LIMITS OF RIEMANN SOLUTIONS FOR ISENTROPIC MHD IN A VARIABLE CROSS-SECTION DUCT AS MAGNETIC FIELD VANISHES*

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Abstract. The stability for magnetic field to the solution of the Riemann problem for the polytropic fluid in a variable cross-section duct is discussed. By the vanishing magnetic field method, the stable solutions are determined by comparing the limit solutions with the solutions of the Riemann problem for the polytropic fluid in a duct obtained by the entropy rate admissibility criterion.

Keywords. Variable duct, magnetogasdynamics, Riemann problem, stability, vanishing magnetic field method.

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1. Introduction

The non-conservative hyperbolic system plays an important role in many areas, such as the laminar flow in compliant tubes [2], the shallow water [9] and the multiphase flows [13]. The main difficulties of the Riemann problem for it are the existence and the uniqueness of the solution. In a recent paper [16], the Riemann problem for the isentropic, inviscid, simple flow of ideal gas, subjected to transverse magnetic field, in a duct with cross-sectional area $a(x) > 0$ in magnetogasdynamics, has been studied. It is governed by the hyperbolic system

$$\begin{cases} (a\rho)_t + (a\rho u)_x = 0, \\ (a\rho u)_t + (a(\rho u^2 + p + \frac{B^2}{2\mu}))_x = (p + \frac{B^2}{2\mu})a_x, \\ a_t = 0, \end{cases} \tag{1.1}$$

with the Riemann initial data

$$(u, \rho, a) = \begin{cases} U_-(u_-, \rho_-, a_-), x < 0, \\ U_+(u_+, \rho_+, a_+), x > 0, \end{cases} \tag{1.2}$$

where $a_+ > a_- > 0, \rho_- > 0, \rho_+ > 0, u_-$ and u_+ are arbitrary constants. Symbols ρ, p, u, B and μ are the specific density, the pressure, the velocity, the transverse magnetic field and the magnetic permeability, resp., see [15]. The pressure function and the transverse magnetic field function are given by $p = \kappa\rho^\gamma$ and $B = k\rho$, resp., where $\gamma \in (1, 2), \kappa, k$ are positive constants. The existence has been obtained for any given initial data. However, for some initial data, there exist multi solutions. By introducing the entropy rate admissibility criterion [5],

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the uniqueness can be guaranteed. In this article, we will select a proper solution mainly by the vanishing magnetic field method, motivated by the vanishing viscosity method [3] and the vanishing pressure method [4].

We call a solution of (1.1) is stable in a vanishing magnetic field, provided that the limit of it, as $k \to 0$, equals to the solution of

$$\begin{cases} (a\rho)_t + (a\rho u)_x = 0, \\ (a\rho u)_t + (a(\rho u^2 + p))_x = pa_x, \\ a_t = 0, \end{cases} \tag{1.3}$$

with the initial data (1.2). System (1.3) describes a compressible polytropic fluid flow in a nozzle and has been studied in [12, 17]. The nonisentropic case has been investigated in [1, 7, 18]. Putting $a_x = 0$, (1.1) can be written in conservation form as

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p + \frac{B^2}{2\mu})_x = 0, \end{cases} \tag{1.4}$$

which describes an unsteady one-dimensional isentropic flow in magnetogasdynamic. The system was studied in [14]. In [6, 11, 19], the authors were concerned with the nonisentropic cases.

This paper is organised as follow. In section 2, the elementary waves and some properties of them are collected. In section 3, we present all the solutions of (1.1) and (1.3), for any given initial data (1.2). In section 4, the unique solution is determined by choosing the stable solution in a vanishing magnetic field, which satisfies the entropy rate admissibility criterion, as it will be seen.

2. Elementary Waves

System (1.1) has three real eigenvalues

$$\lambda_1 = u - \omega, \quad \lambda_2 = 0, \quad \lambda_3 = u + \omega,$$

where $\omega(\rho) = \sqrt{\frac{\mathrm{d}f}{\mathrm{d}\rho}}$, and $f(\rho) = p + \frac{B^2}{2\mu}$. It is strictly hyperbolic in the following three regions

$$\mathrm{I} = \{(u,\rho,a)\big| u < -\omega\}, \quad \mathrm{II} = \{(u,\rho,a)\big| |u| < \omega\}, \quad \mathrm{III} = \{(u,\rho,a)\big| u > \omega\}.$$

The characteristic fields λ_1 and λ_3 are genuinely nonlinear, and the characteristic field of λ_2 is linearly degenerate. For convenience, we set $\Sigma = \{u = -\omega\}$, $\Pi = \{u = \omega\}$, $\mathrm{II}^- = \mathrm{II} \cap \{u < 0\}$, and $\mathrm{II}^+ = \mathrm{II} \cap \{u > 0\}$. There exist three different elementary waves.

2.1. **Rarefaction waves.** Centered rarefaction waves $R_1(U_0, U)$ and $R_3(U_0, U)$(abb. $R_1(U_0)$ and $R_3(U_0)$, resp.) are

$$\begin{cases} R_1(U_0) : u = u_0 - \displaystyle\int_{\rho_0}^{\rho} \frac{\omega}{\rho}\,\mathrm{d}\rho, \ \ \rho < \rho_0, \\ R_3(U_0) : u = u_0 + \displaystyle\int_{\rho_0}^{\rho} \frac{\omega}{\rho}\,\mathrm{d}\rho, \ \ \rho > \rho_0, \\ a = a_0, \end{cases} \tag{2.1}$$

for any given left hand state $U_0(u_0, \rho_0, a_0)$. $R_1(U_0)$ is convex and monotonic decreasing while $R_3(U_0)$ is concave and monotonic increasing.

2.2. **Shock waves.** The Rankine-Hugoniot jump condition of the third equation in (1.1) is

$$\sigma\,[a] = 0, \tag{2.2}$$

where σ is the speed of the discontinuity, and $[a] = a_r - a_l$. The cross-section area a remains constant across shock waves $S_1(U_0, U)$ and $S_3(U_0, U)$(abb. $S_1(U_0)$ and $S_3(U_0)$, resp.) satisfying the Rankine-Hugoniot jump condition

$$\begin{cases} -\sigma\,[\rho] + [\rho u] = 0, \\ -\sigma\,[\rho u] + [g(u, \rho)] = 0, \\ [a] = 0, \end{cases} \tag{2.3}$$

for any given left hand state $U_0(u_0, \rho_0, a_0)$, where

$$g(u, \rho) = \rho u^2 + \kappa\rho^\gamma + \frac{k^2}{2\mu}\rho^2.$$

By Lax entropy conditions [8], $S_i(U_0)$ can be expressed as

$$\begin{cases} S_1(U_0) : u = u_0 - \sqrt{\dfrac{1}{\rho\rho_0}\,[f]\,[\rho]},\ \sigma = u_0 + \rho\dfrac{[u]}{[\rho]},\ \rho > \rho_0,\ u < u_0, \\ S_3(U_0) : u = u_0 - \sqrt{\dfrac{1}{\rho\rho_0}\,[f]\,[\rho]},\ \sigma = u_0 + \rho\dfrac{[u]}{[\rho]},\ \rho < \rho_0,\ u < u_0, \\ a = a_0. \end{cases} \tag{2.4}$$

$S_1(U_0)$ is convex and monotonic decreasing while $S_3(U_0)$ is concave and monotonic increasing. We obtain the following lemma by direct calculations to (2.4).

Lemma 2.1. *On the shock waves $S_1(U_0)$(resp., $S_3(U_0)$), it holds that*
(i) $\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} < 0$*(resp.,* $\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} > 0$*);*
(ii) $u' < -\frac{\omega}{\rho}$*(resp.,* $u' > \frac{\omega}{\rho}$*);*
(iii) *There exists a unique state $U \in \mathrm{I\!I}^+$(resp., $U \in \mathrm{I}$), denoted by $S_1^0(U_0)$(resp., $S_3^0(U_0)$), such that $\sigma(U_0, U) = 0$ if and only if $U_0 \in \mathrm{I\!I\!I}$(resp., $U_0 \in \mathrm{I\!I}^-$).*

For any given left hand states U_0, we define

$$\begin{aligned} &R_i^-(U_0) = R_i(U_0) \cap \{\lambda_i(U) \le 0\},\ S_i^-(U_0) = S_i(U_0) \cap \{\sigma(U_0, U) \le 0\}, \\ &R_i^+(U_0) = R_i(U_0) \cap \{\lambda_i(U) \ge 0\},\ S_i^+(U_0) = S_i(U_0) \cap \{\sigma(U_0, U) \ge 0\}, \\ &W_i^\pm(U_0) = R_i^\pm(U_0) \cup S_i^\pm(U_0),\ W_i(U_0) = R_i(U_0) \cup S_i(U_0), i = 1, 3. \end{aligned}$$

2.3. **Stationary waves.** The jump condition (2.2) also holds when the gas across the discontinuity, which we call stationary waves,

$$W_2 : \begin{cases} [a\rho u] = 0, \\ [h(u, \rho)] = 0, \\ \sigma = 0, \end{cases} \tag{2.5}$$

where

$$h(u,\rho)=\frac{u^2}{2}+\frac{\kappa\gamma}{\gamma-1}\rho^{\gamma-1}+\frac{k^2}{\mu}\rho. \tag{2.6}$$

For any given state $U_0(u_0,\rho_0,a_0)$, once we assume that $a>a_0$, there exist two different solutions of (2.5), denoted by $\overline{U}_0(\overline{u}_0,\overline{\rho}_0,a)$ and $\underline{U_0}(\underline{u_0},\underline{\rho_0},a)$, with

$$|u_0|>|\overline{u}_0|,\rho_0<\overline{\rho}_0,|u_0|<|\underline{u_0}|,\rho_0>\underline{\rho_0}. \tag{2.7}$$

In particular, when $a=a_0$, the two solutions $\overline{U}_0=U_0,\underline{U_0}\in \mathrm{I}\cup\mathrm{III}$ if $U_0\in\mathrm{II}$, while $\overline{U}_0\in\mathrm{II},\underline{U_0}=U_0$ if $U_0\in\mathrm{I}\cup\mathrm{III}$. The fact that there exists no stationary wave solution for (1.4) motivates us to suggest the Stability Stationary Wave Condition to remove the unreasonable solution.

Stability Stationary Wave Condition. The state $U(u,\rho,a)$ is called a stable stationary solution of (2.5), if u and ρ are continuous functions of a, and the two states U and U_0 satisfy the Rankine-Hugoniot jump condition (2.3) when $a=a_0$.

The Stability Stationary Wave Condition leads to the following lemma.

Lemma 2.2. *For any given $U_0(u_0,\rho_0,a_0)$ and $a>a_0$, the two solutions $\overline{U}_0(\overline{u}_0,\overline{\rho}_0,a)$ and $\underline{U_0}(\underline{u_0},\underline{\rho_0},a)$ of (2.5) satisfy (2.7) and*
(i) $\overline{U}_0\in\mathrm{II}^{\pm}$ *is the unique stable stationary solution, if* $U_0\in\mathrm{II}^{\pm}$*;*
(ii) $\underline{U_0}\in\mathrm{I}$*(resp.,* III*) is the unique stable stationary solution, if* $U_0\in\mathrm{I}$*(resp.,* III*);*
(iii) *Both* $\underline{U_0}\in\mathrm{I}$*(resp.,* III*) and* $\overline{U}_0\in\mathrm{II}$ *are the stable stationary solutions, if* $U_0\in\Sigma$*(resp.,* Π*).*

For any given left hand state U_0 and right hand state U, $\overline{W}_2(U_0,U)$ denotes the stationary wave satisfying $U=\overline{U}_0$, while $\underline{W_2}(U_0,U)$ denotes the stationary wave satisfying $U=\underline{U_0}$.

It is clear that a changes only when the gas passes across the stationary wave. When there exists no confusion, symbols denote the projections of themselves on (u,ρ) plane, either. For example, U_- denotes both $U_-(u_-,\rho_-,a_-)$ and $U_-(u_-,\rho_-)$.

3. The Riemann Solutions for $k\ge 0$

For any given $k\ge 0$, Riemann problem (1.1) with (1.2) can be solved constructively by the two cases $U_-\in\Delta_\pm$, which are separated by

$$R_1(U,O): u=-\int_0^{\rho}\frac{\omega(\widetilde{\rho})}{\widetilde{\rho}}\mathrm{d}\widetilde{\rho},$$

see Figure 1 (a).

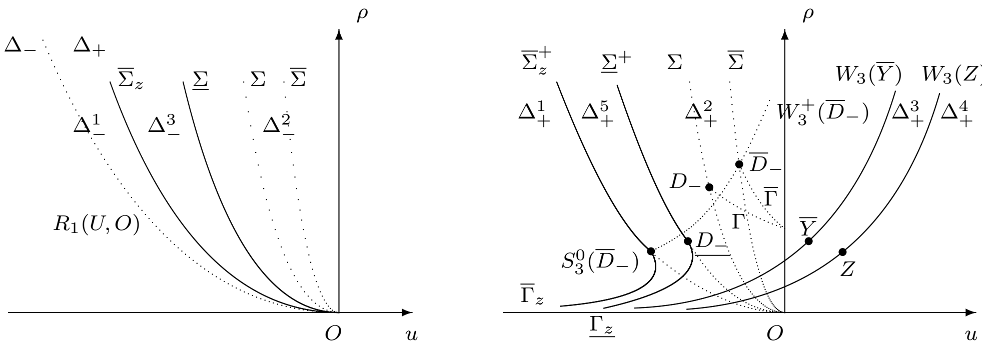
Case 1. $U_-\in\Delta_-$. It follows $u_-\le-\int_0^{\rho_-}\frac{\omega(\rho)}{\rho}\mathrm{d}\rho$. The curves are defined by

$$\underline{\Sigma}=\{U|U=\underline{U_0},U_0\in\Sigma\},\ \overline{\Sigma}=\{U|U=\overline{U}_0,U_0\in\Sigma\},\ \overline{\Sigma}_z=\{U|U=S_3^0(U_0),U_0\in\overline{\Sigma}\}.$$

It can be proved that $\underline{\Sigma}$ is on the right of $\overline{\Sigma}_z$. Thus the solid curves $\overline{\Sigma}_z$ and $\underline{\Sigma}$ in Figure 1 (a) separate the upper half (u,ρ) plan into three regions, Δ_-^1, Δ_-^2 and Δ_-^3 (including $\overline{\Sigma}_z$ and $\underline{\Sigma}$). The Riemann solutions of (1.1) with (1.2) are illustrated as follows.

Subcase 1.1. $U_-\in\Delta_-,U_+\in\Delta_-^1$, the solution is unique and structured in

$$Q_-^1:\quad W_1^-\oplus W_3^-\oplus\underline{W_2}.$$



(a) $U_- \in \Delta_-$. (b) $U_- \in \Delta_+$.

FIGURE 1. Different regions separated by the solid curves.

Subcase 1.2. $U_- \in \Delta_-, U_+ \in \Delta_-^2$, the solution is unique and structured in

$$Q_-^2: \quad W_1^- \oplus R_3^- \oplus \overline{W}_2 \oplus W_3^+.$$

Subcase 1.3. $U_- \in \Delta_-, U_+ \in \Delta_-^3$, The solution is not unique. Besides Q_-^1, Q_-^2, the other solution with two stable stationary waves can be constructed as

$$Q_-^3: \quad W_1^- \oplus R_3^- \oplus \overline{W}_2(U_1,U_2) \oplus S_3(U_2,U_3) \oplus \underline{W_2}(U_3,U_+).$$

The states $U_1(u_1,\rho_1,a_-)$, $U_2(u_2,\rho_2,a)$ and $U_3(u_3,\rho_3,a)$ satisfy that

$$\sigma(U_2,U_3)=0,\ u_1,u_2,u_3<0,\ a\in[a_-,a_+].$$

$a=a_-$ holds if and only if $U_+ \in \underline{\Sigma}$, and then $Q_-^3 = Q_-^1$. $a=a_+$ holds if and only if $U_+ \in \overline{\Sigma}_z$, and then $Q_-^3 = Q_-^2$. When $U_+ \in \Delta_-^2 \backslash (\overline{\Sigma}_z \cup \underline{\Sigma})$, Q_-^3 is unstable, refer to [10]. Because it contains a standing shock wave $S_3(U_2,U_3)(\sigma(U_2,U_3)=0)$, which occurs in contracting duct.

Case 2. $U_- \in \Delta_+$. It follows $u_- > -\int_0^{\rho_-} \frac{\omega}{\rho}\mathrm{d}\rho$. The solid curves $\overline{\Sigma}_z^+$, $\overline{\Gamma}_z$, $\underline{\Sigma}^+$, $\underline{\Gamma_z}$, $W_3(\overline{Y})$ and $W_3(Z)$ in Figure 1 (b) separate the upper half (u,ρ) plan into Δ_+^1, Δ_+^2(including $W_3(\overline{Y})$), Δ_+^3, Δ_+^4(including $W_3(Z)$) and Δ_+^5(including the boundaries). Γ is the part of the curve $W_1(U_-)$, the ends of which are $D_- \in \Sigma$ and the one on ρ axis, resp.. Define

$$\overline{\Gamma} = \left\{U|U=\overline{U}_0,\ U_0\in\Gamma\right\}, \quad \overline{\Gamma}_z = \left\{U|U=S_3^0(U_0), U_0\in\overline{\Gamma}\right\},$$
$$\Gamma_z = \{U|U=S_3^0(U_0),\ U_0\in\Gamma\}, \quad \underline{\Gamma_z} = \left\{U|U=\underline{U_0},\ U_0\in\Gamma_z\right\}.$$

Obviously, $\underline{D_-}$ is an end of $\underline{\Gamma_z}$ and $S_3^0(\overline{D}_-)$ is an end of $\overline{\Gamma}_z$. For convenience, let

$$\overline{\Sigma}_z^+ = \overline{\Sigma}_z \cap \{\rho \geq \rho(S_3^0(\overline{D}_-))\}, \quad \underline{\Sigma}^+ = \underline{\Sigma} \cap \{\rho \geq \rho(\underline{D_-})\},$$

where $\rho(\underline{D_-})$ denotes the ρ coordinate at $\underline{D_-}$, etc. It can be proved that $\overline{\Gamma}_z$ is at the left of $\underline{\Gamma_z}$, and $W_3(\overline{Y})$ is at the left of $W_3(Z)$. Here,

$$Y = S_1^0(U_-), \quad Z = S_1^0(\underline{U_-}),$$

if $U_- \in \mathrm{III}$, otherwise,

$$Y = D_+, \quad Z = S_1^0(\underline{D_+}),$$

where D_+ is the intersection point of Π with $W_1(U_-)$.

Subcase 2.1. $U_- \in \Delta_+, U_+ \in \Delta_+^1$. The solution is unique and structured in

$$Q_+^1: \quad W_1^- \oplus W_3^- \oplus \underline{W_2}.$$

Subcase 2.2. $U_- \in \Delta_+, U_+ \in \Delta_+^2$. The solution is unique and structured in

$$Q_+^2: \quad W_1^-(\oplus R_3^-) \oplus \overline{W}_2 \oplus W_3^+,$$

where R_3^- appears if and only if U_+ is located on the left of $W_3^+(\overline{D}_-)$.

Subcase 2.3. $U_- \in \Delta_+, U_+ \in \Delta_+^3$. The solution is unique and structured in

$$Q_+^3: \quad (R_1^- \oplus)\underline{W_2} \oplus S_1 \oplus \overline{W}_2 \oplus W_3^+,$$

where the speeds of both sides of the standing shock wave S_1 are positive, so S_1 occurs in a compacting duct, and R_1^- appears if and only if $u_+ < \omega(\rho_+)$.

Subcase 2.4. $U_- \in \Delta_+, U_+ \in \Delta_+^4$. The solution is unique and structured in

$$Q_+^4: \quad (R_1^- \oplus)\underline{W_2} \oplus W_1^+ \oplus W_3^+,$$

where R_1^- appears if and only if $u_+ < \omega(\rho_+)$.

Subcase 2.5. $U_- \in \Delta_+, U_+ \in \Delta_+^5$. In this case, solution loses uniqueness. Besides Q_+^1 and Q_+^2, the other solution can be structured in

$$Q_+^5: \quad W_1^-(\oplus R_3^-) \oplus \underline{W_2}(U_1, U_2) \oplus S_3(U_2, U_3) \oplus \overline{W}_2(U_3, U_+),$$

where, R_3^- appears if and only if U_+ is located on the left of $W_3^+(\overline{D}_-)$. The states $U_1(u_1, \rho_1, a_-)$, $U_2(u_2, \rho_2, a)$ and $U_3(u_3, \rho_3, a)$ satisfy that

$$\sigma(U_2, U_3) = 0, \;\; u_1, u_2, u_3 < 0, \;\; a \in [a_-, a_+].$$

$a = a_-$ holds if and only if $U_+ \in \overline{\Sigma}_z^+ \cup \overline{\Gamma}_z$, and then $Q_+^5 = Q_+^1$. $a = a_+$ holds if and only if $U_+ \in \underline{\Sigma}^+ \cup \underline{\Gamma_z}$, and then $Q_+^5 = Q_+^2$. Otherwise, similar to Q_-^3, the solution Q_+^5 will not be considered, either.

The entropy rate admissibility criterion leads to that the unique admissible solutions for (1.1) with (1.2) are $Q_\pm^2$ in subcase 1.3 and 2.5 respectively, in which there are multi solutions.

4. The Behaviour of the Solution As $k \to 0$

It has been declaimed that in subcase 1.3 and 2.5, both $Q_\pm^1$ and $Q_\pm^2$ are the solutions of (1.1) for $k > 0$, respectively. By the entropy rate admissibility criterion, we can construct the solution uniquely for any given initial data (1.2). In this section, we will study with any initial data (1.2), the limit solutions of (1.1) as $k \to 0$, and compare the limit solutions with the solutions of (1.3). We want to check whether the limit solution is the one selected by the entropy rate admissibility criterion. The variation of k leads to the changes of the solid curves in Figure 1. Thus, the structure of the solution may change if k vanishes for the fixed initial data. To study the limit solution for (1.1) with (1.2), we need only to concentrate the case where U_+ is located on the solid curves when $k = 0$ in Figure 1. Then our goal is to investigate the behaviour of the solid curves as $k \to 0$. For simplify calculations, we replace $\frac{k^2}{\mu}$ with k. Then ω, f, g and h are rewritten as

$$\omega(\rho) = \sqrt{\kappa\gamma\rho^{\gamma-1} + k\rho}, \quad f(\rho) = \kappa\rho^\gamma + \tfrac{k}{2}\rho^2,$$
$$g(u,\rho) = \rho u^2 + \kappa\rho^\gamma + \tfrac{k}{2}\rho^2, \quad h(u,\rho) = \tfrac{u^2}{2} + \tfrac{\kappa\gamma}{\gamma-1}\rho^{\gamma-1} + k\rho,$$

respectively, from now on. To be more exactly, $\omega(\rho)$ is the abbreviation of $\omega(\rho,k)$, etc. We define $\Sigma(0) = \{U(u,\rho) | u = -\omega(\rho,0) = -\sqrt{\kappa\gamma\rho^{\gamma-1}}\}$, etc.

4.1. **The behaviour of the solution as $k \to 0$ when $U_- \in \Delta_-(0)$.** We have the following lemma when we investigate the behaviour of $\overline{\Sigma}_z$ and $\underline{\Sigma}$ as $k \to 0$.

Lemma 4.1. *There exists a sufficient small constant $k_0 > 0$ such that, for any $k \in (0, k_0)$,* (i) *$\underline{\Sigma}$ is at the left of $\underline{\Sigma}(0)$;* (ii) *$\overline{\Sigma}_z$ is at the left of $\overline{\Sigma}_z(0)$.*

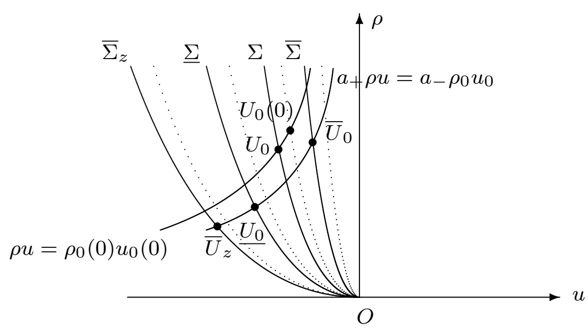


FIGURE 2. Curves, $\overline{\Sigma}_z$, $\underline{\Sigma}$, Σ and $\overline{\Sigma}$, move to the dotted lines $\overline{\Sigma}_z(0)$, $\underline{\Sigma}(0)$, $\Sigma(0)$ and $\overline{\Sigma}(0)$, resp., from their left as $k \to 0$.

Proof. (i) Assume $U_0(0) = (u_0(0), \rho_0(0), a_-) \in \Sigma(0)$ is an arbitrary state. Define $U_0 = (u_0, \rho_0, a_-) \in \Sigma$, $\underline{U_0} = (\underline{u_0}, \underline{\rho_0}, a_+) \in \underline{\Sigma}$, $\overline{U}_0 = (\overline{u}_0, \overline{\rho}_0, a_+) \in \overline{\Sigma}$ and $\overline{U}_z = (\overline{u}_z, \overline{\rho}_z, a_+) \in \overline{\Sigma}_z$ satisfying that

$$u_0^2 = \omega_0^2 = \kappa\gamma\rho_0^{\gamma-1} + k\rho_0, \tag{4.1}$$
$$h(u_0, \rho_0) = h(\underline{u_0}, \underline{\rho_0}) = h(\overline{u}_0, \overline{\rho}_0), \tag{4.2}$$
$$g(\overline{u}_z, \overline{\rho}_z) = g(\overline{u}_0, \overline{\rho}_0), \tag{4.3}$$
$$a_-\rho_0(0)u_0(0) = a_-\rho_0 u_0 = a_+\underline{\rho_0}\,\underline{u_0} = a_+\overline{\rho}_0\overline{u}_0 = a_+\overline{\rho}_z\overline{u}_z. \tag{4.4}$$

See Figure 2. Followed by Lemma 2.2, the following inequalities hold,

$$\overline{\rho}_z < \underline{\rho_0} < \rho_0 < \overline{\rho}_0, \ \overline{u}_z^2 > \overline{\omega}_z^2, \ \underline{u_0}^2 > \underline{\omega_0}^2, \ \overline{u}_0^2 < \overline{\omega}_0^2,$$

where $\overline{\omega}_z = \omega(\overline{\rho}_z)$, etc.

Differentiating (4.4) with respect to k, resp., one gets

$$0 = \rho_0' u_0 + \rho_0 u_0' = \underline{\rho_0}'\underline{u_0} + \underline{\rho_0}\,\underline{u_0}' = \overline{\rho}_0'\overline{u}_0 + \overline{\rho}_0\overline{u}_0' = \overline{\rho}_z'\overline{u}_z + \overline{\rho}_z\overline{u}_z', \tag{4.5}$$

here and from now on, $' = \frac{\mathrm{d}}{\mathrm{d}k}$. From (4.1), (4.2) and (4.5), we get

$$\rho_0 = \frac{\omega_0^2 - u_0^2}{\rho_0}\rho_0' + \rho_0 = \frac{\underline{\omega_0}^2 - \underline{u_0}^2}{\underline{\rho_0}}\underline{\rho_0}' + \underline{\rho_0} = \frac{\overline{\omega}_0^2 - \overline{u}_0^2}{\overline{\rho}_0}\overline{\rho}_0' + \overline{\rho}_0, \tag{4.6}$$

which follows $\underline{\rho_0}' < 0$. Because the intersection point of $\underline{\Sigma}$ with $\rho u = const.$ is unique if k is fixed. We say $\underline{\Sigma}$ is at the left of $\underline{\Sigma}(0)$.

(ii) Differentiating (4.3) with respect to k and from (4.6), we obtain

$$(\overline{\omega}_z^2 - \overline{u}_z^2)\overline{\rho}_z' = (\overline{\omega}_0^2 - \overline{u}_0^2)\overline{\rho}_0' + \frac{\overline{\rho}_0^2 - \overline{\rho}_z^2}{2} = \rho_0\overline{\rho}_0 - \overline{\rho}_0^2 + \frac{\overline{\rho}_0^2 - \overline{\rho}_z^2}{2} = \frac{2\rho_0\overline{\rho}_0 - \overline{\rho}_0^2 - \overline{\rho}_z^2}{2}.$$

We are about to prove that $2\rho_0 > \overline{\rho}_0 + \overline{\rho}_z$ holds at $k = 0$ for any given state $U_0(u_0, \rho_0, a_-)$ with $0 < -u_0 \le \omega_0$. For $a \ge a_-$, we define $\overline{U}_0(\overline{u}_0, \overline{\rho}_0, a)$ and $\overline{U}_z(\overline{u}_z, \overline{\rho}_z, a) = S_3^0(\overline{U}_0)$ satisfying

$$h_0 = \overline{h}_0 \quad \overline{g}_0 = \overline{g}_z \quad a_-\rho_0 u_0 = a\overline{\rho}_0\overline{u}_0 = a\overline{\rho}_z\overline{u}_z. \tag{4.7}$$

It is clear that $\overline{U}_0$ and $\overline{U}_z$ are functions of a, followed by the stable stationary wave condition. The following inequalities hold when $a > a_-$,

$$\overline{\rho}_z < \rho_0 < \overline{\rho}_0, \quad \overline{u}_z < u_0 < \overline{u}_0 < 0, \quad \overline{u}_z^2 > \overline{\omega}_z^2, \quad u_0^2 < \omega_0^2, \quad \overline{u}_0^2 > \overline{\omega}_0^2.$$

We have $\overline{U}_0 = U_0$ and $\rho_0 \ge \overline{\rho}_z$, thus $2\rho_0 \ge \overline{\rho}_0 + \overline{\rho}_z$, if $a = a_-$. From (4.7), it holds that

$$\begin{aligned}
\frac{\mathrm{d}(\overline{\rho}_0 + \overline{\rho}_z)}{\mathrm{d}a} &= \frac{\overline{\rho}_0\overline{u}_0^2}{a(\overline{\omega}_0^2 - \overline{u}_0^2)} + \frac{2\overline{\rho}_z\overline{u}_z^2 - \overline{\rho}_0\overline{u}_0^2}{a(\overline{\omega}_z^2 - \overline{u}_z^2)} \\
&= \frac{\overline{\rho}_0\overline{u}_0^2}{a(\overline{\omega}_0^2 - \overline{u}_0^2)(\overline{\omega}_z^2 - \overline{u}_z^2)}\left(\overline{\omega}_z^2 - \overline{u}_z^2 + (2\frac{\overline{\rho}_0}{\overline{\rho}_z} - 1)(\overline{\omega}_0^2 - \overline{u}_0^2)\right) \\
&= \frac{\overline{\rho}_0\overline{u}_0^2}{a(\overline{\omega}_0^2 - \overline{u}_0^2)(\overline{\omega}_z^2 - \overline{u}_z^2)}\left(\kappa\gamma\overline{\rho}_z^{\gamma-1} - \frac{\overline{\rho}_0}{\overline{\rho}_z}\frac{\overline{f}_0 - f_z}{\overline{\rho}_0 - \overline{\rho}_z} + (2\frac{\overline{\rho}_0}{\overline{\rho}_z} - 1)(\kappa\gamma\overline{\rho}_0^{\gamma-1} - \frac{\overline{\rho}_z}{\overline{\rho}_0}\frac{\overline{f}_0 - f_z}{\overline{\rho}_0 - \overline{\rho}_z})\right) \\
&= \frac{\overline{\rho}_0\overline{u}_0^2\kappa\overline{\rho}_z^{\gamma-1}}{a(\overline{\omega}_0^2 - \overline{u}_0^2)(\overline{\omega}_z^2 - \overline{u}_z^2)}\left(\gamma - \alpha\frac{\alpha^\gamma - 1}{\alpha - 1} + (2\alpha - 1)(\gamma\alpha^{\gamma-1} - \frac{1}{\alpha}\frac{\alpha^\gamma - 1}{\alpha - 1})\right),
\end{aligned}$$

for $k = 0$, where

$$\frac{\overline{f}_0 - f_z}{\overline{\rho}_0 - \overline{\rho}_z} = \frac{\kappa\overline{\rho}_0^\gamma - \kappa\overline{\rho}_z^\gamma}{\overline{\rho}_0 - \overline{\rho}_z} = \kappa\overline{\rho}_z^{\gamma-1}\frac{(\frac{\overline{\rho}_0}{\overline{\rho}_z})^\gamma - 1}{\frac{\overline{\rho}_0}{\overline{\rho}_z} - 1} = \kappa\overline{\rho}_z^{\gamma-1}\frac{\alpha^\gamma - 1}{\alpha - 1},$$

and $\alpha = \frac{\overline{\rho}_0}{\overline{\rho}_z} > 1$ for $a > a_-$. Therefore, $\frac{\mathrm{d}(\overline{\rho}_0 + \overline{\rho}_z)}{\mathrm{d}a}$ has the different sign with the auxiliary function

$$\begin{aligned}
M_1(\alpha) &= \gamma\alpha(\alpha - 1) - \alpha^2(\alpha^\gamma - 1) + (2\alpha - 1)(\gamma\alpha^\gamma(\alpha - 1) - \alpha^\gamma + 1) \\
&= \alpha^{\gamma+2}(2\gamma - 1) - \alpha^{\gamma+1}(3\gamma + 2) + \alpha^2(\gamma + 1) + \alpha^\gamma(\gamma + 1) + \alpha(-\gamma + 2) - 1.
\end{aligned}$$

It is easy to check that $M_1(\alpha) > M_1(1) = 0$ by $\frac{\mathrm{d}M_1}{\mathrm{d}\alpha}(\alpha) > 0$. In fact,

$$\frac{\mathrm{d}^3 M_1}{\mathrm{d}\alpha^3}(\alpha) = \gamma(\gamma + 1)\alpha^{\gamma-3}\left(\alpha^2(2\gamma - 1)(\gamma + 2) - \alpha(3\gamma + 2)(\gamma - 1) + (\gamma - 2)(\gamma - 1)\right) > 0,$$

and

$$\frac{\mathrm{d}^2 M_1}{\mathrm{d}\alpha^2}(\alpha) = (\gamma + 1)\left(\alpha^\gamma(2\gamma - 1)(\gamma + 2) - \alpha^{\gamma-1}(3\gamma + 2)\gamma + \alpha^{\gamma-2}\gamma(\gamma - 1) + 2\right) > 0.$$

Then we obtain

$$\begin{aligned}
\frac{\mathrm{d}M_1}{\mathrm{d}\alpha}(\alpha) &= \alpha^{\gamma+1}(2\gamma - 1)(\gamma + 2) - \alpha^\gamma(3\gamma + 2)(\gamma + 1) + 2\alpha(\gamma + 1) + \alpha^{\gamma-1}(\gamma + 1)\gamma + 2 - \gamma \\
&> \frac{\mathrm{d}M_1}{\mathrm{d}\alpha}(1) = 0.
\end{aligned}$$

As a result, we have $2\rho_0 > \overline{\rho}_0 + \overline{\rho}_z$ holds for any $a > a_-$. Thus

$$2\rho_0\overline{\rho}_0 - \overline{\rho}_0^2 - \overline{\rho}_z^2 > (\overline{\rho}_0+\overline{\rho}_z)\overline{\rho}_0 - \overline{\rho}_0^2 - \overline{\rho}_z^2 = \overline{\rho}_z(\overline{\rho}_0 - \overline{\rho}_z) > 0,$$

which implies $\overline{\rho}_z'(0) < 0$. We complete the proof. □

Corollary 4.1. *When the solution loses uniqueness, we choose $Q^1_\pm$(resp., $Q^2_\pm$) as the unique solution of* (1.1) *with* (1.2) *for any $k \geq 0$. There exists a $k_0 > 0$, such that, for any $k \in (0, k_0)$, we have:*

(i) *$U_- \in \Delta_-$ and $U_+ \in \Delta^3_-$, if $U_- \in \Delta_-(0)\backslash R_1(U,O)|_{k=0}$ and $U_+ \in \overline{\Sigma}_z(0)$. Then for* (1.2), *the unique solution of* (1.1) *is Q^1_-(resp., Q^2_-), while the unique solution of* (1.3) *is $Q^1_-(0)$(resp., $Q^2_-(0)$).*

(ii) *$U_- \in \Delta_-$ and $U_+ \in \Delta^2_-$, if $U_- \in \Delta_-(0)\backslash R_1(U,O)|_{k=0}$ and $U_+ \in \underline{\Sigma}(0)$. Then for* (1.2), *the unique solution of* (1.1) *is Q^2_-, while the unique solution of* (1.3) *is $Q^1_-(0)$(resp., $Q^2_-(0)$).*

(iii) *$U_- \in \Delta_+$ and $U_+ \in \Delta^5_+$ is located on the left of $W_3^+(\overline{D}_-)$, if $U_- \in R_1(U,O)|_{k=0}$ and $U_+ \in \overline{\Sigma}_z(0)$. Then for* (1.2), *the unique solution of* (1.1) *is Q^1_+(resp., $Q^2_+ : W_1^- \oplus R_3^- \oplus W_2 \oplus W_3^+$), while the unique solution of* (1.3) *is $Q^1_-(0)$(resp., $Q^2_-(0)$).*

(iv) *$U_- \in \Delta_+$ and $U_+ \in \Delta^2_+$ is located on the left of $W_3^+(\overline{D}_-)$, if $U_- \in R_1(U,O)|_{k=0}$ and $U_+ \in \underline{\Sigma}(0)$. Then for* (1.2), *the unique solution of* (1.1) *is $Q^2_+ : W_1^- \oplus R_3^- \oplus W_2 \oplus W_3^+$, while the unique solution of* (1.3) *is $Q^1_-(0)$(resp., $Q^2_-(0)$).*

4.2. **The behaviour of the solution as $k \to 0$ when $U_- \in \Delta_+(0)$.** We have the following lemma when we investigate the behaviour of $\overline{\Gamma}_z$ and $\underline{\Gamma_z}$ as $k \to 0$.

Lemma 4.2. *There exists a sufficient small constant $k_0 > 0$ such that, for any $k \in (0, k_0)$,* (i) *$\underline{\Gamma_z}$ is at the left of $\underline{\Gamma_z}(0)$;* (ii) *$\overline{\Gamma}_z$ is at the left of $\overline{\Gamma}_z(0)$.*

Proof. (i) Assume $U_0(0)(u_0(0), \rho(0), a_-) \in \Gamma(0)$ is an arbitrary state. Let $U_0(u_0, \rho_0, a_-) \in \mathrm{II}$ be the intersection point of $\rho u = \rho_0(0)u_0(0)$ with Γ, $\underline{U_z}(\underline{u_z}, \underline{\rho_z}, a_+) \in \mathrm{I}$, $\overline{U}_0(\overline{u}_0, \overline{\rho}_0, a_+) \in \mathrm{I}$, and $\overline{U}_v(\overline{u}_v, \overline{\rho}_v, a_+) \in \mathrm{I}$ satisfy that $\sigma(U_0, U_z) = 0, \sigma(\overline{U}_0, \overline{U}_v) = 0$. More precisely, the following equations hold,

$$a_-\rho_0(0)u_0(0) = a_-\rho_0 u_0 = a_-\rho_z u_z = a_+\underline{\rho_z}\,\underline{u_z} = a_+\overline{\rho}_0\overline{u}_0 = a_+\overline{\rho}_v\overline{u}_v, \tag{4.8}$$

$$g_0 = g_z, h_z = \underline{h_z}, h_0 = \overline{h}_0, \overline{g}_0 = \overline{g}_v, \tag{4.9}$$

with

$$u_0^2 < \omega_0^2,\ \underline{u_z}^2 > \underline{\omega_z}^2,\ \overline{u}_0^2 < \overline{\omega}_0^2,\ \overline{u}_v^2 > \overline{\omega}_z^2,\ u_z^2 > \omega_z^2,\ \overline{\rho}_v < \underline{\rho_z} < \rho_z < \rho_0 < \overline{\rho}_0, \tag{4.10}$$

see Figure 3. Differentiating (4.9) with respect to k, resp., one obtains

$$(\omega_0^2 - u_0^2)\rho_0' + \frac{\rho_0^2}{2} = (\omega_z^2 - u_z^2)\rho_z' + \frac{\rho_z^2}{2}, \quad \frac{\omega_z^2 - u_z^2}{\rho_z}\rho_z' + \rho_z = \frac{\underline{\omega_z}^2 - \underline{u_z}^2}{\underline{\rho_z}}\underline{\rho_z}' + \underline{\rho_z}, \tag{4.11}$$

$$\frac{\omega_0^2 - u_0^2}{\rho_0}\rho_0' + \rho_0 = \frac{\overline{\omega}_0^2 - \overline{u}_0^2}{\overline{\rho}_0}\overline{\rho}_0' + \overline{\rho}_0, \quad (\overline{\omega}_0^2 - \overline{u}_0^2)\overline{\rho}_0' + \frac{\overline{\rho}_0^2}{2} = (\overline{\omega}_v^2 - \overline{u}_v^2)\overline{\rho}_v' + \frac{\overline{\rho}_v^2}{2}. \tag{4.12}$$

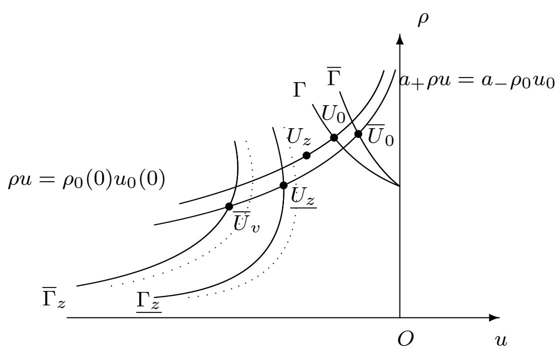


FIGURE 3. $\overline{\Gamma}_z$ and $\underline{\Gamma_z}$ move to the dotted lines $\overline{\Gamma}_z(0)$ and $\underline{\Gamma_z}(0)$, resp., from their left as $k \to 0$.

The sign of $\underline{\rho_z}'$ can be determined by ρ_0', and the value of ρ_0' can be obtained by the following two cases. When $U_0 \in R_1(U_-)$, it holds

$$u_0 = u_- - \int_{\rho_-}^{\rho_0} \frac{\omega}{\rho} \mathrm{d}\rho, \quad \rho_0 < \rho_-.$$

Associating it with the first equation of (4.8), we have

$$u_0' = \frac{u_0}{\omega_0 - u_0} \int_{\rho_-}^{\rho_0} \frac{1}{2\omega} \mathrm{d}\rho, \quad \rho_0' = -\frac{\rho_0}{\omega_0 - u_0} \int_{\rho_-}^{\rho_0} \frac{1}{2\omega} \mathrm{d}\rho. \tag{4.13}$$

Therefore $\rho_0' > 0$. When $U_0 \in S_1(U_-)$, setting $\omega_\theta^2 = \frac{f_0 - f_-}{\rho_0 - \rho_-}$, we achieve $\rho_- < \rho_0$ and

$$u_- - u_0 = \frac{(\rho_0 - \rho_-)\omega_\theta}{\sqrt{\rho_- \rho_0}}, \quad u_0' = \frac{f_0'(\frac{1}{\rho_-} - \frac{1}{\rho_0}) + (f_0 - f_-)\frac{\rho_0'}{\rho_0^2}}{2(u_0 - u_-)}, \tag{4.14}$$

from (2.3). The speed of the shock wave can be rewritten as

$$\sigma = \frac{\rho_0 u_0 - \rho_- u_-}{\rho_0 - \rho_-} = u_0 + \rho_- \frac{u_0 - u_-}{\rho_0 - \rho_-} = u_0 - \sqrt{\frac{\rho_-}{\rho_0}} \omega_\theta.$$

Applying (4.14), we have

$$(u_0^2 - \sigma^2 - \omega_0^2)\rho_0' = \frac{\rho_0^2}{2}. \tag{4.15}$$

From (4.11), determined by

$$\frac{\underline{\omega_z}^2 - \underline{u_z}^2}{\underline{\rho_z}} \underline{\rho_z}' = \frac{\omega_z^2 - u_z^2}{\rho_z} \rho_z' + \rho_z - \underline{\rho_z} = \frac{1}{2\rho_z} \left(2(\omega_0^2 - u_0^2)\rho_0' + (\rho_0^2 + \rho_z^2 - 2\underline{\rho_z}\rho_z) \right),$$

$\underline{\rho_z}' < 0$ holds when $U_0 \in R_1(U_-)$. Since $\rho_0 \geq \underline{\rho_z}$ and $\rho_0' > 0$. When $U_0 \in S_1(U_-)$, it holds that

$$\begin{aligned} \frac{\underline{\omega_z}^2 - \underline{u_z}^2}{\underline{\rho_z}} \underline{\rho_z}' &= \frac{1}{2\rho_z} \left(-\frac{\omega_0^2 - u_0^2}{\omega_0^2 - u_0^2 + \sigma^2} \rho_0^2 + \rho_0^2 + \rho_z^2 - 2\underline{\rho_z}\rho_z \right) \\ &\geq \frac{1}{2\rho_z \omega_0^2} \left(-(\omega_0^2 - u_0^2)\rho_0^2 + \omega_0^2\rho_0^2 + \omega_0^2\rho_z^2 - 2\omega_0^2 \underline{\rho_z}\rho_z \right) \\ &= \frac{1}{2\rho_z \omega_0^2} \left(u_0^2\rho_0^2 + \omega_0^2\rho_z^2 - 2\omega_0^2\underline{\rho_z}\rho_z \right) > \frac{1}{2\rho_z\omega_0^2} 2(\rho_z - \underline{\rho_z})\rho_z\omega_0^2 > 0. \end{aligned}$$

In fact, it is clear that

$$\begin{aligned}\rho_0^2u_0^2-\rho_z^2\omega_0^2&=\frac{\kappa\rho_z}{\rho_0-\rho_z}(\rho_0^{\gamma+1}-\rho_z^{\gamma}\rho_0-\gamma\rho_0^{\gamma}\rho_z+\gamma\rho_0^{\gamma-1}\rho_z^2)\\&=\frac{\kappa\rho_z^{\gamma+1}\rho_0}{\rho_0-\rho_z}\left(\left(\frac{\rho_0}{\rho_z}\right)^{\gamma}-1-\gamma\left(\frac{\rho_0}{\rho_z}\right)^{\gamma-1}+\gamma\left(\frac{\rho_0}{\rho_z}\right)^{\gamma-2}\right)>0,\end{aligned}$$

for $\rho_0>\rho_z$. Thus, we also have $\underline{\rho_z}'<0$.

(ii) When $U_0\in R_1^-(U_-)$, it is apparent that for $k=0$

$$(\overline{\omega}_v^2-\overline{u}_v^2)\overline{\rho}_v'=(\overline{\omega}_0^2-\overline{u}_0^2)\overline{\rho}_0'+\frac{\overline{\rho}_0^2}{2}-\frac{\overline{\rho}_v^2}{2}=\frac{\overline{\rho}_0(\omega_0^2-u_0^2)}{\rho_0}\rho_0'+\rho_0\overline{\rho}_0-\overline{\rho}_0^2-\frac{\overline{\rho}_v^2}{2}>0,$$

followed by (4.12), (4.13) and the proof of Lemma 4.1. Therefore, $\overline{\rho}_v'(0)<0$.

When $U_0\in S_1^-(U_-)$, from (4.12) and (4.15), we have

$$\begin{aligned}(\overline{\omega}_v^2-\overline{u}_v^2)\overline{\rho}_v'&=\frac{\overline{\rho}_0\rho_0}{2}\left(2-\frac{\omega_0^2-u_0^2}{\omega_0^2-u_0^2+\sigma^2}\right)+\frac{-\overline{\rho}_0^2-\overline{\rho}_v^2}{2}\\&\geq\frac{1}{2\omega_0^2}\left((\omega_0^2+u_0^2)\rho_0\overline{\rho}_0-\omega_0^2\overline{\rho}_0^2-\omega_0^2\overline{\rho}_v^2\right).\end{aligned}$$

To declaim $\overline{\rho}_v'(0)<0$, we are about to prove that

$$\rho_0>\frac{\overline{\rho}_0^2+\overline{\rho}_v^2}{\overline{\rho}_0}\frac{\omega_0^2}{\omega_0^2+u_0^2},\tag{4.16}$$

if $\alpha\triangleq\frac{\overline{\rho}_0}{\overline{\rho}_v}\leq\alpha_0$, and

$$\rho_0\geq(\overline{\rho}_0+\frac{\overline{\rho}_v}{\gamma})\frac{\omega_0^2}{\omega_0^2+u_0^2},\tag{4.17}$$

if $\alpha>\alpha_0$, where $\alpha_0>1$ is the root of the equation $\alpha_0^2-2\alpha_0-1=0$. Inequality (4.17) implies (4.16) when $\alpha>\alpha_0$, since $\gamma<\alpha_0$.

For any the given state $U_0(u_0,\rho_0,a_-)$, with $u_0<0$, $u_0^2<\omega_0^2$, we define $\overline{U}_0(\overline{u}_0,\overline{\rho}_0,a)$ and $\overline{U}_v(\overline{u}_v,\overline{\rho}_v,a)=S_3^0(\overline{U}_0)$ for $a\geq a_-$. It is clear that when $a=a_-$ and $k=0$, it holds that

$$\rho_0>\frac{\rho_0^2+\overline{\rho}_v^2}{\rho_0}\frac{\omega_0^2}{\omega_0^2+u_0^2},$$

which is equivalence to

$$\rho_0^2u_0^2=\rho_0\overline{\rho}_v\frac{\kappa\rho_0^{\gamma}-\kappa\overline{\rho}_v^{\gamma}}{\rho_0-\overline{\rho}_v}>\overline{\rho}_v^2\kappa\gamma\rho_0^{\gamma-1}.$$

Since

$$1-(\frac{\overline{\rho}_v}{\rho_0})^{\gamma}>\gamma\frac{\overline{\rho}_v}{\rho_0}(1-\frac{\overline{\rho}_v}{\rho_0})$$

holds when $\rho_0>\overline{\rho}_v$. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{\overline{\rho}_0^2+\overline{\rho}_v^2}{\overline{\rho}_0}\right)&=\frac{(2\overline{\rho}_0^2-\overline{\rho}_0^2-\overline{\rho}_v^2)\frac{\mathrm{d}\overline{\rho}_0}{\mathrm{d}a}+2\overline{\rho}_0\overline{\rho}_v\frac{\mathrm{d}\overline{\rho}_v}{\mathrm{d}a}}{\overline{\rho}_0^2}=\frac{(\overline{\rho}_0^2-\overline{\rho}_v^2)\frac{\mathrm{d}\overline{\rho}_0}{\mathrm{d}a}+2\overline{\rho}_0\overline{\rho}_v\frac{\mathrm{d}\overline{\rho}_v}{\mathrm{d}a}}{\overline{\rho}_0^2}\\&<\frac{(2\overline{\rho}_0\overline{\rho}_v-(\overline{\rho}_0^2-\overline{\rho}_v^2))}{\overline{\rho}_0^2}\frac{\mathrm{d}\overline{\rho}_v}{\mathrm{d}a}=\frac{-(\alpha^2-2\alpha-1)}{\overline{\rho}_0^2\overline{\rho}_v^2}\frac{\mathrm{d}\overline{\rho}_v}{\mathrm{d}a}<0\end{aligned}$$

for $1 \le \alpha < \alpha_0$. So far, (4.16) has been proved.

When $\alpha \ge \alpha_0$, for $a = a_-$ and $k = 0$, it holds that $\overline{U}_0 = U_0, U_z = \underline{U_z} = \overline{U}_v$. Followed by

$$\rho_0 u_0^2 = \overline{\rho}_v \frac{\kappa\rho_0^\gamma - \kappa\overline{\rho}_v^\gamma}{\rho_0 - \overline{\rho}_v} > \overline{\rho}_v \kappa \rho_0^{\gamma-1},$$

we have

$$\rho_0 \ge (\overline{\rho}_0 + \frac{\overline{\rho}_v}{\gamma})\frac{\omega_0^2}{\omega_0^2 + u_0^2}.$$

Direct calculations lead to

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a}\left(\overline{\rho}_0 + \frac{\overline{\rho}_v}{\gamma}\right) &= \frac{\overline{\rho}_0\overline{u}_0^2}{a(\overline{\omega}_0^2 - \overline{u}_0^2)} + \frac{1}{\gamma}\frac{2\overline{\rho}_v\overline{u}_v^2 - \overline{\rho}_0\overline{u}_0^2}{a(\overline{\omega}_v^2 - \overline{u}_v^2)} \\
&= \frac{\overline{\rho}_0\overline{u}_0^2\kappa\overline{\rho}_v^{\gamma-1}}{a\gamma(\overline{\omega}_0^2 - \overline{u}_0^2)(\overline{\omega}_v^2 - \overline{u}_v^2)}\left(\gamma^2 - \gamma\alpha\frac{\alpha^\gamma - 1}{\alpha - 1} + (2\alpha - 1)(\gamma\alpha^{\gamma-1} - \frac{1}{\alpha}\frac{\alpha^\gamma - 1}{\alpha - 1})\right),
\end{aligned}$$

which has different sign with the auxiliary function

$$\begin{aligned}
M_2(\alpha) &= \gamma^2\alpha(\alpha-1) - \gamma\alpha^2(\alpha^\gamma - 1) + (2\alpha - 1)\gamma\alpha^\gamma(\alpha - 1) - (2\alpha - 1)(\alpha^\gamma - 1) \\
&= \gamma\alpha^{\gamma+2} - (3\gamma + 2)\alpha^{\gamma+1} + (\gamma + 1)\alpha^\gamma + (\gamma^2 + \gamma)\alpha^2 + (2 - \gamma^2)\alpha - 1.
\end{aligned}$$

For $\gamma \in (1, 2)$, the third derivative

$$\begin{aligned}
\frac{1}{\gamma+1}\frac{\mathrm{d}^3 M_2}{\mathrm{d}\alpha^3}(\alpha) &= \gamma\alpha^{\gamma-3}(\gamma(\gamma+2)\alpha^2 - (\gamma-1)(3\gamma+2)\alpha + (\gamma-1)(\gamma-2)) \\
&> \gamma\alpha^{\gamma-3}(\gamma^2 + 2\gamma - 3\gamma^2 + \gamma + 2 + \gamma^2 - 3\gamma + 2) = \gamma\alpha^{\gamma-3}(4 - \gamma^2) > 0
\end{aligned}$$

holds for any $\alpha > 1$. By using the equation $\alpha_0^2 - 2\alpha_0 - 1 = 0$, we have

$$\frac{1}{\gamma+1}\frac{\mathrm{d}^2 M_2}{\mathrm{d}\alpha^2}(\alpha) = (\gamma+2)\gamma\alpha^\gamma - (3\gamma+2)\gamma\alpha^{\gamma-1} + \gamma(\gamma-1)\alpha^{\gamma-2} + 2\gamma \ge \frac{1}{\gamma+1}\frac{\mathrm{d}^2 M_2}{\mathrm{d}\alpha^2}(\alpha_0) > 0.$$

Thus, we get

$$\frac{\mathrm{d}M_2}{\mathrm{d}\alpha}(\alpha) > \frac{\mathrm{d}M_2}{\mathrm{d}\alpha}(\alpha_0) = -(\gamma^2 + \gamma + 2)\alpha_0^\gamma + \gamma(2\gamma + 2)\alpha_0^{\gamma-1} + 2(\gamma+1)\gamma\alpha_0 + 2 - \gamma^2 + \gamma\alpha_0^{\gamma-1}.$$

It holds that

$$\frac{\mathrm{d}M_2}{\mathrm{d}\alpha}(\alpha_0) > (2\gamma^2 + 2\gamma - \frac{5}{2}\gamma^2 - \frac{5}{2}\gamma - 5 + 2\gamma^2 + 2\gamma)\alpha_0^{\gamma-1} > 0$$

if $\gamma^2 \ge 2$. Meanwhile

$$\frac{\mathrm{d}M_2}{\mathrm{d}\alpha}(\alpha_0) > (2\gamma^2 + 3\gamma - \frac{5}{2}\gamma^2 - \frac{5}{2}\gamma - 5 + 3\gamma^2 + 3\gamma)\alpha_0^{\gamma-1} = \frac{1}{2}(5\gamma^2 + 7\gamma - 10)\alpha_0^{\gamma-1} > 0$$

holds if $\gamma^2 < 2$. We finally achieve

$$\begin{aligned}
M_2(\alpha) &\ge M_2(\alpha_0) = -(\gamma+2)\alpha_0^{\gamma+1} + (2\gamma+1)\alpha_0^\gamma + (\gamma^2 + 2\gamma + 2)\alpha_0 + (\gamma^2 + \gamma - 1) \\
&> \alpha_0^\gamma(-\tfrac{1}{2}\gamma - 4) + (\gamma^2 + 2\gamma + 2)\alpha_0 + (\gamma^2 + \gamma - 1) \triangleq M_3(\gamma) > 0.
\end{aligned}$$

In fact, $M_3(\gamma)$ is concave in $(1, 2)$, $M_3(1) > 0$ and $M_3(2) = 0$. We complete the proof. □

Corollary 4.2. *Provided that when the initial data satisfy $U_- \in \Delta_+$ and $U_+ \in \Delta_+^5$, we choose Q_+^1 (resp., Q_+^2) as the unique solution of (1.1) with (1.2) for any $k \geq 0$. Lemma 4.1 and 4.2 imply that, there exists a $k_0 > 0$, such that, for any $k \in (0, k_0)$, we have:*
(i) $U_- \in \Delta_+$ and $U_+ \in \Delta_+^5$, if $U_- \in \Delta_+(0), U_+ \in \overline{\Sigma}_z^+(0) \cup \overline{\Gamma}_z(0)$. Then for (1.2), the unique solution of (1.1) is Q_+^1 (resp., Q_+^2), while the unique solution of (1.3) is $Q_+^1(0)$ (resp., $Q_+^2(0)$).
(ii) $U_- \in \Delta_+$ and $U_+ \in \Delta_+^2$, if $U_- \in \Delta_+(0), U_+ \in \underline{\Sigma}^+(0) \cup \underline{\Gamma}_z(0)$. Then for (1.2), the unique solution of (1.1) is Q_+^2, while the unique solution of (1.3) is $Q_+^1(0)$ (resp., $Q_+^2(0)$).

To discuss the behaviour of $W_3(\overline{Y})$ as $k \to 0$, we consider the Riemann initial data satisfying

$$U_- \in \Delta_+(0) \backslash \mathrm{III}(0), \quad U_+ \in W_3(\overline{Y})|_{k=0} \subset \Delta_+^2(0), \tag{4.18}$$

see Figure 4(a). Thus, the solution for (1.3) is structured in

$$Q_+^2(0) : R_1^-(U_-, Y(0)) \oplus W_2(Y(0), \overline{Y}(0)) \oplus W_3^+(\overline{Y}(0), U_+).$$

More precisely, the following equations hold,

$$\begin{cases} u_y(0) = u_- - \displaystyle\int_{\rho_-}^{\rho_y} \frac{\sqrt{\kappa\rho^{\gamma-1}}}{\rho} \mathrm{d}\rho, u_y(0) = \omega_y(0), \\ a_-\rho_y(0)u_y(0) = a_+\overline{\rho}_y(0)\overline{u}_y(0), h_y(0) = h_1(0). \end{cases} \tag{4.19}$$

Here, we use the fact that $Y(0) = D_+(0) = (u_y(0), \rho_y(0), a_-)$ and $\overline{Y}(0) = (\overline{u}_y(0), \overline{\rho}_y(0), a_+)$. That whether U_+ is located on Δ_+^2 or not depends on both U_- and U_+ for $k > 0$. Since as we will see, the value of $\frac{\mathrm{d}\overline{u}_y}{\mathrm{d}\overline{\rho}_y}|_{k=0}$ changes in

$$\left(-\infty, -\frac{\overline{u}_y}{\overline{\rho}_y}\right) \cup \left(\frac{\overline{u}_y(\overline{\omega}_y^2 - u_y^2)}{\overline{\rho}_y(u_y^2 - \overline{u}_y^2)}, +\infty\right).$$

By the definitions of Y and $\overline{Y}$, we have

$$u_y = u_- - \int_{\rho_-}^{\rho_y} \frac{\omega}{\rho} \mathrm{d}\rho = \omega_y, \quad a_-\rho_y u_y = a_+\overline{\rho}_y\overline{u}_y, \quad h_y = \overline{h}_y. \tag{4.20}$$

The equations follow that

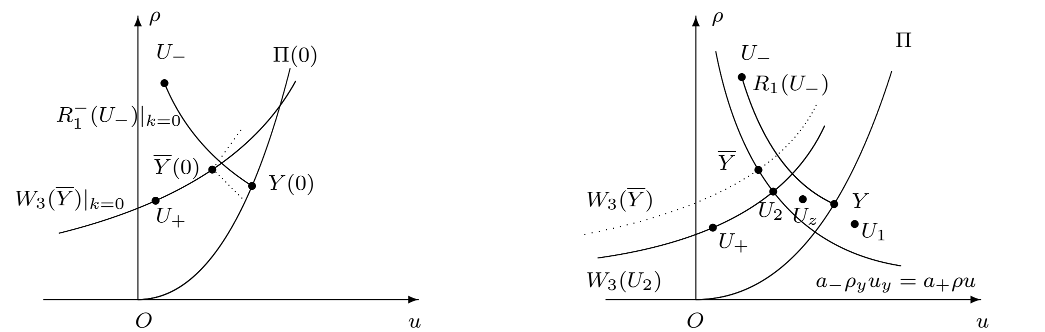
$$\begin{cases} \rho_y' = \dfrac{\omega_y \int_{\rho_y}^{\rho_-} \frac{1}{\omega}\mathrm{d}\rho - \rho_y}{\kappa\gamma\rho_y^{\gamma-2}(\gamma+1) + 3k}, \\ u_y' = \dfrac{\kappa\gamma\rho_y^{\gamma-1}(\gamma+3) + 5k\rho_y}{\kappa\gamma\rho_y^{\gamma-1}(\gamma+1) + 3k\rho_y} \displaystyle\int_{\rho_y}^{\rho_-} \frac{1}{2\omega}\mathrm{d}\rho - \frac{\omega_y}{\kappa\gamma\rho_y^{\gamma-2}(\gamma+1) + 3k}. \end{cases} \tag{4.21}$$

Direct calculations to (4.20) and (4.21) yield

$$\begin{cases} (\overline{\omega}_y^2 - \overline{u}_y^2)\overline{\rho}_y' = \overline{\rho}_y(\rho_y - \overline{\rho}_y) + \dfrac{\overline{\rho}_y}{u_y}(u_y^2 - \overline{u}_y^2) \displaystyle\int_{\rho_y}^{\rho_-} \frac{1}{2\omega}\mathrm{d}\rho, \\ \dfrac{\mathrm{d}\overline{u}_y}{\mathrm{d}\overline{\rho}_y} = \dfrac{\frac{\overline{u}_y\left(\overline{\omega}_y^2 - u_y^2\right)}{u_y} \int_{\rho_y}^{\rho_-} \frac{1}{2\omega}\mathrm{d}\rho + \overline{u}_y(\overline{\rho}_y - \rho_y)}{\frac{\overline{\rho}_y}{u_y}(u_y^2 - \overline{u}_y^2) \int_{\rho_y}^{\rho_-} \frac{1}{2\omega}\mathrm{d}\rho + \overline{\rho}_y(\rho_y - \overline{\rho}_y)}. \end{cases}$$

the value of $\frac{\mathrm{d}\overline{u}_y}{\mathrm{d}\overline{\rho}_y}$ depends on U_- for the fixed Y. It is no doubt that for k is sufficient small, $W_3(\overline{Y})$ is always at the right of $W_3(\overline{Y})|_{k=0}$, if $\overline{\rho}'_y \leq 0$. However, when $\overline{\rho}'_y > 0$, $W_3(\overline{Y})$(at least $S_3(\overline{Y})$) may be at the left of $W_3(\overline{Y})|_{k=0}$. Since the minimum value of $\frac{\mathrm{d}\overline{u}_y}{\mathrm{d}\overline{\rho}_y}$ satisfies that

$$\frac{\overline{u}_y(\overline{\omega}_y^2 - u_y^2)}{\overline{\rho}_y(u_y^2 - \overline{u}_y^2)} < \frac{\overline{u}_y}{\overline{\rho}_y} < \frac{\overline{\omega}_y}{\overline{\rho}_y}.$$



(a) $k = 0$, $Q_+^2(0) : R_1^- \oplus \overline{W}_2 \oplus W_3^+$. (b) $k > 0$, $Q_+^3 : R_1^- \oplus \underline{W_2} \oplus S_1 \oplus \overline{W}_2 \oplus W_3^+$.

FIGURE 4. The Riemann solution for (1.1) with (1.2) satisfying (4.18).

Likewise, when the initial data satisfy that

$$U_- \in \Delta_+(0), \quad U_+ \in W_3(Z)|_{k=0} \subset \Delta_+^4(0),$$

the solution of (1.1) with (1.2) may change from Q_+^3 to $Q_+^4(0)$ as $k \to 0$.

4.3. **The stability of the limit solution.** Even though that U_+ is located on either Δ_+^2 or Δ_+^3 can not be determined, when we discuss the limit solution of (1.1) with (1.2) satisfying (4.18), we have the following lemma.

Lemma 4.3. *As $k \to 0$, the limit solution of* (1.1) *equals to the solution of* (1.3)*, provided the initial data* (1.2) *satisfy the condition* (4.18)*.*

Proof. As an example, we now prove that the solution of (1.1) with (1.2)

$$Q_+^3 : \quad R_1^-(U_-, Y) \oplus W_2(Y, U_1) \oplus S_1(U_1, U_z) \oplus W_2(U_z, U_2) \oplus W_3^+(U_2, U_+),$$

tends to the solution of (1.3) with (1.2)

$$Q_+^2(0) : \quad R_1^-(U_-, Y(0)) \oplus W_2(Y(0), \overline{Y}(0)) \oplus S_3^+(\overline{Y}(0), U_+).$$

as $k \to 0$. The situation is that $U_- \in \Delta_+(0)$, $U_+ \in S_3(\overline{Y})|_{k=0}$, and for some small k_0, $U_+ \in \Delta_+^3$ if $k \in (0, k_0)$. More precisely, we have (4.19) and

$$\begin{cases} u_y = u_- - \displaystyle\int_{\rho_-}^{\rho_y} \frac{\omega}{\rho} \mathrm{d}\rho = \omega_y, \ u_+ = u_2 - \sqrt{\frac{(\kappa\rho_2^\gamma + \frac{k}{2}\rho_2^2 - \kappa\rho_+^\gamma - \frac{k}{2}\rho_+^2)(\rho_2 - \rho_+)}{\rho_2\rho_+}}, \\ a_-\rho_y u_y = a\rho_1 u_1 = a\rho_z u_z = a_+\rho_2 u_2, \ h_y = h_1, \ g_1 = g_z, \ h_z = h_2, \\ u_+ = \overline{u}_y(0) - \sqrt{\dfrac{(\kappa\overline{\rho}_y^\gamma(0) - \kappa\rho_+^\gamma)(\overline{\rho}_y(0) - \rho_+)}{\overline{\rho}_y(0)\rho_+}}, \ a \in (a_-, a_+), \end{cases} \tag{4.22}$$

see Figure 4. Our problem reduces to prove that $\rho_y \to \rho_y(0)$, $u_y \to u_y(0)$, $\rho_2 \to \rho_2(0) = \overline{\rho}_y(0)$, $u_2 \to u_2(0) = \overline{u}_y(0)$, $a \to a_-$ as $k \to 0$. To this end, we now show that $\rho_y'(0)$, $u_y'(0)$, $\rho_2'(0)$ and $u_2'(0)$ are finite. Direct calculations to (4.22) yield that

$$\begin{aligned} &a_+\rho_2' u_2 + a_+\rho_2 u_2' = a_-\rho_y' u_y + a_-\rho_y u_y' = -a_-\rho_y \int_{\rho_-}^{\rho_y} \frac{1}{2\omega}\mathrm{d}\rho, \\ &0 = u_2' + \rho_2'\left(\frac{\omega_2^2}{2(u_+ - u_2)}\frac{\rho_2 - \rho_+}{\rho_+\rho_2} + \frac{(u_+ - u_2)\rho_+}{2\rho_2(\rho_2 - \rho_+)}\right) + \frac{\rho_2^2 - \rho_+^2}{4(u_+ - u_2)}(\frac{1}{\rho_+} - \frac{1}{\rho_2}) \end{aligned}$$

where u_y' and ρ_y' are given by (4.21), resp., which are finite at $k = 0+$ for the given state $U_- \in \Delta_+$. Thus

$$\rho_2'\left(\frac{\omega_2^2}{2(u_+ - u_2)}\frac{\rho_2 - \rho_+}{\rho_+\rho_2} + \frac{(u_+ - u_2)\rho_+}{2\rho_2(\rho_2 - \rho_+)} - \frac{u_2}{\rho_2}\right) = \frac{a_-\rho_y}{a_+\rho_2}\int_{\rho_-}^{\rho_y} \frac{1}{2\omega}\mathrm{d}\rho - \frac{\rho_2^2 - \rho_+^2}{4(u_+ - u_2)}\frac{\rho_2 - \rho_+}{\rho_+\rho_2},$$

which implies that $\rho_2'(0)$ is finite, and so is $u_2'(0)$. Because the coefficient of $\rho_2'(0)$ is not greater than $-\frac{\omega_2 + u_2}{\rho_2} < 0$. Hence we have $h_y \to h_y(0)$ and $h_2 \to h_2(0)$ as $k \to 0$. From $h_y(0) = \overline{h}_y(0) = h_2(0)$, $h_1 = h_y$, we notice that $h_1 \to h_y(0)$ and $h_z \to h_2(0)$ as $k \to 0$. Associating with $g_1 = g_z$, ones obtain $\rho_1 \to \rho_z$, $u_1 \to u_z$ and $a \to a_-$ as $k \to 0$. For the other cases, the lemma can be obtained similarly. We complete the proof. □

A similar argument shows that, the limit solution for (1.1) as $k \to 0$ equals to the solution for (1.3), with (1.2) satisfying neither $U_+ \in \underline{\Sigma}(0)$ when $U_- \in \Delta_-(0)$ nor $U_+ \in \underline{\Sigma}^+(0) \cup \underline{\Gamma_z}(0)$ when $U_- \in \Delta_+(0)$, by Corollary 4.1 and Corollary 4.2. We achieve the following two theorems.

Theorem 4.3. *The solution for* (1.1) *with* (1.2) *is stable in a vanishing magnetic field, provided that, the unique solution is defined as* Q_-^2 *(resp.,* Q_+^2*), when* $U_- \in \Delta_-$ *and* $U_+ \in \Delta_-^3$ *(resp.,* $U_- \in \Delta_+$ *and* $U_+ \in \Delta_+^5$*).*

Theorem 4.4. *The solution for* (1.1) *with* (1.2) *is unstable in a vanishing magnetic field, provided that, the unique solution is defined as* Q_-^1 *(resp.,* Q_+^1*), when* $U_- \in \Delta_-$ *and* $U_+ \in \Delta_-^3$ *(resp.,* $U_- \in \Delta_+$ *and* $U_+ \in \Delta_+^5$*).*

Proof. When the initial data satisfy $U_- \in \Delta_-(0)$ and $U_+ \in \underline{\Sigma}(0)$, the solution for (1.3) is

$$Q_-^1(0): \quad W_1^-(U_-, U_2) \oplus R_3(U_2, D_2) \oplus \underline{W_2}(D_2, U_+),$$

where $D_2(u_{D_2}, \rho_{D_2}) \in \Sigma(0)$. By Corollary 4.2, we know that $U_+ \in \Delta_3$. Similar as we have done in Lemma 4.3, as $k \to 0$, the limit solution of (1.1) with (1.2) is

$$Q^2_-(0): \quad W_1^-(U_-, U_1) \oplus R_3(U_1, D_1) \oplus W_2(D_1, \overline{D}_1) \oplus S_1^+(\overline{D}_1, U_+),$$

where $D_1(u_{D_1}, \rho_{D_1}) \in \Sigma(0)$, $\overline{D}_1(u_{\overline{D}_1}, \rho_{\overline{D}_1}, a_+) \in \overline{\Sigma}(0)$. Figure 5 shows the two solutions in (u, ρ) plane, and Figure 6 shows them in (x, t) plane.

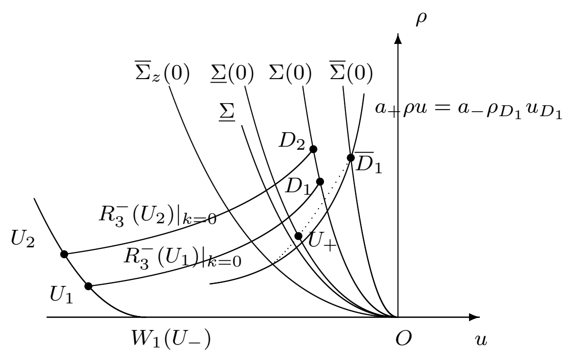
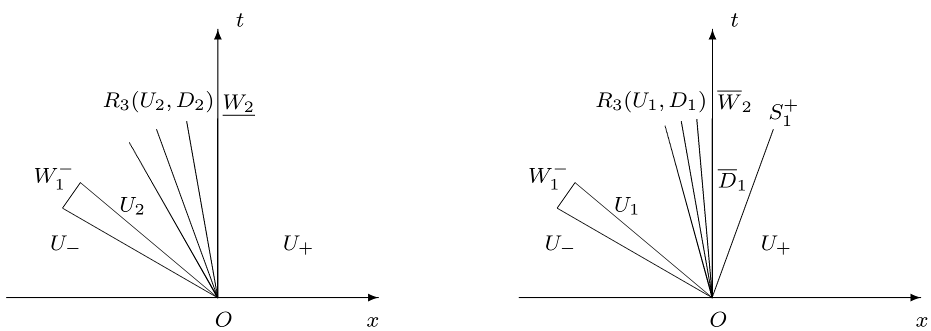


FIGURE 5. The Riemann solutions in (u, ρ) plane. The dotted line is $S_3^+(\overline{D}_1)|_{k=0}$.



a. Solution $Q^1_-(0)$. b. The limit solution $Q^2_-(0)$.

FIGURE 6. The Riemann solutions in (x, t) plane.

It holds that

$$a_+\rho_+u_+ = a_-\rho_{D_2}u_{D_2}, \quad h_+ = h_{D_2}, \quad \sigma(\overline{D}_1, U_+) = \frac{\rho_+u_+ - \rho_{\overline{D}_1}u_{\overline{D}_1}}{\rho_+ - \rho_{\overline{D}_1}} > 0.$$

In fact, by the definition of $\overline{\Sigma}_z$, the unique state $S_3^0(\overline{D}_1)$ is on the curve $\overline{\Sigma}_z(0)$, which is located on the left of $\underline{\Sigma}(0)$. Thus

$$a_-\rho_{D_2}u_{D_2} = a_+\rho_+u_+ < a_+\rho_{\overline{D}_1}u_{\overline{D}_1} = a_-\rho_{D_1}u_{D_1},$$

which imply that $\rho_{D_2} > \rho_{D_1}$ since both D_1 and D_2 are on the curve $\Sigma(0)$. It is clear that $\rho_2 > \rho_1$ and $\lambda_3(U_1) > \lambda_3(U_2)$. As far, we have proved that $Q^1_-(0)$ and $Q^2_-(0)$ are totally different. Likewise, the case where the initial data (1.2) satifying

$$U_- \in \Delta_+(0), \quad U_+ \in \underline{\Sigma}^+(0) \cup \underline{\Gamma_z}(0),$$

shows the solution for (1.1) is unstable in a vanishing magnetic field. We complete the proof. $\square$

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