that for n and m sufficiently large and $\sigma \leq t$, we have

$$\begin{aligned}
\mathbb{E} \left\| \int_{-\infty}^t F(n, s, t) dW(s) - \int_{-\infty}^t F(m, s, t) dW(s) \right\|^p &\leq \mathbb{E} \left\| \int_{-\infty}^{\sigma} F(n, s, t) dW(s) + \int_{\sigma}^t F(n, s, t) dW(s) \right. \\
&\quad \left. - \int_{-\infty}^{\sigma} F(m, s, t) dW(s) - \int_{\sigma}^t F(m, s, t) dW(s) \right\|^p \\
&\leq 3^{p-1} \mathbb{E} \left\| \int_{-\infty}^{\sigma} F(n, s, t) dW(s) \right\|^p + 3^{p-1} \mathbb{E} \left\| \int_{-\infty}^{\sigma} F(m, s, t) dW(s) \right\|^p \\
&\quad + 3^{p-1} \mathbb{E} \left\| \int_{\sigma}^t F(n, s, t) dW(s) - \int_{\sigma}^t F(m, s, t) dW(s) \right\|^p \\
&\leq 2 \times 3^{p-1} \gamma e^{-\omega p(t-\sigma)} \\
&\quad + 3^{p-1} \mathbb{E} \left\| \int_{\sigma}^t F(n, s, t) dW(s) - \int_{\sigma}^t F(m, s, t) dW(s) \right\|^p
\end{aligned}$$

Since $\lim_{n \rightarrow +\infty} \mathbb{E} \left\| \int_{\sigma}^t F(n, s, t) dW(s) \right\|^p$ exists, then

$$\limsup_{n, m \rightarrow +\infty} \mathbb{E} \left\| \int_{-\infty}^t F(n, s, t) dW(s) - \int_{-\infty}^t F(m, s, t) dW(s) \right\|^p \leq 2 \times 3^{p-1} \gamma e^{-\omega p(t-\sigma)}.$$

If $\sigma \rightarrow -\infty$, then

$$\limsup_{n, m \rightarrow +\infty} \mathbb{E} \left\| \int_{-\infty}^t F(n, s, t) dW(s) - \int_{-\infty}^t F(m, s, t) dW(s) \right\|^p = 0.$$

We deduce that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left\| \int_{-\infty}^t F(n, s, t) dW(s) \right\|^p = \lim_{n \rightarrow \infty} \mathbb{E} \left\| \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_n X_0 g(s)) dW(s) \right\|^p$$

exists.

Therefore the limit $\lim_{n \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_n X_0 g(s)) dW(s)$ exists. In addition, one can see from the equation (4.1) that the function

$$\eta_1 : t \rightarrow \lim_{n \rightarrow +\infty} \mathbb{E} \left\| \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_n X_0 g(s)) dW(s) \right\|^p$$

is bounded on \mathbb{R} . Similarly, we can show that the function

$$\eta_2 : t \rightarrow \lim_{n \rightarrow +\infty} \mathbb{E} \left\| \int_t^{+\infty} \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_n X_0 g(s)) dW(s) \right\|^p$$

is well defined and bounded on \mathbb{R} . ■

Proposition 4.20. Assume that (H_5) holds. Let $f, g \in AA(\mathbb{R}, X)$ and Γ be the mapping defined for $t \in \mathbb{R}$ by

$$\begin{aligned}
\Gamma(f, g)(t) &= \left[\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 f(s)) ds + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 f(s)) ds \right. \\
&\quad \left. + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) dW(s) + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 g(s)) dW(s) \right](0).
\end{aligned}$$

Then $\Gamma(f, g) \in AA(\mathbb{R}, L^p(\Omega, H))$.

Proof. The proof of this Proposition will be in two steps.

Step 1 : We will show that $\Gamma(f, g)$ is continuous. For $t_0 \in \mathbb{R}$, we have

$$\begin{aligned}
\mathbb{E}\|\Gamma(f, g)(t) - \Gamma(f, g)(t_0)\|^p &= \mathbb{E}\left\|\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\tilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s)\Pi^u(\tilde{B}_\lambda X_0 f(s))ds \right. \\
&\quad + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\tilde{B}_\lambda X_0 g(s))dW(s) + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s)\Pi^u(\tilde{B}_\lambda X_0 g(s))dW(s) \\
&\quad - \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{t_0} \mathcal{U}^s(t_0-s)\Pi^s(\tilde{B}_\lambda X_0 f(s))ds - \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^{t_0} \mathcal{U}^u(t_0-s)\Pi^u(\tilde{B}_\lambda X_0 f(s))ds \\
&\quad \left. - \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{t_0} \mathcal{U}^s(t_0-s)\Pi^s(\tilde{B}_\lambda X_0 g(s))dW(s) - \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^{t_0} \mathcal{U}^u(t_0-s)\Pi^u(\tilde{B}_\lambda X_0 g(s))dW(s)\right\|^p \\
&\leq 4^{p-1}\mathbb{E}\left\|\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\tilde{B}_\lambda X_0 f(s))ds - \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{t_0} \mathcal{U}^s(t_0-s)\Pi^s(\tilde{B}_\lambda X_0 f(s))ds\right\|^p \\
&\quad + 4^{p-1}\mathbb{E}\left\|\lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s)\Pi^u(\tilde{B}_\lambda X_0 f(s))ds - \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^{t_0} \mathcal{U}^u(t_0-s)\Pi^u(\tilde{B}_\lambda X_0 f(s))ds\right\|^p \\
&\quad + 4^{p-1}\mathbb{E}\left\|\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\tilde{B}_\lambda X_0 g(s))dW(s) - \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{t_0} \mathcal{U}^s(t_0-s)\Pi^s(\tilde{B}_\lambda X_0 g(s))dW(s)\right\|^p \\
&\quad + 4^{p-1}\mathbb{E}\left\|\lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s)\Pi^u(\tilde{B}_\lambda X_0 g(s))dW(s) - \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^{t_0} \mathcal{U}^u(t_0-s)\Pi^u(\tilde{B}_\lambda X_0 g(s))dW(s)\right\|^p \\
&= 4^{p-1}(I_1 + I_2 + I_3 + I_4).
\end{aligned}$$

We have

$$I_1 = \mathbb{E}\left\|\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\tilde{B}_\lambda X_0 f(s))ds - \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{t_0} \mathcal{U}^s(t_0-s)\Pi^s(\tilde{B}_\lambda X_0 f(s))ds\right\|^p.$$

Let $\sigma = s - t + t_0$ and by Hölder inequality, we have

$$\begin{aligned}
I_1 &= \mathbb{E}\left\|\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{t_0} \mathcal{U}^s(t_0-\sigma)\Pi^s(\tilde{B}_\lambda X_0 f(\sigma+t-t_0))d\sigma - \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{t_0} \mathcal{U}^s(t_0-s)\Pi^s(\tilde{B}_\lambda X_0 f(s))ds\right\|^p \\
&\leq \mathbb{E}\left\|\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{t_0} \mathcal{U}^s(t_0-s)\Pi^s(\tilde{B}_\lambda X_0 [f(s+t-t_0) - f(s)])ds\right\|^p \\
&\leq \mathbb{E}\left(\overline{M}\widetilde{M} \int_{-\infty}^{t_0} e^{-\omega(t_0-s)} |\Pi^s| \|f(s+t-t_0) - f(s)\| ds\right)^p \\
&\leq \mathbb{E}\left(\overline{M}\widetilde{M} |\Pi^s| \int_{-\infty}^{t_0} e^{-\frac{\omega(p-1)(t_0-s)}{p}} \times e^{-\frac{\omega(t-s)}{p}} \|f(s+t-t_0) - f(s)\| ds\right)^p \\
&\leq (\overline{M}\widetilde{M} |\Pi^s|)^p \mathbb{E}\left[\left(\int_{-\infty}^t \left(e^{-\frac{\omega(p-1)(t_0-s)}{p}}\right)^{\frac{p}{p-1}} ds\right)^{\frac{p-1}{p}} \times \left(\int_{-\infty}^t \left(e^{-\frac{\omega(t-s)}{p}} \|f(s+t-t_0) - f(s)\|\right)^p ds\right)^{\frac{1}{p}}\right]^p \\
&\leq (\overline{M}\widetilde{M} |\Pi^s|)^p \left(\int_{-\infty}^{t_0} e^{-\omega(t_0-s)} ds\right)^{p-1} \times \int_{-\infty}^{t_0} e^{-\omega(t_0-s)} \mathbb{E}\|f(s+t-t_0) - f(s)\|^p ds \\
&\leq \frac{(\overline{M}\widetilde{M} |\Pi^s|)^p}{\omega^{p-1}} \times \int_{-\infty}^{t_0} e^{-\omega(t_0-s)} \mathbb{E}\|f(s+t-t_0) - f(s)\|^p ds.
\end{aligned}$$

For an arbitrary sequence of real $\{t_n\}$ with $t_n \rightarrow t$ as $n \rightarrow +\infty$. By Lemma 4.4 and the definition of $AA(\mathbb{R}, L^p(\Omega, H))$ we deduce that $f \in BC(\mathbb{R}, L^p(\Omega, H))$. So

$$e^{-\omega(t_0-s)}\mathbb{E}\|f(s+t_n-t_0)-f(s)\|^p \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Hence

$$e^{-\omega(t_0-s)}\mathbb{E}\|f(s+t_n-t_0)-f(s)\|^p \leq 2^p e^{-\omega(t_0-s)}\|f\|_\infty^p,$$

for every n sufficiently large. Note that

$$\int_{-\infty}^{t_0} 2^p e^{-\omega(t_0-s)}\|f\|_\infty^p ds < \infty.$$

Then according to Lebesgue dominated convergence theorem, we have

$$\lim_{n \rightarrow +\infty} \int_{-\infty}^{t_0} e^{-\omega(t_0-s)}\mathbb{E}\|f(s+t_n-t_0)-f(s)\|^p ds = 0.$$

Since the arbitrariness of $\{t_n\}$, we deduce that

$$\lim_{t \rightarrow t_0} \int_{-\infty}^{t_0} e^{-\omega(t_0-s)}\mathbb{E}\|f(s+t-t_0)-f(s)\|^p ds = 0,$$

which implies that

$$\lim_{t \rightarrow t_0} \mathbb{E} \left\| \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 f(s)) ds - \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{t_0} \mathcal{U}^s(t_0-s) \Pi^s(\tilde{B}_\lambda X_0 f(s)) ds \right\|^p = 0 \quad (4.2)$$

Similarly, we can see that

$$\lim_{t \rightarrow t_0} \mathbb{E} \left\| \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 f(s)) ds - \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^{t_0} \mathcal{U}^u(t_0-s) \Pi^u(\tilde{B}_\lambda X_0 f(s)) ds \right\|^p = 0. \quad (4.3)$$

Let $\tilde{W}(\tau) = W(\tau+t-t_0) - W(t-t_0)$. One can see that \tilde{W} is a Wiener process and has the same distribution as W . Let $\tau = s-t+t_0$. Then by Lemma 4.18 and Hölder inequality, we have

$$\begin{aligned} I_3 &= \mathbb{E} \left\| \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) dW(s) - \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{t_0} \mathcal{U}^s(t_0-s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) dW(s) \right\|^p \\ &= \mathbb{E} \left\| \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{t_0} \mathcal{U}^s(t_0-s) \Pi^s(\tilde{B}_\lambda X_0 g(\tau+t-t_0)) dW(\tau+t-t_0) - \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{t_0} \mathcal{U}^s(t_0-s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) dW(s) \right\|^p \\ &= \mathbb{E} \left\| \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{t_0} \mathcal{U}^s(t_0-s) \Pi^s(\tilde{B}_\lambda X_0 g(\tau+t-t_0)) d\tilde{W}(\tau) - \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{t_0} \mathcal{U}^s(t_0-s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) d\tilde{W}(s) \right\|^p \\ &= \mathbb{E} \left\| \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{t_0} \mathcal{U}^s(t_0-s) \Pi^s(\tilde{B}_\lambda X_0 [g(s+t-t_0)-g(s)]) d\tilde{W}(s) \right\|^p \\ &\leq C_p (\overline{MM} |\Pi^s|)^p \left(\int_{-\infty}^{t_0} e^{-2\omega(t_0-s)} ds \right)^{\frac{p-2}{2}} \times \int_{-\infty}^{t_0} e^{-2\omega(t_0-s)} \mathbb{E} \|g(s+t-t_0)-g(s)\|^p ds. \end{aligned}$$

By the similar arguments as above, we obtain

$$\lim_{t \rightarrow t_0} \int_{-\infty}^{t_0} e^{-2\omega(t_0-s)} \mathbb{E} \|g(s+t-t_0)-g(s)\|^p ds = 0,$$

which implies that

$$\lim_{t \rightarrow t_0} \mathbb{E} \left\| \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) dW(s) - \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{t_0} \mathcal{U}^s(t_0-s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) dW(s) \right\|^p = 0. \quad (4.4)$$

Similarly, we can see that

$$\lim_{t \rightarrow t_0} \mathbb{E} \left\| \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 g(s)) dW(s) - \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^{t_0} \mathcal{U}^u(t_0-s) \Pi^u(\tilde{B}_\lambda X_0 g(s)) dW(s) \right\|^p = 0. \quad (4.5)$$

From equations (4.2), (4.3), (4.4) and (4.5), we deduce that

$$\lim_{t \rightarrow t_0} \mathbb{E} \left\| \Gamma(f, g)(t) - \Gamma(f, g)(t_0) \right\|^p = 0$$

and yield the continuity of $\Gamma(f, g)$.

Step 2 : Since $f, g \in AA(\mathbb{R}, L^p(\Omega, H))$. Thus, for every sequence of real numbers $(s_m)_{m \in \mathbb{N}}$, there exists a subsequence $(s_n)_{n \in \mathbb{N}}$ and stochastic processes $\tilde{f}, \tilde{g} : \mathbb{R} \rightarrow L^p(\Omega, H)$ which each $t \in \mathbb{R}$ such that

$$\lim_{n \rightarrow \infty} \mathbb{E} \|f(t + s_n) - \tilde{f}(t)\|^p = 0, \quad \lim_{n \rightarrow \infty} \mathbb{E} \|\tilde{f}(t - s_n) - f(t)\|^p = 0$$

and

$$\lim_{n \rightarrow \infty} \mathbb{E} \|g(t + s_n) - \tilde{g}(t)\|^p = 0, \quad \lim_{n \rightarrow \infty} \mathbb{E} \|\tilde{g}(t - s_n) - g(t)\|^p = 0.$$

Let

$$\begin{aligned} w(t + s_n) &= \left[\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 f(s + s_n)) ds + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 f(s + s_n)) ds \right. \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 g(s + s_n)) dW(s + s_n) \\ &\quad \left. + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 g(s + s_n)) dW(s + s_n) \right] \end{aligned}$$

and

$$\begin{aligned} \tilde{w}(t) &= \left[\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 f(s)) ds + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 f(s)) ds \right. \\ &\quad \left. + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) dW(s) + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 g(s)) dW(s) \right]. \end{aligned}$$

Then we have

$$\begin{aligned}
\mathbb{E}\|w(t+s_n) - \tilde{w}(t)\|^p &\leq 4^{p-1}\mathbb{E}\left\|\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\tilde{B}_\lambda X_0[f(s+s_n) - f(s)]ds\right\|^p \\
&\quad + 4^{p-1}\mathbb{E}\left\|\lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s)\Pi^u(\tilde{B}_\lambda X_0[f(s+s_n) - f(s)]ds\right\|^p \\
&\quad + 4^{p-1}\mathbb{E}\left\|\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\tilde{B}_\lambda X_0[g(s+s_n) - g(s)]d\tilde{W}(s)\right\|^p \\
&\quad + 4^{p-1}\mathbb{E}\left\|\lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s)\Pi^u(\tilde{B}_\lambda X_0[g(s+s_n) - g(s)]d\tilde{W}(s)\right\|^p,
\end{aligned}$$

where $\tilde{W}(s) = W(s+s_n) - W(s)$. Note that W and \tilde{W} are two Wiener processes and have the same distribution. Then we have

$$\begin{aligned}
\mathbb{E}\|w(t+s_n) - \tilde{w}(t)\|^p &\leq 4^{p-1}(\overline{M}\tilde{M}|\Pi^s|)^p \left(\int_{-\infty}^t e^{-\omega(t-s)} ds \right)^{p-1} \times \int_{-\infty}^t e^{-\omega(t-s)} \mathbb{E}\|f(s+s_n) - f(s)\|^p ds \\
&\quad + 4^{p-1}(\overline{M}\tilde{M}|\Pi^u|)^p \left(\int_{+\infty}^t e^{\omega(t-s)} ds \right)^{p-1} \times \int_{+\infty}^t e^{-\omega(t-s)} \mathbb{E}\|f(s+s_n) - f(s)\|^p ds \\
&\quad + 4^{p-1}C_p(\overline{M}\tilde{M}|\Pi^s|)^p \left(\int_{-\infty}^t e^{-2\omega(t-s)} ds \right)^{\frac{p-2}{2}} \times \int_{-\infty}^t e^{-2\omega(t-s)} \mathbb{E}\|g(s+s_n) - g(s)\|^p ds \\
&\quad + 4^{p-1}C_p(\overline{M}\tilde{M}|\Pi^u|)^p \left(\int_{+\infty}^t e^{2\omega(t-s)} ds \right)^{\frac{p-2}{2}} \times \int_{+\infty}^t e^{2\omega(t-s)} \mathbb{E}\|g(s+s_n) - g(s)\|^p ds.
\end{aligned}$$

By similarly arguments as above, we have

$$\begin{aligned}
\lim_{n \rightarrow +\infty} \int_{-\infty}^t e^{-\omega(t-s)} \mathbb{E}\|f(s+s_n) - f(s)\|^p ds &= 0, \quad \lim_{n \rightarrow +\infty} \int_{+\infty}^t e^{\omega(t-s)} \mathbb{E}\|f(s+s_n) - f(s)\|^p ds = 0, \\
\lim_{n \rightarrow +\infty} \int_{-\infty}^t e^{-2\omega(t-s)} \mathbb{E}\|g(s+s_n) - g(s)\|^p ds &= 0 \text{ and } \lim_{n \rightarrow +\infty} \int_{+\infty}^t e^{2\omega(t-s)} \mathbb{E}\|g(s+s_n) - g(s)\|^p ds = 0.
\end{aligned}$$

Thus

$$\lim_{n \rightarrow +\infty} \mathbb{E}\|w(t+s_n) - \tilde{w}(t)\|^p = 0.$$

Similarly, we have

$$\lim_{n \rightarrow +\infty} \mathbb{E}\|\tilde{w}(t-s_n) - w(t)\|^p = 0.$$

Therefore by **Steps 1** and **2**, we proved that $\Gamma(f, g) \in AA(\mathbb{R}, L^p(\Omega, H))$.

Theorem 4.21. Assume that (H_3) and (H_5) hold. Let $f, g \in \mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$, then $\Gamma(f, g) \in \mathcal{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$.

Proof. We have

$$\begin{aligned}
\Gamma(f, g)(t) &= \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\tilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s)\Pi^u(\tilde{B}_\lambda X_0 f(s))ds \\
&\quad + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\tilde{B}_\lambda X_0 g(s))dW(s) + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s)\Pi^u(\tilde{B}_\lambda X_0 g(s))dW(s).
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathbb{E}\|\Gamma(f, g)(\theta)\|^p &= \mathbb{E}\left\|\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{\theta} \mathcal{U}^s(\theta - s) \Pi^s(\tilde{B}_\lambda X_0 f(s)) ds + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^{\theta} \mathcal{U}^u(\theta - s) \Pi^u(\tilde{B}_\lambda X_0 f(s)) ds \right. \\
&\quad + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{\theta} \mathcal{U}^s(\theta - s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) dW(s) \\
&\quad \left. + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^{\theta} \mathcal{U}^u(\theta - s) \Pi^u(\tilde{B}_\lambda X_0 g(s)) dW(s) \right\|^p.
\end{aligned}$$

Then for $\tau > 0$, using Lemma 4.18 we have

$$\begin{aligned}
\int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left(\mathbb{E}\|\Gamma(f, g)(\theta)\|^p \right) d\mu(t) &\leq \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} 4^{p-1} \mathbb{E} \left(\overline{M} \widetilde{M} \int_{-\infty}^{\theta} e^{-\omega(\theta-s)} |\Pi^s| \|f(s)\| ds \right)^p d\mu(t) \\
&\quad + \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} 4^{p-1} \mathbb{E} \left(\overline{M} \widetilde{M} \int_{+\infty}^{\theta} e^{\omega(\theta-s)} |\Pi^u| \|f(s)\| ds \right)^p d\mu(t) \\
&\quad + \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} 4^{p-1} C_p \mathbb{E} \left(\overline{M}^2 \widetilde{M}^2 \int_{-\infty}^{\theta} e^{-2\omega(\theta-s)} |\Pi^s|^2 \|g(s)\|^2 ds \right)^{p/2} d\mu(t) \\
&\quad + \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} 4^{p-1} C_p \mathbb{E} \left(\overline{M}^2 \widetilde{M}^2 \int_{-\infty}^{\theta} e^{-2\omega(\theta-s)} |\Pi^u|^2 \|g(s)\|^2 ds \right)^{p/2} d\mu(t)
\end{aligned}$$

By using Höder inequality, we obtain

$$\begin{aligned}
\mathbb{E}\|\Gamma(f, g)(\theta)\|^p &= \mathbb{E}\left\|\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{\theta} \mathcal{U}^s(\theta - s) \Pi^s(\tilde{B}_\lambda X_0 f(s)) ds + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^{\theta} \mathcal{U}^u(\theta - s) \Pi^u(\tilde{B}_\lambda X_0 f(s)) ds \right. \\
&\quad + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{\theta} \mathcal{U}^s(\theta - s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) dW(s) \\
&\quad \left. + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^{\theta} \mathcal{U}^u(\theta - s) \Pi^u(\tilde{B}_\lambda X_0 g(s)) dW(s) \right\|^p.
\end{aligned}$$

Then for $\tau > 0$, using Lemma 4.18 again, we have

$$\begin{aligned}
&\int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left(\mathbb{E}\|\Gamma(f, g)(\theta)\|^p \right) d\mu(t) \\
&\leq 4^{p-1} (\overline{M} \widetilde{M} |\Pi^s|)^p \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left[\left(\int_{-\infty}^{\theta} e^{-\omega(\theta-s)} ds \right)^{p-1} \times \int_{-\infty}^{\theta} e^{-\omega(\theta-s)} \mathbb{E} \|f(s)\|^p ds \right] d\mu(t) \\
&\quad + 4^{p-1} (\overline{M} \widetilde{M} |\Pi^u|)^p \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left[\left(\int_{+\infty}^{\theta} e^{\omega(\theta-s)} ds \right)^{p-1} \times \int_{+\infty}^{\theta} e^{\omega(\theta-s)} \mathbb{E} \|f(s)\|^p ds \right] d\mu(t) \\
&\quad + 4^{p-1} (\overline{M} \widetilde{M} |\Pi^s|)^p \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left[\left(\int_{-\infty}^{\theta} e^{-2\omega(\theta-s)} ds \right)^{\frac{p-2}{2}} \times \int_{-\infty}^{\theta} e^{-2\omega(\theta-s)} \mathbb{E} \|g(s)\|^p ds \right] d\mu(t) \\
&\quad + 4^{p-1} (\overline{M} \widetilde{M} |\Pi^u|)^p \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left[\left(\int_{+\infty}^{\theta} e^{2\omega(\theta-s)} ds \right)^{\frac{p-2}{2}} \times \int_{+\infty}^{\theta} e^{2\omega(\theta-s)} \mathbb{E} \|g(s)\|^p ds \right] d\mu(t).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
& \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left(\mathbb{E} \|\Gamma(f, g)(\theta)\|^p \right) d\mu(t) \\
& \leq \frac{4^{p-1} (\overline{M} \widetilde{M} |\Pi^s|)^p}{\omega^{p-1}} \times \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left(e^{\omega r} \int_{-\infty}^{\theta} e^{-\omega(t-s)} \mathbb{E} \|f(s)\|^p ds \right) d\mu(t) \\
& \leq \frac{4^{p-1} (\overline{M} \widetilde{M} |\Pi^u|)^p}{\omega^{p-1}} \times \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left(\int_{\theta}^{+\infty} e^{\omega(t-s)} \mathbb{E} \|f(s)\|^p ds \right) d\mu(t) \\
& \leq \frac{4^{p-1} (\overline{M} \widetilde{M} |\Pi^s|)^p}{(2\omega)^{\frac{p-2}{2}}} \times \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left(e^{2\omega r} \int_{-\infty}^{\theta} e^{-2\omega(t-s)} \mathbb{E} \|g(s)\|^p ds \right) d\mu(t) \\
& \leq \frac{4^{p-1} (\overline{M} \widetilde{M} |\Pi^u|)^p}{(2\omega)^{\frac{p-2}{2}}} \times \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left(\int_{\theta}^{+\infty} e^{2\omega(t-s)} \mathbb{E} \|g(s)\|^p ds \right) d\mu(t).
\end{aligned}$$

On the one hand using Fubini's theorem, we have

$$\begin{aligned}
\int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left(e^{\omega r} \int_{-\infty}^{\theta} e^{-\omega(t-s)} \mathbb{E} \|f(s)\|^p ds \right) d\mu(t) & \leq \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left(e^{\omega r} \int_{-\infty}^t e^{-\omega(t-s)} \mathbb{E} \|f(s)\|^p ds \right) d\mu(t) \\
& \leq e^{\omega r} \int_{-\tau}^{\tau} \int_{-\infty}^t e^{-\omega(t-s)} \mathbb{E} \|f(s)\|^p ds d\mu(t) \\
& \leq e^{\omega r} \int_{-\tau}^{\tau} \int_0^{\infty} e^{-\omega s} \mathbb{E} \|f(t-s)\|^p ds d\mu(t) \\
& \leq e^{\omega r} \int_0^{\infty} e^{-\omega s} \int_{-\tau}^{\tau} \mathbb{E} \|f(t-s)\|^p d\mu(t) ds.
\end{aligned}$$

By Theorem 3.15, we deduce that

$$\lim_{\tau \rightarrow +\infty} \frac{e^{-\omega s}}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \mathbb{E} \|f(t-s)\|^p d\mu(t) \rightarrow 0 \text{ for all } s \in \mathbb{R}_+$$

and

$$\frac{e^{-\omega s}}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \mathbb{E} \|f(t-s)\|^p d\mu(t) \leq \frac{e^{-\omega s} \nu([(\tau, \tau)])}{\nu([- \tau, \tau])} \|f\|_{\infty}^p.$$

Similarly, we have

$$\lim_{\tau \rightarrow +\infty} \frac{e^{-2\omega s}}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \mathbb{E} \|g(t-s)\|^p d\mu(t) \rightarrow 0 \text{ for all } s \in \mathbb{R}_+$$

and

$$\frac{e^{-2\omega s}}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \mathbb{E} \|f(t-s)\|^p d\mu(t) \leq \frac{e^{-\omega s} \nu([(\tau, \tau)])}{\nu([- \tau, \tau])} \|g\|_{\infty}^p.$$

Since f and g are two bounded functions, then the functions $s \mapsto \frac{e^{-\omega s} \nu([\tau, \tau])}{\nu([- \tau, \tau])} \|f\|_\infty^p$ and $s \mapsto \frac{e^{-2\omega s} \nu([\tau, \tau])}{\nu([- \tau, \tau])} \|g\|_\infty^p$ belong to $L^1(]0, \infty[)$ in view of the Lebesgue dominated convergence theorem, it follows that

$$e^{\omega r} \lim_{\tau \rightarrow +\infty} \int_0^{+\infty} \frac{e^{-\omega s}}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \mathbb{E} \|f(t-s)\|^p d\mu(t) ds = 0$$

and

$$e^{2\omega r} \lim_{\tau \rightarrow +\infty} \int_0^{+\infty} \frac{e^{-2\omega s}}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \mathbb{E} \|g(t-s)\|^p d\mu(t) ds = 0.$$

On the other hand, we have

$$\begin{aligned} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left(\int_{\theta}^{+\infty} e^{\omega(\theta-s)} \mathbb{E} \|f(s)\|^p ds \right) d\mu(t) &\leq \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left(\int_{t-r}^{+\infty} e^{\omega(\theta-s)} \mathbb{E} \|f(s)\|^p ds \right) d\mu(t) \\ &\leq \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left(\int_{t-r}^{+\infty} e^{\omega(t-s)} \mathbb{E} \|f(s)\|^p ds \right) d\mu(t) \\ &\leq \int_{-\tau}^{\tau} \int_{-\infty}^r e^{\omega(\theta-s)} \mathbb{E} \|f(s)\|^p ds d\mu(t) \\ &\leq \int_0^{+\infty} e^{\omega s} \int_{-\tau}^{\tau} \mathbb{E} \|f(t-s)\|^p d\mu(t) ds \end{aligned}$$

By the same arguments, we have

$$\int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left(\int_{\theta}^{+\infty} e^{2\omega(\theta-s)} \mathbb{E} \|f(s)\|^p ds \right) d\mu(t) \leq \int_0^{+\infty} e^{2\omega s} \int_{-\tau}^{\tau} \mathbb{E} \|g(t-s)\|^p d\mu(t) ds$$

Similarly as above, we have the functions $s \mapsto \frac{e^{\omega s} \nu([\tau, \tau])}{\nu([- \tau, \tau])} \|f\|_\infty^p$ and $s \mapsto \frac{e^{2\omega s} \nu([\tau, \tau])}{\nu([- \tau, \tau])} \|g\|_\infty^p$ belong to $L^1(]0, \infty[)$ in view of the Lebesgue dominated convergence theorem, it follows that

$$\lim_{\tau \rightarrow +\infty} \int_{-\infty}^r \frac{e^{\omega s}}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \mathbb{E} \|f(t-s)\|^p d\mu(t) ds = 0$$

and

$$\lim_{\tau \rightarrow +\infty} \int_{-\infty}^r \frac{e^{2\omega s}}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \mathbb{E} \|g(t-s)\|^p d\mu(t) ds = 0.$$

Consequently

$$\lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \left(\mathbb{E} \|\Gamma(f, g)(\theta)\|^p \right) d\mu(t) = 0.$$

Thus, we obtain the desired result. ■

Our next objective is to show the existence of p -th ($p \geq 2$) mean (μ, ν) -pseudo almost auto-morphic solution of class r for the following problem

$$dx(t) = [Ax(t) + L(x_t) + f(t, x_t)]dt + g(t, x_t)dW(t), \text{ for } t \in \mathbb{R}, \quad (4.6)$$

where $f : \mathbb{R} \times \mathcal{C} \rightarrow L^p(\Omega, H)$ and $g : \mathbb{R} \times \mathcal{C} \rightarrow L^p(\Omega, H)$ are two processes.

For the sequel we make the following assumptions.

(H₆) Let $\mu, \nu \in \mathcal{M}$ and $f : \mathbb{R} \times C([-r, 0], L^p(\Omega, H)) \rightarrow L^p(\Omega, H)$ p -th mean $cl(\mu, \nu)$ -pseudo almost automorphic of class r such that there exists a positive constant L_f such that

$$\mathbb{E}\|f(t, \phi_1) - f(t, \phi_2)\|^p \leq L_f \mathbb{E}\|\phi_1 - \phi_2\|^p \text{ for all } t \in \mathbb{R} \text{ and } \phi_1, \phi_2 \in C([-r, 0], L^p(\Omega, H)).$$

(H₇) Let $\mu, \nu \in \mathcal{M}$ and $g : \mathbb{R} \times C([-r, 0], L^p(\Omega, H)) \rightarrow L^p(\Omega, H)$ p -th mean $cl(\mu, \nu)$ pseudo almost automorphic of class r such that there exists a positive constant L_g such that

$$\mathbb{E}\|g(t, \phi_1) - g(t, \phi_2)\|^p \leq L_g \mathbb{E}\|\phi_1 - \phi_2\|^p \text{ for all } t \in \mathbb{R} \text{ and } \phi_1, \phi_2 \in C([-r, 0], L^p(\Omega, H)).$$

(H₈) The instable space $U \equiv \{0\}$.

Theorem 4.22. *Let $p \geq 2$, assume that (H₀), (H₁), (H₄), (H₆) (H₇) and (H₈) hold. If*

$$(\overline{M}\widetilde{M}|\Pi|^s)^p \left[\frac{L_f}{\omega^p} + \frac{L_g C_p}{(2\omega)^{\frac{p}{2}}} \right] < \frac{1}{2^{p-1}},$$

then equation (4.6) has a unique p -th mean $cl(\mu, \nu)$ -pseudo almost automorphic solution of class r .

Proof. Let x be a function in $PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$. From Theorem 4.14 the function $t \rightarrow x_t$ belongs to $PAA(C([-r, 0]); L^p(\Omega, H), \mu, \nu, r)$. Hence Theorem 4.16 implies that $g(\cdot) = f(\cdot, x)$ is in $PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$. Since the unstable space $U \equiv \{0\}$, then $\Pi^u \equiv 0$. Consider the following mapping $H : PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r) \rightarrow PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ defined for $t \in \mathbb{R}$ by

$$(\mathcal{H}x)(t) = \left[\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 f(s, x_s)) ds + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 g(s, x_s)) dW(s) \right](0).$$

Let $x_1, x_2 \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$, we have

$$\begin{aligned} \mathbb{E}\|(\mathcal{H}x_1)(t) - (\mathcal{H}x_2)(t)\|^p &= \mathbb{E}\left\| \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 [f(s, x_{1s}) - f(s, x_{2s})]) ds \right. \\ &\quad \left. + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 [g(s, x_{1s}) - g(s, x_{2s})]) dW(s) \right\|^p \\ &\leq 2^{p-1} \mathbb{E}\left\| \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 [f(s, x_{1s}) - f(s, x_{2s})]) ds \right\| \\ &\quad + 2^{p-1} \mathbb{E}\left\| \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 [g(s, x_{1s}) - g(s, x_{2s})]) dW(s) \right\|^p \\ &\leq 2^{p-1} (I_1 + I_2). \end{aligned}$$

By Hölder inequality, it follows that

$$\begin{aligned}
I_1 &= \mathbb{E} \left\| \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0[f(s, x_{1s}) - f(s, x_{2s})]) ds \right\| \\
&\leq \mathbb{E} \left[(\overline{M} \widetilde{M} |\Pi|^s)^p \left(\int_{-\infty}^t e^{-\omega(t-s)} ds \right)^{p-1} \times \int_{-\infty}^t e^{-\omega(t-s)} \|f(s, x_{1s}) - f(s, x_{2s})\|^p ds \right] \\
&\leq (\overline{M} \widetilde{M} |\Pi|^s)^p \left(\int_{-\infty}^t e^{-\omega(t-s)} ds \right)^{p-1} \times \int_{-\infty}^t e^{-\omega(t-s)} \mathbb{E} \|f(s, x_{1s}) - f(s, x_{2s})\|^p ds \\
&\leq \frac{(\overline{M} \widetilde{M} |\Pi|^s)^p}{\omega^{p-1}} \int_{-\infty}^t e^{-\omega(t-s)} L_f \mathbb{E} \|x_{1s} - x_{2s}\|^p ds \\
&\leq \frac{(\overline{M} \widetilde{M} |\Pi|^s)^p}{\omega^{p-1}} L_f \sup_{t \in \mathbb{R}} \|x_1(t) - x_2(t)\|^p \left(\int_{-\infty}^t e^{-\omega(t-s)} ds \right) \\
&\leq \frac{(\overline{M} \widetilde{M} |\Pi|^s)^p}{\omega^p} L_f \|x_1 - x_2\|_\infty^p.
\end{aligned}$$

By Hölder inequality and by Lemma 4.18, we have

$$\begin{aligned}
I_2 &= \mathbb{E} \left\| \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0[g(s, x_{1s}) - g(s, x_{2s})]) dW(s) \right\|^p \\
&\leq C_p \mathbb{E} \left[\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \left\| \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0[g(s, x_{1s}) - g(s, x_{2s})]) \right\|^2 ds \right]^{p/2} \\
&\leq (\overline{M} \widetilde{M} |\Pi|^s)^p C_p \left(\int_{-\infty}^t e^{-\omega(t-s)} ds \right)^{\frac{p-2}{2}} \times \int_{-\infty}^t e^{-2\omega(t-s)} \mathbb{E} \|g(s, x_{1s}) - g(s, x_{2s})\|^p ds \\
&\leq \frac{C_p (\overline{M} \widetilde{M} |\Pi|^s)^p}{(2\omega)^{\frac{p-2}{2}}} \int_{-\infty}^t e^{-2\omega(t-s)} L_g \mathbb{E} \|x_{1s} - x_{2s}\|^p ds \\
&\leq \frac{C_p (\overline{M} \widetilde{M} |\Pi|^s)^p}{(2\omega)^{\frac{p-2}{2}}} L_g \sup_{t \in \mathbb{R}} \mathbb{E} \|x_1(t) - x_2(t)\|^p \left(\int_{-\infty}^t e^{-2\omega(t-s)} ds \right) \\
&\leq \frac{C_p (\overline{M} \widetilde{M} |\Pi|^s)^p}{(2\omega)^{\frac{p}{2}}} L_g \|x_1 - x_2\|_\infty^p.
\end{aligned}$$

Thus we have

$$\mathbb{E} \|(\mathcal{H}x_1)(t) - (\mathcal{H}x_2)(t)\|^p \leq 2^{p-1} (\overline{M} \widetilde{M} |\Pi|^s)^p \left[\frac{L_f}{\omega^p} + \frac{L_g C_p}{(2\omega)^{\frac{p}{2}}} \right] \|x_1 - x_2\|_\infty^p$$

This means that \mathcal{H} is a strict contraction. Thus by Banach's fixed point theorem, \mathcal{H} has a unique fixed point u in $PAA(\mathbb{R}; L^p(\Omega, H), \mu, \nu, r)$. We conclude that equation (4.6), has one and only one p -th mean $cl(\mu, \nu)$ -pseudo almost automorphic solution of class r . ■

Proposition 4.23. *Let $p \geq 2$, assume that (\mathbf{H}_0) , (\mathbf{H}_1) and (\mathbf{H}_4) hold, f, g are lipschitz continuous with respect the second argument. If*

$$Lip(f) = Lip(g) < \frac{1}{2^{p-1} (\overline{M} \widetilde{M} |\Pi|^s)^p \left[\frac{1}{\omega^p} + \frac{1}{(2\omega)^{\frac{p}{2}}} \right]},$$

then (4.6) has a unique p -th mean $cl(\mu, \nu)$ -pseudo almost automorphic of class r , where $Lip(f)$ and $Lip(g)$ are respectively the lipschitz constants of f and g .

Proof. Let us pose $k = Lip(f) = Lip(g)$, we have

$$\begin{aligned} \mathbb{E}\|(\mathcal{H}x_1)(t) - (\mathcal{H}x_2)(t)\|^p &\leq 2^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p \left[\frac{k}{\omega^p} + \frac{k}{(2\omega)^{\frac{p}{2}}} \right] \sup_{t \in \mathbb{R}} \mathbb{E}\|x_1(t) - x_2(t)\|^p \\ &\leq 2^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p k \left[\frac{1}{\omega^p} + \frac{1}{(2\omega)^{\frac{p}{2}}} \right] \sup_{t \in \mathbb{R}} \mathbb{E}\|x_1(t) - x_2(t)\|^p \end{aligned}$$

Consequently \mathcal{H} is a strict contraction if

$$k < \frac{1}{2^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p \left[\frac{1}{\omega^p} + \frac{1}{(2\omega)^{\frac{p}{2}}} \right]}$$

■

5. Application

For illustration, we propose to study the existence of solutions for the following model

$$\left\{ \begin{array}{l} dz(t, x) = -\frac{\partial^2}{\partial x^2} z(t, x) dt + \left[\int_{-r}^0 G(\theta) z(t + \theta, x) d\theta + \sin\left(\frac{1}{2 + \cos t + \cos \sqrt{2}t}\right) + \arctan(t) + \int_{-r}^0 h(\theta, z(t + \theta, x)) d\theta \right] dt \\ \quad + \left[\sin\left(\frac{1}{2 + \cos t + \cos \sqrt{3}t}\right) + \arctan(t) + \int_{-r}^0 h(\theta, z(t + \theta, x)) d\theta \right] dW(t) \text{ for } t \in \mathbb{R}, \text{ and } x \in [0, \pi], \\ z(t, 0) = z(t, \pi) = 0 \text{ for } t \in \mathbb{R}, \text{ and } x \in [0, \pi], \end{array} \right. \quad (5.1)$$

where $G : [-r, 0] \rightarrow \mathbb{R}$ is continuous function and $h : [-r, 0] \rightarrow \mathbb{R}$ is lipschitz continuous with the respect of the second argument. $W(t)$ is a two-sided standard Brownian motion with values in separable Hilbert space H . To rewrite equation (5.1) in abstract form, we introduce the space $H = L^2((0, \pi))$. Let $A : D(A) \rightarrow L^2((0, \pi))$ defined by

$$\left\{ \begin{array}{l} D(A) = H^2(0, \pi) \cap H^1(0, \pi) \\ Ay(t) = y''(t) \text{ for } t \in (0, \pi) \text{ and } y \in D(A). \end{array} \right.$$

Then A generates a C_0 -semigroup $(\mathcal{U}(t))_{t \geq 0}$ on $L^2((0, \pi))$ given by

$$(\mathcal{U}(t)x)(r) = \sum_{n=1}^{+\infty} e^{-n^2 \pi^2 t} \langle x, e_n \rangle_{L^2} e_n(r),$$

where $e_n(r) = \sqrt{2} \sin(n\pi r)$ for $n = 1, 2, \dots$ and $\|\mathcal{U}(t)\| \leq e^{-\pi^2 t}$ for all $t \geq 0$. Thus $\overline{M} = 1$ and $\omega = \pi^2$. Then A satisfies the Hille-Yosida conditions in $L^2(0, \pi)$. Moreover the part A_0 of A in $\overline{D(A)}$ is the generator of compact semigroup. It follows that (\mathbf{H}_0) and (\mathbf{H}_1) are satisfied.

We define $f : \mathbb{R} \times \mathcal{C} \rightarrow L^2((0, \pi))$ and $L : \mathcal{C} \rightarrow L^2(\Omega, H)$ as follows

$$f(t, \phi)(x) = \sin\left(\frac{1}{2 + \cos t + \cos \sqrt{2}t}\right) + \arctan(t) + \int_{-r}^0 h(\theta, \phi(\theta)(x)) d\theta \text{ for } x \in (0, \pi) \text{ and } t \in \mathbb{R},$$

$$g(t, \phi)(x) = \sin\left(\frac{1}{2 + \cos t + \cos \sqrt{3}t}\right) + \arctan(t) + \int_{-r}^0 h(\theta, \phi(\theta)(x))d\theta \text{ for } x \in (0, \pi) \text{ and } t \in \mathbb{R},$$

and

$$L(\phi)(x) = \int_{-r}^0 G(\theta)(\phi(\theta)(x)) \text{ for } -r \leq \theta \text{ and } x \in (0, \phi).$$

Let us pose $v(t) = z(t, x)$. Then equation (5.1) takes the following abstract form

$$dv(t) = [Av(t) + L(v_t) + f(t, v_t)]dt + g(t, v_t)dW(t) \text{ for } t \in \mathbb{R}. \quad (5.2)$$

Consider the measure μ and ν where its Randon-Nikodym derivates are respectively ρ_1 and ρ_2

$$\rho_1(t) = \begin{cases} 1 & \text{for } t > 0 \\ e^t & \text{for } t \leq 0 \end{cases}$$

and

$$\rho_2(t) = |t| \text{ for } t \in \mathbb{R}$$

i.e $d\mu(t) = \rho_1(t)dt$ and $d\nu(t) = \rho_2(t)dt$, where dt denotes the Lebesgue measure on \mathbb{R} and

$$\mu(A) = \int_A \rho_1(t)dt \text{ for } \nu(A) = \int_A \rho_2(t)dt \text{ for } A \in \mathcal{N}$$

From[14] $\mu, \nu \in \mathcal{M}$ satisfies Hypothesis **(H₄)**.

$$\lim_{\tau \rightarrow +\infty} \frac{\mu([- \tau, \tau])}{\nu([- \tau, \tau])} = \limsup_{\tau \rightarrow +\infty} \frac{\int_{-r}^0 e^t dt + \int_0^\tau dt}{2 \int_0^\tau t dt} = \limsup_{\tau \rightarrow +\infty} \frac{1 + e^{-\tau} + \tau}{\tau^2} = 0 < \infty,$$

which implies that **(H₂)** is satisfied.

For $t \in \mathbb{R}$, $-\frac{\pi}{2} \leq \arctan(\theta) \leq \frac{\pi}{2}$, therefore for all $\theta \in [t-r, t]$ $\arctan(t-r) \leq \arctan(t)$. It follows that $|\arctan \theta - \frac{\pi}{2}| = \frac{\pi}{2} - \arctan \theta \leq |\arctan(t-r) - \frac{\pi}{2}| = \frac{\pi}{2} - \arctan(t-r)$ which implies that $|\arctan \theta - \frac{\pi}{2}|^p \leq |\arctan(t-r) - \frac{\pi}{2}|^p$, hence, we have

$$\sup_{\theta \in [t-r, t]} \mathbb{E} |\arctan \theta - \frac{\pi}{2}|^p \leq \mathbb{E} |\arctan(t-r) - \frac{\pi}{2}|^p.$$

On one hand, we have

$$\begin{aligned} \frac{1}{\nu([- \tau, \tau])} \int_0^\tau \mathbb{E} \sup_{\theta \in [t-r, t]} \mathbb{E} |\arctan \theta - \frac{\pi}{2}|^p dt &\leq \frac{1}{\nu([- \tau, \tau])} \int_0^\tau \mathbb{E} |\arctan(t-r) - \frac{\pi}{2}|^p dt \\ &\leq \frac{1}{\nu([- \tau, \tau])} \int_0^\tau \mathbb{E} \left(\frac{\pi}{2} - \arctan(t-r)\right)^p dt \\ &\leq \frac{1}{\nu([- \tau, \tau])} \int_0^\tau \frac{\pi^p}{2^p} dt \\ &\leq \frac{\pi^p}{2^{p+1}\tau} \rightarrow 0, \text{ as } \tau \rightarrow +\infty. \end{aligned}$$

On other hand, we have

$$\begin{aligned} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^0 \mathbb{E} \sup_{\theta \in [t-r, t]} \mathbb{E} \left| \arctan \theta - \frac{\pi}{2} \right|^p dt &\leq \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^0 \frac{\pi^p}{2^p} e^t dt \\ &\leq \frac{\pi^p (1 - e^{-\tau})}{2^{p+1} \tau} \rightarrow 0, \text{ as } \tau \rightarrow +\infty. \end{aligned}$$

Consequently

$$\lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \mathbb{E} \sup_{\theta \in [t-r, t]} \mathbb{E} \left| \arctan \theta - \frac{\pi}{2} \right|^p d\mu(t) = 0.$$

It follows that $t \mapsto \arctan(t) - \frac{\pi}{2}$ is p -th mean (μ, ν) -ergodic of class r , consequently f is uniformly p -th mean (μ, ν) -pseudo almost automorphic of class r . Moreover L is bounded linear operator from \mathcal{C} to $L^2(\Omega, H)$.

Let k be the lipschitz constant of h . Then by using Hölder-inequality for every $\varphi_1, \varphi_2 \in \mathcal{C}$ and $t \geq 0$, we have

$$\begin{aligned} \mathbb{E} \|f(t, \varphi_1)(x) - f(t, \varphi_2)(x)\|^p &= \mathbb{E} \left\| \int_{-r}^0 \left(h(\theta, \varphi_1(\theta)(x)) - h(\theta, \varphi_2(\theta)(x)) \right) d\theta \right\|^p \\ &\leq \left[\int_{-r}^0 \|h(\theta, \varphi_1(\theta)(x)) - h(\theta, \varphi_2(\theta)(x))\|^p d\theta \right]^p \\ &\leq \mathbb{E} \left[\left(\int_{-r}^0 d\theta \right)^{\frac{p-1}{p}} \times \left(\int_{-r}^0 \|h(\theta, \varphi_1(\theta)(x)) - h(\theta, \varphi_2(\theta)(x))\|^p d\theta \right)^{\frac{1}{p}} \right]^p \\ &\leq r^{p-1} \int_{-r}^0 \mathbb{E} \|h(\theta, \varphi_1(\theta)(x)) - h(\theta, \varphi_2(\theta)(x))\|^p d\theta \\ &\leq r^{p-1} k \int_{-r}^0 \mathbb{E} \|\varphi_1(\theta)(x) - \varphi_2(\theta)(x)\|^p d\theta \\ &\leq r^p k \sup_{-r \leq \theta \leq 0} \mathbb{E} \|\varphi_1(\theta)(x) - \varphi_2(\theta)(x)\|^p \\ &\leq r^p k \alpha \sup_{-r \leq \theta \leq 0} \mathbb{E} \|\varphi_1(x) - \varphi_2(x)\|^p \text{ for a certain } \alpha \in \mathbb{R}_+. \end{aligned}$$

Consequently, we conclude that f and g are Lipschitz continuous and $cl(\mu, \nu)$ -pseudo almost automorphic in p -th mean sense. Moreover, since h is stochastically bounded in p -th mean

sense, i.e $\mathbb{E}\|h(t, \phi(t))\|^p \leq \beta$. By Hölder inequality, we have

$$\begin{aligned}
\mathbb{E}\|g(t, \varphi(x))\|^p &= 1 + \frac{\pi}{2} + \mathbb{E}\left\|\int_{-r}^0 h(\theta, \varphi(\theta)(x)) d\theta\right\|^p \\
&\leq \frac{2+\pi}{2} + \left(\int_{-r}^0 d\theta\right)^{p-1} \times \int_{-r}^0 \mathbb{E}\|h(\theta, \varphi(\theta)(x))\|^p d\theta \\
&\leq \frac{2+\pi}{2} + r^{p-1} \int_{-r}^0 \beta d\theta \\
&\leq \frac{2+\pi}{2} + r^p \beta \\
&\leq \beta_1, \text{ with } \beta_1 = \frac{2+\pi}{2} + r^p \beta,
\end{aligned}$$

which implies that g satisfies (\mathbf{H}_5) .

For hyperbolicity, we suppose that

$$(\mathbf{H}_8) \int_{-r}^0 |G(\theta)| d\theta < 1.$$

Proposition 5.1. [18] *Assume that (\mathbf{H}_6) and (\mathbf{H}_7) hold. The the seùigroup $(\mathcal{U}(t))_{t \geq 0}$ is hyperbolic.*

Then by Proposition 4.23, we deduce the following result

Theorem 5.2. *Under above assumptions, if $\text{Lip}(h)$ is small enough large, then equation (5.1) has unique p -th mean $cl(\mu, \nu)$ -pseudo almost automorphic solution of class r .*

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